# ON ELEMENTS OF FINITE ORDER IN FREE CENTRAL EXTENSIONS OF GROUPS

A THESIS SUBMITTED TO THE UNIVERSITY OF MANCHESTER INSTITUTE OF SCIENCE AND TECHNOLOGY

FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

IN THE MATHEMATICS

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# Contents

# Abstract

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**ABSTRACT OF THESIS** submitted by **Ali M. Sager** for the Degree of Doctor of Philosophy and entitled **On elements of finite order in free central extensions of groups**

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Let $F$ be a free group on $x_1, \cdots, x_d$. Consider the group $F/[\gamma_p(F'), F]F'''$ where $p$ is any prime number. A peculiar feature of this group is the occurrence of torsion in its center, this means that the torsion elements in this quotient form a subgroup of the abelian group $\gamma_p(F')F'''/[\gamma_p(F'), F]F'''$. In this thesis we give a complete description of this torsion subgroup in terms of generators and this is based on computing certain connecting homomorphisms. Furthermore this description can be exploited to obtain a complete description of the torsion subgroup of $F/[\gamma_p(F'), F]$ in terms of generators of $F$ as well. For the group $F/[\gamma_{p^n}(F'), F]F'''$ where $n \geq 2$, we describe just rank 4 torsion subgroups of this group in terms of generators of $F$.

# Declaration

No portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institution of learning.

# Dedication

TO MY PARENTS, WHO TAUGHT ME THE VALUE OF EDUCATION

# Acknowledgements

I would like to express my deep appreciation and gratitude to my supervisor Dr. R. Stöhr for his helpful advice and his useful comments, which made this work possible. I would like to thank all my family members and all my friends for their moral support. My thanks also go to the pure Mathematics group in both UMIST and Manchester University for their kindness and their friendly attitude. Finally, I would like to thank my Country Libya for the financial support, without which this work would never have been accomplished.

# Chapter 1

# Introduction

The purpose of this thesis is to investigate the elements of finite order in certain free central extensions of groups. To be more specific, we need to introduce some notions first.

Let $a$ and $b$ be elements of a group $G$; then the commutator $[a, b] = a^{-1}b^{-1}ab$. The left-normed commutator $[a_1, ...., a_n]$ is defined for $n > 2$ by setting $[a_1, \cdots, a_n] = [[a_1, \cdots, a_{n-1}], a_n]$. If $H$ and $K$ are subgroups of $G$, then $[H, K]$ is the subgroup generated by all $[h, k]$ with $h$ in $H$ and $k$ in $K$. In particular, the commutator subgroup or derived group of $G$ is $G' = [G, G]$. The lower central series of $G$ is the chain of its normal subgroups

$$G = \gamma_1(G) \geq \gamma_2(G) \geq \cdots \geq \gamma_i(G) \geq \gamma_{i+1}(G) \geq \cdots$$

where $\gamma_{i+1}(G) = [\gamma_i(G), G]$. The conjugate of $a$ by $x$ is $a^x = x^{-1}ax$.

Let $F$ be the free group on $X = \{x_1, x_2, \cdots, x_d\}$, $N \trianglelefteq F$ a normal subgroup of $F$, the subgroup $N''$ is the second term of the derived series of $N$ and $\gamma_c N$ is the $c$–term of the lower central series of the subgroup $N$ ($c \geq 2$).

Consider the quotient

$$F/[\gamma_c(N),\, F]N'' \tag{1.1}$$

Then we have an exact sequence of groups

$$1 \longrightarrow \gamma_c(N)N''/[\gamma_c(N),\, F]N'' \longrightarrow F/[\gamma_c(N),\, F]N'' \longrightarrow F/\gamma_c(N)N'' \longrightarrow 1. \tag{1.2}$$

Since $\gamma_c(N)N''/[\gamma_c(N),\, F]N''$ is in the centre of (1.1), our quotient is a central extension of $F/\gamma_c(N)N''$.

For the last twenty three years such groups have been extensively studied by many authors such as C.K.Gupta, Kuz'min, Stöhr and others.

A peculiar feature of this group is the occurrence of torsion in its center. This phenomenon was studied in some detail in [2] , [13] and [14]. The original motivation for these and other related investigations came from C.K. Gupta's pioneering work on the free centre–by– metabelian groups. Indeed, if $c = 2$ and $N = F'$, the quotient (1.1) turns into $F/[F'',\, F]$, the free center–by–metabelian group of rank $d$, and the exact sequence (1.2) turns into

$$1 \longrightarrow F''/[F'',\, F] \longrightarrow F/[F'',\, F] \longrightarrow F/F'' \longrightarrow 1.$$

In 1973 C.K. Gupta [1] proved that this group is torsion–free for $d = 2$ and $d = 3$, and she discovered that if $d \geq 4$, then $F/[F'',\, F]$ contains an elementary abelian 2–group of rank $C_4^d$ ($d$ choose 4) in its centre. Moreover, she proved that the elements

$$\begin{gathered}
\big[[x_{i_1}, x_{i_2}], [x_{i_3}^{-1}, x_{i_4}^{-1}]\big]\ \big[[x_{i_3}, x_{i_4}], [x_{i_1}^{-1}, x_{i_2}^{-1}]\big] \\
\big[[x_{i_1}, x_{i_3}], [x_{i_4}^{-1}, x_{i_2}^{-1}]\big]\ \big[[x_{i_4}, x_{i_2}], [x_{i_1}^{-1}, x_{i_3}^{-1}]\big] \\
\big[[x_{i_1}, x_{i_4}], [x_{i_2}^{-1}, x_{i_3}^{-1}]\big]\ \big[[x_{i_2}, x_{i_3}], [x_{i_1}^{-1}, x_{i_4}^{-1}]\big]
\end{gathered}$$

where ($1 \leq i_1 < i_2 < i_3 < i_4 \leq d$) form a basis for this torsion subgroup. This remarkable and at the time surprising result initiated a series of investigations about torsion in free central extensions (see , [2], [3], [4], [5], [6], [7], [8], [9], [10], [14] and [16] ). While Gupta's proof was purely group–theoretic, Kuz'min [4] introduced homological methods for discussing this torsion subgroup, and the later papers on this subject make extensive use of homological methods. Now when $N = F'$, the quotient (1.1) turns into

$$F/[\gamma_c(F'), F]F'''. \tag{1.3}$$

It was pointed out, in ([2], Theorem 7.1) that the order of any torsion element in $F/[\gamma_c(F'), F]F'''$ divides $c$ if $c$ is odd and $2c$ if $c$ is even. Of course this does not answer the question of whether or not there are any torsion elements in (1.3). Now, let $c = p$,where $p$ is a prime. In this case, Hannebauer and Stöhr showed in [2] that the group (1.3) is torsion–free for $d = 2$ and $d = 3$, and if $d \geq 4$, then $F/[\gamma_p(F'), F]F'''$ contains an elementary abelian $p$–group of rank $C_4^d$ in its centre, and the quotient of $F/[\gamma_p(F'), F]F'''$ by its torsion subgroup is torsion free. In fact, Hannebauer and Stöhr proved that the torsion subgroup of $F/[\gamma_p(F'), F]F'''$ can be identified with $H_4(F/F', \mathbf{Z}_p)$, the fourth homology group of $F/F'$ with coefficients in $\mathbf{Z}_p = \mathbf{Z}/p\mathbf{Z}$, that is

$$t(F/[\gamma_p(F'), F]F''') \cong H_4(F/F', \mathbf{Z}_p). \tag{1.4}$$

On the other hand, it has been shown in [13], that,

$$t(F/[\gamma_4(F'), F]F''') \cong H_4(F/F', \mathbf{Z}_2) \oplus H_6(F/F', \mathbf{Z}_4) \oplus H_7(F/F', \mathbf{Z}_2) \qquad (1.5)$$

Later on, in [14], when $c = p^\alpha$, $\alpha > 1$, $p$ is any prime and $F$ is of rank 4, Stöhr showed that

$$t_p(F/[\gamma_{p^\alpha}(F'), F]F''') \cong H_4(F/F', \mathbf{Z}_p). \qquad (1.6)$$

It should be pointed out that the results 1.4, 1.5 and 1.6 are special cases of more general results which have been shown in [2], [13], and [14]. Now, these results provide a description of the torsion subgroups as an abstract group in homological terms. On the other hand, the problem of describing these subgroups in group theoretic terms, [i.e. in terms of explicit generators], remained open except for the case $p = 2$ where Gupta's result applies. It would be desirable to have a complete description of these torsion subgroups in terms of generators.

The main aim of this thesis is to give a full description of the torsion subgroup of $F/[\gamma_p(F'), F]F'''$, and a partial description to the torsion subgroup of $F/[\gamma_{p^\alpha}(F'), F]F'''$ in terms of generators. The arrangement of this thesis is as follows.

In chapter 2, we introduce basic notions and some preliminary material required in this research, such as the concept of complexes, connecting homomorphisms, relation modules, symmetric powers. The homology of free abelian groups play an important role in this work, so in this chapter we compute the homology of the free abelian group with coefficients in the trivial $G$–module $\mathbf{Z}_p$, where $G$ is any free abelian group of a finite rank. In section 2.4, we recall the notions and concepts of Lie rings, metabelian Lie powers and we give some general results concerning metabelian Lie powers, we recall briefly some facts which explain the connection between the centre of $F/[\gamma_c(F'), F]F'''$ and the metabelian Lie powers of the relation module, and we also introduce two important chain complexes which play a crucial role in our computation. In the last section of this chapter we introduce some elementary facts about binomial coefficients, which will be used in our calculation. In the end of this section we briefly introduce the notion of localization at a prime $p$.

In chapter 3 we exhibit the generators for the torsion subgroup of $F/[\gamma_p(F'), F]F'''$ where $p$ is any prime. The main result reads as follows.

If $p$ is an odd prime, then the torsion subgroup of $F/[\gamma_p(F'), F]F'''$ is generated by $W_p(x_{\tau_1}, x_{\tau_2}, x_{\tau_3}, x_{\tau_4})$, where $W_p(x_{\tau_1}, x_{\tau_2}, x_{\tau_3}, x_{\tau_4})$ is given by:

$$\prod_{(i,j,k)\in I_1}\left\{\begin{array}{c}
[[x_{\tau_2}x_{\tau_1}x_{\tau_3}x_{\tau_4}],\ [x_{\tau_2}x_{\tau_1}x_{\tau_4}]^i,[x_{\tau_2}x_{\tau_1}]^j,\ [x_{\tau_2}x_{\tau_1}x_{\tau_3}]^k,\ [x_{\tau_2}x_{\tau_1}x_{\tau_3}x_{\tau_4}]^{\overline{p}-1}] \\
[[x_{\tau_2}x_{\tau_3}x_{\tau_1}x_{\tau_4}],\ [x_{\tau_3}x_{\tau_2}x_{\tau_4}]^i,[x_{\tau_3}x_{\tau_2}]^j,\ [x_{\tau_3}x_{\tau_2}x_{\tau_1}]^k,\ [x_{\tau_3}x_{\tau_2}x_{\tau_1}x_{\tau_4}]^{\overline{p}-1}] \\
[[x_{\tau_4}x_{\tau_3}x_{\tau_1}x_{\tau_2}],\ [x_{\tau_4}x_{\tau_3}x_{\tau_2}]^i,[x_{\tau_4}x_{\tau_3}]^j,\ [x_{\tau_4}x_{\tau_3}x_{\tau_1}]^k,\ [x_{\tau_4}x_{\tau_3}x_{\tau_1}x_{\tau_2}]^{\overline{p}-1}] \\
[[x_{\tau_1}x_{\tau_4}x_{\tau_3}x_{\tau_2}],\ [x_{\tau_1}x_{\tau_4}x_{\tau_2}]^i,[x_{\tau_1}x_{\tau_4}]^j,[x_{\tau_1}x_{\tau_4}x_{\tau_3}]^k,[x_{\tau_1}x_{\tau_4}x_{\tau_3}x_{\tau_2}]^{\overline{p}-1}] \\
[[x_{\tau_1}x_{\tau_3}x_{\tau_2}x_{\tau_4}],\ [x_{\tau_1}x_{\tau_3}x_{\tau_4}]^i,[x_{\tau_1}x_{\tau_3}]^j,[x_{\tau_1}x_{\tau_3}x_{\tau_2}]^k,[x_{\tau_1}x_{\tau_3}x_{\tau_2}x_{\tau_4}]^{\overline{p}-1}] \\
[[x_{\tau_2}x_{\tau_4}x_{\tau_1}x_{\tau_3}],\ [x_{\tau_2}x_{\tau_4}x_{\tau_3}]^i,[x_{\tau_2}x_{\tau_4}]^j,[x_{\tau_2}x_{\tau_4}x_{\tau_1}]^k,[x_{\tau_2}x_{\tau_4}x_{\tau_1}x_{\tau_3}]^{\overline{p}-1}]
\end{array}\right\}^{\gamma_1}$$

$$\prod_{(i,j,k)\in I_2}\left\{\begin{array}{c}
[[x_{\tau_2}x_{\tau_1}x_{\tau_3}],\ [x_{\tau_2}x_{\tau_1}x_{\tau_3}x_{\tau_4}]^{\overline{p}},\ [x_{\tau_2}x_{\tau_1}x_{\tau_4}]^i,[x_{\tau_2}x_{\tau_1}]^j,\ [x_{\tau_2}x_{\tau_1}x_{\tau_3}]^{k-1}] \\
[[x_{\tau_3}x_{\tau_2}x_{\tau_1}],\ [x_{\tau_3}x_{\tau_2}x_{\tau_1}x_{\tau_4}]^{\overline{p}},\ [x_{\tau_3}x_{\tau_2}x_{\tau_4}]^i,[x_{\tau_3}x_{\tau_2}]^j,\ [x_{\tau_3}x_{\tau_2}x_{\tau_1}]^{k-1}] \\
[[x_{\tau_4}x_{\tau_3}x_{\tau_1}],\ [x_{\tau_4}x_{\tau_3}x_{\tau_1}x_{\tau_2}]^{\overline{p}},\ [x_{\tau_4}x_{\tau_3}x_{\tau_2}]^i,[x_{\tau_4}x_{\tau_3}]^j,\ [x_{\tau_4}x_{\tau_3}x_{\tau_1}]^{k-1}] \\
[[x_{\tau_1}x_{\tau_4}x_{\tau_3}],\ [x_{\tau_1}x_{\tau_4}x_{\tau_3}x_{\tau_2}]^{\overline{p}},\ [x_{\tau_1}x_{\tau_4}x_{\tau_2}]^i,[x_{\tau_1}x_{\tau_4}]^j,\ [x_{\tau_1}x_{\tau_4}x_{\tau_3}]^{k-1}] \\
[[x_{\tau_1}x_{\tau_3}x_{\tau_2}],\ [x_{\tau_1}x_{\tau_3}x_{\tau_2}x_{\tau_4}]^{\overline{p}},\ [x_{\tau_1}x_{\tau_3}x_{\tau_4}]^i,[x_{\tau_1}x_{\tau_3}]^j,\ [x_{\tau_1}x_{\tau_3}x_{\tau_2}]^{k-1}] \\
[[x_{\tau_2}x_{\tau_4}x_{\tau_1}],\ [x_{\tau_2}x_{\tau_4}x_{\tau_1}x_{\tau_3}]^{\overline{p}},\ [x_{\tau_2}x_{\tau_4}x_{\tau_3}]^i,[x_{\tau_2}x_{\tau_4}]^j,\ [x_{\tau_2}x_{\tau_4}x_{\tau_1}]^{k-1}]
\end{array}\right\}^{\gamma_2}$$

$$\prod_{(i,j,k)\in I_3}\left\{\begin{array}{c}
[[x_{\tau_2}x_{\tau_1}x_{\tau_3}x_{\tau_4}],\ [x_{\tau_1}x_{\tau_3}x_{\tau_2}x_{\tau_4}]^k,[x_{\tau_1}x_{\tau_3}x_{\tau_2}]^{\overline{p}},[x_{\tau_2}x_{\tau_1}x_{\tau_3}x_{\tau_4}]^{i-1},[x_{\tau_2}x_{\tau_1}x_{\tau_3}]^j] \\
[[x_{\tau_2}x_{\tau_3}x_{\tau_4}x_{\tau_1}],\ [x_{\tau_2}x_{\tau_4}x_{\tau_1}x_{\tau_3}]^k,[x_{\tau_2}x_{\tau_4}x_{\tau_3}]^{\overline{p}},[x_{\tau_3}x_{\tau_2}x_{\tau_1}x_{\tau_4}]^{i-1},[x_{\tau_3}x_{\tau_2}x_{\tau_4}]^j] \\
[[x_{\tau_4}x_{\tau_3}x_{\tau_1}x_{\tau_2}],\ [x_{\tau_3}x_{\tau_1}x_{\tau_4}x_{\tau_2}]^k,[x_{\tau_3}x_{\tau_1}x_{\tau_4}]^{\overline{p}},[x_{\tau_4}x_{\tau_3}x_{\tau_1}x_{\tau_2}]^{i-1},[x_{\tau_4}x_{\tau_3}x_{\tau_1}]^j] \\
[[x_{\tau_4}x_{\tau_1}x_{\tau_2}x_{\tau_3}],\ [x_{\tau_4}x_{\tau_2}x_{\tau_1}x_{\tau_3}]^k,[x_{\tau_4}x_{\tau_2}x_{\tau_1}]^{\overline{p}},[x_{\tau_1}x_{\tau_4}x_{\tau_3}x_{\tau_2}]^{i-1},[x_{\tau_1}x_{\tau_4}x_{\tau_2}]^j]
\end{array}\right\}^{\gamma_3}$$

$$\prod_{(i,j,k)\in I_4}\left\{\begin{array}{c}
[[x_{\tau_1}x_{\tau_3}x_{\tau_2}x_{\tau_4}],[x_{\tau_1}x_{\tau_3}x_{\tau_2}]^{\overline{p}},[x_{\tau_2}x_{\tau_1}x_{\tau_3}]^j,[x_{\tau_2}x_{\tau_1}x_{\tau_3}x_{\tau_4}]^i,\ [x_{\tau_1}x_{\tau_3}x_{\tau_2}x_{\tau_4}]^{k-1} \\
[[x_{\tau_4}x_{\tau_2}x_{\tau_1}x_{\tau_3}],[x_{\tau_2}x_{\tau_4}x_{\tau_3}]^{\overline{p}},[x_{\tau_3}x_{\tau_2}x_{\tau_4}]^j,[x_{\tau_3}x_{\tau_2}x_{\tau_4}x_{\tau_1}]^i,\ [x_{\tau_2}x_{\tau_4}x_{\tau_3}x_{\tau_1}]^{k-1} \\
[[x_{\tau_3}x_{\tau_1}x_{\tau_4}x_{\tau_2}],[x_{\tau_3}x_{\tau_1}x_{\tau_4}]^{\overline{p}},[x_{\tau_4}x_{\tau_3}x_{\tau_1}]^j,[x_{\tau_4}x_{\tau_3}x_{\tau_1}x_{\tau_2}]^i,\ [x_{\tau_3}x_{\tau_1}x_{\tau_4}x_{\tau_2}]^{k-1} \\
[[x_{\tau_2}x_{\tau_4}x_{\tau_1}x_{\tau_3}],[x_{\tau_4}x_{\tau_2}x_{\tau_1}]^{\overline{p}},[x_{\tau_1}x_{\tau_4}x_{\tau_2}]^j,[x_{\tau_1}x_{\tau_4}x_{\tau_3}x_{\tau_2}]^i,\ [x_{\tau_4}x_{\tau_2}x_{\tau_1}x_{\tau_3}]^{k-1}
\end{array}\right\}^{\gamma_4}$$

$$\prod_{i=0}^{p-2}\prod_{k=-1}^{p-2-i}\left\{\begin{array}{c}
[[x_{\tau_2}x_{\tau_1}x_{\tau_3}x_{\tau_4}],\ [x_{\tau_4}x_{\tau_1}x_{\tau_3}x_{\tau_2}]^{k+1},\ [x_{\tau_1}x_{\tau_3}x_{\tau_2}x_{\tau_4}]^i,\ [x_{\tau_2}x_{\tau_1}x_{\tau_3}x_{\tau_4}]^{p-2-k-i}] \\
[[x_{\tau_1}x_{\tau_3}x_{\tau_2}x_{\tau_4}],\ [x_{\tau_4}x_{\tau_1}x_{\tau_3}x_{\tau_2}]^{p-1-i},\ [x_{\tau_1}x_{\tau_3}x_{\tau_2}x_{\tau_4}]^i]
\end{array}\right\}^{\omega}$$

where

$$\gamma_1 = \tfrac{1}{p-j-k} C_{k+j}^{p-1} C_k^{k+j} C_{i-1}^{p-1-k-j}, \quad \gamma_2 = \tfrac{1}{p-j-i} C_{i+j}^{p-1} C_i^{i+j} C_{k-1}^{p-1-i-j}$$

$$\gamma_3 = \tfrac{1}{i+k} C_{i-1}^{p-1} C_{j+k}^{p-i} C_k^{j+k}, \quad \gamma_4 = \tfrac{1}{p-i-j} C_{i+j+k}^{p-1} C_{k+i}^{i+j+k} C_i^{k+i}; \quad \omega = \tfrac{1}{i+1} C_i^{p-1}$$

$$I_1 = \{(i,j,k) : i \neq 0\,,\ j+k \neq 0\ ,\ \ i+j+k \leq p-1\,\}$$

$$I_2 = \{(i,j,k) : k \neq 0\,,\ i+j \neq 0,\ j+k \neq p,\ \ i+j+k \leq p\,,\}$$

$$I_3 = \{(i,j,k) : i \neq 0\,,\ i+j+k \leq p\,,\ i+k \neq p\}$$

$$I_4 = \{(i,j,k) : k \neq 0,\ ,\ i+j \neq 0\,,\ \ i+j+k < p\}$$

$$1 \leq \tau_1 < \tau_2 < \tau_3 < \tau_4 \leq d\ ;\ \ \overline{p} = p-i-j-k\ \ .$$

Our proof makes use of the approach developed in [2], which will be outlined in the first section of chapter 3, where we also introduce some auxiliary results . In particular , we outline the proof of a special case of the main result of [2], which gives a homological description of the torsion subgroup of central extensions of type (1.1) where $G = F/N$ is $p$–torsion free. In fact, we simplify the original proof from [2] in this special case by replacing a spectral sequence argument with an elementary dimension shifting argument. This enables us to compute an isomorphism between the homology group and the torsion subgroup explicitly as a dimension shifting isomorphism, and hence to obtain the generators given in theorem 3.1.1. Also in this chapter we apply our method to give another proof of Gupta's result, (i.e. we compute the generators of the torsion subgroup of $(F/[F'', F])$). Furthermore, we prove that our result for $p = 2$ is consistent with Gupta's result (i.e. our elements generate the same group as her). In the end of this chapter, we discuss briefly an important application to our main result, namely our description of torsion can be exploited to obtain a complete description of the torsion subgroup of $F/[\gamma_p(F'), F]$ in terms of generators of $F$ as well, using homomorphisms introduced in [29].

In chapter 4 we give an alternative proof to the following result, ([14],Theorem 2).

Let $G$ be a $p$–torsion–free such that $H_s(G, \mathbf{Z}_p) = 0$ for all $s \geq 5$. Then

$$t(F/[\gamma_{p^\alpha}(N), F]N'') \cong H_4(G, \mathbf{Z}_p).$$

Again this provides a description of the torsion subgroup as an abstract group in homological terms. Moreover, in chapter 3, our main result was about describing the torsion subgroup of $F/[\gamma_p(F'), F]F'''$, where $p$ is any prime, in terms of generators. A similar result holds for the torsion subgroup of $F/[\gamma_{p^\alpha}(F'), F]F'''$, where $F$ is of rank 4. If the rank of $F$ is greater then 4, then any four of the free generators $x_1, \cdots, x_d$ generate a rank 4 subgroup of $F/[\gamma_{p^\alpha}(F'), F]F'''$, and hence the rank 4 torsion elements appear in all higher ranks. However, these elements form only a subgroup of the higher rank torsion subgroup, and (1.5), for example, implies that the rank 4 torsion elements form a proper subgroup of the rank 6 torsion subgroup, where $p = 2$ and $\alpha = 2$. As in chapter 3, we obtain the torsion elements by computing the isomorphism

$$H_4(F/F', \mathbf{Z}_p) \longrightarrow t(F/[\gamma_{p^\alpha}(F'), F]F'''),$$

explicitly. The proof of this result makes use of the approach developed in [14] as modified in the first section of this chapter. Here we consider two cases, the first when $p$ is any odd prime. In this case we find that the computation of the connecting homomorphism is from a certain stage onwards very similar to the first one [i.e. the one in chapter 3]. In the case where $p = 2$, the calculation is slightly different.

Throughout this thesis the notation and terminology are mostly standard and the reader is referred to the book of P. Hilton and U. Stammbach : A course in Homological Algebra.

A list of symbols used, and their meanings is provided below.

**Notation :**

| | | |
|---|---|---|
| $\exists$ | : | There exists |
| $\forall$ | : | For all |
| $\cong$ | : | Isomorphic to |
| s.e.s. | : | Short exact sequence |
| $\otimes$ | : | Tensor product |
| $\oplus$ | : | Direct sum |
| $\prod$ | : | Direct product |
| $\mathrm{Ker}\alpha$ | : | The kernel of the map $\alpha$ |
| $\mathrm{Im}\alpha$ | : | The image of $\alpha$ |
| $S_n$ | : | Symmetric group of degree $n$ |
| $tA$ | : | Torsion subgroup of $A$, where $A$ is any abelain group |
| $G' = [G, G]$ | : | Commutator subgroup of $G$ |
| $G_{ab}$ | : | $G/G'$ |
| $\gamma_c N$ | : | The $c$–th term of the lower central series of $N$; note that $\gamma_2(N) = [N, N] = N'$, the commutator subgroup of $N$ |
| $H_n(G, B)$ | : | The $n$-th homology of $G$ with coefficients in $B$ |
| $H_n(G)$ | : | The integral homology of $G$ |
| $\mathbf{Z}$ | : | The ring of integers |
| $\mathbf{Z}_n$ | : | $\mathbf{Z}/\mathbf{nZ}$ |
| $\mathbf{Z}G$ | : | Group ring of $G$ over $\mathbf{Z}$ |
| $IG$ | : | Augmentation ideal |

| | | |
|---|---|---|
| $\mathbf{R}$ | : | Integers localized at $p$ |
| $\mathbf{R}G$ | : | Group ring of $G$ with coefficients in $\mathbf{R}$ |
| $\Delta$ | : | Augmentation ideal of $\mathbf{R}G$ |
| $M$ | : | The localized relation module |
| $A^n$ | : | The n–th symmetric powers of the module $A$ |
| $Gr(N)$ | : | Associated graded group of a group $N$ |
| $[x_1, \cdots, x_n]$ | : | $= [[x_1, \cdots, x_{n-1}], x_n]$, a simple commutator of weight $n \geq 2$, where by convention $[x_1] = x_1$ and $[x_1, x_2]$ is the commutator of $x_1$ and $x_2$ |
| $\mathcal{L}A$ | : | The free Lie ring on $A$ |
| $\mathcal{M}A$ | : | The free metabelian Lie ring on $A$ |

Lastly, we explain the numbering system which we have used. Equations are numbered by chapter, e.g. equation 3.2, while Theorems, Lemmas, etc., are numbered by chapter and section, e.g., Theorem 3.2.1.

# Chapter 2

# Preliminaries and notations

Occasionally, well-known facts will be drawn from homological algebra without citing special references; these however can easily be found, e.g. in Hilton and Stammbach [17].

For convenience we shall write $B \otimes_G A$, $Tor_n^G$ for $B \otimes_{\mathbf{Z}G} A$, $Tor_n^{\mathbf{Z}G}$ respectively.

## 2.1 General notations and some basic facts

### 2.1.1 Complexes

Let $\mathbf{S}$ be a ring with 1, we begin with following definition.

**Definition 2.1.1** A chain complex $\mathbf{C}$ of $\mathbf{S}$-modules is a family $\{C_i : i \in \mathbf{Z}\}$ of $\mathbf{S}$-modules, together with $\mathbf{S}$-module maps $d_n : C_n \longrightarrow C_{n-1}$ such that each composite

$d_{n-1} \circ d_n : C_n \longrightarrow C_{n-2}$ is zero. The maps $d_n$ are called the differentials of $\mathbf{C}$ . The kernel of $d_n$ is the module of $n$–cycles of $\mathbf{C}$ . The image of $d_{n+1} : C_{n+1} \longrightarrow C_n$ is the module of $n$–boundaries of $\mathbf{C}$ .

From this definition it is to easy see that every exact sequence is a complex. On the other hand if $\mathcal{F}$ is any functor and $\mathbf{C}$ is a complex, then

$$\mathcal{F}(\mathbf{C}): \cdots \longrightarrow \mathcal{F}(C_n) \xrightarrow{\mathcal{F}d_n} \mathcal{F}(C_{n-1}) \longrightarrow \cdots$$

is also a complex. In particular, if $\mathbf{C}$ is an exact sequence, then $\mathcal{F}(\ \mathbf{C})$ is a complex.

Now if $\mathbf{C}$ and $\mathbf{C}'$ are complexes, a chain map $f : \mathbf{C} \longrightarrow \mathbf{C}'$ is a sequence of maps $C_n \longrightarrow C'_n$, for all $n \in \mathbf{Z}$, such that the following diagram commutes

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & C_{n+1} & \xrightarrow{d_{n+1}} & C_n & \xrightarrow{d_n} & C_{n-1} & \longrightarrow & \cdots \\ & & \downarrow f_{n+1} & & \downarrow f_n & & \downarrow f_{n-1} & & \\ \cdots & \longrightarrow & C'_{n+1} & \xrightarrow{d'_{n+1}} & C'_n & \xrightarrow{d'_n} & C'_{n-1} & \longrightarrow & \cdots \end{array}$$

**Remark 2.1.2:**

1. $0 \subseteq \mathrm{Im} d_{n+1} \subseteq \mathrm{Ker} d_n \subseteq C_n$.
2. For any $n$, the $n^{th}$ homology module of $\mathbf{C}$ is defined by $H_n(\mathbf{C}) = \mathrm{Ker} d_n / \mathrm{Im} d_{n+1}$.

Thus $\mathbf{C}$ is exact if and only if all the homology groups of $\mathbf{C}$ vanish. As $H_n$ is really a functor, we need to define its action on chain maps. If $f : \mathbf{C} \longrightarrow \mathbf{C}'$ is a chain map, we define

$$H_n(f) : H_n(\mathbf{C}) \longrightarrow H_n(\mathbf{C}')$$

by $z + \mathrm{Im} d_{n+1} \longmapsto f_n(z) + \mathrm{Im} d'_{n+1}$.

An S–module $P$ is projective if if it satisfies the following universal lifting property: Given a $\beta : P \longrightarrow C$ and $\alpha : B \longrightarrow C$ where $\alpha$ is a surjective, then there exist $\gamma : P \longrightarrow B$ such that $\gamma\alpha = \beta$.

$$\begin{array}{ccccc} & & P & & \\ & \swarrow & \downarrow \beta & & \\ B & \xrightarrow{\alpha} & C & \rightarrow & 0 \end{array}$$

**Definition 2.1.3.** A projective resolution of the left S–module $B$ is a complex,

$$\mathcal{P} : \cdots \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow 0$$

with the following properties:

1. $P_n$ is a projective module for all $n \geq 0$.
2. $H_i(\mathcal{P}) = 0 \ \forall i \geq 1$.
3. $H_0(\mathcal{P}) \cong B$.

If we include the module $B$ in the resolution in this case we will denote the resolution by $\underline{\mathcal{P}}$

$$\underline{\mathcal{P}} : \cdots \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow B \rightarrow 0$$

we notice that $\underline{\mathcal{P}}$ is an exact sequence.

Now we briefly recall how the abelian groups $Tor_n^{\mathbf{S}}(B, A)$ and $H_n(G, B)$ are defined and calculated (where $B$ is right S–module and $A$ is left S–module).

Given $\mathcal{T} = B \otimes_{\mathbf{S}}$ is a functor from the category of S–modules to the category of abelian groups, we now describe its left derived functors $Tor_n^{\mathbf{S}}(A,\, B)$ as follows.

First we choose any projective resolution of $A$ (this can be done, because it is well known that every module has a projective resolution)

$$\mathcal{P} : \cdots \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow 0.$$

By tensoring each module of $\mathcal{P}$ with $B$ over $\mathbf{S}$ and taking the natural induced maps we produce the following complex

$$B \otimes_{\mathbf{S}} \mathcal{P} : \cdots \longrightarrow B \otimes_{\mathbf{S}} P_n \longrightarrow \cdots \longrightarrow B \otimes_{\mathbf{S}} P_1 \longrightarrow B \otimes_{\mathbf{S}} P_0 \longrightarrow 0,$$

then we define $Tor_n^{\mathbf{S}}(B, A) = H_n(B \otimes_{\mathbf{S}} \underline{\mathbf{P}})$ =Ker$(d_n \otimes 1)$/Im$(d_{n+1} \otimes 1)$.

We notice that if $A$ is projective module, then $Tor_n^{\mathbf{S}}(B, A) = 0$ $\forall n \geq 1$ and for all modules $B$.

That is because, as $A$ is projective, it has resolution

$$\mathcal{P} : \cdots 0 \longrightarrow P_0 \longrightarrow 0$$

with $P_0 = A$ is a projective resolution of $A$, hence $Tor_n^{\mathbf{S}}(A, B) = 0$ $\forall n \geq 1$.

Let $G$ be any group written multiplicatively. As usual, $\mathbf{Z}G$ denotes the integral group ring of $G$. The underlying abelian group of $\mathbf{Z}G$ is a free $\mathbf{Z}$–module with $\mathbf{Z}$–basis $= \{1.g : g \in G\}$, where the rank $(\mathbf{Z}G) = |G|$. A module over $\mathbf{Z}G$ will be referred to simply as a $G$–module. The tensor product over $\mathbf{Z}$ will be denoted by $\otimes$ instead of $\otimes_{\mathbf{Z}}$. If $B, C$ are $G$–modules, the tensor product $B \otimes C$ can be endowed with a $G$–module structure by defining $(b \otimes c).g = bg \otimes cg$ $(b \in B,\ c \in C,\ g \in G)$. This type of action is called diagonal action. By forgetting the $G$–module structure of $B$, $B \otimes C$ becomes $G$–module, using only the structure of $C$, by defining $(b \otimes c).g = b \otimes cg$ $(b \in B,\ c \in C,\ g \in G)$. This type of action is called single action.

**Remark 2.1.4**: The $\mathbf{Z}$–tensor product of $\mathbf{Z}$–free $G$–modules is a free $G$–module if at least one of the tensor factors is $G$–free.

A $G$–module $A$ is called trivial if $ga = a$, $\forall a \in A$, $\forall g \in G$. the ring of integers $\mathbf{Z}$ will always be regarded as a trivial $G$–module. By $\mathbf{Z}_p$ we denote the quotient $\mathbf{Z}/p\mathbf{Z}$, which is also viewed as a trivial $G$–module.

**Definition 2.1.5.** Let $B$ be a right $G$–module, then the n$^{\text{th}}$ homology group of $G$ with coefficients in $B$, denoted by $H_n(G, B)$, is defined as follows

Take any $G$–projective resolution of the trivial $G$–module $\mathbf{Z}$

$$\mathcal{P}: \cdots \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow 0,$$

by tensoring each module of $\mathcal{P}$ with $B$ over $\mathbf{Z}G$ and taking the natural induced maps we produce a complex of $\mathbf{Z}$–modules

$$B \otimes_G \mathcal{P}: \longrightarrow B \otimes_G P_n \longrightarrow B \otimes_G P_{n-1} \cdots \longrightarrow B \otimes_G P_0 \longrightarrow 0,$$

then we define the n$^{\text{th}}$ homology group of $G$ with coefficients in $B$ to be the abelian group $H_n(G, B) = H_n(B \otimes_G \mathcal{P})$. In fact this definition describes how to compute the homology of the groups via resolutions. If $B = \mathbf{Z}$ we get the integral homology, which is denoted by $H_n(G)$.

**Remark 2.1.6**

1. $H_n(G, B) = Tor_n^G(B, \mathbf{Z})$.

2. If $B$ is projective, then $H_n(G, B) = 0 \;\; \forall n \geq 1$.

3. If $B$ is a right $G$–module, then $H_0(G, B) = B \otimes_G \mathbf{Z}$.

### 2.1.2 Connecting homomorphisms

Since most of our work in chapters 3 and 4 involves computing rather complicated connecting homomorphisms, we need to give here a list of useful results concerning connecting homomorphism, so in chapter 3 will be able to develop or produce a method of computing complicated connecting homomorphisms.

**Theorem 2.1.7** : If $0 \to \mathbf{C}' \xrightarrow{\iota} \mathbf{C} \xrightarrow{\rho} \mathbf{C}'' \longrightarrow 0$ is an exact sequence of complexes, then for each $n$ there is a connecting homomorphism

$$\partial_n : H_n(\mathbf{C}'') \longrightarrow H_{n-1}(\mathbf{C}').$$

In fact, $\partial_n$ is computed as follows.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C'_n & \longrightarrow & C_n & \longrightarrow & C''_n & \longrightarrow & 0 \\
 & & \downarrow d & & \downarrow d & & \downarrow d & & \\
0 & \longrightarrow & C'_{n-1} & \longrightarrow & C_{n-1} & \longrightarrow & C''_{n-1} & \longrightarrow & 0
\end{array}$$

If $z''$ is a cycle in $C''_n$, choose an element $c_n \in C_n$ projecting onto $z''$; $(c_n)d$ is an element of $C_{n-1}$. By commutativity,

$$(c_n)d \in \mathrm{Ker}(C_{n-1} \longrightarrow C''_{n-1}) = \mathrm{Im}\iota.$$

By exactness of the bottom sequence, there is unique $c'_{n-1} \in C'_n$ with $(c'_{n-1})\iota = (c_n)d$. It follows that

$$z'' \longmapsto z''\rho^{-1}d\iota^{-1}.$$

The long exact homology sequence is of course linked to the connecting homomorpism, and we use this long exact homology sequence quite often, so it is worth while to mention it.

**Theorem 2.1.8** : If $0 \longrightarrow \mathbf{C}' \xrightarrow{\iota} \mathbf{C} \xrightarrow{\rho} \mathbf{C}'' \longrightarrow 0$ is an exact sequence of complexes, then there is an exact sequence of modules

$$\cdots \to H_n(\mathbf{C}') \xrightarrow{\iota_*} H_n(\mathbf{C}) \xrightarrow{\rho_*} H_n(\mathbf{C}'') \xrightarrow{\partial} H_{n-1}(\mathbf{C}') \xrightarrow{\iota_*} H_{n-1}(\mathbf{C}) \to \cdots.$$

**Lemma 2.1.9**:

If $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ is a short exact sequence of right **S**–modules, then for any left **S**–module $C$, there is a long exact sequence

$$.... \to Tor^{\mathbf{S}}_{n+1}(A'', C) \xrightarrow{\partial} Tor^{\mathbf{S}}_{n}(A', C) \longrightarrow Tor^{\mathbf{S}}_{n}(A, C) \longrightarrow Tor^{\mathbf{S}}_{n}(A'', C) \to .....$$
$$..... \to Tor^{\mathbf{S}}_{1}(A'', C) \longrightarrow A' \otimes_{\mathbf{S}} C \longrightarrow A \otimes_{\mathbf{S}} C \longrightarrow A'' \otimes_{\mathbf{S}} C \to 0.$$

we observe that the functor $(Tor)$ repairs the exactness we may have lost by tensoring.

**Remark 2.1.10**

(a). To any s.e.s. $0 \longrightarrow B' \longrightarrow B \longrightarrow B'' \longrightarrow 0$ of right $G$–modules, there is a long exact sequence,

$$.... \to H_{n+1}(G, B'') \xrightarrow{\partial} H_n(G, B') \longrightarrow H_n(G, B) \longrightarrow H_n(G, B'') \to ....$$
$$.... \to H_1(G, B'') \longrightarrow H_0(G, B') \longrightarrow H_0(G, B) \longrightarrow H_0(G, B'') \to 0.$$

(b). If $0 \longrightarrow B' \longrightarrow B \longrightarrow B'' \longrightarrow 0$ is a s.e.s. of right $G$–modules, where $B$ is projective, then

$$H_{n+1}(G, B'') \cong H_n(G, B') \ \forall n \geq 1.$$

The result just described is called dimension shifting.

(c). Let

$$0 \longrightarrow K \longrightarrow P_k \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \mathbf{Z} \longrightarrow 0$$

be an exact sequence of right $G$–modules, with $P_0, P_1, \cdots, P_k$ projective. Then the following sequence is exact: $0 \longrightarrow H_{k+1}(G, B) \longrightarrow K \otimes_G B \longrightarrow P_k \otimes_G B$.

(d). Let

$$\mathbf{K} : 0 \longrightarrow A \longrightarrow K_n \longrightarrow \cdots \longrightarrow K_1 \longrightarrow B \longrightarrow 0.$$

If $\mathbf{K}$ is exact and $H_k(G, K_i) = 0$ for $k \geq 1$ and $i = 1, 2, \cdots, n$; then $H_k(G, A) \cong H_{k+n}(G, B)$ for all $k \geq 1$, and the connecting homomorphism $H_n(G, B) \longrightarrow H_0(G, A)$ can be computed as follows:

Take projective resolution

$$\underline{\mathcal{P}} : \cdots \longrightarrow P_n \longrightarrow \cdots \rightarrow P_0 \longrightarrow \mathbf{Z} \longrightarrow 0,$$

of the trivial $G$–module $\mathbf{Z}$, then we form the double complex $\mathbf{K} \otimes_G \underline{\mathcal{P}}$:

$$
\begin{tikzcd}
 & & & & B \otimes_G P_{n-1} & B \otimes_G P_n \arrow[l] & {} \arrow[l] \\
 & & & & K_1 \otimes_G P_{n-1} & K_1 \otimes_G P_n \arrow[l] \arrow[u] & \\
 & & & {} & {} \arrow[l] \arrow[u] & & \\
 & & {} & {} \arrow[l] \arrow[u] & & & \\
 & K_n \otimes_G P_0 & \cdots \arrow[l] & & & & \\
 A \otimes_G \mathbf{Z} & A \otimes_G P_0 \arrow[l] \arrow[u] & & & & &
\end{tikzcd}
$$

We start with any cycle in $B \otimes_G P_n$ and then we have to go along the calligraphic arrows up to $A \otimes_G \mathbf{Z}$.

We conclude this subsection with two questions concerning the complex $\mathbf{K}$.

Now suppose that $\mathbf{K}$ is not exact but the other condition remains unchanged.

Do we still have $H_k(G, A) \cong H_{k+n}(G, B)$ ?

Can we still compute the connecting homomorphism $H_n(G, B) \longrightarrow H_0(G, A)$ ?

Under some additional conditions these questions have a positive answer, as we will see in the beginning of chapter 3. Here we should mention that this kind of connecting homomorphism plays a crucial role in the proof of our main results.

## 2.2 Some specific modules and symmetric powers

### 2.2.1 Relation modules and augmentation ideal

For the material of this subsection we refer to [17].

The map $\varepsilon : \mathbf{Z}G \longrightarrow \mathbf{Z}$ defined by

$$\left(\sum_{g\in G} m_g g\right)\varepsilon = \sum_{g\in G} m_g$$

is called the augmentation map. This map is a ring homomorphism; its kernel is denoted by $IG$, and is called the augmentation ideal. The exact sequence

$$0 \longrightarrow \mathbf{I}G \longrightarrow \mathbf{Z}G \longrightarrow \mathbf{Z} \longrightarrow 0$$

will be referred to as the augmentation sequence.

The short exact sequence

$$1 \longrightarrow N \longrightarrow F \xrightarrow{\pi} G \longrightarrow 1, \tag{2.1}$$

where $N$ and $F$ are free groups is known as a free presentation for the group $G$.

Now suppose that $G$ is given by this presentation where $F$ is a free group with free basis $X = \{x_1, \cdots, x_n\}$. The augmentation ideal $IF$ turns out to be a free $F$-module on the set $\{x - 1 : x \in X\}$ (see e.g., [17], Theorem 5.5). The group ring $\mathbf{Z}G$ carries $F$-module structure via $\pi$.

Given such a presentation, the abelian group $N_{ab} = N/N'$ carries, by conjugation, the structure of an $F$–module. Since $N$ operates trivially, $N_{ab}$ may be regarded as a right $G$–module by defining $rN'.g = (x^{-1}rx)N'$ where $r \in N$ , $x \in F$ , $g \in G$ with $g = (x)\pi$, which is called the relation module of $G$ associated with (2.1). Now we turn to the action: this action is well defined, indeed, let $n \in N$, $g \in G$ and if $x, y \in F$ such that $(x)\pi = (y)\pi = g$, then

$$x^{-1}nx = y^{-1}ny(\mathrm{mod} N').$$

Now, the tensor product $IF \otimes_F \mathbf{Z}G$ becomes a right $G$–module by a single action. In fact the module $P = IF \otimes_F \mathbf{Z}G$ is a free $G$–module on $\{(x_i - 1) \otimes 1; x_i \in X\}$. For it is plain that

$$\begin{aligned} IF \otimes_F \mathbf{Z}G &\cong \oplus_{x \in X}(x-1)\mathbf{Z}F \otimes_F \mathbf{Z}G \\ &\cong \oplus_{x \in X}(x-1)(\mathbf{Z}F \otimes_F \mathbf{Z}G) \\ &\cong \oplus_{x \in X}(x-1)\mathbf{Z}G. \end{aligned}$$

Furthermore the module $P$ is contained in the exact $\mathbf{Z}$–split $G$–module sequence

$$0 \longrightarrow N_{ab} \xrightarrow{\mu} P \xrightarrow{\sigma} IG \longrightarrow 0, \tag{2.2}$$

which is usually called the relation sequence stemming from (2.1).

The embedding $\mu$ is given by $(nN')\mu = (n-1) \otimes 1$ for any $n \in N$.

Now, if $g \in G$, $n \in N$ and $f \in F$, where $g = (f)\pi$, then we have

$$[nN'.g]\mu = [f^{-1}nfN]\mu = f^{-1}nf - 1 \otimes 1 \tag{2.3}$$

as $(f-1)(f^{-1}nf-1)=0$ in $IF \otimes_F \mathbf{Z}G$, then from (2.3) we get,

$$\begin{aligned}
[nN'.g]\mu &= [f^{-1}nf - 1 + (f-1)(f^{-1}nf-1)] \otimes 1 \\
&= (n-1)f \otimes 1 \\
&= (n-1) \otimes (f)\pi \\
&= (n-1) \otimes g \\
&= [(n-1) \otimes 1].g \\
&= [(nN')\mu].g
\end{aligned}$$

So we have got the following well-known result, (see e.g., [[17], pp. 198–199, Theorem 6.3]).

__Lemma 2.2.1__ The map $\mu : N_{ab} \longrightarrow P$ given by $(nN')\mu = (n-1) \otimes 1$ is $G$–module embedding.

This embedding $\mu$ is called the Magnus embedding and the Lemma is due to Magnus.

Finally the map $\sigma : P \rightarrow IG$ is defined by $((x-1) \otimes 1)\sigma = (x)\pi - 1$, which is also compatible with the $G$–action.

For our applications we consider the following special case. We put $G = F/F'$. This means that $G$ is a free abelian group with free basis $\{b_1, b_2, \cdots, b_n\}$, where $b_i = x_i F'$, in this case the relation module is $F'/F''$. It is (as a $G$–module) generated by the commutators $[y_i, y_j]$, where $y_i = x_i F''$, $1 \leq i < j \leq n$ :

$$F'/F'' = \{[x_i, x_j]F'' : 1 \leq i < j \leq n\}.$$

For simplicity, we write as follows $F'/F'' = \{[x_i, x_j] : 1 \leq i < j \leq n\}$

On the other hand $F'/F''$ as a $\mathbf{Z}$–module is generated by

$$\{[x_i, x_j]g : 1 \leq i < j \leq n, \, g \in G\}.$$

Let us now write the Magnus embedding and the epimorphism $\sigma$ in a form convenient for our computation in chapters 3 and 4. The epimorphism $P \longrightarrow IG$ defined by $e_i \longrightarrow b_i - 1$, where $e_i = x_i - 1 \otimes 1$ are the free generators of $P$. For the Magnus embedding $\mu : F'/F'' \longrightarrow P$ will be as the following:

$$[x_i, x_j] \longrightarrow e_i(b_j - 1) - e_j(b_i - 1)$$

where $b_i$ are the free generators of the free abelian group $G$, and $e_i$ free generators of $P$. This is because

$$\begin{aligned}
(x_i^{-1}x_j^{-1}x_ix_j - 1) \otimes 1 &= [(x_i^{-1}x_j^{-1} - 1)x_ix_j + (x_ix_j - 1)] \otimes 1 \\
&= (x_i^{-1}x_j^{-1} - 1) \otimes b_ib_j + (x_ix_j - 1) \otimes 1 \\
&= [(x_i^{-1} - 1)x_j^{-1} + (x_j^{-1} - 1)] \otimes b_ib_j + [(x_i - 1)x_j + (x_j - 1)] \otimes 1 \\
&= (x_i^{-1} - 1) \otimes b_i + (x_j^{-1} - 1) \otimes b_ib_j + (x_i - 1) \otimes b_j + (x_j - 1) \otimes 1 \\
&= (1 - x_i) \otimes 1 + (1 - x_j) \otimes b_i + (x_i - 1) \otimes b_j + (x_j - 1) \otimes 1 \\
&= (x_i - 1) \otimes (b_j - 1) - (x_j - 1) \otimes (b_i - 1) \\
&= [(x_i - 1) \otimes 1](b_j - 1) - [(x_j - 1) \otimes 1](b_i - 1) \\
&= e_i(b_j - 1) - e_j(b_i - 1)
\end{aligned}$$

Before closing this subsection we recall the following basic results which are called the reduction theorems for homology (see e.g. [17], p 213 and p 214).

<u>**Theorem 2.2.2**</u> For $n \geq 2$ we have

$$H_n(G, B) \cong H_{n-1}(G, B \otimes IG),$$

where $B \otimes IG$ is $G$–module with diagonal action.

__Theorem 2.2.3.__ Let $G = F/N$ with $F$ free. For $n \geq 3$ and $B$ is any $G$–module, we have

$$H_n(G, B) \cong H_{n-2}(G, B \otimes N_{ab}).$$

### 2.2.2 Symmetric powers

In this subsection we record some well–known facts concerning symmetric powers.

For a free $\mathbf{Z}$–module $A$, we denote the $\mathrm{n}^{\mathrm{th}}$ tensor power of $A$ by $T^n A$. Now, the $\mathrm{n}^{\mathrm{th}}$ symmetric powers of $A$ is defined by

$$A^n = T^n A/\{a_1 \otimes \cdots \otimes a_n - a_{1\rho} \otimes \cdots \otimes a_{n\rho}\}$$

where $\rho$ ranges over all permutations of $\{1, 2, \cdots, n\}$ and $a_1, \cdots, a_n \in A$. In particular,

$$A^2 = T^2 A/\{a_1 \otimes a_2 - a_2 \otimes a_1\}.$$

For $a_1, \cdots, a_n \in A$ we write $a_1 \circ \cdots \circ a_n$ for the corresponding symmetric tensor in $A^n$. If $A$ is a $G$–module, $A^n$ will be regarded as a $G$–module with diagonal action.

We begin with the following facts from [2].

First the map:

$$a_1 \circ a_2 \circ \cdots \circ a_n \longrightarrow \sum_{i=1}^{n} a_i \otimes (a_1 \circ a_2 \circ \cdots \circ \hat{a_i} \circ \cdots \circ a_n) \tag{2.4}$$

extends to an embedding $\nu_n : A^n \longrightarrow A \otimes A^{n-1}$ of $G$–modules, where the $a_i$ are arbitrary elements of $A$ and the circumflex denotes that $a_i$ is omitted. We also have a projection, $\rho_n : A \otimes A^{n-1} \longrightarrow A^n$ given by

$$(a_1 \otimes (a_2 \circ a_3 \circ \cdots \circ a_n))\rho_n = a_1 \circ a_2 \circ \cdots \circ a_n \,. \tag{2.5}$$

It is clear that the composite of $\nu_n$ and $\rho_n$ is the n$^{\text{th}}$ multiple map on $A^n$ :

$$\nu_n \rho_n = n.$$

Now for our examination of the $G$-module $A^n$, we need to recall the following construction, from [14]. Let

$$0 \longrightarrow B \longrightarrow A \xrightarrow{\beta} D \longrightarrow 0$$

be a short exact sequence of $\mathbf{Z}$-free $G$-modules. We identify $B$ with its image in $A$. Let $K_l^n$ be the submodule of $A^n$ spanned by the elements

$$b_1 \circ \cdots b_{l+1} \circ a_{l+2} \circ \cdots a_n$$

where $b_1, \cdots, b_{l+1} \in B$, $a_{l+2}, \cdots, a_n \in A$ and $n > l \geq 0$. We put $K_{-1}^n = A^n$ and $K_n^n = 0$. The submodules $K_l^n$ form a chain

$$0 < K_{n-1}^n < K_{n-2}^n < \cdots < K_0^n < K_{-1}^n = A^n$$

in $A^n$ with quotients

$$K_{l-1}^n / K_l^n \cong B^l \otimes D^{n-l}$$

which will be referred to as the $(B, D)$-filtration of $A^n$.

In particular, $K_{n-1}^n$ is the canonical image of $B^n$ in $A^n$ and $K_0^n$ is the kernel of the canonical epimorphism $\beta^n : A^n \longrightarrow D^n$. Specifically, the relation sequence yields a finite filtration

$$0 < Y_{n-1}^n < Y_{n-2}^n < \cdots < Y_0^n < Y_{-1}^n = P^n$$

with quotients $Y_{l-1}^n/Y_l^n \cong (N_{ab})^l \otimes (IG)^{n-l}$, for the symmetric power of the free $G$–module $P$, [ we notice here $Y_0^n$ is the kernel of the epimorphism : $P^n \longrightarrow (IG)^n$].

**Theorem 2.2.4** ([11] , Corollary 3.13). Let $G$ be a group without $n$–torsion, $A$ is a free $G$–module. Then $A^n$ is a free $G$–module.

Also we need the following result.

**Lemma 2.2.5** [[19], Lemma 2.3]. Let $p$ be a prime, $A$ a free $\mathbf{S}G$–module. Suppose that $G$ is $p$–torsion free, and every prime $q \neq p$ is invertible in $\mathbf{S}$, where $\mathbf{S}$ is a commutative ring with 1. Then $H_k(G,\, D \otimes_S A^c) = 0$ for every $\mathbf{S}G$–module $D$, $k \geq 1$ and $c \geq 1$.

## 2.3 Homology of free abelian groups

The homology of free abelian groups is very important topic in this work. However the integral homology groups of free abelian groups are well–known (see e.g., [18], chapter 6). Nevertheless in this section we include the computation of these homology groups, because some details will be needed later. Most of the following material of this section is from [4] and [15].

To compute the homology of free abelian groups and also later on the connecting homomorphism $H_4(G,\, \mathbf{Z}) \longrightarrow t(\gamma_p NN''/\gamma_{p+1}NN'' \otimes_G \mathbf{Z})$, we need to recall the following construction of the free resolution of the trivial $G$–module $\mathbf{Z}$.

Let $G$ be free abelian group of rank $n$, with free generators $b_1, b_2, \cdots, b_n$. Let $\Lambda^k P$ be the $k$–fold exterior power of the free $G$–module $P$, where $e_1, e_2, \cdots, e_n$ are free generators of $P$. For $k > 1$, $\Lambda^k P$ is a free $G$–module of rank $C_k^n$, with free generators $e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}$ , $(1 \leq i_1 < \cdots < i_k \leq n)$ in particular, $\Lambda^n P$ is a free $G$–module of rank 1 with basis

$e_1 \wedge e_2 \wedge \cdots \wedge e_n$, we extend the definition of $\Lambda^k P$ to the cases $k = 0$, and $k = 1$ by setting $\Lambda^1 P = P$, and $\Lambda^0 P = \mathbf{Z}G$.

We define $d_k : \Lambda^k P \longrightarrow \Lambda^{k-1} P$ by

$$( e_{i_1} \wedge \cdots \wedge e_{i_k})d_k = \sum_{j=1}^{k} (-1)^{j-1}(b_{i_j} - 1)(e_{i_1} \wedge \cdots \wedge \hat{e_{i_j}} \cdots \wedge e_{i_k})$$

and $( e_j)d_1 = (b_j - 1)$, (here $\hat{e_{i_j}}$ indicates the omission of $e_{i_j}$ ). It is easy to see that $d_k d_{k-1} = 0$; therefore we obtain a complex of free modules,

$$0 \longrightarrow \Lambda^n P \xrightarrow{d_n} \Lambda^{n-1} P \xrightarrow{d_{n-1}} \cdots \cdot \Lambda^2 P \xrightarrow{d_2} P \xrightarrow{d_1} \mathbf{Z}G \longrightarrow 0. \qquad (2.6)$$

If we supplement the complex (2.6) by the homomorphism $\varepsilon : \mathbf{Z}G \longrightarrow \mathbf{Z}$ , we obtain a free resolution (the **Koszul complex**) of the trivial module $\mathbf{Z}$ :

$$\underline{\mathcal{P}} : 0 \rightarrow \Lambda^n P \xrightarrow{d_n} \Lambda^{n-1} P \xrightarrow{d_{n-1}} \cdots \cdot \Lambda^2 P \xrightarrow{d_2} P \xrightarrow{d_1} \mathbf{Z}G \longrightarrow \mathbf{Z} \rightarrow 0,$$

A proof of this fact can be found in [18]. To simplify the notation we put $\Lambda^i P = P_i$, $i = 1, 2, \cdots, n$, so $\underline{\mathcal{P}}$ becomes as

$$\underline{\mathcal{P}} : 0 \rightarrow P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \cdot P_2 \xrightarrow{d_2} P \xrightarrow{d_1} \mathbf{Z}G \longrightarrow \mathbf{Z} \rightarrow 0. \qquad (2.7)$$

By tensoring $\underline{\mathcal{P}}$ with $\mathbf{Z}$ over $G$ we get the complex

$$\mathbf{Z} \otimes_G \underline{\mathcal{P}} : 0 \rightarrow \mathbf{Z} \otimes_G P_n \xrightarrow{d_n} \cdots \cdot \mathbf{Z} \otimes_G P \xrightarrow{d_1} \mathbf{Z} \otimes_G \mathbf{Z}G \xrightarrow{d_0} \mathbf{Z} \otimes_G \mathbf{Z} \rightarrow 0$$

As an application of this we get,

$$H_k(G, \mathbf{Z}) = H_k(G, \mathbf{Z} \otimes_G \underline{P}).$$

On the other hand we note that, $(P_k)\partial_k \subseteq IG(P_{k-1})$ which implies the induced differentials on $\mathbf{Z} \otimes_G \underline{P}$ are all zero maps. Hence $H_k(G, \mathbf{Z}) = \mathbf{Z} \otimes_G P_k$ . Since $P_k$ is a free $G-$module of rank $C_k^n$, $H_k(G, \mathbf{Z})$ is a free abelian group of the same rank.

Likewise, by tensoring the complex $\underline{P}$ with $\mathbf{Z}_p$ over $G$ we get,

$$\mathbf{Z}_p \otimes_G \underline{P} : 0 \to \mathbf{Z}_p \otimes_G P_n \xrightarrow{d_n} \cdots \mathbf{Z}_p \otimes_G P \xrightarrow{d_1} \mathbf{Z}_p \otimes_G \mathbf{Z}G \xrightarrow{d_0} \mathbf{Z}_p \otimes_G \mathbf{Z} \to 0 \quad (2.8)$$

Again $\mathbf{Z}_p$ is a trivial $G$–module, so as before the induced differentials on $\mathbf{Z}_p \otimes_G \underline{P}$ are all zero maps. Hence $H_k(G, \mathbf{Z}_p) = \mathbf{Z}_p \otimes_G P_k$. Since $\mathbf{Z}_p \otimes_G P_k$ is an elementary abelian $p$–group of rank $C_k^n$, $H_k(G, \mathbf{Z}_p)$ is an elementary abelian $p$–group of the same rank. So we have the following

**Proposition 2.3.1** The homology group of the free abelian group $G$ with coefficients in the trivial $G$–module $\mathbf{Z}_p$ is an elementary abelian $p$–group of rank $C_k^n$ (i.e. $H_k(G, \mathbf{Z}_p)$ $\cong \mathbf{Z}_p^{C_k^n}$ ).

## 2.4 Metabelian Lie powers

Most of the material of this section are taken from [2]. For more general reference we refer to [21], [22] and [24].

### 2.4.1 Notation and some general results

In this subsection we shall collect some basic concepts and some general results concerning metabelian Lie powers.

Suppose $\mathcal{A}(X)$ is a free associative algebra on a set $X$ over $\mathbf{S}$ where $\mathbf{S}$ is a commutative ring with 1, and let $I$ be the two–sided ideal of $\mathcal{A}(X)$ generated by the elements of the form $aa$, $a \in \mathcal{A}(X)$ and $a(bc) + b(ca) + c(ab)$, where $a$, $b$, $c \in \mathcal{A}(X)$. The quotient algebra $\mathcal{A}(X)/I$ is called the free Lie algebra on $X$, and is denoted by $\mathcal{L}X$ .

**Remark:** Any a free Lie algebra over $\mathbf{Z}$ is called free Lie ring.

For an (additively written) free abelian group $A$, let $\mathcal{X}$ be a free $\mathbf{Z}$–basis on $A$. We write $\mathcal{LX}$ for the free Lie ring over $\mathbf{Z}$ on $\mathcal{X}$. The free abelian group $A$ can be identified with $\mathbf{Z}$–submodule of $\mathcal{LX}$ spanned by $\mathcal{X}$. We define $\mathcal{L}A = \mathcal{LX}$. Now $\mathcal{L}A$ has a $\mathbf{Z}$–module decomposition

$$\mathcal{L}A = \bigoplus_{n \geq 1} \mathcal{L}^n A$$

where $\mathcal{L}^n A$ is spanned by the left–normed Lie monomials $[x_{i_1}, \cdots, x_{i_n}]$ with $x_{i_1}, \cdots, x_{i_n} \in \mathcal{X}$.

Moreover, if $A$ carries the structure of a right $G$–module, then the $G$–action on $A$ extends uniquely to $G$–action on $\mathcal{L}A$ turning the Lie powers $\mathcal{L}^n A$ into $G$–modules, and the induced action is, in degree $n$, given by

$$[a_1, a_2, \cdots, a_n].g = [a_1 g, a_2 g, \cdots, a_n g] \ , \ g \in G \ , \ a_i \in A \, .$$

Now we turn our attention to another class of free Lie rings, consisting of soluble Lie rings of soluble length at most 2, called the free metabelian Lie ring $\mathcal{M}A$ on $A$ which is defined by

$$\mathcal{M}A = \mathcal{L}A/[[\mathcal{L}A, \mathcal{L}A], [\mathcal{L}A, \mathcal{L}A]].$$

(i.e. $\mathcal{M}A$ is isomorphic to $\mathcal{L}A$ factored out by its second derived ring). Like $\mathcal{L}A$ the Z–span of $\mathcal{X}$ in $\mathcal{M}A$ can be identified with $A$, and also the lie ring $\mathcal{M}A$ is a graded $G$–module $\mathcal{M}A = \bigoplus_{i \geq 1} \mathcal{M}^i A$, with

$$\mathcal{M}^n A = \mathcal{L}^n A/\mathcal{L}^n A \cap [[\mathcal{L}A, \mathcal{L}A], [\mathcal{L}A, \mathcal{L}A]].$$

It is known [see, e.g., [21]] that $\mathcal{M}^n A$ is generated by the left normed commutators

$$[a_1, a_2, \cdots, a_n], a_1, a_2, \cdots, a_n \in A,$$

and these commutators are subject to relations

(i). $[a_1, a_2, a_3, \cdots, a_n] = -[a_2, a_1, a_3, \cdots, a_n]$

(ii). $[a_1, a_2, a_3, \cdots, a_n] + [a_3, a_1, a_2, \cdots, a_n] + [a_2, a_3, a_1, \cdots, a_n] = 0$

(iii). $[a_1, \cdots, a_i, a_{i+1}, \cdots, a_n] = [a_1, \cdots, a_{i+1}, a_i, \cdots, a_n]$ , $3 \leq i \leq n-1$.

Moreover, if $\mathcal{X}$ is a totally ordered free Z–basis of $A$ then the left normed commutators

$$\{[a_1, a_2, \cdots, a_n], a_1, a_2, \cdots, a_n \in \mathcal{X}, a_1 > a_2 \leq a_3 \leq \cdots \leq a_n\}$$

form a free Z–basis for $\mathcal{M}^n A$. These facts can be found in the literature (for example in [21]).

The following result describes an embedding of $\mathcal{M}^n A$ into $A \otimes A^{n-1}$.

This embedding is essential since our results in chapter 3 and 4 depend at some stages on this embedding.

**Lemma 2.4.2.** [[2], Theorem 3.1]. Let $A$ be $\mathbf{Z}$–free $G$–module, $n \geq 2$. Then the map

$$[a_1, \cdots, a_n] \longrightarrow a_1 \otimes (a_2 \circ \cdots \circ a_n) - a_2 \otimes (a_1 \circ a_3 \circ \cdots \circ a_n)$$

extends to an embedding $\varphi_n : \mathcal{M}^n A \longrightarrow A \otimes A^{n-1}$.

As result of this Lemma we have the following corollary which is also from [2].

**Corollary 2.4.3.** There is a short exact sequence

$$0 \longrightarrow \mathcal{M}^n A \xrightarrow{\varphi_n} A \otimes A^{n-1} \xrightarrow{\rho_n} A^n \longrightarrow 0, \tag{2.9}$$

of $G$–modules.

**Theorem 2.4.4**. Let $G$ be a group without $n$–torsion, $A$ a free $G$–module. Then $\mathcal{M}^n A$ is a free $G$–module. (For the proof see e.g. [11], Theorem 3.11).

To say that a group $G$ has a finite exponent $m$ means that $g^m = 1$ for all $g \in G$. The next result,which is from [2], shows that each homology group $H_k(G, \mathcal{M}^n N_{ab})$ is a periodic abelian group of finite exponent, and gives an upper bound for the exponent.

**Theorem 2.4.5**. For any odd $n \geq 3$, $H_k(G, \mathcal{M}^n N_{ab})$ $k \geq 1$ is a periodic abelian group of finite exponent dividing $n$ and $H_0(G, \mathcal{M}^n N_{ab})$ is a direct sum of a free abelian group and a periodic abelian group of finite exponent dividing $n$, and for any even $n \geq 2$, $H_k(G, \mathcal{M}^n N_{ab})$ $k \geq 1$ is periodic abelian group of finite exponent dividing $2n$ and $H_0(G, \mathcal{M}^n N_{ab})$ is a direct sum of a free abelian group and a periodic abelian group of finite exponent dividing $2n$.

To obtain more precise information about the homology groups $H_k(G, \mathcal{M}^n N_{ab})$, we need to recall the following complex $\mathcal{M}$ from [2], which also plays a crucial role in our computations.

Let $n \geq 2$ be a fixed integer and let

$$\mathcal{M} : 0 \to \mathcal{M}^n N_{ab} \xrightarrow{\delta_5} P \otimes N_{ab}^{n-1} \xrightarrow{\delta_4} P^n \xrightarrow{\delta_3} \mathbf{Z}G^n \xrightarrow{\delta_2} \mathbf{Z}G \xrightarrow{\delta_1} \mathbf{Z}_n \xrightarrow{\delta_0} 0, \tag{2.10}$$

where the differentials are given as follows :

$\delta_1$ is the composite of the augmentation map $\varepsilon : \mathbf{Z}G \longrightarrow \mathbf{Z}$ and the projection $\mathbf{Z} \longrightarrow \mathbf{Z}_n$.

$\delta_2$ is the composite of $\vartheta_n : \mathbf{Z}G^n \longrightarrow \mathbf{Z}G \otimes \mathbf{Z}G^{n-1}$ and $1 \otimes \varepsilon^{n-1} : \mathbf{Z}G \otimes \mathbf{Z}G^{n-1} \longrightarrow \mathbf{Z}G$ $(\mathbf{Z}G \otimes \mathbf{Z}^{n-1} \cong \mathbf{Z}G)$; thus for $a_1, a_2, \cdots, a_p \in \mathbf{Z}G$, $(a_1 \circ \cdots \circ a_p)\delta_2 = \sum_{i=1}^{n} (\prod_{j \neq i} a_j \varepsilon) a_i$, in particular for $g_1, \cdots, g_p \in G$, $(g_1 \circ \cdots \circ g_n)\delta_2 = g_1 + g_2 + \cdots + g_n$.

$\delta_3$ is the composite of $\sigma^n : P^n \longrightarrow (IG)^n$ and the injection $(IG)^n \longrightarrow (\mathbf{Z}G)^n$.

$\delta_4$ is the composite of $1 \otimes \mu^{n-1} : P \otimes N_{ab}^{n-1} \longrightarrow P \otimes P^{n-1}$ and $\rho_n : P \otimes P^{n-1} \longrightarrow P^n$.

$\delta_5$ is the composite of $\varphi_n : \mathcal{M}^n N_{ab} \longrightarrow N_{ab} \otimes N_{ab}^{n-1}$ and $\mu \otimes 1 : N_{ab} \otimes N_{ab}^{n-1} \longrightarrow P \otimes N_{ab}^{n-1}$, where $\varphi_n$ is defined by

$$[m_1, m_2, \ldots, m_n] \longrightarrow m_1 \otimes (m_2 \circ \ldots \circ m_n) - m_2 \otimes (m_1 \circ m_3 \circ \ldots \circ m_n)$$

where $m_1, m_2, \ldots, m_n \in N_{ab}$.

Lemma 6.1 in [2], tells us that this complex is exact in dimensions 0, 1, 4 and 5 (i.e.

$H_0(\mathcal{M}) = H_1(\mathcal{M}) = H_4(\mathcal{M}) = H_5(\mathcal{M}) = 0)$.

We also need the following corollary.

**Lemma 2.4.6.** [[2], Corollary 6.2]. The homology $H_i(\mathcal{M})$ $(i = 0, 1, \cdots, 5)$ of the chain complex (2.10), has the following property:

$H_k(G,\, H_i(\mathcal{M}))$ $(k \geq 1)$ is a periodic abelian group of finite exponent dividing $((n-1)!)^q$ for some positive integer $q$.

To obtain a characterization of the torsion in the zero–dimensional homology group of $G$ with coefficients in $\mathcal{M}^n N_{ab}$, we focus in the case when $n$ is a prime or power of prime.

We can now state the main result of [2].

**Theorem 2.4.7.** Let $p$ be a prime, $\sigma : P \longrightarrow \mathbf{Z}G$ a homomorphism from the relation sequence and $\sigma^p : P^p \longrightarrow (\mathbf{Z}G)^p$ its $\mathrm{p^{th}}$ symmetric power. Then there is a long exact sequence

$$\cdots \to H_{k+1}(G,\, \mathcal{M}^p N_{ab}) \longrightarrow \mathrm{coker} H_{k+3}(\sigma^p) \longrightarrow H_{k+4}(G,\, \mathbf{Z}_p) \longrightarrow H_k(G,\, \mathcal{M}^p N_{ab})$$
$$\to \ker H_{k+1}(\sigma^p) \longrightarrow H_{k+3}(G,\, \mathbf{Z}_p) \longrightarrow H_{k-1}(G,\, \mathcal{M}^p N_{ab}) \longrightarrow \mathrm{coker} H_{k+1}(\sigma^p) \to \cdots$$

$(k = 2,\, 4,\, 6, \cdots)$ terminating at

$$\cdots \to \mathrm{coker} H_3(\sigma^p) \longrightarrow H_4(G,\, \mathbf{Z}_p) \longrightarrow tH_0(G,\, \mathcal{M}^p N_{ab}) \longrightarrow \ker H_1(\sigma^p) \to H_3(G,\, \mathbf{Z}_p).$$

If $G$ has no $p$–torsion, then $(\mathbf{Z}G)^p$ and $P^p$ are free $G$–modules. Consequently $H_k(G, (\mathbf{Z}G)^p) = H_k(G,\, P^p) = 0$ for $k \geq 1$. In particular, $\mathrm{coker} H_3(\sigma^p) = \ker H_1(\sigma^p) = 0$, and the exactness of the sequence gives $H_4(G,\, \mathbf{Z}_p) \cong tH_0(G,\, \mathcal{M}^p N_{ab})$.

Now, for our computation in chapter 4, we need to modify another complex $\overline{\mathcal{M}}$.

First, from ([19], Lemma 2.1 (iv)), we have a 4–term exact sequence

$$0 \to (IG)^{p^n} \longrightarrow (\mathbf{Z}G)^{p^n} \xrightarrow{\pi^{p^n}_{p^n-1}} (\mathbf{Z}G)^{p^n-1} \longrightarrow \mathrm{coker}\pi^{p^n}_{p^n-1} \to 0 \tag{2.11}$$

where $\pi^{p^n}_{p^n-1}$ is defined by

$$(\alpha_1 \circ \alpha_2 \circ \ldots . \circ \alpha_{p^n})\, \pi^{p^n}_{p^n-1} = \sum_{i=1}^{p^n} (\alpha_i \varepsilon)\alpha_1 \circ \ldots . \circ \hat{\alpha_i} \circ \ldots \circ \alpha_{p^n}$$

where $\alpha_1, \cdots \alpha_n \in \mathbf{Z}G$, the circumflex denotes that $\alpha_i$ is omitted, and $\varepsilon$ is the augmentation map $\mathbf{Z}G \longrightarrow \mathbf{Z}$. In particular,

$$\underbrace{(1 \circ 1 \circ \cdots \circ 1)}_{p^n} \pi^{p^n}_{p^n-1} \longrightarrow p^n \underbrace{(1 \circ 1 \circ \cdots \circ 1)}_{p^n-1}$$

$$((g_1 - 1) \circ \cdots \circ (g_l - 1) \circ 1 \circ \cdots \circ 1)\pi^{p^n}_{p^n-1} \to (p^n - l)(g_1 - 1) \circ \cdots \circ (g_l - 1) \circ 1 \circ \cdots \circ 1$$

for $l < p^n$, and for $l = p^n$ we get

$$((g_1 - 1) \circ \cdots \circ (g_{p^n} - 1)\pi^{p^n}_{p^n-1} = 0$$

Combining (2.10) and (2.11), we obtain the following complex $\overline{\mathcal{M}}$

$$\overline{\mathcal{M}} : 0 \to \mathcal{M}^{p^n} N_{ab} \xrightarrow{\delta_5} P \otimes N_{ab}^{p^n-1} \xrightarrow{\delta_4} P^{p^n} \xrightarrow{\delta_3} \mathbf{Z}G^{p^n} \xrightarrow{\pi^{p^n}_{p^n-1}} \mathbf{Z}G^{p^n-1} \to \mathrm{co\,ker}\,\pi^{p^n}_{p^n-1} \to 0. \tag{2.12}$$

where the differentials $\delta_5$, $\delta_4$, $\delta_3$ are as in (2.10) and $\pi^{p^n}_{p^n-1}$ as it is defined above.

### 2.4.2 Central series of subgroups and metabelian Lie powers

The whole point of this subsection is to relate Lie rings associated with certain descending series of subgroups of a free group to free Lie rings and free metabelian Lie rings.

Let $F$ be a free group on a set $X = \{x_1, x_2, \cdots x_n\}$ and $G$ is given by

$$1 \to N \to F \to G \to 1.$$

The subgroup $N$ is free by Schreier's theorem. We consider the lower central quotient $\gamma_c N/\gamma_{c+1} N$ of the free group $N$. This quotient (free abelian group) is generated by the commutators of weight $i$ (i.e. generated by $\{[a_1, a_2, \cdots a_i]\gamma_{i+1}(N) : a_i \in N\}$. On the other hand it carries, by conjugation, the structure of a $F$–module. Since $N$ operates trivially, it may be regarded as a right $G$–module by defining

$$[a_1, a_2, \cdots a_i]\gamma_{i+1}(N).b_i = [a_1, a_2, \cdots a_i]^{x_i}\gamma_{i+1}(N)$$

where $a_1, a_2, \cdots a_i \in N$, $b_i \in G$, $x_i \in F$. The modules $\gamma_i(N)/\gamma_{i+1}(N)$ are called the higher relation modules.

Now we consider the family $\{\gamma_n(N)/\gamma_{n+1}(N)\}_{n\geq 1}$. We introduce an additive abelian group of the form

$$\mathrm{Gr}(N) = \bigoplus_{i=1}^{\infty} \gamma_i(N)/\gamma_{i+1}(N)$$

which is an associated graded group. We will turn this graded group into a Lie ring by defining the commutator of two homogeneous elements $a\gamma_{i+1}(N) \in \gamma_i(N)/\gamma_{i+1}(N)$, $b\gamma_{j+1}(N) \in \gamma_j(N)/\gamma_{j+1}(N)$ as follows:

$$[a\gamma_{i+1}(N), b\gamma_{j+1}(N)] = [a, b]\gamma_{i+j+1}(N)$$

where $a \in \gamma_i(N)$, $b \in \gamma_j(N)$. Using the following relations

$$[ab, cd] = [a, d]^b[a, c]^{db}[b, d][b, c]^d$$

$$[[a, b], c^a]\,[[c, a], b^c]\,[[b, c], a^b] = 1$$

the conditions of the Lie ring can be verified. Moreover, the graded Lie ring associated with the subgroup $N$, is the free Lie ring on the free abelian group $N_{ab} = N/N'$ (see, e.g. [22], chapter 4).

Let $n_1, \cdots, n_c \in N$ and let $a_i = n_i N'$ $(i = 1, 2, \cdots, c)$ be their images in the relation module $N_{ab}$. For any $c = 1, 2, 3, \cdots$ we have

$$\gamma_c(N)/\gamma_{c+1}(N) \cong \mathcal{L}^c(N_{ab}),$$

as abelian groups, where the map is defined by

$$\theta : [n_1, \cdots, n_c]\gamma_{c+1}(N) \longrightarrow [a_1, \cdots, a_c]$$

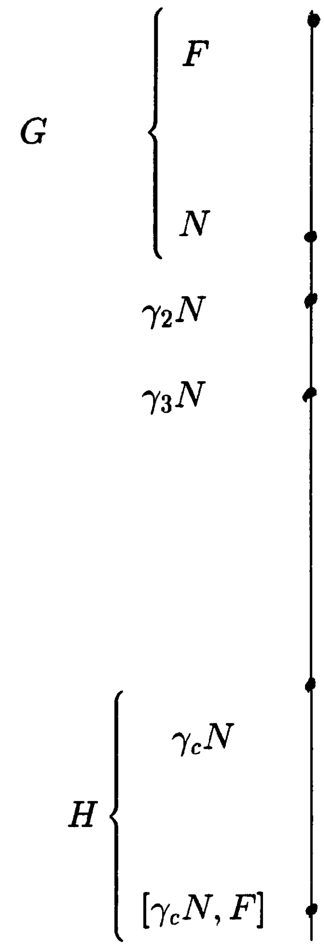
[for more details see, [22], chapter 4 and [21]].

**Proposition 2.4.8** (Baumslag, Strebel, Thomson [23]). The map

$$[n_1, \cdots, n_c][\gamma_c(N), F] \longrightarrow [a_1, \cdots, a_c] \otimes 1$$

extends to an isomorphism

$$\gamma_c(N)/[\gamma_c(N), F] \xrightarrow{\cong} \mathcal{L}^c(N_{ab}) \otimes_G \mathbf{Z}.$$




where $H = \gamma_c(N)/[\gamma_c(N), F]$.

Now we focus in the metabelian case, because our main results are connected with the metabelian Lie powers of the relation module.

Consider the quotient $F/[\gamma_c N, F]N''$. As was mentioned in the introduction, this quotient is characterized by the exact sequence

$$1 \longrightarrow \gamma_c(N)N''/[\gamma_c N, F]N'' \longrightarrow F/[\gamma_c N, F]N'' \longrightarrow F/\gamma_c(N)N'' \longrightarrow 1$$

i.e. $F/[\gamma_c N, F]N''$ is central extension of $F/\gamma_c(N)N''$.

First we consider the lower central quotient $\gamma_c(N)N''/\gamma_{c+1}(N)N''$ of the free metabelian group $N/N''$. This quotient is generated by the left normed commutators of weight $c$ (i.e. generated by $\{[a_1, a_2, \cdots a_c]\gamma_{c+1}(N)N'' : a_i \in N\}$. On the other hand it carries, by conjugation, the structure of a $F$–module. Since $N$ operates trivially, it may be regarded as a right $G$–module by defining

$$[a_1, a_2, \cdots a_c]\gamma_{i+1}(N)N''.b_i = [a_1, a_2, \cdots a_c]^{x_i}\gamma_{c+1}(N)N''$$

where $a_1, a_2, \cdots a_c \in N$, $b_i \in G$, $x_i \in F$.

Exactly as in the case of the higher relation module, we consider the following family $\{\gamma_c(N)N''/\gamma_{c+1}(N)N''\}_{c\geq 1}$. Again we introduce an additive abelian group of the form

$$\mathrm{Gr}(N/N'') = \bigoplus_{c=1}^{\infty} \gamma_c(N)N''/\gamma_{c+1}(N)N''$$

which is an associated graded group. As before we will turn this graded group into a Lie ring. Moreover, the graded metabelian Lie ring on the group $N/N''$, is the free metabelian Lie ring on the free metabelian group $N/N''$.

The next result is very important, so it may be justified to include the proof of this result in this chapter. Here we should mention that Hannebauer and Stöhr outline the proof of this result in ([2], section 7).

**Proposition 2.4.9**. For any natural number $n > 1$, there is a $G$–module isomorphism

$$\gamma_c(N)N''/[\gamma_c(N), F]N'' \cong \mathcal{M}^c(N_{ab}) \otimes_G \mathbf{Z}.$$

**Proof.** The normal subgroup $N$ is free by Schreier' theorem, then $N/N''$ is a free metabelian group. This implies that

$$\mathrm{Gr}(N/N'') = \oplus_{i=1}^{\infty} \gamma_i(N)N''/\gamma_{i+1}(N)N''$$

is graded Lie ring on the group $N/N''$. We notice that $\gamma_c(N)N''/\gamma_{c+1}(N)N''$ is the c–th homogeneous component of the graded Lie ring $\mathrm{Gr}(N/N'')$. By Theorem 3.2 of [27], there is an isomorphism between lower central quotients of free metabelian groups and metabelian lie rings

$$\gamma_c NN''/\gamma_{c+1}NN'' \cong \mathcal{M}^c N_{ab} \tag{2.13}$$

as abelian groups. On the other hand, as we mentioned before, conjugation in $F$ induces on $\gamma_c(N)N''/\gamma_{c+1}(N)N''$ the structure of $F/N$–module. Also $\mathcal{M}^c N_{ab}$ is $G$–module with diagonal action. Moreover, it is easy to see that this isomorphism is compatible with the $G$–action.

Trivializing the action on both sides in (2.13) gives

$$\mathcal{M}^c N_{ab} \otimes_G \mathbf{Z} \cong \gamma_c NN''/\gamma_{c+1}NN'' \otimes_G \mathbf{Z} \cong \frac{\gamma_c NN''/\gamma_{c+1}NN''}{(\gamma_c NN''/\gamma_{c+1}NN'').IG}. \tag{2.14}$$

For $m \in \gamma_c N$, and $(x)\pi = g \in G$ we consider

$$\begin{aligned} m\gamma_{c+1}NN''.(g-1) &= (m\gamma_{c+1}NN'').g\,(m\gamma_{c+1}NN'')^{-1} \\ &= x^{-1}m\,x\gamma_{c+1}NN''m^{-1}\gamma_{c+1}N\,N'' \\ &= x^{-1}mxm^{-1}\gamma_{c+1}NN'' \\ &= [x,\, m^{-1}]\gamma_{c+1}NN''. \end{aligned}$$

Therefore $(\gamma_c NN''/\gamma_{c+1}NN'').IG = [F,\, \gamma_c N]N''/\gamma_{c+1}NN'' = [\gamma_c N,\, F]N''/\gamma_{c+1}NN''$.

Thus we obtain

$$\frac{\gamma_c NN''/\gamma_{c+1}NN''}{(\gamma_c NN''/\gamma_{c+1}NN'').IG} \cong \frac{\gamma_c NN''/\gamma_{c+1}NN''}{[\gamma_c N, F]N''/\gamma_{c+1}NN''}$$
$$\cong \gamma_c NN''/[\gamma_c N, F]N'',$$

and by combining this with (2.14) we get

$$\gamma_c NN''/[\gamma_c N, F]N'' \cong \mathcal{M}^c N_{ab} \otimes_G \mathbf{Z} = H_0(G, \mathcal{M}^c N_{ab}).$$

## 2.5 Some other related topics

### 2.5.1 Binomial coefficients

We collect and introduce some results on binomial coefficients that are needed in our calculation. Our conventions are the standard ones: the binomial coefficient $C_k^n = 0$ if $n < 0$ or $k < 0$ or $n < k$.

**Lemma 2.5.1.** If $\alpha$, $\beta$, $\gamma$ are non–negative integers, where $\alpha \geq \gamma + \beta$ and $\beta \leq \alpha - \gamma$, then $C^{\alpha}_{\gamma+\beta} C^{\gamma+\beta}_{\beta} = C^{\alpha}_{\alpha-\gamma} C^{\alpha-\gamma}_{\beta}$

**Lemma 2.5.2.** Let $n$ be any positive integer. Then $\sum\limits_{i=0}^{n} (-1)^i C_i^n = 0.$

**Proof.** :

$$0 = (1-1)^n = C_0^n - C_1^n + C_2^n - C_3^n + \ldots\ldots (-1)^n C_n^n = \sum_{i=0}^{n} (-1)^i C_i^n.$$

Most of the following observations are easily proved by induction, using the following recursion.

Let $n$ be any natural number and $i$ is integer where $0 \leq i \leq n$ ,then

$$C_{i+1}^{n-1} + C_i^{n-1} = C_{i+1}^{n} \tag{2.15}$$

**Lemma 2.5.3.** Let $j$ , $k$ , $n$ be fixed non–negative integers such that $n-2-j-k \geq 0$.

Then
$$\sum_{i=0}^{n-2-j-k} (-1)^{j+k+i} C_j^{n-2-k-i} C_i^{n-1-k} = (-1)^n.$$

**Proof.** Induction on $n-2-j-k$. If $n-2-j-k=0$, then $\sum\limits_{i=0}^{0} (-1)^{j+k} = (-1)^{j+k} = (-1)^{n-2}$.

Induction step : $\sum\limits_{i=0}^{n-1-j-k} (-1)^{j+k+i} C_j^{n-1-k-i} C_i^{n-k}$

$$\begin{aligned}
&= \left\{ \begin{array}{c} (-1)^{j+k} C_j^{n-1-k} + (-1)^{j+k+1} C_j^{n-2-k} C_1^{n-k} + (-1)^{j+k+2} C_j^{n-3-k} C_2^{n-k} + \cdots \\ + (-1)^{n-2} C_j^{j+1} C_{j+2}^{n-k} + (-1)^{n-1} C_{j+1}^{n-k} \end{array} \right\} \\
&= \left\{ \begin{array}{c} (-1)^{j+k} C_j^{n-1-k} + (-1)^{j+k+1} C_j^{n-2-k} (C_1^{n-1-k} + C_0^{n-1-k}) \\ +(-1)^{j+k+2} C_j^{n-3-k} (C_2^{n-1-k} + C_1^{n-1-k}) + \cdots \\ + (-1)^{n-1} (C_{j+1}^{n-1-k} + C_j^{n-1-k}) \end{array} \right\} \\
&= \sum_{i=0}^{n-1-j-k} (-1)^{j+k+i} C_j^{n-1-k-i} C_i^{n-1-k} - \sum_{i=0}^{n-2-j-k} (-1)^{j+k+i} C_j^{n-2-k-i} C_i^{n-1-k} \\
&= \sum_{i=0}^{n-1-j-k} (-1)^{j+k+i} C_j^{n-1-k-i} C_i^{n-1-k} - (-1)^n \quad \text{by induction} \\
&= (-1)^{j+k} C_j^{n-1-k} \sum_{i=0}^{n-1-j-k} (-1)^i C_i^{n-1-j-k} - (-1)^n \\
&= 0 - (-1)^n \quad \text{by lemma 2.5.2.} \\
&= (-1)^{n+1}
\end{aligned}$$

This completes the inductive step and therefore the proof of the Lemma.

**Lemma 2.5.4:** Let $j$ , $k$ , $n$ be fixed non–negative integers, $n-2-j \geq 0$. Then

$$\sum_{i=k}^{n-2-j} (-1)^{j+i} C_i^{n-1} C_j^{n-2-i} C_k^i = (-1)^n C_k^{n-1}.$$

**Proof.** :

$$\begin{aligned}\sum_{i=k}^{n-2-j} (-1)^{j+i} C_i^{n-1} C_j^{n-2-i} C_k^i &= C_k^{n-1} \{ \sum_{i=0}^{n-2-j-k} (-1)^{k+j+i} C_j^{n-2-k-i} C_i^{n-1-k} \} \\ &= (-1)^n C_k^{n-1} \text{ by lemma 2.5.3.}\end{aligned}$$

**Lemma 2.5.5.** For a fixed natural number $n$, the integer $C_i^{2^n-1}$ is odd, for any $0 \leq i < 2^n - 1$.

**Proof.** The prove is by induction on $i$. It is obviously true for $i = 0, 1$. By induction hypothesis, we assume that the Lemma is true for $i = m$.

From (2.15) we have $C_{m+1}^{2^n-1} = C_{m+1}^{2^n} - C_m^{2^n-1}$, by induction the second term in the right hand side is odd, on the other hand the first term is even. Therefore $C_{m+1}^{2^n-1}$ is odd. This completes the induction.

### 2.5.2 Localization

Finally, in this subsection we introduce briefly the concept of localization.

By localization of the ring $\mathbf{Z}$ with respect to the prime $p$, we mean a subring of $\mathbf{Q}$ consisting of those rational numbers which can be written as fractions whose denominators are relatively prime to $p$, and we denote it by $\mathbf{R}$ . If $A$ is $G$–module its localization

$A \otimes \mathbf{R}$ may be regarded as $\mathbf{R}G$–module, where $\mathbf{R}G$ is the group ring of $G$ with coefficients in $\mathbf{R}$, and recall that tensoring with $\mathbf{R}$, that is localizing at $p$, is an exact functor on the category of abelian groups. Thus $H_k(G, A \otimes \mathbf{R}) \cong H_k(G, A) \otimes \mathbf{R}$ for any $G$–module $A$. From that we can deduce the following observation (see [19], lemma 2.4), but before that we need to introduce the following notation: for any abelian group $A$, we denote by $t_pA$ the subgroup of all those elements whose order divides some power of $p$.

**Lemma 2.5.6** $t_pH_k(G, A) \cong tH_k(G, A \otimes \mathbf{R})$ for any $G$–module $A$.

If $A$ is any abelian torsion group, then $A = \bigoplus_p A_p$ where $A_p$ is a subgroup of $A$ generated by elements of order $p^\alpha$. Then $A \otimes \mathbf{R} \cong A_p$, in particular if $A$ is of exponent $q$ where $(p, q) = 1$, then $A \otimes \mathbf{R} = 0$.

For these reasons we will work over $\mathbf{R}$ instead of $\mathbf{Z}$. The following notation will be used. By $\triangle$ we denote the augmentation ideal of $\mathbf{R}G$, and we write $M$ for the localized relation module (i.e. $M = N_{ab} \otimes R$).

Before closing this subsection, we should mention that the main source for this is ([25], chapter 8).

For convenience of our further discussions, we introduce the following notation:

$$[a, \underbrace{b, b, \cdots, b}_{n}] = [a, b^n].$$

# Chapter 3

# Torsion in free central extensions

For a group $G$ and a free presentation

$$1 \longrightarrow N \longrightarrow F \xrightarrow{\pi} G \longrightarrow 1$$

let $\gamma_c N$ denote the $c$–th term of the lower central series of $N$. Consider the free central extension

$$1 \longrightarrow \gamma_c N.N''/[\gamma_c N,\, F]N'' \longrightarrow F/[\gamma_c N,\, F]N'' \longrightarrow F/\gamma_c N N'' \longrightarrow 1. \qquad (3.1)$$

The group $F/\gamma_c N N''$ is torsion free for any normal subgroup $N$, see [28]. Hence, the torsion elements form a subgroup of the kernel of (3.1).

In this chapter will give a description of this torsion subgroup where $c = p$, $p$ a prime number. First we give a homological description to this torsion subgroup. Then for the case $N = F'$ we give an explicit description to this torsion subgroup, in group theoretic terms.

Now, by Proposition 2.4.9, we have

$$\gamma_p N N'' / [\gamma_p N, F] N'' \cong \mathcal{M}^p N_{ab} \otimes_G \mathbf{Z} = H_0(G, \mathcal{M}^p N_{ab}). \tag{3.2}$$

Thus, questions about elements of finite order in $F/[\gamma_p N, F]N''$ are equivalent to the questions about torsion in the zero-dimensional homology group of $G$ with coefficients in $\mathcal{M}^p N_{ab}$. So we have transformed our problem to the problem of describing the torsion subgroup of $H_0(G, \mathcal{M}^p N_{ab})$.

## 3.1 Description of $tH_0(G, \mathcal{M}^p M)$ in homological terms

To obtain a characterization of the torsion in the zero-dimensional homology group of $G$ with coefficients in $\mathcal{M}^p N_{ab}$ where $p$ is any prime, we use the complex $\mathcal{M}$, which is introduced in chapter 2,

$$\mathcal{M} : \mathcal{M}^p N_{ab} \xrightarrow{\delta_5} P \otimes N_{ab}^{p-1} \xrightarrow{\delta_4} P^p \xrightarrow{\delta_3} \mathbf{Z}G^p \xrightarrow{\delta_2} \mathbf{Z}G \xrightarrow{\delta_1} \mathbf{Z}_p \to 0. \tag{3.3}$$

First, let us remind ourselves of the two questions which were raised in chapter 2, about dimension shifting and computing connecting homomorphism.

Do we still have $H_k(G, \mathcal{M}^p N_{ab}) \cong H_{k+4}(G, \mathbf{Z}_p)$, $k \geq 1$? and can we still compute the connecting homomorphism $H_4(G, \mathbf{Z}_p) \longrightarrow \mathcal{M}^p N_{ab} \otimes_G \mathbf{Z}$ ?.

Now we state an easy version of the main result of [2].

**Theorem 3.1.1:** Let $p$ be a prime and let $G$ be a $p$–torsion–free group, given by a free presentation $1 \longrightarrow N \longrightarrow F \xrightarrow{\pi} G \longrightarrow 1$ .Then there are isomorphisms

1. $t(\gamma_p NN''/[\gamma_p N, F]N'') \cong H_4(G, \mathbf{Z}_p)$.

2. $H_k(G, \mathcal{M}^p M) \cong H_{k+4}(G, \mathbf{Z}_p)$, $k \geq 1$, where $M$ is the localization of the relation module $N_{ab}$.

In order to outline a simplified version of the original proof given in [2], we need the following results, which enables us to avoid using a spectral sequence argument.

**Lemma 3.1.2** Let $G$ be any group and let

$$0 \longrightarrow A \xrightarrow{\alpha} K \xrightarrow{\delta} B \longrightarrow 0\ , \tag{3.4}$$

be a chain complex of $G$–modules, with the following properties,

1. $\alpha$ is injective and $\delta$ is surjective.

2. $H_j(G, K) = 0$, $\forall\ j \geq 1$.

3. $H_j(G, \mathrm{Ker}\delta/\mathrm{Im}\alpha) = 0$, $\forall\ j \geq 1$.

   Then $H_k(G, A) \cong H_{k+1}(G, B)$, $k \geq 1$.

**Proof.** From the chain complex (3.4) we get this exact sequence

$$0 \longrightarrow A \xrightarrow{\alpha} K \xrightarrow{\lambda} K/\mathrm{Im}\alpha \longrightarrow 0\,, \tag{3.5}$$

where $\lambda$ is the natural epimorphism of $K$ onto $K/\mathrm{Im}\alpha$. By Remark 2.1.10 (a), we get a long exact homology sequence,

$$\cdots \longrightarrow H_{k+1}(G,K) \longrightarrow H_{k+1}(G,K/im\alpha) \longrightarrow H_k(G,A) \longrightarrow H_k(G,K) \longrightarrow \cdots .$$

By assumption the outside terms are zero for $k \geq 1$, then we get

$$H_{k+1}(G,K/\mathrm{Im}\alpha) \cong H_k(G,A). \tag{3.6}$$

On the other hand

$$0 \longrightarrow \mathrm{Ker}\delta/\mathrm{Im}\alpha \longrightarrow K/\mathrm{Im}\alpha \longrightarrow B \longrightarrow 0$$

is an exact sequence, and this gives long exact homology sequence,

$$\cdots \to H_{k+1}(G,\mathrm{Ker}\delta/\mathrm{Im}\alpha) \to H_{k+1}(G,K/\mathrm{Im}\alpha) \to H_{k+1}(G,B) \to H_k(G,\mathrm{Ker}\delta/\mathrm{Im}\alpha) \to \cdots$$

The outside terms are zero for $k \geq 1$

$$H_{k+1}(G,K/\mathrm{Im}\alpha) \cong H_{k+1}(G,B),\ k \geq 1 \tag{3.7}$$

Hence, from (3.6) and (3.7) we get , $H_k(G,A) \cong H_{k+1}(G,B)$, for $k \geq 1$, and this completes the proof of the Lemma.

**Corollary 3.1.3** Let $G$ be any group and let

$$\underline{\mathbf{K}} : 0 \longrightarrow A \xrightarrow{\alpha} K_n \xrightarrow{\delta_n} \cdots \cdots \to K_1 \xrightarrow{\delta_1} B \longrightarrow 0 ,$$

be a chain complex of $G$–modules, with the following properties,

1. $\alpha$ is injective and $\delta_1$ is surjective.

2. $H_j(G, K_i) = 0, \forall\ j \geq 1$ and $1 \leq i \leq n$ .

3. $H_j(G, H(\underline{\mathbf{K}})) = 0,\ \forall j \geq 1.$

   Then $H_j(G, A) \cong H_{j+n}(G, B),\ \ j \geq 1.$

**Proof.** The claim follows by induction, namely by breaking the complex $\underline{\mathbf{K}}$ into two complexes.

$$0 \longrightarrow A \longrightarrow K_n \longrightarrow \cdots \longrightarrow K_2 \longrightarrow \mathrm{Im}\delta_2 \longrightarrow 0\,,$$
$$0 \longrightarrow \mathrm{Im}\delta_2 \longrightarrow K_1 \xrightarrow{\delta_1} B \to 0\ ,$$

Applying Lemma 3.1.2 to the second complex we have

$$H_{j-1+n}(G, \mathrm{Im}\delta_2) \cong H_{j+n}(G, B). \tag{3.8}$$

By induction we get from the first complex the following

$$H_j(G, A) \cong H_{j-1+n}(G, \mathrm{Im}\delta_2). \tag{3.9}$$

Hence, (3.8) and (3.9) yields the desired isomorphism.

**Lemma 3.1.4.** Let $G$ and $\underline{\mathbf{K}}$ be as above, and let $\mathrm{Ker}\delta_n = \mathrm{Im}\alpha$. Then $H_n(G, B)$ is isomorphic to the kernel of the map: $A \otimes_G \mathbf{Z} \longrightarrow K_n \otimes_G \mathbf{Z}$.

**Proof.** From the complex $\underline{\mathbf{K}}$ we obtain a short exact sequence

$$0 \longrightarrow A \xrightarrow{\alpha} K_n \longrightarrow K_n/\mathrm{Im}\alpha \longrightarrow 0.$$

By applying the homology functor we get an exact sequence

$$0 \longrightarrow H_1(G, K_n/\mathrm{Im}\alpha) \longrightarrow A \otimes_G \mathbf{Z} \longrightarrow K_n \otimes_G \mathbf{Z} \tag{3.10}$$

Now we consider the following complex

$$0 \longrightarrow K_n/\mathrm{Im}\alpha \rightarrow K_{n-1} \cdots \longrightarrow K_1 \longrightarrow B \longrightarrow 0\ , \tag{3.11}$$

since $\mathrm{Ker}\delta_n = \mathrm{Im}\alpha$, it follows that the complex (3.11) satisfies the hypothesis of Lemma 3.1.3, so we get

$$H_{j+n-1}(G, B) \cong H_j(G, K_n/\mathrm{Im}\alpha) \quad j \geq 1, \tag{3.12}$$

hence (3.12) and (3.10) gives that $H_n(G, B)$ is isomorphic to the kernel of the map

$$A \otimes_G \mathbf{Z} \longrightarrow K_n \otimes_G \mathbf{Z},$$

which proves the lemma.

**Remark 3.1.5.** For an explicit computation of the connecting homomorphism

$$H_n(G, B) \longrightarrow A \otimes_G \mathbf{Z},$$

we consider the double complex $\underline{\mathbf{K}} \otimes_G \underline{\mathcal{P}}$ , where $\underline{\mathcal{P}}$ is any projective resolution of the trivial $G$-module $\mathbf{Z}$, and by induction it is sufficient to do the computation when $n = 2$.

Suppose $n = 2$, then $\underline{\mathbf{K}}$ has the form

$$0 \longrightarrow A \xrightarrow{\alpha} K_2 \xrightarrow{\delta_2} K_1 \xrightarrow{\delta_1} B \longrightarrow 0. \tag{3.13}$$

where the chain complex (3.13) satisfies the hypothesis of Lemma 3.1.4.

We use the following diagram to compute the connecting homomorphism

$$H_2(G, B) \longrightarrow A \otimes_G \mathbf{Z},$$

$$\begin{tikzcd}
 & & B \otimes_G P_1 & B \otimes_G P_2 \arrow[l] \\
 & & K_1 \otimes_G P_1 & K_1 \otimes_G P_2 \arrow[l] \arrow[u] \\
 & K_2 \otimes_G \mathbf{Z}G & K_2 \otimes_G P_1 \arrow[l] \arrow[u] & \\
A \otimes_G \mathbf{Z} & A \otimes_G \mathbf{Z}G \arrow[l] \arrow[u] & &
\end{tikzcd}$$

As tensoring is a right exact functor, we get this exact sequence,

$$K_1 \otimes_G P_2 \xrightarrow{\delta_1^*} B \otimes_G P_2 \longrightarrow 0. \qquad (3.14)$$

Let $z$ be a cycle in $B \otimes_G P_2$ , then by exactness of (3.14) we may lift $z$ to $\overline{y} \in (K_1 \otimes_G P_2)$. As we know, the inverse image $\overline{y}$ is not unique. But the crucial thing here is to get a right one (i.e. an inverse image when we apply the homomorphism $K_1 \otimes_G P_2 \longrightarrow K_1 \otimes_G P_1$, to it we obtain an element belongs to $\mathrm{Im}\delta_2^*$ ).

Now we try to get a right choice: if $\overline{y}$ is non zero element in $K_1/\mathrm{Im}\delta_2^* \otimes_G P_2$, then we push to $(\overline{y})\partial_2 \in (K_1/\mathrm{Im}\delta_2^* \otimes_G P_1)$. By commutativity of this diagram

$$\begin{tikzcd}
B \otimes_G P_1 & B \otimes_G P_2 \arrow[l] \\
K_1/\mathrm{Im}\delta_2^* \otimes_G P_1 \arrow[u] & K_1/\mathrm{Im}\delta_2^* \otimes_G P_2 \arrow[l] \arrow[u]
\end{tikzcd}$$

$(\overline{y})\partial_2 \in \mathrm{Ker}\,(K_1/\mathrm{Im}\delta_2^* \otimes_G P_1 \longrightarrow B \otimes_G P_1) = \mathrm{Ker}\delta_1^*/\mathrm{Im}\delta_2^* \otimes_G P_1$. On the other hand from the commutativity of this diagram,

$$\begin{array}{ccc} K_1/\mathrm{Im}\delta_2^* \otimes_G \mathbf{Z}G & \longleftarrow & K_1/\mathrm{Im}\delta_2^* \otimes_G P_1 \\ \uparrow & & \uparrow \\ \mathrm{Ker}\delta_1^*/\mathrm{Im}\delta_2^* \otimes_G \mathbf{Z}G & \longleftarrow & \mathrm{Ker}\delta_1^*/\mathrm{Im}\delta_2^* \otimes_G P_1 \end{array}$$

and from the injectivity of the map $[\mathrm{Ker}\delta_1^*/\mathrm{Im}\delta_2^* \otimes_G \mathbf{Z}G \longrightarrow K_1/\mathrm{Im}\delta_2^* \otimes_G \mathbf{Z}G]$, we can see that $(\overline{y})\partial_2$ is a cycle in $(\mathrm{Ker}\delta_1^*/\mathrm{Im}\delta_2^* \otimes_G P_1)$. But from the hypothesis of lemma 3.1.4, we have $H_1(G, \mathrm{Ker}\delta_1^*/\mathrm{Im}\delta_2^*) = 0$, thus $(\overline{y})\partial_2$ is a boundary in $\mathrm{Ker}\delta_1^*/\mathrm{Im}\delta_2^* \otimes_G P_1$. So there exists an element $x \in \mathrm{Ker}\delta_1^*/\mathrm{Im}\delta_2^* \otimes_G P_2$ such that $(x)\partial_2 = (\overline{y})\partial_2$ modulo $\mathrm{Im}\delta_2^*$.

Now it is easy to see that $(\overline{y} - x)\partial_2 \in \mathrm{Im}\delta_2^*$. We choose$(\overline{y} - x)$ as an inverse image of $z$. On the other hand, from the first diagram, we can lift $(\overline{y} - x)\partial_2$ to $y \in K_2 \otimes_G P_1$, then we push to $(y)\partial_1 \in (K_2 \otimes_G \mathbf{Z}G)$. The commutativity of this diagram,

$$\begin{array}{ccc} K_1 \otimes_G \mathbf{Z}G & \longleftarrow & K_1 \otimes_G P_1 \\ \uparrow & & \uparrow \\ K_2 \otimes_G \mathbf{Z}G & \longleftarrow & K_2 \otimes_G P_1 \end{array}$$

implies that $(y)\partial_1 \in \mathrm{Ker}(K_2 \otimes_G \mathbf{Z}G \longrightarrow K_1 \otimes_G \mathbf{Z}G)$, and by exactness we may lift $(y)\partial_1$ to $a \in A \otimes_G \mathbf{Z}G$, then we push to $(a)\partial_0 \in A \otimes_G \mathbf{Z}$, and this completes the computation of the connecting homomorphism.

Now we proceed to our actual concern in this section, namely, to simplify the proof of the main result of [2], which gives a homological description to the torsion subgroup of

$$F/[\gamma_p(F'), F]F'''.$$

**Proof of Theorem 3.1.1.** We consider the localized version of the complex $\mathcal{M}$ . Now, by remark 2.1.4, $P \otimes M^{p-1}$ is free, then by remark 2.1.6 part 2, $H_k(G, P \otimes M^{p-1}) = 0$, $\forall k \geq 1$. Moreover, if $G$ is $p$-torsion-free, then by Theorem 2.2.4, $H_k(G, P^p) = H_k(G, RG^p) = H_k(G, RG) = 0$, $\forall\ k \geq 1$.

On the other hand, by Lemma 2.4.6, we have that $H_k(G, H_i(\mathcal{M}))$ ($k \geq 1$, $i = 0, 1, \dots, 5$), is a periodic abelian group of finite exponent dividing $((p-1)!)^q$ for some positive integer $q$, and this number is relatively prime to $p$. Then as we localized at $p$, $H_k(G, H_i(\mathcal{M})) = 0$ for all $k \geq 1$. Furthermore, as was mentioned in chapter 2, $\mathcal{M}$ is exact in dimension 4 (i.e. $\mathrm{Im}\delta_5 = \mathrm{Ker}\delta_4$). Consequently, Corollary 3.1.3 and Lemma 3.1.4, can be applied to the complex $\mathcal{M}$ , and we get

$$H_k(G, \mathcal{M}^p M) \cong H_{k+4}(G, \mathbf{Z}_p)\ ,\ k \geq 1$$

$$0 \to H_4(G, \mathbf{Z}_p) \to \mathcal{M}^p M \otimes_G \mathbf{R} \to P \otimes M^{p-1} \otimes_G \mathbf{R},$$

where the sequence is exact, on the other hand the group $P \otimes M^{p-1} \otimes_G \mathbf{R}$ is free as an $\mathbf{R}$-module, therefore

$$t(\mathcal{M}^p M \otimes_G R) \subseteq H_4(G, \mathbf{Z}_p),$$

but $H_4(G, \mathbf{Z}_p)$ is a torsion group. Thus $t(\mathcal{M}^p M \otimes_G \mathbf{R}) \cong H_4(G, \mathbf{Z}_p)$ and this completes the proof of the theorem.

Enough background information is already available, thereby enabling us to establish our first result concerning torsion elements in $F/[\gamma_p F', F]F'''$.

## 3.2 Description of $t(F/[\gamma_p F', F]F''')$ in terms of generators

From now on we will assume that $G = F/F'$, where $F$ is the free group on $X = \{x_1, \cdots, x_d\}$, $d \geq 2$. This means that $G$ is a free abelian group of rank $d$, with free generators $\{b_1, b_2, \cdots, b_d\}$, where $b_i = x_i F'$. We have already mentioned in chapter 2 that the homology of $G$ with coefficients in the trivial $G$-module $\mathbf{Z}_p$ is an elementary abelian group of rank $C_k^d$. Consequently $F/[\gamma_p(F'), F]F'''$ is torsion-free for $d \leq 3$, and for $d \geq 4$ its torsion subgroup is an elementary abelian group of rank $C_4^d$. In this case we give a complete description to this torsion subgroup, in terms of generators, where $p$ is any prime.

### 3.2.1 Description of $t(F/[\gamma_p F', F]F''')$ where $p$ is any odd prime

Let $W_p(x_{\tau_1}, x_{\tau_2}, x_{\tau_3}, x_{\tau_4})$, be as in the introduction. Then the main result of this chapter reads as follows.

**Theorem 3.2.1.** Let $p$ be any odd prime. Then the torsion subgroup of $F/[\gamma_p F', F]F'''$ is generated by the elements $W_p(x_{\tau_1}, x_{\tau_2}, x_{\tau_3}, x_{\tau_4})$.

**Proof.** The proof is by computing the connecting homomorphism

$$H_4(G, \mathbf{Z}_p) \longrightarrow t(\mathcal{M}^p M \quad \otimes_G \mathbf{R})$$

where $p$ is an odd prime number, $\mathbf{R}$ is the integers localized at $p$, and $G = F/F'$.

For computing the connecting homomorphism, we use the localized version of the double complex $\mathcal{M} \otimes_G \underline{\mathcal{P}}$, where $\mathcal{M}$ is the chain complex (3.3) and $\underline{\mathcal{P}}$ is the Koszul complex which is introduced in chapter 2. As we see in the proof of Theorem 3.1.1, $\mathcal{M}$ satisfies the hypothesis of lemma 3.1.4. Then the theoretical justification of our computation of the connecting homomorphism (which is outlined in Remark 3.1.5) can be applied to the double complex $\mathcal{M} \otimes_G \underline{\mathcal{P}}$ .

Now we consider the following diagram:

$$\begin{tikzcd}
\mathbf{Z}_p \otimes R & \mathbf{Z}_p \otimes RG \arrow[l] & \mathbf{Z}_p \otimes P_1 \arrow[l] & \mathbf{Z}_p \otimes P_2 \arrow[l] & \mathbf{Z}_p \otimes P_3 \arrow[l] & \mathbf{Z}_p \otimes P_4 \arrow[l] \\
RG \otimes R \arrow[u] & & & & RG \otimes P_3 & RG \otimes P_4 \arrow[l] \arrow[u] \\
RG^p \otimes R \arrow[u] & & & RG^p \otimes P_2 & RG^p \otimes P_3 \arrow[l] \arrow[u] & \\
P^p \otimes R \arrow[u] & & P^p \otimes P_1 & P^p \otimes P_2 \arrow[l] \arrow[u] & & \\
A \otimes R \arrow[u] & A \otimes RG & A \otimes P_1 \arrow[l] \arrow[u] & & & \\
\mathcal{M}^p M \otimes R \arrow[u] & \mathcal{M}^p M \otimes RG \arrow[l] \arrow[u] & & & &
\end{tikzcd}$$

where $A = P \otimes_{\mathbf{R}} M^{p-1}$; in this double complex we use $\otimes$ instead of $\otimes_G$.

We have to start in $\mathbf{Z}_p \otimes_G P_4$, and then we go along the arrows down to $\mathcal{M}^p M \otimes_G \mathbf{R}$.

The group $\mathbf{Z}_p \otimes_G P_4$ is free abelian with basis

$$\{1 \otimes e_{\tau_1} \wedge e_{\tau_2} \wedge e_{\tau_3} \wedge e_{\tau_4}; 1 \leq \tau_1 < \cdots < \tau_4 \leq d\}.$$

For simplicity, we consider the element

$$1 \otimes e_1 \wedge e_2 \wedge e_3 \wedge e_4 \ . \tag{3.15}$$

An inverse image of (3.15) in $\mathbf{R}G \otimes_G P_4$ is

$$1 \otimes e_1 \wedge e_2 \wedge e_3 \wedge e_4 \tag{3.16}$$

By applying the homomorphism $\mathbf{R}G \otimes_G P_4 \longrightarrow \mathbf{R}G \otimes_G P_3$, to (3.16) we obtain,

$$\sum_{i=1}^{4} (-1)^{i+1}(b_i - 1) \otimes (e_1 \wedge \cdots \wedge \hat{e_i} \wedge \cdots \wedge e_4). \tag{3.17}$$

In order to get an inverse image of (3.17) in $\mathbf{R}G^p \otimes_G P_3$, we consider the element

$$(1 \circ 1 \circ \cdots \circ 1).(b_i - 1) = b_i \circ b_i \circ \cdots \circ b_i - 1 \circ 1 \circ \cdots \circ 1.$$

By writing $b_i = (b_i - 1) + 1$, and expanding, we get

$$(1 \circ 1 \circ \cdots \circ 1).(b_i - 1) = \sum_{j=0}^{p-1} C_j^p (b_i - 1)^{p-j} \circ 1^j.$$

By subtracting $(b_i - 1)^p$ from both sides, we get

$$[(1 \circ 1 \circ \cdots \circ 1).(b_i - 1) - (b_i - 1)^p] = \sum_{j=1}^{p-1} C_j^p (b_i - 1)^{p-j} \circ 1^j .$$

Note that all the binomial coefficients $C_j^p$ ($j = 1, 2 \cdots, p-1$) on the right hand side are divisible by $p$. Hence the expression

$$\frac{1}{p}[(1 \circ 1 \circ \cdots \circ 1).(b_i - 1) - (b_i - 1)^p]$$

makes sense in our situation. We claim that

$$\frac{1}{p} \sum_{i=1}^{4} (-1)^{i+1}[(1 \circ 1 \circ \cdots \circ 1).(b_i - 1) - (b_i - 1)^p] \otimes (e_1 \wedge \cdots \wedge \hat{e_i} \cdots \wedge e_4) \tag{3.18}$$

is an inverse image of (3.17) in $\mathbf{R}G^p \otimes_G P_3$. This is because $(b_i - 1)^p$ is in the kernel of the map $\mathbf{R}G^p \longrightarrow \mathbf{R}G$, and the image of element $\underbrace{(1 \circ 1 \circ \cdots \circ 1)}_{p}$ under this map is $p$.

After applying the homomorphism $\mathbf{R}G^p \otimes_G P_3 \longrightarrow \mathbf{R}G^p \otimes_G P_2$ to (3.18) and rearranging the resulting element of $\mathbf{R}G^p \otimes_G P_2$, we get:

$$\frac{1}{p} \sum_{\eta} (-1)^{\eta} \{(b_{2\eta} - 1)^p . (b_{1\eta} - 1) - (b_{1\eta} - 1)^p . (b_{2\eta} - 1)\} \otimes e_{3\eta} \wedge e_{4\eta}, \tag{3.19}$$

where $\eta$ ranges over all permutations of $\{1, 2, 3, 4\}$ with $1\eta < 2\eta$ and $3\eta < 4\eta$.

In order to get an inverse image of (3.19) in $P^p \otimes_G P_2$, we consider the following element:

$$e_2^p (b_1 - 1) - e_1^p (b_2 - 1) + [e_1 (b_2 - 1) - e_2 (b_1 - 1)]^p \in P^p. \tag{3.20}$$

Now,

$$\begin{aligned} e_2^p (b_1 - 1) &= (e_2 \circ e_2 \circ \cdots \circ e_2) . (b_1 - 1) \\ &= e_2 b_1 \circ e_2 b_1 \circ \cdots \circ e_2 b_1 - e_2 \circ e_2 \circ \cdots \circ e_2 . \end{aligned}$$

By writing $b_1 = (b_1 - 1) + 1$, and expanding, we get

$$\begin{aligned} e_2^p (b_1 - 1) &= \sum_{i=0}^{p} C_i^p e_2^{p-i} (b_1 - 1) \circ e_2^i - e_2^p \\ &= \sum_{i=0}^{p-1} C_i^p e_2^{p-i} (b_1 - 1) \circ e_2^i \end{aligned} \tag{3.21}$$

Similarly,

$$e_1^p (b_2 - 1) = \sum_{i=0}^{p-1} C_i^p e_1^{p-i} (b_2 - 1) \circ e_1^i \tag{3.22}$$

For the last term in (3.20) we have

$$[e_1 (b_2 - 1) - e_2 (b_1 - 1)]^p = \sum_{i=0}^{p} (-1)^i C_i^p e_1 (b_2 - 1)^{p-i} \circ e_2 (b_1 - 1)^i . \tag{3.23}$$

In view of (3.21), (3.22) and (3.23) the element (3.20) can be rewritten as

$$\left\{\begin{array}{c} \sum_{i=1}^{p-1} C_i^p e_2^{p-i}(b_1-1)\circ e_2^i - \sum_{i=1}^{p-1} C_i^p e_1^{p-i}(b_2-1)\circ e_1^i \\ + \sum_{i=1}^{p-1}(-1)^i C_i^p e_1(b_2-1)^{p-i}\circ e_2(b_1-1)^i \end{array}\right\}$$

Again we note that all the binomial coefficients $C_i^p$ $(i = 1, 2, \cdots, p-1)$, on these terms are divisible by $p$. Hence the expression

$$\frac{1}{p}\{e_{2\eta}^p(b_{1\eta}-1) - e_{1\eta}^p(b_{2\eta}-1) + [e_{1\eta}(b_{2\eta}-1) - e_{2\eta}(b_{1\eta}-1)]^p\},$$

makes sense. Remember that $e_{1\eta}(b_{2\eta}-1) - e_{2\eta}(b_{1\eta}-1) = [x_{2\eta}, x_{1\eta}]\mu$ where $\mu$ is the Magnus embedding, therefore $e_{1\eta}(b_{2\eta}-1) - e_{2\eta}(b_{1\eta}-1)$ can be identified with $[x_{2\eta}, x_{1\eta}]$. Also we put $[x_{2\eta}, x_{1\eta}] = [\ ]_{2\eta 1\eta}$. On the other hand we notice that

$$\frac{1}{p}\sum_{\eta}(-1)^{\eta}[e_{2\eta}^p(b_{1\eta}-1) - e_{1\eta}^p(b_{2\eta}-1) + [\ ]_{2\eta 1\eta}^p] \otimes e_{3\eta}\wedge e_{4\eta} \qquad (3.24)$$

is an inverse image of (3.19) in $P^p \otimes_G P_2$, where

$$[e_{1\eta}(b_{2\eta}-1) - e_{2\eta}(b_{1\eta}-1)]^p \cong [x_{2\eta}, x_{1\eta}]^p = [\ ]_{2\eta 1\eta}^p.$$

Applying the homomorphism $P^p \otimes_G P_2 \longrightarrow P^p \otimes_G P_1$ to (3.24), we get

$$\sum_{\eta}(-1)^{\eta}\left\{\begin{array}{c} \frac{1}{p}\{e_{2\eta}^p(b_{1\eta}-1)(b_{3\eta}-1) - e_{1\eta}^p(b_{2\eta}-1)(b_{3\eta}-1) + [\,]_{2\eta 1\eta}^p(b_{3\eta}-1)\}\otimes e_{4\eta} \\ \frac{1}{p}\{e_{1\eta}^p(b_{2\eta}-1)(b_{4\eta}-1) - e_{2\eta}^p(b_{1\eta}-1)(b_{4\eta}-1) - [\,]_{2\eta 1\eta}^p(b_{4\eta}-1)\}\otimes e_{3\eta} \end{array}\right\}$$

and this gives

$$\left\{\begin{array}{l}
\frac{1}{p}\{e_2^p(b_1-1)(b_3-1)-e_1^p(b_2-1)(b_3-1)+[\,]_{21}^p(b_3-1)\}\otimes e_4\\
-\frac{1}{p}\{e_3^p(b_1-1)(b_2-1)-e_1^p(b_3-1)(b_2-1)+[\,]_{31}^p(b_2-1)\}\otimes e_4\\
\frac{1}{p}\{e_3^p(b_2-1)(b_1-1)-e_2^p(b_3-1)(b_1-1)+[\,]_{32}^p(b_1-1)\}\otimes e_4\\
\frac{1}{p}\{e_4^p(b_3-1)(b_1-1)-e_3^p(b_4-1)(b_1-1)+[\,]_{43}^p(b_1-1)\}\otimes e_2\\
-\frac{1}{p}\{e_4^p(b_2-1)(b_1-1)-e_2^p(b_4-1)(b_1-1)+[\,]_{42}^p(b_1-1)\}\otimes e_3\\
\frac{1}{p}\{e_4^p(b_1-1)(b_2-1)-e_1^p(b_4-1)(b_2-1)+[\,]_{41}^p(b_2-1)\}\otimes e_3\\
\frac{1}{p}\{e_1^p(b_2-1)(b_4-1)-e_2^p(b_1-1)(b_4-1)-[\,]_{21}^p(b_4-1)\}\otimes e_3\\
-\frac{1}{p}\{e_1^p(b_3-1)(b_4-1)-e_3^p(b_1-1)(b_4-1)-[\,]_{31}^p(b_4-1)\}\otimes e_2\\
\frac{1}{p}\{e_1^p(b_4-1)(b_3-1)-e_4^p(b_1-1)(b_3-1)-[\,]_{41}^p(b_3-1)\}\otimes e_2\\
\frac{1}{p}\{e_2^p(b_3-1)(b_4-1)-e_3^p(b_2-1)(b_4-1)-[\,]_{32}^p(b_4-1)\}\otimes e_1\\
-\frac{1}{p}\{e_2^p(b_4-1)(b_3-1)-e_4^p(b_3-1)(b_3-1)-[\,]_{42}^p(b_3-1)\}\otimes e_1\\
\frac{1}{p}\{e_3^p(b_4-1)(b_2-1)-e_4^p(b_3-1)(b_2-1)-[\,]_{43}^p(b_4-1)\}\otimes e_1
\end{array}\right\} \tag{3.25}$$

but (3.25) is equal to the following:

$$\begin{array}{l}
\frac{1}{p}\{[\,]_{21}^p.(b_3-1)+[\,]_{13}^p.(b_2-1)+[\,]_{32}^p.(b_1-1)\}\otimes e_4\\
-\frac{1}{p}\{[\,]_{32}^p.(b_4-1)+[\,]_{24}^p.(b_3-1)+[\,]_{43}^p.(b_2-1)\}\otimes e_1\\
\frac{1}{p}\{[\,]_{43}^p.(b_1-1)+[\,]_{31}^p.(b_4-1)+[\,]_{14}^p.(b_3-1)\}\otimes e_2\\
-\frac{1}{p}\{[\,]_{14}^p.(b_2-1)+[\,]_{42}^p.(b_1-1)+[\,]_{21}^p.(b_4-1)\}\otimes e_3
\end{array} \tag{3.26}$$

We notice that (3.26) belongs to $\mathrm{Im}\delta_4 \otimes_G P_1$, this means that (3.24) is a right inverse image.

In order to get an inverse image of (3.26) in $P \otimes M^{p-1} \otimes_G P_1$, we consider the element

$$([\,]_{21} \otimes [\,]_{21}^{p-1}).(b_3-1) = [\,]_{21}b_3 \otimes [\,]_{21}b_3 \circ \cdots \circ [\,]_{21}b_3 - [\,]_{21} \otimes [\,]_{21}^{p-1}.$$

By writing $b_3 = (b_3 - 1) + 1$ and expanding on the right hand side, we get

$([\ ]_{21}(b_3 - 1) + [\ ]_{21}) \otimes \sum_{i=0}^{p-1} C_i^{p-1}[\ ]_{21}(b_3 - 1)^{p-1-i} \circ [\ ]_{21}^{i} - [\ ]_{21} \otimes [\ ]_{21}^{p-1}$. Therefore we obtain

$$([\ ]_{21} \otimes [\ ]_{21}^{p-1}).(b_3 - 1) = \left\{ \begin{array}{c} [\ ]_{21}(b_3 - 1) \otimes [\ ]_{21}(b_3 - 1)^{p-1} \\ +[\ ]_{21}(b_3 - 1) \otimes \sum_{i=1}^{p-1} C_i^{p-1}[\ ]_{21}(b_3 - 1)^{p-1-i} \circ [\ ]_{21}^{i} \\ +[\ ]_{21} \otimes \sum_{i=0}^{p-2} C_i^{p-1}[\ ]_{21}(b_3 - 1)^{p-1-i} \circ [\ ]_{21}^{i}. \end{array} \right\} \tag{3.27}$$

On the other hand, the element

$$[\ ]_{21}(b_3 - 1) \otimes \sum_{i=0}^{p-2} C_i^{p-1}[\ ]_{21}(b_3 - 1)^{p-2-i} \circ [\ ]_{21}^{i+1} - [\ ]_{21} \otimes \sum_{i=0}^{p-2} C_i^{p-1}[\ ]_{21}(b_3 - 1)^{p-1-i} \circ [\ ]_{21}^{i}$$

belongs to the kernel of $(P \otimes M^{p-1} \longrightarrow P^p)$. By adding this element to (3.27) we obtain, by using (2.15), the following:

$$[\ ]_{21}(b_3 - 1) \otimes [\ ]_{21}(b_3 - 1)^{p-1} + [\ ]_{21}(b_3 - 1) \otimes \sum_{i=1}^{p-1} C_i^{p}[\ ]_{21}(b_3 - 1)^{p-1-i} \circ [\ ]_{21}^{i}.$$

Also, here we note that all the binomial coefficients $C_i^p$ $(i = 1, 2, \cdots, p-1)$ in the second term are divisible by $p$.

Now we do the same thing for $([\ ]_{13} \otimes [\ ]_{13}^{p-1}).(b_2 - 1)$ and $([\ ]_{32} \otimes [\ ]_{32}^{p-1}).(b_1 - 1)$. The coefficients of the first term in each one of them is not divisible by $p$. Consider those three terms :

$$\begin{array}{l} [\ ]_{21}(b_3 - 1) \otimes [\ ]_{21}(b_3 - 1)^{p-1} \\ +[\ ]_{13}(b_2 - 1) \otimes [\ ]_{13}(b_2 - 1)^{p-1} \\ +[\ ]_{32}(b_1 - 1) \otimes [\ ]_{32}(b_1 - 1)^{p-1}. \end{array} \tag{3.28}$$

By the Jacobi identity, we can write the element $[\ ]_{32}(b_1-1)\otimes[\ ]_{32}(b_1-1)^{p-1}$ as

$$= \left\{ \begin{array}{l} -[\ ]_{13}(b_2-1)\otimes \sum_{i=0}^{p-1} C_i^{p-1}[\ ]_{13}(b_2-1)^{p-1-i}\circ[\ ]_{21}(b_3-1)^i \\ -[\ ]_{21}(b_3-1)\otimes \sum_{i=0}^{p-1} C_i^{p-1}[\ ]_{13}(b_2-1)^{p-1-i}\circ[\ ]_{21}(b_3-1)^i \end{array} \right. \tag{3.29}$$

By substituting (3.29) into (3.28) we obtain

$$\begin{array}{l} -[\ ]_{13}(b_2-1)\otimes \sum_{i=1}^{p-1} C_i^{p-1}[\ ]_{13}(b_2-1)^{p-1-i}\circ[\ ]_{21}(b_3-1)^i \\ -[\ ]_{21}(b_3-1)\otimes \sum_{i=0}^{p-2} C_i^{p-1}[\ ]_{13}(b_2-1)^{p-1-i}\circ[\ ]_{21}(b_3-1)^i \end{array} \tag{3.30}$$

On the other hand, the element

$$\begin{array}{l} [\ ]_{21}(b_3-1)\otimes \sum_{i=0}^{p-2} C_i^{p-1}[\ ]_{13}(b_2-1)^{p-1-i}\circ[\ ]_{21}(b_3-1)^i \\ -[\ ]_{13}(b_2-1)\otimes \sum_{i=0}^{p-2} C_i^{p-1}[\ ]_{13}(b_2-1)^{p-2-i}\circ[\ ]_{21}(b_3-1)^{i+1} \end{array}$$

belongs to the kernel of the map $(P\otimes M^{p-1} \longrightarrow P^p)$, and when we added to (3.30), we obtain the element

$$-[\ ]_{13}(b_2-1)\otimes \sum_{i=1}^{p-1} C_i^{p}[\ ]_{13}(b_2-1)^{p-1-i}\circ[\ ]_{21}(b_3-1)^i.$$

Note that all the binomial coefficients $C_i^p(i=1,2,\cdots,p-1)$ are divisible by $p$ .

As a result of our previous discussion, we note that the expression

$$\frac{1}{p}\{([\,]_{21}\otimes[\,]_{21}^{p-1})(b_3-1)+([\,]_{13}\otimes[\,]_{13}^{p-1})(b_2-1)+([\,]_{32}\otimes[\,]_{32})(b_1-1)\}+$$

$$\frac{1}{p}\sum_{i=0}^{p-2} C_i^{p-1} \left\{ \begin{array}{c} [\,]_{21}(b_3-1)\otimes[\,]_{21}(b_3-1)^{p-2-i}\circ[\,]_{21}^{i+1} - [\,]_{21}\otimes[\,]_{21}(b_3-1)^{p-1-i}\circ[\,]_{21}^{i} \\ +[\,]_{13}(b_2-1)\otimes[\,]_{13}(b_2-1)^{p-2-i}\circ[\,]_{13}^{i+1} - [\,]_{13}\otimes[\,]_{13}(b_2-1)^{p-1-i}\circ[\,]_{13}^{i} \\ +[\,]_{32}(b_1-1)\otimes[\,]_{32}(b_1-1)^{p-2-i}\circ[\,]_{32}^{i+1} - [\,]_{32}\otimes[\,]_{32}(b_1-1)^{p-1-i}\circ[\,]_{32}^{i} \\ +[\ ]_{21}(b_3-1)\otimes[\ ]_{13}(b_2-1)^{p-1-i}\circ[\ ]_{21}(b_3-1)^i \\ -[\ ]_{13}(b_2-1)\otimes[\ ]_{13}(b_2-1)^{p-2-i}\circ[\ ]_{21}(b_3-1)^{i+1} \end{array} \right\}$$

makes sense. Moreover, it is not hard to see that the image of this expression under the map

$$P \otimes M^{p-1} \longrightarrow P^p$$

is $\frac{1}{p}\{[\ ]_{21}^p.(b_3-1)+[\ ]_{13}^p.(b_2-1)+[\ ]_{32}^p.(b_1-1)\}$. We do the same thing for the rest of the other terms in (3.26). Hence the following element (the sum of four symmetrical terms) is an inverse image of (3.26), in $P \otimes M^{p-1} \otimes_G P_1$ :

$$\frac{1}{p}\left\{\begin{array}{c} ([\,]_{21}\otimes[\,]_{21}^{p-1})(b_3-1)+([\,]_{13}\otimes[\,]_{13}^{p-1})(b_2-1) \\ +([\,]_{32}\otimes[\,]_{32})(b_1-1) \\ +\sum\limits_{i=0}^{p-2} C_i^{p-1}\left\{\begin{array}{c} [\ ]_{21}(b_3-1)\otimes[\ ]_{21}(b_3-1)^{p-2-i}\circ[\ ]_{21}^{i+1} \\ -[\ ]_{21}\otimes[\ ]_{21}(b_3-1)^{p-1-i}\circ[\ ]_{21}^{i} \\ +[\ ]_{13}(b_2-1)\otimes[\ ]_{13}(b_2-1)^{p-2-i}\circ[\ ]_{13}^{i+1} \\ -[\,]_{13}\otimes[\,]_{13}(b_2-1)^{p-1-i}\circ[\,]_{13}^{i} \\ +[\ ]_{32}(b_1-1)\otimes[\ ]_{32}(b_1-1)^{p-2-i}\circ[\,]_{32}^{i+1} \\ -[\,]_{32}\otimes[\ ]_{32}(b_1-1)^{p-1-i}\circ[\,]_{32}^{i} \\ +[\,]_{21}(b_3-1)\otimes[\,]_{13}(b_2-1)^{p-1-i}\circ[\,]_{21}(b_3-1)^{i} \\ -[\,]_{13}(b_2-1)\otimes[\,]_{13}(b_2-1)^{p-2-i}\circ[\,]_{21}(b_3-1)^{i+1} \end{array}\right\} \end{array}\right\}\otimes e_4 \tag{3.31}$$

$$
-\frac{1}{p}\left\{\begin{array}{c}
([\,]_{32}\otimes[\,]_{32}^{p-1})(b_4-1)+([\,]_{24}\otimes[\,]_{24}^{p-1})(b_3-1) \\
+([\,]_{43}\otimes[\,]_{43})(b_2-1) \\
+\sum\limits_{i=0}^{p-2} C_i^{p-1}\left\{\begin{array}{c}
[\,]_{32}(b_4-1)\otimes[\,]_{32}(b_4-1)^{p-2-i}\circ[\,]_{32}^{i+1} \\
-[\,]_{32}\otimes[\,]_{32}(b_4-1)^{p-1-i}\circ[\,]_{32}^{i} \\
+[\,]_{24}(b_3-1)\otimes[\,]_{24}(b_3-1)^{p-2-i}\circ[\,]_{24}^{i+1} \\
-[\,]_{24}\otimes[\,]_{24}(b_3-1)^{p-1-i}\circ[\,]_{24}^{i} \\
+[\,]_{43}(b_2-1)\otimes[\,]_{43}(b_2-1)^{p-2-i}\circ[\,]_{43}^{i+1} \\
-[\,]_{43}\otimes[\,]_{43}(b_2-1)^{p-1-i}\circ[\,]_{43}^{i} \\
+[\,]_{32}(b_4-1)\otimes[\,]_{24}(b_3-1)^{p-1-i}\circ[\,]_{32}(b_4-1)^{i} \\
-[\,]_{24}(b_3-1)\otimes[\,]_{24}(b_3-1)^{p-2-i}\circ[\,]_{32}(b_4-1)^{i+1}
\end{array}\right\}
\end{array}\right\}\otimes e_1 \tag{3.32}
$$

$$
+\frac{1}{p}\left\{\begin{array}{c}
([\,]_{43}\otimes[\,]_{43}^{p-1})(b_1-1)+([\,]_{31}\otimes[\,]_{31}^{p-1})(b_4-1) \\
+([\,]_{14}\otimes[\,]_{14})(b_3-1) \\
+\sum\limits_{i=0}^{p-2} C_i^{p-1}\left\{\begin{array}{c}
[\,]_{43}(b_1-1)\otimes[\,]_{43}(b_1-1)^{p-2-i}\circ[\,]_{43}^{i+1} \\
-[\,]_{43}\otimes[\,]_{43}(b_1-1)^{p-1-i}\circ[\,]_{43}^{i} \\
+[\,]_{31}(b_4-1)\otimes[\,]_{31}(b_4-1)^{p-2-i}\circ[\,]_{31}^{i+1} \\
-[\,]_{31}\otimes[\,]_{31}(b_4-1)^{p-1-i}\circ[\,]_{31} \\
+[\,]_{14}(b_3-1)\otimes[\,]_{14}(b_3-1)^{p-2-i}\circ[\,]_{14}^{i+1} \\
-[\,]_{14}\otimes[\,]_{14}(b_3-1)^{p-1-i}\circ[\,]_{14}^{i} \\
+[\,]_{43}(b_1-1)\otimes[\,]_{31}(b_4-1)^{p-1-i}\circ[\,]_{43}(b_1-1)^{i} \\
-[\,]_{31}(b_4-1)\otimes[\,]_{31}(b_4-1)^{p-2-i}\circ[\,]_{43}(b_1-1)^{i+1}
\end{array}\right\}
\end{array}\right\}\otimes e_2 \tag{3.33}
$$

$$
-\frac{1}{p}\left\{\begin{array}{c}
([\ ]_{14}\otimes[\ ]_{14}^{p-1})(b_2-1)+([\ ]_{42}\otimes[\ ]_{42}^{p-1})(b_1-1) \\
+([\ ]_{21}\otimes[\ ]_{21})(b_4-1) \\
+\sum\limits_{i=0}^{p-2} C_i^{p-1}\left\{\begin{array}{c}
[\ ]_{14}(b_2-1)\otimes[\ ]_{14}(b_2-1)^{p-2-i}\circ[\ ]_{14}^{i+1} \\
-[\ ]_{14}\otimes[\ ]_{14}(b_2-1)^{p-1-i}\circ[\ ]_{14}^{i} \\
+[\ ]_{42}(b_1-1)\otimes[\ ]_{42}(b_1-1)^{p-2-i}\circ[\ ]_{42}^{i+1} \\
-[\ ]_{42}\otimes[\ ]_{42}(b_1-1)^{p-1-i}\circ[\ ]_{42} \\
+[\ ]_{21}(b_4-1)\otimes[\ ]_{21}(b_4-1)^{p-2-i}\circ[\ ]_{21}^{i+1} \\
-[\ ]_{21}\otimes[\ ]_{21}(b_4-1)^{p-1-i}\circ[\ ]_{21}^{i} \\
+[\ ]_{14}(b_2-1)\otimes[\ ]_{42}(b_1-1)^{p-1-i}\circ[\ ]_{14}(b_2-1)^{i} \\
-[\ ]_{42}(b_1-1)\otimes[\ ]_{42}(b_1-1)^{p-2-i}\circ[\ ]_{14}(b_2-1)^{i+1}
\end{array}\right\}
\end{array}\right\}\otimes e_3 \tag{3.34}
$$

After applying the homomorphism $P\otimes M^{p-1}\otimes_G P_1 \longrightarrow P\otimes M^{p-1}\otimes_G \mathbf{R}G$ to (3.31), (3.32), (3.33) and (3.34), we get

$$
\frac{1}{p}\sum_{i=0}^{p-2} C_i^{p-1}
\left\{
\begin{array}{c}
\{[\,]_{21}(b_3-1)\otimes[\,]_{21}(b_3-1)^{p-2-i}\circ[\,]_{21}^{i+1}-[\,]_{21}\otimes[\,]_{21}(b_3-1)^{p-1-i}\circ[\,]_{21}^{i} \\
+[\,]_{13}(b_2-1)\otimes[\,]_{13}(b_2-1)^{p-2-i}\circ[\,]_{13}^{i+1}-[\,]_{13}\otimes[\,]_{13}(b_2-1)^{p-1-i}\circ[\,]_{13}^{i} \\
+[\,]_{32}(b_1-1)\otimes[\,]_{32}(b_1-1)^{p-2-i}\circ[\,]_{32}^{i+1}-[\,]_{32}\otimes[\,]_{32}(b_1-1)^{p-1-i}\circ[\,]_{32}^{i} \\
+[\,]_{21}(b_3-1)\otimes[\,]_{13}(b_2-1)^{p-1-i}\circ[\,]_{21}(b_3-1)^{i} \\
-[\,]_{13}(b_2-1)\otimes[\,]_{13}(b_2-1)^{p-2-i}\circ[\,]_{21}(b_3-1)^{i+i}\}.(b_4-1) \\
-\{[\,]_{32}(b_4-1)\otimes[\,]_{32}(b_4-1)^{p-2-i}\circ[\,]_{32}^{i+1}-[\,]_{32}\otimes[\,]_{32}(b_4-1)^{p-1-i}\circ[\,]_{32}^{i} \\
+[\,]_{24}(b_3-1)\otimes[\,]_{24}(b_3-1)^{p-2-i}\circ[\,]_{24}^{i+1}-[\,]_{24}\otimes[\,]_{24}(b_3-1)^{p-1-i}\circ[\,]_{24}^{i} \\
+[\,]_{43}(b_2-1)\otimes[\,]_{43}(b_2-1)^{p-2-i}\circ[\,]_{43}^{i+1}-[\,]_{43}\otimes[\,]_{43}(b_2-1)^{p-1-i}\circ[\,]_{43}^{i} \\
+[\,]_{32}(b_4-1)\otimes[\,]_{24}(b_3-1)^{p-1-i}\circ[\,]_{32}(b_4-1)^{i} \\
-[\,]_{24}(b_3-1)\otimes[\,]_{24}(b_3-1)^{p-2-i}\circ[\,]_{32}(b_4-1)^{i+i}\}.(b_1-1) \\
+\{[\,]_{43}(b_1-1)\otimes[\,]_{43}(b_1-1)^{p-2-i}\circ[\,]_{43}^{i+1}-[\,]_{43}\otimes[\,]_{43}(b_1-1)^{p-1-i}\circ[\,]_{43}^{i} \\
+[\,]_{31}(b_4-1)\otimes[\,]_{31}(b_4-1)^{p-2-i}\circ[\,]_{31}^{i+1}-[\,]_{31}\otimes[\,]_{31}(b_4-1)^{p-1-i}\circ[\,]_{31} \\
+[\,]_{14}(b_3-1)\otimes[\,]_{14}(b_3-1)^{p-2-i}\circ[\,]_{14}^{i+1}-[\,]_{14}\otimes[\,]_{14}(b_3-1)^{p-1-i}\circ[\,]_{14}^{i} \\
+[\,]_{43}(b_1-1)\otimes[\,]_{31}(b_4-1)^{p-1-i}\circ[\,]_{43}(b_1-1)^{i} \\
-[\,]_{31}(b_4-1)\otimes[\,]_{31}(b_4-1)^{p-2-i}\circ[\,]_{43}(b_1-1)^{i+i}\}.(b_2-1) \\
-\{[\,]_{14}(b_2-1)\otimes[\,]_{14}(b_2-1)^{p-2-i}\circ[\,]_{14}^{i+1}-[\,]_{14}\otimes[\,]_{14}(b_2-1)^{p-1-i}\circ[\,]_{14}^{i} \\
+[\,]_{42}(b_1-1)\otimes[\,]_{42}(b_1-1)^{p-2-i}\circ[\,]_{42}^{i+1}-[\,]_{42}\otimes[\,]_{42}(b_1-1)^{p-1-i}\circ[\,]_{42} \\
+[\,]_{21}(b_4-1)\otimes[\,]_{21}(b_4-1)^{p-2-i}\circ[\,]_{21}^{i+1}-[\,]_{21}\otimes[\,]_{21}(b_4-1)^{p-1-i}\circ[\,]_{21}^{i} \\
+[\,]_{14}(b_2-1)\otimes[\,]_{42}(b_1-1)^{p-1-i}\circ[\,]_{14}(b_2-1)^{i} \\
-[\,]_{42}(b_1-1)\otimes[\,]_{42}(b_1-1)^{p-2-i}\circ[\,]_{14}(b_2-1)^{i+i}\}.(b_3-1)
\end{array}
\right\}
\tag{3.35}
$$

Here we identify $P\otimes M^{p-1}\otimes_G \mathbf{R}G$ with $P\otimes M^{p-1}$.

Now, the inverse image of (3.35), in $(\mathcal{M}^pM \otimes_G RG)$ is

$$\frac{1}{p}\sum_{i=0}^{p-2} C_i^{p-1} \left\{ \begin{array}{c} [[\,]_{21}(b_3-1),[\,]_{21}^{i+1},[\,]_{21}(b_3-1)^{p-2-i}](b_4-1) \\ +[[\,]_{13}(b_2-1),[\,]_{13}^{i+1},[\,]_{13}(b_2-1)^{p-2-i}](b_4-1) \\ +[[\,]_{32}(b_1-1),[\,]_{32}^{i+1},[\,]_{32}(b_1-1)^{p-2-i}](b_4-1) \\ +[[\,]_{21}(b_3-1),[\,]_{13}(b_2-1)^{p-1-i},[\,]_{21}(b_3-1)^{i}](b_4-1) \\ -[[\,]_{32}(b_4-1),[\,]_{32}^{i+1},[\,]_{32}(b_4-1)^{p-2-i}](b_1-1) \\ -[[\,]_{24}(b_3-1),[\,]_{24}^{i+1},[\,]_{24}(b_3-1)^{p-2-i}](b_1-1) \\ -[[\,]_{43}(b_2-1),[\,]_{43}^{i+1},[\,]_{43}(b_2-1)^{p-2-i}](b_1-1) \\ -[[\,]_{32}(b_4-1),[\,]_{24}(b_3-1)^{p-1-i},[\,]_{32}(b_4-1)^{i}](b_1-1) \\ +[[\,]_{43}(b_1-1),[\,]_{43}^{i+1},[\,]_{43}(b_1-1)^{p-2-i}](b_2-1) \\ +[[\,]_{31}(b_4-1),[\,]_{31}^{i+1},[\,]_{31}(b_4-1)^{p-2-i}](b_2-1) \\ +[[\,]_{14}(b_3-1),[\,]_{14}^{i+1},[\,]_{14}(b_3-1)^{p-2-i}](b_2-1) \\ +[[\,]_{43}(b_1-1),[\,]_{31}(b_4-1)^{p-1-i},[\,]_{43}(b_1-1)^{i}](b_2-1) \\ -[[\,]_{14}(b_2-1),[\,]_{14}^{i+1},[\,]_{14}(b_2-1)^{p-2-i}](b_3-1) \\ -[[\,]_{42}(b_1-1),[\,]_{42}^{i+1},[\,]_{42}(b_1-1)^{p-2-i}](b_3-1) \\ -[[\,]_{21}(b_4-1),[\,]_{21}^{i+1},[\,]_{21}(b_4-1)^{p-2-i}](b_3-1) \\ -[[\,]_{14}(b_2-1),[\,]_{42}(b_1-1)^{p-1-i},[\,]_{14}(b_2-1)^{i}](b_3-1) \end{array} \right\} \qquad (3.36)$$

Our next goal is to write $p$ times the element (3.36) as a linear combination of terms with coefficients multiples of $p$, and this will be established in the following steps.

**Step 1.** In (3.36), we have four symmetric sums. Looking more closely at the first three terms in each sum we find some similarity between each two terms, for example the first term in the first sum and the third term in the fourth sum.

Now, we collect the similar terms together and we consider them simultaneously,

$$\sum_{i=0}^{p-2} C_i^{p-1} \left\{ \begin{array}{c} \left[[\,]_{21}(b_3-1),[\,]_{21}^{i+1},[\,]_{21}(b_3-1)^{p-2-i}\right].(b_4-1) \\ -\left[[\,]_{21}(b_4-1),[\,]_{21}^{i+1},[\,]_{21}(b_4-1)^{p-2-i}\right].(b_3-1) \end{array} \right\} \tag{3.37}$$

$$\sum_{i=0}^{p-2} C_i^{p-1} \left\{ \begin{array}{c} -\left[[\,]_{32}(b_4-1),[\,]_{32}^{i+1},[\,]_{32}(b_4-1)^{p-2-i}\right].(b_1-1) \\ +\left[[\,]_{32}(b_1-1),[\,]_{32}^{i+1},[\,]_{32}(b_1-1)^{p-2-i}\right].(b_4-1) \end{array} \right\} \tag{3.38}$$

$$\sum_{i=0}^{p-2} C_i^{p-1} \left\{ \begin{array}{c} +\left[[\,]_{43}(b_1-1),[\,]_{43}^{i+1},[\,]_{43}(b_1-1)^{p-2-i}\right].(b_2-1) \\ -\left[[\,]_{43}(b_2-1),[\,]_{43}^{i+1},[\,]_{43}(b_2-1)^{p-2-i}\right].(b_1-1) \end{array} \right\} \tag{3.39}$$

$$\sum_{i=0}^{p-2} C_i^{p-1} \left\{ \begin{array}{c} -\left[[\,]_{14}(b_2-1),[\,]_{14}^{i+1},[\,]_{14}(b_2-1)^{p-2-i}\right].(b_3-1) \\ +\left[[\,]_{14}(b_3-1),[\,]_{14}^{i+1},[\,]_{14}(b_3-1)^{p-2-i}\right].(b_2-1) \end{array} \right\} \tag{3.40}$$

$$\sum_{i=0}^{p-2} C_i^{p-1} \left\{ \begin{array}{c} \left[[\,]_{13}(b_2-1),[\,]_{13}^{i+1},[\,]_{13}(b_2-1)^{p-2-i}\right].(b_4-1) \\ -\left[[\,]_{13}(b_4-1),[\,]_{13}^{i+1},[\,]_{13}(b_4-1)^{p-2-i}\right].(b_2-1) \end{array} \right\} \tag{3.41}$$

$$\sum_{i=0}^{p-2} C_i^{p-1} \left\{ \begin{array}{c} -\left[[\,]_{24}(b_3-1),[\,]_{24}^{i+1},[\,]_{24}(b_3-1)^{p-2-i}\right].(b_1-1) \\ +\left[[\,]_{24}(b_1-1),[\,]_{24}^{i+1},[\,]_{24}(b_1-1)^{p-2-i}\right].(b_3-1) \end{array} \right\} \tag{3.42}$$

The terms in (3.37) can be rewritten as

$$\sum_{i=0}^{p-2} C_i^{p-1} \left\{ \begin{array}{c} [[\,]_{21}(b_3-1)b_4,([\,]_{21}b_4)^{i+1},([\,]_{21}(b_3-1)b_4)^{p-2-i}] \\ -[[\,]_{21}(b_3-1),[\,]_{21}^{i+1},[\,]_{21}(b_3-1)^{p-2-i}] \\ -[[\,]_{21}(b_4-1)b_3,([\,]_{21}b_3)^{i+1},([\,]_{21}(b_4-1)b_3)^{p-2-i}] \\ +[[\,]_{21}(b_4-1),[\,]_{21}^{i+1},[\,]_{21}(b_4-1)^{p-2-i}] \end{array} \right\} \tag{3.43}$$

By writing $b_4 = (b_4 - 1) + 1$ and $b_3 = (b_3 - 1) + 1$, we obtain from (3.43), the following

$$\sum_{i=0}^{p-2} C_i^{p-1} \left\{ \begin{array}{c} [[\,]_{21}(b_3\text{-}1)(b_4\text{-}1), ([\,]_{21}(b_4\text{-}1) + [\,]_{21})^{i+1}, ([\,]_{21}(b_3\text{-}1)(b_4\text{-}1) + [\,]_{21}(b_3\text{-}1))^{p-2-i}] \\ +[[\,]_{21}(b_3\text{-}1), ([\,]_{21}(b_4\text{-}1) + [\,]_{21})^{i+1}, ([\,]_{21}(b_3\text{-}1)(b_4\text{-}1) + [\,]_{21}(b_3\text{-}1))^{p-2-i}] \\ -[[\,]_{21}(b_3 - 1), [\,]_{21}^{i+1}, [\,]_{21}(b_3 - 1)^{p-2-i}] \\ -[[\,]_{21}(b_4\text{-}1)(b_3\text{-}1), ([\,]_{21}(b_3\text{-}1) + [\,]_{21})^{i+1}, ([\,]_{21}(b_4\text{-}1)(b_3\text{-}1) + [\,]_{21}(b_4\text{-}1))^{p-2-i}] \\ -[[\,]_{21}(b_4\text{-}1), ([\,]_{21}(b_3\text{-}1) + [\,]_{21})^{i+1}, ([\,]_{21}(b_4\text{-}1)(b_3\text{-}1) + [\,]_{21}(b_4\text{-}1))^{p-2-i}] \\ +[[\,]_{21}(b_4 - 1), [\,]_{21}^{i+1}, [\,]_{21}(b_4 - 1)^{p-2-i}] \end{array} \right\} \tag{3.44}$$

Now our aim is to write (3.44), as a linear combination of terms with coefficients divisible by $p$. We simplify notation by setting

$$[\,]_{21} = a_{21}\,,\ [\,]_{21}(b_3 - 1) = a_{213}\,,\ [\,]_{21}(b_4 - 1) = a_{214}\,,\ [\,]_{21}(b_3 - 1)(b_4 - 1) = a_{2134}$$

$$[\,]_{13} = a_{13}\,,\ [\,]_{13}(b_2 - 1) = a_{132}\,,\ [\,]_{13}(b_4 - 1) = a_{134}\,,\ [\,]_{13}(b_2 - 1)(b_4 - 1) = a_{1324}$$

$$[\,]_{32} = a_{32}\,,\ [\,]_{32}(b_1 - 1) = a_{321}\,,\ [\,]_{32}(b_4 - 1) = a_{324}\,,\ [\,]_{32}(b_1 - 1)(b_4 - 1) = a_{3214}$$

$$[\,]_{43} = a_{43}\,,\ [\,]_{43}(b_1 - 1) = a_{431}\,,\ [\,]_{43}(b_2 - 1) = a_{432}\,,\ [\,]_{43}(b_1 - 1)(b_2 - 1) = a_{4312}$$

$$[\,]_{41} = a_{41}\,,\ [\,]_{41}(b_3 - 1) = a_{413}\,,\ [\,]_{41}(b_2 - 1) = a_{412}\,,\ [\,]_{41}(b_3 - 1)(b_2 - 1) = a_{4132}$$

$$[\,]_{24} = a_{24}\,,\ [\,]_{24}(b_3 - 1) = a_{243}\,,\ [\,]_{24}(b_1 - 1) = a_{241}\,,\ [\,]_{24}(b_3 - 1)(b_1 - 1) = a_{2431}$$

Using these notations, (3.44) turns into

$$\sum_{i=0}^{p-2} C_i^{p-1} \left\{ \begin{array}{c} [a_{2134}, a_{214}, (a_{214}+a_{21})^i, (a_{2134}+a_{213})^{p-2-i}] \\ +[a_{2134}, a_{21}, (a_{214}+a_{21})^i, (a_{2134}+a_{213})^{p-2-i}] \\ +[a_{213}, a_{214}, (a_{214}+a_{21})^i, (a_{2134}+a_{213})^{p-2-i}] \\ +[a_{213}, a_{21}, (a_{214}+a_{21})^i, (a_{2134}+a_{213})^{p-2-i}] \\ -[a_{213}, a_{21}^{i+1}, a_{213}^{p-2-i}] \\ -[a_{2134}, a_{213}, (a_{213}+a_{21})^i, (a_{2134}+a_{214})^{p-2-i}] \\ -[a_{2134}, a_{21}, (a_{213}+a_{21})^i, (a_{2134}+a_{214})^{p-2-i}] \\ -[a_{214}, a_{213}, (a_{213}+a_{21})^i, (a_{2134}+a_{214})^{p-2-i}] \\ -[a_{214}, a_{21}, (a_{213}+a_{21})^i, (a_{2134}+a_{214})^{p-2-i}] \\ +[a_{214}, a_{21}^{i+1}, a_{214}^{p-2-i}] \end{array} \right\} \tag{3.45}$$

After expanding, (3.45) can be written as a sum of two terms, $A+B$, where

$$A = \left\{ \begin{array}{c} \sum_{i=0}^{p-2}\sum_{j=0}^{i}\sum_{k=0}^{p-2-i} C_i^{p-1}C_{i-j}^{i}C_k^{p-2-i}[a_{2134}, a_{214}^{j+1}, a_{21}^{i-j}, a_{213}^{k}, a_{2134}^{p-2-i-k}] \\ +\sum_{i=0}^{p-2}\sum_{j=0}^{i}\sum_{k=0}^{p-2-i} C_i^{p-1}C_{i-j}^{i}C_k^{p-2-i}[a_{2134}, a_{21}^{i+1-j}, a_{214}^{j}, a_{213}^{k}, a_{2134}^{p-2-i-k}] \\ +\sum_{i=0}^{p-2}\sum_{j=0}^{i}\sum_{k=0}^{p-2-i} C_i^{p-1}C_{i-j}^{i}C_k^{p-2-i}[a_{213}, a_{214}^{j+1}, a_{21}^{i-j}, a_{213}^{k}, a_{2134}^{p-2-i-k}] \\ +\sum_{i=0}^{p-2}\sum_{j=0}^{i}\sum_{k=0}^{p-2-i} C_i^{p-1}C_{i-j}^{i}C_k^{p-2-i}[a_{213}, a_{21}^{i+1-j}, a_{214}^{j}, a_{213}^{k}, a_{2134}^{p-2-i-k}] \\ -\sum_{i=0}^{p-2} C_i^{p-1}[a_{213}, a_{21}^{i+1}, a_{213}^{p-2-i}] \end{array} \right\}$$

$$B = \left\{ \begin{array}{c} -\sum_{i=0}^{p-2}\sum_{k=0}^{i}\sum_{j=0}^{p-2-i} C_i^{p-1}C_{i-k}^{i}C_j^{p-2-i}[a_{2134}, a_{213}^{k+1}, a_{21}^{i-k}, a_{214}^{j}, a_{2134}^{p-2-i-j}] \\ -\sum_{i=0}^{p-2}\sum_{k=0}^{i}\sum_{j=0}^{p-2-i} C_i^{p-1}C_{i-k}^{i}C_j^{p-2-i}[a_{2134}, a_{21}^{i+1-k}, a_{213}^{k}, a_{214}^{j}, a_{2134}^{p-2-i-j}] \\ -\sum_{i=0}^{p-2}\sum_{k=0}^{i}\sum_{j=0}^{p-2-i} C_i^{p-1}C_{i-k}^{i}C_j^{p-2-i}[a_{214}, a_{213}^{k+1}, a_{21}^{i-k}, a_{214}^{j}, a_{2134}^{p-2-i-j}] \\ -\sum_{i=0}^{p-2}\sum_{k=0}^{i}\sum_{j=0}^{p-2-i} C_i^{p-1}C_{i-k}^{i}C_j^{p-2-i}[a_{214}, a_{21}^{i+1-k}, a_{213}^{k}, a_{214}^{j}, a_{2134}^{p-2-i-j}] \\ +\sum_{i=0}^{p-2} C_i^{p-1}[a_{214}, a_{21}^{i+1}, a_{214}^{p-2-i}] \end{array} \right\}$$

In order to collect the similar terms, we rearrange $A$ and $B$, as the following

$$\left\{\begin{array}{l}
\sum_{\substack{i=1\\ i+j+k\le p-1}}^{p-1}\sum_{j=0}^{p-2}\sum_{k=0}^{p-2} C_{i+j-1}^{p-1}\, C_j^{i+j-1} C_k^{p-1-i-j}\,[a_{2134}, a_{214}^{i}, a_{21}^{j}, a_{213}^{k}, a_{2134}^{p-1-i-j-k}]\\
+\sum_{\substack{i=0\\ i+j+k\le p-1}}^{p-2}\sum_{j=1}^{p-1}\sum_{k=0}^{p-2} C_{i+j-1}^{p-1}\, C_{j-1}^{i+j-1} C_k^{p-1-i-j}\,[a_{2134}, a_{21}^{j}, a_{214}^{i}, a_{213}^{k}, a_{2134}^{p-1-i-j-k}]\\
+\sum_{\substack{i=1\\ i+j+k\le p}}^{p-1}\sum_{j=0}^{p-2}\sum_{k=1}^{p-1} C_{i+j-1}^{p-1}\, C_j^{i+j-1} C_{k-1}^{p-1-i-j}\,[a_{213}, a_{214}^{i}, a_{21}^{j}, a_{213}^{k-1}, a_{2134}^{p-i-j-k}]\\
+\sum_{\substack{i=0\\ i+j+k\le p}}^{p-2}\sum_{j=1}^{p-1}\sum_{k=1}^{p-1} C_{i+j-1}^{p-1}\, C_{j-1}^{i+j-1} C_{k-1}^{p-1-i-j}\,[a_{213}, a_{21}^{j}, a_{214}^{i}, a_{213}^{k-1}, a_{2134}^{p-i-j-k}]\\
\qquad\qquad\qquad -\sum_{j=1}^{p-1} C_{j-1}^{p-1}[a_{213}, a_{21}^{j}, a_{213}^{p-1-j}]
\end{array}\right\} \tag{3.46}$$

$$\left\{\begin{array}{l}
-\sum_{\substack{i=0\\ i+j+k\le p-1}}^{p-2}\sum_{j=0}^{p-2}\sum_{k=1}^{p-1} C_{k+j-1}^{p-1} C_j^{k+j-1}\, C_i^{p-1-j-k}[a_{2134}, a_{213}^{k}, a_{21}^{j}, a_{214}^{i}, a_{2134}^{p-1-i-j-k}]\\
-\sum_{\substack{i=0\\ i+j+k\le p-1}}^{p-2}\sum_{j=1}^{p-1}\sum_{k=0}^{p-2} C_{k+j-1}^{p-1} C_{j-1}^{k+j-1}\, C_i^{p-1-j-k}\,[a_{2134}, a_{21}^{j}, a_{213}^{k}, a_{214}^{i}, a_{2134}^{p-1-i-j-k}]\\
-\sum_{\substack{i=1\\ i+j+k\le p}}^{p-1}\sum_{j=0}^{p-2}\sum_{k=1}^{p-1} C_{k+j-1}^{p-1}\, C_j^{k+j-1}\, C_{i-1}^{p-1-j-k}\,[a_{214}, a_{213}^{k}, a_{21}^{j}, a_{214}^{i-1}, a_{2134}^{p-i-j-k}]\\
-\sum_{\substack{i=1\\ i+j+k\le p}}^{p-1}\sum_{j=1}^{p-1}\sum_{k=0}^{p-2} C_{k+j-1}^{p-1} C_{j-1}^{k+j-1}\, C_{i-1}^{p-1-j-k}\,[a_{214}, a_{21}^{j}, a_{213}^{k}, a_{214}^{i-1}, a_{2134}^{p-i-j-k}]\\
\qquad\qquad\qquad +\sum_{j=1}^{p-1} C_{j-1}^{p-1}[a_{214}, a_{21}^{j}, a_{214}^{p-1-j}]
\end{array}\right\} \tag{3.47}$$

From (3.46) and (3.47), we obtain

$$\left\{\begin{array}{l}\left\{\begin{array}{l}\sum\limits_{\substack{i+j+k<p\\ i\neq 0}} C^{p-1}_{i+j-1}\, C^{i+j-1}_{j} C^{p-1-i-j}_{k}\, [a_{2134}, a^{i}_{214}, a^{j}_{21}, a^{k}_{213}, a^{p-1-i-j-k}_{2134}] \\ + \sum\limits_{\substack{i+j+k<p\\ j\neq 0}} C^{p-1}_{i+j-1}\, C^{i+j-1}_{j-1} C^{p-1-i-j}_{k}\, [a_{2134}, a^{j}_{21}, a^{i}_{214}, a^{k}_{213}, a^{p-1-i-j-k}_{2134}] \\ + \sum\limits_{\substack{i+j+k\leq p\\ i\neq 0,\, k\neq 0}} C^{p-1}_{i+j-1}\, C^{i+j-1}_{j}\, C^{p-1-i-j}_{k-1}\, [a_{213}, a^{i}_{214}, a^{j}_{21}, a^{k-1}_{213}, a^{p-i-j-k}_{2134}] \\ + \sum\limits_{\substack{i+j+k\leq p\\ k\neq 0,\, j\neq 0\\ k+j\neq p}} C^{p-1}_{i+j-1}\, C^{i+j-1}_{j-1} C^{p-1-i-j}_{k-1}\, [a_{213}, a^{j}_{21}, a^{i}_{214}, a^{k-1}_{213}, a^{p-i-j-k}_{2134}] \end{array}\right\} \\ \left\{\begin{array}{l} - \sum\limits_{\substack{i+j+k<p\\ k\neq 0}} C^{p-1}_{k+j-1} C^{k+j-1}_{j}\, C^{p-1-j-k}_{i} [a_{2134}, a^{k}_{213}, a^{j}_{21}, a^{i}_{214}, a^{p-1-i-j-k}_{2134}] \\ - \sum\limits_{\substack{i+j+k<p\\ j\neq 0}} C^{p-1}_{k+j-1} C^{k+j-1}_{j-1}\, C^{p-1-j-k}_{i}\, [a_{2134}, a^{j}_{21}, a^{k}_{213}, a^{i}_{214}, a^{p-1-i-j-k}_{2134}] \\ - \sum\limits_{\substack{i+j+k\leq p\\ i\neq 0,\, k\neq 0}} C^{p-1}_{k+j-1}\, C^{k+j-1}_{j}\, C^{p-1-j-k}_{i-1} [a_{214}, a^{k}_{213}, a^{j}_{21}, a^{i-1}_{214}, a^{p-i-j-k}_{2134}] \\ - \sum\limits_{\substack{i+j+k\leq p\\ i\neq 0,\, j\neq 0\\ i+j\neq p}} C^{p-1}_{k+j-1} C^{k+j-1}_{j-1}\, C^{p-1-j-k}_{i-1}\, [a_{214}, a^{j}_{21}, a^{k}_{213}, a^{i-1}_{214}, a^{p-i-j-k}_{2134}] \end{array}\right\} \end{array}\right\} \tag{3.48}$$

Using the Jacobi identity to relate some terms to others in (3.48), enables us to collect coefficients of similar terms, and then we get terms with coefficients divisible by $p$.

1. $C^{p-1}_{i+j-1}\, C^{i+j-1}_{j} C^{p-1-i-j}_{k}\, [a_{2134}, a^{i}_{214}, a^{j}_{21}, a^{k}_{213}, a^{\overline{p}-1}_{2134}]$, where $\overline{p} = p - i - j - k$.

2. $-C^{p-1}_{k+j-1} C^{k+j-1}_{j}\, C^{p-1-j-k}_{i} [a_{2134}, a^{k}_{213}, a^{j}_{21}, a^{i}_{214}, a^{\overline{p}-1}_{2134}]$.

3. $(C^{p-1}_{i+j-1}\, C^{i+j-1}_{j-1}\, C^{p-1-i-j}_{k} - C^{p-1}_{k+j-1} C^{k+j-1}_{j-1}\, C^{p-1-j-k}_{i})[a_{2134}, a^{j}_{21}, a^{i}_{214}, a^{k}_{213}, a^{\overline{p}-1}_{2134}]$.

4. $(C^{p-1}_{i+j-1}\, C^{i+j-1}_{j} C^{p-1-i-j}_{k-1} + C^{p-1}_{k+j-1}\, C^{k+j-1}_{j}\, C^{p-1-j-k}_{i-1})[a_{213}, a^{i}_{214}, a^{j}_{21}, a^{k-1}_{213}, a^{\overline{p}}_{2134}]$,

   here we have two cases,

a. If $\overline{p} \neq 0$, then by Jacobi identity $[a_{213}, a_{214}^i, a_{21}^j, a_{213}^{k-1}, a_{2134}^{\overline{p}}]$

$$= [a_{2134}, a_{214}^i, a_{21}^j, a_{213}^k, a_{2134}^{\overline{p}-1}] - [a_{2134}, a_{213}^k, a_{21}^j, a_{214}^i, a_{2134}^{\overline{p}-1}]$$

b. If $\overline{p} = 0$, then we get

$$(C_{i+j-1}^{p-1} C_j^{i+j-1} + C_{p-1-i}^{p-1} C_j^{p-1-i})[a_{213}, a_{214}^i, a_{21}^j, a_{213}^{p-1-i-j}]$$

5. $C_{i+j-1}^{p-1} C_{j-1}^{i+j-1} C_{k-1}^{p-1-i-j} [a_{213}, a_{21}^j, a_{214}^i, a_{213}^{k-1}, a_{2134}^{\overline{p}}]$, again we have two cases.

a. If $\overline{p} \neq 0$, then 5 becomes:

$$C_{i+j-1}^{p-1} C_{j-1}^{i+j-1} C_{k-1}^{p-1-i-j} \left\{ \begin{array}{c} [a_{2134}, a_{21}^j, a_{214}^i, a_{213}^k, a_{2134}^{\overline{p}-1}] \\ -[a_{2134}, a_{213}^k, a_{21}^j, a_{214}^i, a_{2134}^{\overline{p}-1}] \end{array} \right\}$$

b. If $\overline{p} = 0$, we get

$$C_{i+j-1}^{p-1} C_{j-1}^{i+j-1} [a_{213}, a_{21}^j, a_{214}^i, a_{213}^{p-1-i-j}]$$

6. $-C_{k+j-1}^{p-1} C_{j-1}^{k+j-1} C_{i-1}^{p-1-j-k} [a_{214}, a_{21}^j, a_{213}^k, a_{214}^{i-1}, a_{2134}^{\overline{p}}]$, again we have two cases.

a. If $\overline{p} \neq 0$, then we have

$$C_{k+j-1}^{p-1} C_{j-1}^{k+j-1} C_{i-1}^{p-1-j-k} \left\{ \begin{array}{c} [a_{2134}, a_{214}^i, a_{21}^j, a_{213}^k, a_{2134}^{\overline{p}-1}] \\ -[a_{2134}, a_{21}^j, a_{214}^i, a_{213}^k, a_{2134}^{\overline{p}-1}] \end{array} \right\}$$

b. If $\overline{p} = 0$, we get

$$-C_{p-1-i}^{p-1} C_{j-1}^{p-1-i} [a_{214}, a_{21}^j, a_{213}^{p-i-j}, a_{214}^{i-1}]$$

By Lemma 2.5.1 $C_{i+j-1}^{p-1} C_{j-1}^{i+j-1} = C_{p-1-i}^{p-1} C_{j-1}^{p-1-i}$. Then using the Jacobi identity we get from 5(b) and 6(b) the following:

7. $\sum_{\substack{i=1 \\ i+j \leq p-1}}^{p-2} \sum_{j=1}^{p-2} C_{p-1-i}^{p-1} C_{j-1}^{p-1-i} [a_{213}, a_{214}^i, a_{21}^j, a_{213}^{p-1-i-j}]$

Now we collect the coefficients of similar terms, and we start with the coefficient of

$$[a_{2134}, a_{214}^{i}, a_{21}^{j}, a_{213}^{k}, a_{2134}^{\overline{p-1}}].$$

From 1, 4(a) and 6(a) we get

$$\begin{aligned}
&= \left\{ \begin{array}{c} C_{i+j-1}^{p-1} C_j^{i+j-1} C_k^{p-1-i-j} + C_{i+j-1}^{p-1} C_j^{i+j-1} C_{k-1}^{p-1-i-j} \\ + C_{k+j-1}^{p-1} C_j^{k+j-1} C_{i-1}^{p-1-j-k} + C_{k+j-1}^{p-1} C_{j-1}^{k+j-1} C_{i-1}^{p-1-j-k} \end{array} \right\} \\
&= \left\{ \begin{array}{c} C_{i+j-1}^{p-1} C_j^{i+j-1} \{ C_k^{p-1-i-j} + C_{k-1}^{p-1-i-j} \} \\ + C_{k+j-1}^{p-1} C_{i-1}^{p-1-j-k} \{ C_j^{k+j-1} + C_{j-1}^{k+j-1} \} \end{array} \right\} \\
&= C_{i+j-1}^{p-1} C_j^{i+j-1} C_k^{p-i-j} + C_{k+j-1}^{p-1} C_{i-1}^{p-1-j-k} C_j^{k+j} \\
&= \frac{(p-1)!}{j!k!(i-1)!(p-i-j-k)!} + \frac{(p-1)!(k+j)}{(p-k-j)!(i-1)!(p-i-j-k)!j!k!} \\
&= \frac{(p-1)!}{j!k!(i-1)!(p-i-j-k)!} \left\{ 1 + \frac{k+j}{p-k-j} \right\} \\
&= \frac{p(p-1)!}{j!k!(i-1)!(p-i-j-k)!(p-k-j)} \\
&= \frac{p}{(p-k-j)} C_{k+j}^{p-1} C_k^{k+j} C_{i-1}^{p-1-k-j}
\end{aligned}$$

From 2, 4(a) and 5(a), the coefficient of $[a_{2134}, a_{213}^{k}, a_{21}^{j}, a_{214}^{i}, a_{2134}^{\overline{p-1}}]$ is

$$\begin{aligned}
&= \left\{ \begin{array}{c} -C_{k+j-1}^{p-1} C_j^{k+j-1} C_i^{p-1-k-j} - C_{i+j-1}^{p-1} C_j^{i+j-1} C_{k-1}^{p-1-i-j} \\ -C_{k+j-1}^{p-1} C_j^{k+j-1} C_{i-1}^{p-1-j-k} - C_{i+j-1}^{p-1} C_{j-1}^{i+j-1} C_{k-1}^{p-1-j-i} \end{array} \right\} \\
&= -C_{k+j-1}^{p-1} C_j^{k+j-1} \{ C_i^{p-1-k-j} + C_{i-1}^{p-1-j-k} \} - C_{i+j-1}^{p-1} C_{k-1}^{p-1-i-j} \{ C_j^{i+j-1} + C_{j-1}^{i+j-1} \} \\
&= -C_{k+j-1}^{p-1} C_j^{k+j-1} C_i^{p-k-j} - C_{i+j-1}^{p-1} C_{k-1}^{p-1-i-j} C_j^{i+j} \\
&= -\frac{(p-1)!}{j!i!(k-1)!(p-i-j-k)!} - \frac{(p-1)!(i+j)}{(p-i-j)!(k-1)!(p-i-j-k)!j!i!} \\
&= -\frac{(p-1)!}{j!i!(k-1)!(p-i-j-k)!} \left\{ 1 + \frac{i+j}{p-i-j} \right\} \\
&= -\frac{p(p-1)!}{j!i!(k-1)!(p-i-j-k)!(p-i-j)} \\
&= -\frac{p}{(p-i-j)} C_{i+j}^{p-1} C_i^{i+j} C_{k-1}^{p-1-i-j}
\end{aligned}$$

From 3, 5(a) and 6(a) the coefficient of $[a_{2134}, a_{21}^{j}, a_{214}^{i}, a_{213}^{k}, a_{2134}^{\overline{p}-1}]$ is

$$
\begin{aligned}
&= \left\{ \begin{array}{l} C_{i+j-1}^{p-1} C_{j-1}^{i+j-1} C_{k}^{p-1-i-j} - C_{j+k-1}^{p-1} C_{j-1}^{j+k-1} C_{i}^{p-1-j-k} \\ +C_{i+j-1}^{p-1} C_{j-1}^{i+j-1} C_{k-1}^{p-1-i-j} - C_{j+k-1}^{p-1} C_{j-1}^{j+k-1} C_{i-1}^{p-1-k-j} \end{array} \right\} \\
&= C_{i+j-1}^{p-1} C_{j-1}^{i+j-1} \{C_{k}^{p-1-i-j} + C_{k-1}^{p-1-i-j}\} - C_{j+k-1}^{p-1} C_{j-1}^{j+k-1} \{C_{i}^{p-1-j-k} + C_{i-1}^{p-1-k-j}\} \\
&= C_{i+j-1}^{p-1} C_{j-1}^{i+j-1} C_{k}^{p-i-j} - C_{j+k-1}^{p-1} C_{j-1}^{j+k-1} C_{i}^{p-j-k} \\
&= 0
\end{aligned}
$$

Finally from 4(b) and 7 the coefficient of $[a_{213}, a_{214}^{i}, a_{21}^{j}, a_{213}^{p-1-i-j}]$ is

$$
\begin{aligned}
&= C_{i+j-1}^{p-1} C_{j}^{i+j-1} + C_{p-1-i}^{p-1} C_{j}^{p-1-i} + C_{i+j-1}^{p-1} C_{j-1}^{i+j-1} \\
&= C_{i+j-1}^{p-1} \{C_{j}^{i+j-1} + C_{j-1}^{i+j-1}\} + C_{p-1-i}^{p-1} C_{j}^{p-1-i} \\
&= C_{i+j-1}^{p-1} C_{j}^{i+j} + C_{p-1-i}^{p-1} C_{j}^{p-1-i} \\
&= \frac{p(p-1)!}{i!j!(p-1-i-j)!(p-i-j)} \\
&= \frac{p(p-1)!}{i!j!(p-i-j)!} = \frac{p}{i+j} \overset{p-1}{\underset{i+j-1}{C}} \overset{i+j}{\underset{j}{C}}
\end{aligned}
$$

Therefore, as result of this we obtain

$$
\left\{ \begin{array}{l}
\sum\limits_{\substack{i+j+k<p \\ i\neq 0}} \frac{p}{(p-k-j)} C_{k+j}^{p-1} C_{k}^{k+j} C_{i-1}^{p-1-k-j} [a_{2134}, a_{214}^{i}, a_{21}^{j}, a_{213}^{k}, a_{2134}^{\overline{p}-1}] \\
+ \sum\limits_{\substack{i+j+k<p \\ j+k\neq p,\, k\neq 0}} \frac{p}{(p-i-j)} C_{i+j}^{p-1} C_{i}^{i+j} C_{k-1}^{p-1-i-j} [a_{213}, a_{2134}^{\overline{p}}, a_{214}^{i}, a_{21}^{j}, a_{213}^{k-1}] \\
\quad + \sum\limits_{\substack{i+j<p \\ i\neq 0}} \frac{p}{i+j} C_{i+j-1}^{p-1} C_{j}^{i+j} [a_{213}, a_{214}^{i}, a_{21}^{j}, a_{213}^{p-1-i-j}]
\end{array} \right\}
$$

and this gives the following:

$$
\left\{ \begin{array}{l}
\sum\limits_{\substack{i+j+k<p \\ i\neq 0}} \frac{p}{(p-k-j)} C_{k+j}^{p-1} C_{k}^{k+j} C_{i-1}^{p-1-k-j} [a_{2134}, a_{214}^{i}, a_{21}^{j}, a_{213}^{k}, a_{2134}^{\overline{p}-1}] \\
+ \sum\limits_{\substack{i+j+k\leq p \\ j+k\neq p,\, k\neq 0}} \frac{p}{(p-i-j)} C_{i+j}^{p-1} C_{i}^{i+j} C_{k-1}^{p-1-i-j} [a_{213}, a_{2134}^{\overline{p}}, a_{21}^{j}, a_{214}^{i}, a_{213}^{k-1}]
\end{array} \right\} \tag{3.49}
$$

Exactly by same computation we obtain from (3.38), (3.39), (3.40), (3.41) and (3.42) the following five summands:

$$\left\{\begin{aligned} &\sum_{\substack{i+j+k<p\\ i\neq 0}} \frac{p}{(p-k-j)} C_{k+j}^{p-1}\, C_k^{k+j}\, C_{i-1}^{p-1-k-j}[a_{3214}, a_{324}^{i}, a_{32}^{j}, a_{321}^{k}, a_{3214}^{\overline{p}-1}] \\ &+ \sum_{\substack{i+j+k\leq p\\ j+k\neq p,\, k\neq 0}} \frac{p}{(p-i-j)} C_{i+j}^{p-1} C_i^{i+j} C_{k-1}^{p-1-i-j}[a_{321}, a_{3214}^{\overline{p}}, a_{324}^{i}, a_{32}^{j}, a_{321}^{k-1}] \end{aligned}\right\} \tag{3.50}$$

$$\left\{\begin{aligned} &\sum_{\substack{i+j+k<p\\ i\neq 0}} \frac{p}{(p-k-j)} C_{k+j}^{p-1}\, C_k^{k+j}\, C_{i-1}^{p-1-k-j}[a_{4321}, a_{432}^{i}, a_{43}^{j}, a_{431}^{k}, a_{4321}^{\overline{p}-1}] \\ &+ \sum_{\substack{i+j+k\leq p\\ j+k\neq p,\, k\neq 0}} \frac{p}{(p-i-j)} C_{i+j}^{p-1} C_i^{i+j} C_{k-1}^{p-1-i-j}[a_{431}, a_{4321}^{\overline{p}}, a_{432}^{i}, a_{43}^{j}, a_{431}^{k-1}] \end{aligned}\right\} \tag{3.51}$$

$$\left\{\begin{aligned} &\sum_{\substack{i+j+k<p\\ i\neq 0}} \frac{p}{(p-k-j)} C_{k+j}^{p-1}\, C_k^{k+j}\, C_{i-1}^{p-1-k-j}[a_{1432}, a_{142}^{i}, a_{14}^{j}, a_{143}^{k}, a_{1432}^{\overline{p}-1}] \\ &+ \sum_{\substack{i+j+k\leq p\\ j+k\neq p,\, k\neq 0}} \frac{p}{(p-i-j)} C_{i+j}^{p-1} C_i^{i+j} C_{k-1}^{p-1-i-j}[a_{143}, a_{1432}^{\overline{p}}, a_{142}^{i}, a_{14}^{j}, a_{143}^{k-1}] \end{aligned}\right\} \tag{3.52}$$

$$\left\{\begin{aligned} &\sum_{\substack{i+j+k<p\\ i\neq 0}} \frac{p}{(p-k-j)} C_{k+j}^{p-1}\, C_k^{k+j}\, C_{i-1}^{p-1-k-j}[a_{1324}, a_{134}^{i}, a_{13}^{j}, a_{132}^{k}, a_{1324}^{\overline{p}-1}] \\ &+ \sum_{\substack{i+j+k\leq p\\ j+k\neq p,\, k\neq 0}} \frac{p}{(p-i-j)} C_{i+j}^{p-1} C_i^{i+j} C_{k-1}^{p-1-i-j}\, [a_{132}, a_{1324}^{\overline{p}}, a_{134}^{i}, a_{13}^{j}, a_{132}^{k-1}] \end{aligned}\right\} \tag{3.53}$$

$$\left\{\begin{aligned} &\sum_{\substack{i+j+k<p\\ i\neq 0}} \frac{p}{(p-k-j)} C_{k+j}^{p-1}\, C_k^{k+j}\, C_{i-1}^{p-1-k-j}[a_{2431}, a_{243}^{i}, a_{24}^{j}, a_{241}^{k}, a_{2431}^{\overline{p}-1}] \\ &+ \sum_{\substack{i+j+k\leq p\\ j+k\neq p,\, k\neq 0}} \frac{p}{(p-i-j)} C_{i+j}^{p-1} C_i^{i+j} C_{k-1}^{p-1-i-j}[a_{241}, a_{2431}^{\overline{p}}, a_{243}^{i}, a_{24}^{j}, a_{241}^{k-1}] \end{aligned}\right\} \tag{3.54}$$

By substituting (3.49), (3.50), (3.51), (3.52), (3.53) and (3.54) into (3.36) we obtain

$$
\frac{1}{p}\left\{\begin{array}{l}
\sum\limits_{\substack{i+j+k<p\\ i\neq 0}} \overline{\gamma}_1 \left\{\begin{array}{c}
[a_{2134}, a^{i}_{214}, a^{j}_{21}, a^{k}_{213}, a^{\overline{p}-1}_{2134}] + [a_{3214}, a^{i}_{324}, a^{j}_{32}, a^{k}_{321}, a^{\overline{p}-1}_{3214}] \\
+[a_{4312}, a^{i}_{432}, a^{j}_{43}, a^{k}_{431}, a^{\overline{p}-1}_{4312}] + [a_{1432}, a^{i}_{142}, a^{j}_{14}, a^{k}_{143}, a^{\overline{p}-1}_{1432}] \\
+[a_{1324}, a^{i}_{134}, a^{j}_{13}, a^{k}_{132}, a^{\overline{p}-1}_{1324}] + [a_{2413}, a^{i}_{243}, a^{j}_{24}, a^{k}_{241}, a^{\overline{p}-1}_{2413}]
\end{array}\right\} \\
\sum\limits_{\substack{i+j+k\leq p\\ j+k\neq p\\ k\neq 0}} \overline{\gamma}_2 \left\{\begin{array}{c}
[a_{213}, a^{\overline{p}}_{2134}, a^{i}_{214}, a^{j}_{21}, a^{k-1}_{213}] + [a_{321}, a^{\overline{p}}_{3214}, a^{i}_{324}, a^{j}_{32}, a^{k-1}_{321}] \\
+[a_{431}, a^{\overline{p}}_{4312}, a^{i}_{432}, a^{j}_{43}, a^{k-1}_{431}] + [a_{143}, a^{\overline{p}}_{1432}, a^{i}_{142}, a^{j}_{14}, a^{k-1}_{143}] \\
+[a_{132}, a^{\overline{p}}_{1324}, a^{i}_{134}, a^{k-1}_{132}, a^{j}_{13}] + [a_{241}, a^{\overline{p}}_{2413}, a^{i}_{243}, a^{j}_{24}, a^{k-1}_{241}]
\end{array}\right\} \\
\sum\limits_{i=0}^{p-2} C^{p-1}_{i} \left\{\begin{array}{c}
[[\,]_{21}(b_3-1), [\,]_{13}(b_2-1)^{p-1-i}, [\,]_{21}(b_3-1)^{i}](b_4-1) \\
-[[\,]_{32}(b_4-1), [\,]_{24}(b_3-1)^{p-1-i}, [\,]_{32}(b_4-1)^{i}](b_1-1) \\
+[[\,]_{43}(b_1-1), [\,]_{31}(b_4-1)^{p-1-i}, [\,]_{43}(b_1-1)^{i}](b_2-1) \\
-[[\,]_{14}(b_2-1), [\,]_{42}(b_1-1)^{p-1-i}, [\,]_{14}(b_2-1)^{i}](b_3-1)
\end{array}\right\}
\end{array}\right\} \tag{3.55}
$$

where, $\overline{\gamma}_1 = \frac{p}{(p-k-j)} C^{p-1}_{k+j}\, C^{k+j}_{k}\, C^{p-1-k-j}_{i-1}$ and $\overline{\gamma}_1 = \frac{p}{(p-i-j)} C^{p-1}_{i+j} C^{i+j}_{i} C^{p-1-i-j}_{k-1}$.

Now when $j = k = 0$ the coefficient $\overline{\gamma}_1$ is not divisible by $p$, and when $i = j = 0$, the coefficient $\overline{\gamma}_2$ is also not divisible by $p$. So it is convenient to rewrite (3.55), as the sum of two summands

$$
\left\{\begin{array}{l}
\sum\limits_{\substack{i+j+k<p\\ i\neq 0,\, k+j\neq 0}} \gamma_1 \left\{\begin{array}{c}
[a_{2134}, a^{i}_{214}, a^{j}_{21}, a^{k}_{213}, a^{\overline{p}-1}_{2134}] + [a_{3214}, a^{i}_{324}, a^{j}_{32}, a^{k}_{321}, a^{\overline{p}-1}_{3214}] \\
+[a_{4312}, a^{i}_{432}, a^{j}_{43}, a^{k}_{431}, a^{\overline{p}-1}_{4312}] + [a_{1432}, a^{i}_{142}, a^{j}_{14}, a^{k}_{143}, a^{\overline{p}-1}_{1432}] \\
[a_{1324}, a^{i}_{134}, a^{j}_{13}, a^{k}_{132}, a^{\overline{p}-1}_{1324}] + [a_{2413}, a^{i}_{243}, a^{j}_{24}, a^{k}_{241}, a^{\overline{p}-1}_{2413}]
\end{array}\right\} \\
\sum\limits_{\substack{i+j+k\leq p\\ k\neq 0,\, i+j\neq 0\\ j+k\neq p}} \gamma_2 \left\{\begin{array}{c}
[a_{213}, a^{\overline{p}}_{2134}, a^{i}_{214}, a^{j}_{21}, a^{k-1}_{213}] + [a_{321}, a^{\overline{p}}_{3214}, a^{i}_{324}, a^{j}_{32}, a^{k-1}_{321}] \\
[a_{431}, a^{\overline{p}}_{4312}, a^{i}_{432}, a^{j}_{43}, a^{k-1}_{431}] + [a_{143}, a^{\overline{p}}_{1432}, a^{i}_{142}, a^{j}_{14}, a^{k-1}_{143}] \\
[a_{132}, a^{\overline{p}}_{1324}, a^{i}_{134}, a^{k-1}_{132}, a^{j}_{13}] + [a_{241}, a^{\overline{p}}_{2413}, a^{i}_{243}, a^{j}_{24}, a^{k-1}_{241}]
\end{array}\right\}
\end{array}\right\} \tag{3.56}
$$

$$\frac{1}{p}\left\{\begin{array}{c}\sum_{i=0}^{p-2} C_i^{p-1}\left\{\begin{array}{c}[a_{2134}, a_{214}^{i+1}, a_{2134}^{p-2-i}] + [a_{3214}, a_{324}^{i+1}, a_{3214}^{p-2-i}] \\ +[a_{4321}, a_{432}^{i+1}, a_{4321}^{p-2-i}] + [a_{1432}, a_{142}^{i+1}, a_{1432}^{p-2-i}] \\ +[a_{1324}, a_{134}^{i+1}, a_{1324}^{p-2-i}] + [a_{2413}, a_{243}^{i+1}, a_{2413}^{p-2-i}]\end{array}\right\} \\ +\sum_{k=0}^{p-2} C_k^{p-1}\left\{\begin{array}{c}-[a_{2134}, a_{213}^{k+1}, a_{2134}^{p-2-k}] - [a_{3214}, a_{321}^{k+1}, a_{3214}^{p-2-k}] \\ -[a_{4321}, a_{431}^{k+1}, a_{4321}^{p-2-k}] - [a_{1432}, a_{143}^{k+1}, a_{1432}^{p-2-k}] \\ -[a_{1324}, a_{132}^{k+1}, a_{1324}^{p-2-k}] - [a_{2413}, a_{241}^{k+1}, a_{2413}^{p-2-k}]\end{array}\right\} \\ +\sum_{i=0}^{p-2} C_i^{p-1}\left\{\begin{array}{c}[[\,]_{21}(b_3-1), [\,]_{13}(b_2-1)^{p-1-i}, [\,]_{21}(b_3-1)^i](b_4-1) \\ -[[\,]_{32}(b_4-1), [\,]_{24}(b_3-1)^{p-1-i}, [\,]_{32}(b_4-1)^i](b_1-1) \\ +[[\,]_{43}(b_1-1), [\,]_{31}(b_4-1)^{p-1-i}, [\,]_{43}(b_1-1)^i](b_2-1) \\ -[[\,]_{14}(b_2-1), [\,]_{42}(b_1-1)^{p-1-i}, [\,]_{14}(b_2-1)^i](b_3-1)\end{array}\right\}\end{array}\right\} \quad (3.57)$$

where $\gamma_1 = C_{k+j}^{p-1} C_k^{k+j} C_{i-1}^{p-1-k-j}$ and $\gamma_2 = \frac{1}{(p-i-j)} C_{i+j}^{p-1} C_i^{i+j} C_{k-1}^{p-1-i-j}$.

**Step 2.** Our aim in this step is to write $p$ times the element (3.57), as a linear combination of terms with coefficients multiple of $p$. In order to do so, we consider the following terms:

$$\sum_{i=0}^{p-2} C_i^{p-1}\left\{\begin{array}{c}[[\,]_{21}(b_3-1), [\,]_{13}(b_2-1)^{p-1-i}, [\,]_{21}(b_3-1)^i].(b_4-1) \\ -[a_{3214}, a_{321}^{i+1}, a_{3214}^{p-2-i}]\end{array}\right\} \quad (3.58)$$

$$\sum_{i=0}^{p-2} C_i^{p-1}\left\{\begin{array}{c}-[[\,]_{32}(b_4-1), [\,]_{24}(b_3-1)^{p-1-i}, [\,]_{32}(b_4-1)^i].(b_1-1) \\ +[a_{4321}, a_{432}^{i+1}, a_{4321}^{p-2-i}]\end{array}\right\} \quad (3.59)$$

$$\sum_{i=0}^{p-2} C_i^{p-1}\left\{\begin{array}{c}[[\,]_{43}(b_1-1), [\,]_{31}(b_4-1)^{p-1-i}, [\,]_{43}(b_1-1)^i].(b_2-1) \\ -[a_{1432}, a_{143}^{i+1}, a_{1432}^{p-2-i}]\end{array}\right\} \quad (3.60)$$

$$\sum_{i=0}^{p-2} C_i^{p-1} \left\{ \begin{array}{c} -[[\,]_{14}(b_2-1),[\,]_{42}(b_1-1)^{p-1-i},[\,]_{14}(b_2-1)^i].(b_3-1) \\ +[a_{2134},a_{214}^{i+1},a_{2134}^{p-2-i}] \end{array} \right\} \tag{3.61}$$

We start our computation in these terms with (3.58), where the computation in the first term of (3.58) is exactly as before, and for the second term we apply the Jacobi identity to each component of this term. Hence, from (3.58), we obtain

$$\begin{array}{l} \sum_{i=0}^{p-2} C_i^{p-1} \left\{ \begin{array}{c} [a_{2134},a_{1324},(a_{1324}+a_{132})^{p-2-i},(a_{2134}+a_{213})^i] \\ [a_{2134},a_{132},(a_{1324}+a_{132})^{p-2-i},(a_{2134}+a_{213})^i] \\ [a_{213},a_{1324},(a_{1324}+a_{132})^{p-2-i},(a_{2134}+a_{213})^i] \\ [a_{2134},a_{132},(a_{1324}+a_{132})^{p-2-i},(a_{2134}+a_{213})^i] \\ -[a_{213},a_{132}^{p-1-i},a_{213}^i] \end{array} \right\} \\ \sum_{i=0}^{p-2} C_i^{p-1} \left\{ \begin{array}{c} [a_{2134},a_{213},(a_{213}+a_{132})^i,(a_{2134}+a_{1324})^{p-2-i}] \\ [a_{2134},a_{132},(a_{213}+a_{132})^i,(a_{2134}+a_{1324})^{p-2-i}] \\ [a_{1324},a_{213},(a_{213}+a_{132})^i,(a_{2134}+a_{1324})^{p-2-i}] \\ [a_{2134},a_{132},(a_{213}+a_{132})^i,(a_{2134}+a_{1324})^{p-2-i}] \end{array} \right\} \end{array} \tag{3.62}$$

After expanding, (3.62) can be written as a sum of two terms, $C+D$, where

$$C = \left\{ \begin{array}{c} \sum_{i=0}^{p-2}\sum_{j=0}^{i}\sum_{k=0}^{p-2-i} C_i^{p-1}C_{i-j}^{i}C_k^{p-2-i}[a_{2134},a_{214}^{j+1},a_{21}^{i-j},a_{213}^k,a_{2134}^{p-2-i-k}] \\ +\sum_{i=0}^{p-2}\sum_{j=0}^{i}\sum_{k=0}^{p-2-i} C_i^{p-1}C_{i-j}^{i}C_k^{p-2-i}[a_{2134},a_{21}^{i+1-j},a_{214}^j,a_{213}^k,a_{2134}^{p-2-i-k}] \\ +\sum_{i=0}^{p-2}\sum_{j=0}^{i}\sum_{k=0}^{p-2-i} C_i^{p-1}C_{i-j}^{i}C_k^{p-2-i}[a_{213},a_{214}^{j+1},a_{21}^{i-j},a_{213}^k,a_{2134}^{p-2-i-k}] \\ +\sum_{i=0}^{p-2}\sum_{j=0}^{i}\sum_{k=0}^{p-2-i} C_i^{p-1}C_{i-j}^{i}C_k^{p-2-i}[a_{213},a_{21}^{i+1-j},a_{214}^j,a_{213}^k,a_{2134}^{p-2-i-k}] \\ -\sum_{i=0}^{p-2} C_i^{p-1}[a_{213},a_{21}^{i+1},a_{213}^{p-2-i}] \end{array} \right.$$

$$D = \left\{ \begin{array}{l} \sum_{i=0}^{p-2}\sum_{j=0}^{i}\sum_{k=0}^{p-2-i} C_i^{p-1} C_k^{p-2-i}\, C_j^i\, [a_{2134}, a_{213}^{i+1-j}, a_{132}^{j}, a_{2134}^{k}, a_{1324}^{p-2-i-k}] \\ \sum_{i=0}^{p-2}\sum_{j=0}^{i}\sum_{k=0}^{p-2-i} C_i^{p-1} C_k^{p-2-i}\, C_j^i\, [a_{2134}, a_{132}^{j+1}, a_{213}^{i-j}, a_{2134}^{k}, a_{1324}^{p-2-i-k}] \\ \sum_{i=0}^{p-2}\sum_{j=0}^{i}\sum_{k=0}^{p-2-i} C_i^{p-1} C_k^{p-2-i}\, C_j^i\, [a_{1324}, a_{213}^{i+1-j}, a_{132}^{j}, a_{2134}^{k}, a_{1324}^{p-2-i-k}] \\ \sum_{i=0}^{p-2}\sum_{j=0}^{i}\sum_{k=0}^{p-2-i} C_i^{p-1} C_k^{p-2-i}\, C_j^i\, [a_{1324}, a_{132}^{j+1}, a_{213}^{i-j}, a_{2134}^{k}, a_{1324}^{p-2-i-k}] \end{array} \right\}$$

In order to collect the similar terms, we rearrange $C$ and $D$, as follows

$$\left\{ \begin{array}{l} \sum_{\substack{i=1\\ i+j+k\le p}}^{p-1}\sum_{j=0}^{p-2}\sum_{k=1}^{p-1} C_{i+j-1}^{p-1}\, C_j^{i+j-1} C_{k-1}^{p-1-i-j}\, [a_{2134}, a_{1324}^{k}, a_{132}^{\overline{p}}, a_{2134}^{i-1}, a_{213}^{j}] \\ \sum_{\substack{i=1\\ i+j+k\le p-1}}^{p-1}\sum_{j=0}^{p-2}\sum_{k=0}^{p-2} C_{i+j-1}^{p-1}\, C_j^{i+j-1} C_{k}^{p-1-i-j} [a_{2134}, a_{132}^{\overline{p}}, a_{1324}^{k}, a_{2134}^{i-1}, a_{213}^{j}] \\ \sum_{\substack{i=0\\ i+j+k\le p}}^{p-2}\sum_{j=1}^{p-1}\sum_{k=1}^{p-1} C_{i+j-1}^{p-1} C_{j-1}^{i+j-1} C_{k-1}^{p-1-i-j}\, [a_{213}, a_{1324}^{k}, a_{132}^{\overline{p}}, a_{2134}^{i}, a_{213}^{j-1}] \\ \sum_{\substack{i=0\\ i+j+k\le p-1}}^{p-2}\sum_{j=1}^{p-1}\sum_{k=0}^{p-2} C_{i+j-1}^{p-1} C_{j-1}^{i+j-1} C_{k}^{p-1-i-j}\, [a_{213}, a_{132}^{\overline{p}}, a_{1324}^{k}, a_{2134}^{i}, a_{213}^{j-1}] \\ \qquad\qquad - \sum_{j=0}^{p-2} C_j^{p-1}\, [a_{213}, a_{132}^{p-1-j}, a_{213}^{j}] \\ \sum_{\substack{i=1\\ i+j+k\le p}}^{p-1}\sum_{j=1}^{p-1}\sum_{k=0}^{p-2} C_{p-1-i-k}^{p-1}\, C_{j-1}^{p-1-i-k} C_k^{i+k-1} [a_{2134}, a_{213}^{j}, a_{132}^{\overline{p}}, a_{2134}^{i-1}, a_{1324}^{k}] \\ \sum_{\substack{i=1\\ i+j+k\le p-1}}^{p-1}\sum_{j=0}^{p-2}\sum_{k=0}^{p-2} C_{p-1-i-k}^{p-1}\, C_{j}^{p-1-i-k} C_k^{i+k-1} [a_{2134}, a_{132}^{\overline{p}}, a_{213}^{j}, a_{1324}^{k}, a_{2134}^{i-1}] \\ \sum_{\substack{i=0\\ i+j+k\le p}}^{p-2}\sum_{j=1}^{p-1}\sum_{k=1}^{p-1} C_{p-1-i-k}^{p-1}\, C_{j-1}^{p-1-i-k} C_{k-1}^{i+k-1}\, [a_{1324}, a_{213}^{j}, a_{132}^{\overline{p}}, a_{2134}^{i}, a_{1324}^{k-1}] \\ \sum_{\substack{i=0\\ i+j+k\le p-1}}^{p-2}\sum_{j=0}^{p-2}\sum_{k=1}^{p-1} C_{p-1-i-k}^{p-1}\, C_{j}^{p-1-i-k} C_{k-1}^{i+k-1}\, [a_{1324}, a_{132}^{\overline{p}}, a_{213}^{j}, a_{2134}^{i}, a_{1324}^{k-1}] \end{array} \right\} \tag{3.63}$$

Now using the Jacobi identity to relate some terms to others, this enables us to collect the coefficients of similar terms,

1. $C_{i+j-1}^{p-1}\, C_j^{i+j-1} C_{k-1}^{p-1-i-j}\, [a_{2134}, a_{1324}^{k}, a_{132}^{\overline{p}}, a_{2134}^{i-1}, a_{213}^{j}]$.

2. $C^{p-1}_{p-1-i-k}\, C^{p-1-i-k}_{j-1} C^{i+k-1}_{k} [a_{2134}, a^{j}_{213}, a^{\overline{p}}_{132}, a^{i-1}_{2134}, a^{k}_{1324}]$,

   a. If $k \neq 0$, then by Jacobi identity $[a_{2134}, a^{j}_{213}, a^{\overline{p}}_{132}, a^{i-1}_{2134}, a^{k}_{1324}]$

   $$= [a_{2134}, a^{k}_{1324}, a^{\overline{p}}_{132}, a^{i-1}_{2134}, a^{j}_{213}] + [a_{1324}, a^{j}_{213}, a^{\overline{p}}_{132}, a^{i}_{2134}, a^{k-1}_{1324}]$$

   b. If $k = 0$,we get

   $$C^{p-1}_{p-1-i}\; C^{p-1-i}_{j-1} [a_{2134}, a^{j}_{213}, a^{p-i-j}_{132}, a^{i-1}_{2134}]$$

3. $(C^{p-1}_{i+j-1}\, C^{i+j-1}_{j} C^{p-1-i-j}_{k} + C^{p-1}_{p-1-i-k}\, C^{p-1-i-k}_{j} C^{i+k-1}_{k}) [a_{2134}, a^{\overline{p}}_{132}, a^{k}_{1324}, a^{i-1}_{2134}, a^{j}_{213}]$

   a. If $k \neq 0$, then by Jacobi identity $[a_{2134}, a^{\overline{p}}_{132}, a^{k}_{1324}, a^{i-1}_{2134}, a^{j}_{213}]$

   $$= [a_{2134}, a^{k}_{1324}, a^{\overline{p}}_{132}, a^{i-1}_{2134}, a^{j}_{213}] + [a_{1324}, a^{\overline{p}}_{132}, a^{j}_{213}, a^{i}_{2134}, a^{k-1}_{1324}]$$

   b. If $k = 0$, we consider two cases,

   (i). If $j \neq 0$, we have by the Jacobi identity

   $$(C^{p-1}_{i+j-1}\; C^{i+j-1}_{j} + C^{p-1}_{p-1-i}\; C^{p-1-i}_{j}) \left\{ \begin{array}{c} [a_{2134}, a^{j}_{213}, a^{p-i-j}_{132} a^{i-1}_{2134}] \\ +[a_{213}, a^{p-i-j}_{132}, a^{j-1}_{213}, a^{i}_{2134}] \end{array} \right\}$$

   (ii). If $j = 0$, then we have

   $$(C^{p-1}_{i-1} + C^{p-1}_{p-1-i})[a_{2134}, a^{p-i}_{132}, a^{i-1}_{2134}] = C^{p}_{i}[a_{2134}, a^{p-i}_{132}, a^{i-1}_{2134}] \qquad (3.64)$$

4. $(C^{p-1}_{p-1-i-k}\, C^{p-1-i-k}_{j-1} C^{i+k-1}_{k-1} - C^{p-1}_{i+j-1} C^{i+j-1}_{j-1} C^{p-1-i-j}_{k-1}) [a_{1324}, a^{j}_{213}, a^{\overline{p}}_{132}, a^{i}_{2134}, a^{k-1}_{1324}]$.

5. $C^{p-1}_{i+j-1} C^{i+j-1}_{j-1} C^{p-1-i-j}_{k}\; [a_{213}, a^{\overline{p}}_{132}, a^{k}_{1324}, a^{i}_{2134}, a^{j-1}_{213}]$,

   a. If $k \neq 0$, we have

   $$C^{p-1}_{i+j-1} C^{i+j-1}_{j-1} C^{p-1-i-j}_{k} \left\{ \begin{array}{c} [a_{1324}, a^{\overline{p}}_{132}, a^{j}_{213}, a^{i}_{2134}, a^{k-1}_{1324}] \\ -[a_{1324}, a^{j}_{213}, a^{\overline{p}}_{132}, a^{i}_{2134}, a^{k-1}_{1324}] \end{array} \right\}$$

b. If $k = 0$, we get

$$C^{p-1}_{i+j-1} C^{i+j-1}_{j-1} \; [a_{213}, a_{132}^{p-i-j}, a_{2134}^{i}, a_{213}^{j-1}]$$

6. $C^{p-1}_{p-1-i-k} C^{p-1-i-k}_{j} C^{i+k-1}_{k-1} \; [a_{1324}, a_{132}^{\overline{p}}, a_{213}^{j}, a_{2134}^{i}, a_{1324}^{k-1}]$

7. $-C^{p-1}_{j} \, [a_{213}, a_{132}^{p-1-j}, a_{213}^{j}]$, this term cancels with 5(b) when $i = 0$.

Now we collect the coefficients of similar terms, and we begin with the first term.

From 1, 2(a) and 3(a) the coefficient of $[a_{2134}, a_{1324}^{k}, a_{132}^{\overline{p}}, a_{2134}^{i-1}, a_{213}^{j}]$ is

$$\begin{aligned}
= \;& C^{p-1}_{i+j-1} C^{i+j-1}_{j} C^{p-1-i-j}_{k-1} + C^{p-1}_{p-1-i-k} \; C^{p-1-i-k}_{j-1} C^{i+k-1}_{k} \\
& + C^{p-1}_{i+j-1} C^{i+j-1}_{j} C^{p-1-i-j}_{k} + \; C^{p-1}_{p-1-i-k} C^{p-1-i-k}_{j} C^{i+k-1}_{k} \\
= \;& C^{p-1}_{i+j-1} C^{i+j-1}_{j} (C^{p-1-i-j}_{k-1} + C^{p-1-i-j}_{k}) + C^{p-1}_{p-1-i-k} \; C^{i+k-1}_{k} (C^{p-1-i-k}_{j-1} + C^{p-1-i-k}_{j}) \\
= \;& C^{p-1}_{i+j-1} C^{i+j-1}_{j} C^{p-i-j}_{k} + C^{p-1}_{p-1-i-k} \; C^{i+k-1}_{k} C^{p-i-k}_{j} \\
= \;& \frac{p(p-1)!}{j!k!(i-1)!(p-i-j-k)!(i+k)} \\
= \;& \tfrac{p}{i+k} C^{p-1}_{i-1} C^{p-i}_{j+k} C^{j+k}_{k}
\end{aligned} \tag{3.65}$$

From 2(a), 5(a) and 4 the coefficient of $[a_{1324}, a_{213}^{j}, a_{132}^{\overline{p}}, a_{2134}^{i}, a_{1324}^{k-1}]$ is

$$\begin{aligned}
= \;& C^{p-1}_{p-1-i-k} \; C^{p-1-i-k}_{j-1} C^{i+k-1}_{k} - C^{p-1}_{i+j-1} C^{i+j-1}_{j-1} C^{p-1-i-j}_{k} \\
& + C^{p-1}_{p-1-i-k} \; C^{p-1-i-k}_{j-1} C^{i+k-1}_{k-1} - C^{p-1}_{i+j-1} C^{i+j-1}_{j-1} C^{p-1-i-j}_{k-1} \\
= \;& C^{p-1}_{p-1-i-k} \; C^{p-1-i-k}_{j-1} (C^{i+k-1}_{k} + C^{i+k-1}_{k-1}) - C^{p-1}_{i+j-1} C^{i+j-1}_{j-1} (C^{p-1-i-j}_{k-1} + C^{p-1-i-j}_{k}) \\
= \;& C^{p-1}_{p-1-i-k} \; C^{p-1-i-k}_{j-1} C^{i+k}_{k} - C^{p-1}_{i+j-1} C^{i+j-1}_{j-1} C^{p-i-j}_{k} \\
= \;& 0
\end{aligned} \tag{3.66}$$

From 3(a), 5(a) and 6, the coefficient of $[a_{1324}, a_{132}^{\overline{p}}, a_{213}^{j}, a_{2134}^{i}, a_{1324}^{k-1}]$ is

$$\begin{aligned}
= \quad & C_{i+j-1}^{p-1} C_j^{i+j-1} C_k^{p-1-i-j} + C_{p-1-i-k}^{p-1} C_j^{p-1-i-k} C_k^{i+k-1} \\
& + C_{i+j-1}^{p-1} C_{j-1}^{i+j-1} C_k^{p-1-i-j} + C_{p-1-i-k}^{p-1} C_j^{p-1-i-k} C_{k-1}^{i+k-1} \\
= \quad & C_{i+j-1}^{p-1} C_k^{p-1-i-j}(C_j^{i+j-1} + C_{j-1}^{i+j-1}) + C_{p-1-i-k}^{p-1} C_j^{p-1-i-k}(C_k^{i+k-1} + C_{k-1}^{i+k-1}) \\
= \quad & C_{i+j-1}^{p-1} C_k^{p-1-i-j} C_j^{i+j} + C_{p-1-i-k}^{p-1} C_j^{p-1-i-k} C_k^{i+k} \\
= \quad & \tfrac{p(p-1)!}{i!j!k!(p-1-i-j-k)!(p-i-j)} \\
= \quad & \tfrac{p}{(p-i-j)} C_{i+j+k}^{p-1} C_{k+i}^{i+j+k} C_i^{k+i}
\end{aligned} \tag{3.67}$$

From 2(b) and 3(b)(i), the coefficient of $[a_{2134}, a_{213}^{j}, a_{132}^{p-i-j}, a_{2134}^{i-1}]$ is

$$\begin{aligned}
= \quad & C_{p-1-i}^{p-1} C_{j-1}^{p-1-i} + C_{i+j-1}^{p-1} C_j^{i+j-1} + C_{p-1-i}^{p-1} C_j^{p-1-i} \\
= \quad & C_{p-1-i}^{p-1}(C_{j-1}^{p-1-i} + C_j^{p-1-i}) + C_{i+j-1}^{p-1} C_j^{i+j-1} \\
= \quad & C_{p-1-i}^{p-1} C_j^{p-i} + C_{i+j-1}^{p-1} C_j^{i+j-1} \\
= \quad & \tfrac{p(p-1)!}{i!j!(p-i-j)!} \\
= \quad & \tfrac{p}{i} C_{i+j-1}^{p-1} C_{i-1}^{i+j-1}
\end{aligned} \tag{3.68}$$

Also, if $p = i + j$, in 2(b), we get

$$C_{j-1}^{p-1}[a_{2134}, a_{213}^{j}, a_{2134}^{p-1-j}] \tag{3.69}$$

From 3(b)(i) and 5(b), the coefficient of $[a_{213}, a_{132}^{p-i-j}, a_{2134}^{i}, a_{213}^{j-1}]$ is

$$\begin{aligned}
= \quad & C_{i+j-1}^{p-1} C_j^{i+j-1} + C_{p-1-i}^{p-1} C_j^{p-1-i} + C_{i+j-1}^{p-1} C_{j-1}^{i+j-1} \\
= \quad & C_{i+j-1}^{p-1}(C_j^{i+j-1} + C_{j-1}^{i+j-1}) + C_{p-1-i}^{p-1} C_j^{p-1-i} \\
= \quad & C_{i+j-1}^{p-1} C_j^{i+j} + C_{p-1-i}^{p-1} C_j^{p-1-i} \\
= \quad & \tfrac{p(p-1)!}{i!j!(p-i-j)!} \\
= \quad & \tfrac{p}{i} C_{i+j-1}^{p-1} C_{i-1}^{i+j-1}
\end{aligned} \tag{3.70}$$

But, $[a_{213}, a_{132}^{p-i-j}, a_{2134}^{i}, a_{213}^{j-1}] + [a_{2134}, a_{213}^{j}, a_{132}^{p-i-j}, a_{2134}^{i-1}] = [a_{2134}, a_{132}^{p-i-j}, a_{213}^{j}, a_{2134}^{i-1}]$.

Therefore from (3.68) and (3.70), we get

$$\frac{p}{i} C_{i+j-1}^{p-1} C_{i-1}^{i+j-1} [a_{2134}, a_{132}^{p-i-j}, a_{213}^{j}, a_{2134}^{i-1}] \tag{3.71}$$

From (3.64), (3.65), (3.66), (3.67), (3.69) and (3.71), we obtain the following:

$$\left\{ \begin{array}{c} \sum\limits_{\substack{i=1 \\ i+j+k \leq p}}^{p-1} \sum\limits_{j=0}^{p-2} \sum\limits_{k=1}^{p-1} \frac{p}{i+k} C_{i-1}^{p-1} C_{j+k}^{p-i} C_{k}^{j+k} [a_{2134}, a_{1324}^{k}, a_{132}^{\overline{p}}, a_{2134}^{i-1}, a_{213}^{j}] \\ \sum\limits_{\substack{i=0 \\ i+j+k<p}}^{p-2} \sum\limits_{j=0}^{p-2} \sum\limits_{k=1}^{p-1} \frac{p}{(p-i-j)} C_{i+j+k}^{p-1} C_{k+i}^{i+j+k} C_{i}^{k+i} [a_{1324}, a_{132}^{\overline{p}}, a_{213}^{j}, a_{2134}^{i}, a_{1324}^{k-1}] \\ \sum\limits_{\substack{i=1 \\ i+j \leq p-1}}^{p-2} \sum\limits_{j=1}^{p-2} \frac{p}{i} C_{i+j-1}^{p-1} C_{i-1}^{i+j-1} [a_{2134}, a_{132}^{p-i-j}, a_{213}^{j}, a_{2134}^{i-1}] \\ \sum\limits_{j=1}^{p-1} C_{j-1}^{p-1} [a_{2134}, a_{213}^{j}, a_{2134}^{p-1-j}] + \sum\limits_{i=1}^{p-2} C_{i}^{p} [a_{2134}, a_{132}^{p-i}, a_{2134}^{i-1}] \end{array} \right\}$$

and this gives

$$\left\{ \begin{array}{c} \sum\limits_{\substack{i=1 \\ i+j+k \leq p}}^{p-1} \sum\limits_{j=0}^{p-2} \sum\limits_{k=0}^{p-1} \frac{p}{i+k} C_{i-1}^{p-1} C_{j+k}^{p-i} C_{k}^{j+k} [a_{2134}, a_{1324}^{k}, a_{132}^{\overline{p}}, a_{2134}^{i-1}, a_{213}^{j}] \\ + \sum\limits_{\substack{i=0 \\ i+j+k<p}}^{p-2} \sum\limits_{j=0}^{p-2} \sum\limits_{k=1}^{p-1} \frac{p}{(p-i-j)} C_{i+j+k}^{p-1} C_{k+i}^{i+j+k} C_{i}^{k+i} [a_{1324}, a_{132}^{\overline{p}}, a_{213}^{j}, a_{2134}^{i}, a_{1324}^{k-1}] \\ + \sum\limits_{j=1}^{p-1} C_{j-1}^{p-1} [a_{2134}, a_{213}^{j}, a_{2134}^{p-1-j}] \end{array} \right\} \tag{3.72}$$

Now, by similar computation the terms (3.53), (3.54), (3.55), give the following respectively

$$\left\{ \begin{array}{c} -\sum\limits_{\substack{i=1 \\ i+j+k \leq p}}^{p-1} \sum\limits_{j=0}^{p-2} \sum\limits_{k=0}^{p-1} \frac{p}{i+k} C_{i-1}^{p-1} C_{j+k}^{p-i} C_{k}^{j+k} [a_{3241}, a_{2431}^{k}, a_{243}^{\overline{p}}, a_{3241}^{i-1}, a_{324}^{j}] \\ -\sum\limits_{\substack{i=0 \\ i+j+k \leq p-1}}^{p-2} \sum\limits_{j=0}^{p-2} \sum\limits_{k=1}^{p-1} \frac{p}{(p-i-j)} C_{i+j+k}^{p-1} C_{k+i}^{i+j+k} C_{i}^{k+i} [a_{2431}, a_{243}^{\overline{p}}, a_{324}^{j}, a_{3241}^{i}, a_{2431}^{k-1}] \\ -\sum\limits_{j=1}^{p-1} C_{j-1}^{p-1} [a_{3241}, a_{324}^{j}, a_{3241}^{p-1-j}] \end{array} \right\} \tag{3.73}$$

$$\left\{\begin{array}{l} \displaystyle\sum_{i=1}^{p-1}\sum_{j=0}^{p-2}\sum_{\substack{k=0\\ i+j+k\leq p}}^{p-1} \tfrac{p}{i+k} C_{i-1}^{p-1} C_{j+k}^{p-i} C_k^{j+k} [a_{4312}, a_{3142}^k, a_{314}^{p}, a_{4312}^{i-1}, a_{431}^j] \\ \displaystyle + \sum_{i=1}^{p-1}\sum_{j=0}^{p-2}\sum_{\substack{k=0\\ i+j+k\leq p,\, j+k\neq 0}}^{p-1} \tfrac{p}{(p-i-j)} C_{i+j+k}^{p-1} C_{k+i}^{i+j+k} C_i^{k+i} [a_{3142}, a_{314}^{p}, a_{431}^j, a_{4312}^i, a_{3142}^{k-1}] \\ \displaystyle \qquad\qquad + \sum_{j=1}^{p-1} C_{j-1}^{p-1} [a_{4312}, a_{431}^j, a_{4312}^{p-1-j}] \end{array}\right\} \tag{3.74}$$

$$\left\{\begin{array}{l} \displaystyle -\sum_{i=1}^{p-1}\sum_{j=0}^{p-2}\sum_{\substack{k=0\\ i+j+k\leq p}}^{p-1} \tfrac{p}{i+k} C_{i-1}^{p-1} C_{j+k}^{p-i} C_k^{j+k} [a_{1423}, a_{4213}^k, a_{421}^{p}, a_{1423}^{i-1}, a_{142}^j] \\ \displaystyle - \sum_{i=0}^{p-2}\sum_{j=0}^{p-2}\sum_{\substack{k=1\\ i+j+k\leq p-1}}^{p-1} \tfrac{p}{(p-i-j)} C_{i+j+k}^{p-1} C_{k+i}^{i+j+k} C_i^{k+i} [a_{4213}, a_{421}^{p}, a_{142}^j, a_{1423}^i, a_{4213}^{k-1}] \\ \displaystyle \qquad\qquad - \sum_{j=1}^{p-1} C_{j-1}^{p-1} [a_{1423}, a_{142}^j, a_{1423}^{p-1-j}] \end{array}\right\} \tag{3.75}$$

By substituting (3.72), (3.73), (3.74) and (3.75) into (3.57), we obtain

$$\frac{1}{p}\left\{\begin{array}{l} \displaystyle\sum_{\substack{i+j+k\leq p\\ i\neq 0,\, j+k\neq 0}} \overline{\gamma}_3 \left\{\begin{array}{l} [a_{2134}, a_{1324}^k, a_{132}^{p}, a_{2134}^{i-1}, a_{213}^j] - [a_{3241}, a_{2431}^k, a_{243}^{p}, a_{3241}^{i-1}, a_{324}^j] \\ [a_{4312}, a_{3142}^k, a_{314}^{p}, a_{4312}^{i-1}, a_{431}^j] - [a_{1423}, a_{4213}^k, a_{421}^{p}, a_{1423}^{i-1}, a_{142}^j] \end{array}\right\} \\ \displaystyle\sum_{\substack{i+j+k< p\\ k\neq 0}} \overline{\gamma}_4 \left\{\begin{array}{l} [a_{1324}, a_{132}^{p}, a_{213}^j, a_{2134}^i, a_{1324}^{k-1}] - [a_{2431}, a_{243}^{p}, a_{324}^j, a_{3241}^i, a_{2431}^{k-1}] \\ [a_{3142}, a_{314}^{p}, a_{431}^j, a_{4312}^i, a_{3142}^{k-1}] - [a_{4213}, a_{421}^{p}, a_{142}^j, a_{1423}^i, a_{4213}^{k-1}] \end{array}\right\} \\ \displaystyle\qquad \sum_{i=1}^{p-1} C_{i-1}^{p-1} \left\{\begin{array}{l} [a_{1324}, a_{134}^i, a_{1324}^{p-1-i}] + [a_{2413}, a_{243}^i, a_{2413}^{p-1-i}] \\ -[a_{1324}, a_{132}^i, a_{1324}^{p-1-i}] - [a_{2413}, a_{241}^i, a_{2413}^{p-1-i}] \end{array}\right\} \end{array}\right\} \tag{3.76}$$

where $\overline{\gamma}_3 = \frac{p}{i+k} C_{i-1}^{p-1} C_{j+k}^{p-i} C_k^{j+k}$ and $\overline{\gamma}_4 = \frac{p}{(p-i-j)} C_{i+j+k}^{p-1} C_{k+i}^{i+j+k} C_i^{k+i}$.

Now when $i+k=p$ the coefficient of the first term does not divisible by $p$, also when $i=j=0$ the coefficient of the second term does not divisible by $p$. So we decompose

(3.76) into two summands

$$\left.\begin{array}{l} \displaystyle\sum_{\substack{i=1\\ i+j+k\leq p\\ i+k\neq p}}^{p-1}\sum_{j=0}^{p-2}\sum_{k=1}^{p-1} \gamma_3 \left\{\begin{array}{l} [a_{2134}, a_{1324}^{k}, a_{132}^{\overline{p}}, a_{2134}^{i-1}, a_{213}^{j}] - [a_{3241}, a_{2431}^{k}, a_{243}^{\overline{p}}, a_{3241}^{i-1}, a_{324}^{j}] \\ [a_{4312}, a_{3142}^{k}, a_{314}^{\overline{p}}, a_{4312}^{i-1}, a_{431}^{j}] - [a_{1423}, a_{4213}^{k}, a_{421}^{\overline{p}}, a_{1423}^{i-1}, a_{142}^{j}] \end{array}\right\} \\ \displaystyle\sum_{\substack{i=0\\ i+j+k<p\\ i+j\neq 0}}^{p-2}\sum_{j=0}^{p-2}\sum_{k=1}^{p-1} \gamma_4 \left\{\begin{array}{l} [a_{1324}, a_{132}^{\overline{p}}, a_{213}^{j}, a_{2134}^{i}, a_{1324}^{k-1}] - [a_{2431}, a_{243}^{\overline{p}}, a_{324}^{j}, a_{3241}^{i}, a_{2431}^{k-1}] \\ [a_{3142}, a_{314}^{\overline{p}}, a_{431}^{j}, a_{4312}^{i}, a_{3142}^{k-1}] - [a_{4213}, a_{421}^{\overline{p}}, a_{142}^{j}, a_{1423}^{i}, a_{4213}^{k-1}] \end{array}\right\} \end{array}\right\} \tag{3.77}$$

$$\frac{1}{p}\left\{\begin{array}{l} \displaystyle\sum_{i=1}^{p-1} C_{i-1}^{p-1} \left\{\begin{array}{l} [a_{2134}, a_{1324}^{p-i}, a_{2134}^{i-1}] - [a_{3241}, a_{2431}^{p-i}, a_{3241}^{i-1}] \\ [a_{4312}, a_{3142}^{p-i}, a_{4312}^{i-1}] - [a_{1423}, a_{4213}^{p-i}, a_{1423}^{i-1}] \end{array}\right\} \\ \displaystyle +\sum_{k=1}^{p-1} C_{k}^{p-1} \left\{\begin{array}{l} [a_{1324}, a_{132}^{p-k}, a_{1324}^{k-1}] - [a_{2431}, a_{243}^{p-k}, a_{2431}^{k-1}] \\ [a_{3142}, a_{314}^{p-k}, a_{3142}^{k-1}] - [a_{4213}, a_{421}^{p-k}, a_{4213}^{k-1}] \end{array}\right\} \\ \displaystyle\sum_{i=1}^{p-1} C_{i-1}^{p-1} \left\{\begin{array}{l} [a_{1324}, a_{134}^{i}, a_{1324}^{p-1-i}] + [a_{2413}, a_{243}^{i}, a_{2413}^{p-1-i}] \\ -[a_{1324}, a_{132}^{i}, a_{1324}^{p-1-i}] - [a_{2413}, a_{241}^{i}, a_{2413}^{p-1-i}] \end{array}\right\} \end{array}\right\} \tag{3.78}$$

where $\gamma_3 = \frac{1}{i+k} C_{i-1}^{p-1} C_{j+k}^{p-i} C_{k}^{j+k}$ and $\gamma_4 = \frac{1}{(p-i-j)} C_{i+j+k}^{p-1} C_{k+i}^{i+j+k} C_{i}^{k+i}$.

Now, it is easy to see that (3.78) can be rewritten as

$$\frac{1}{p}\sum_{i=1}^{p-1} C_{i-1}^{p-1} \left\{\begin{array}{l} [a_{2134}, a_{1324}^{p-i}, a_{2134}^{i-1}] - [a_{3241}, a_{2431}^{p-i}, a_{3241}^{i-1}] \\ [a_{4312}, a_{3142}^{p-i}, a_{4312}^{i-1}] - [a_{1423}, a_{4213}^{p-i}, a_{1423}^{i-1}] \end{array}\right\} \tag{3.79}$$

**Step 3.** Our final move is to write

$$\sum_{i=1}^{p-1} C_{i-1}^{p-1} \left\{\begin{array}{l} [a_{2134}, a_{1324}^{p-i}, a_{2134}^{i-1}] - [a_{3241}, a_{2431}^{p-i}, a_{3241}^{i-1}] \\ [a_{4312}, a_{3142}^{p-i}, a_{4312}^{i-1}] - [a_{1423}, a_{4213}^{p-i}, a_{1423}^{i-1}] \end{array}\right\}$$

as a linear combination of terms with coefficients a multiple of $p$. By the Jacobi identity we get from (3.79) the following

$$\frac{1}{p}\sum_{j=0}^{p-2} C_j^{p-1}\left\{\begin{array}{c} [a_{2134}, a_{1324}^{p-1-j}, a_{2134}^{j}] \\ +[a_{1324}, (a_{2134}-a_{4123})^{p-1-j}, (-a_{1324}-a_{2134})^{j}] \\ +[a_{2134}, (a_{2134}-a_{4123})^{p-1-j}, (-a_{1324}-a_{2134})^{j}] \\ +[a_{4123}, -a_{1324}^{p-1-j}, (a_{4123}+a_{1324})^{j}] \\ +[a_{4123}, (a_{4123}-a_{2134})^{p-1-j}, -a_{4123}^{j}] \end{array}\right\}$$

This gives the following

$$\frac{1}{p}\sum_{j=0}^{p-2} C_j^{p-1}\left\{\begin{array}{c} [a_{2134}, a_{1324}^{p-1-j}, a_{2134}^{j}] \\ +[a_{1324}, a_{2134}, (a_{2134}-a_{4123})^{p-2-j}, (-a_{1324}-a_{2134})^{j}] \\ +[a_{1324}, -a_{4123}, (a_{2134}-a_{4123})^{p-2-j}, (-a_{1324}-a_{2134})^{j}] \\ +[a_{2134}, -a_{4123}, (a_{2134}-a_{4123})^{p-2-j}, (-a_{1324}-a_{2134})^{j}] \\ +[a_{4123}, -a_{1324}^{p-1-j}, (a_{4123}+a_{1324})^{j}] \\ +[a_{4123}, -a_{2134}, (a_{4123}-a_{2134})^{p-2-j}, -a_{4123}^{j}] \end{array}\right\} \tag{3.80}$$

From (3.80), we obtain

1. $\frac{1}{p}\sum_{j=0}^{p-2} C_j^{p-1}[a_{2134}, a_{1324}^{p-1-j}, a_{2134}^{j}]$

2. $\frac{1}{p}\sum_{j=0}^{p-2} C_j^{p-1}[a_{1324}, a_{2134}, (a_{2134}-a_{4123})^{p-2-j}, (-a_{1324}-a_{2134})^{j}]$

   $= \frac{1}{p}\sum_{j=0}^{p-2}\sum_{i=0}^{j}\sum_{k=0}^{p-2-j} (-1)^{k+j} C_j^{p-1} C_i^j C_k^{p-2-j}[a_{1324}, a_{2134}^{p-1-j-k}, a_{4132}^{k}, a_{1324}^{i}, a_{2134}^{j-i}]$

   Changing the order of summation, so that $i$ runs from 0 to $p-2$, for fixed $i$, $k$ runs from 0 to $p-2-i$, and for $i$ and $k$ fixed, $j$ runs from $i$ to $p-2-k$, then we get

   $= \frac{1}{p}\sum_{i=0}^{p-2}\sum_{k=0}^{p-2-i}\sum_{j=i}^{p-2-k} (-1)^{k+j} C_j^{p-1} C_i^j C_k^{p-2-j}[a_{1324}, a_{2134}^{p-1-j-k}, a_{4132}^{k}, a_{1324}^{i}, a_{2134}^{j-i}]$

$$= \frac{1}{p}\sum_{i=0}^{p-2}\sum_{k=0}^{p-2-i}\left(\sum_{j=i}^{p-2-k}(-1)^{k+j}C_j^{p-1}C_i^jC_k^{p-2-j}\right)[a_{1324}, a_{2134}^{p-1-i-k}, a_{4132}^k, a_{1324}^i]$$

$$= \frac{1}{p}\sum_{i=0}^{p-2}\sum_{k=0}^{p-2-i}((-1)^pC_i^{p-1})[a_{1324}, a_{2134}^{p-1-i-k}, a_{4132}^k, a_{1324}^i] \text{ by Lemma 2.5.4,}$$

$$= -\frac{1}{p}\sum_{i=0}^{p-2}\sum_{k=0}^{p-2-i} C_i^{p-1}[a_{1324}, a_{2134}^{p-1-i-k}, a_{4132}^k, a_{1324}^i]$$

3. $\frac{1}{p}\sum_{j=0}^{p-2} C_j^{p-1}[a_{1324}, -a_{4123}, (a_{2134}-a_{4123})^{p-2-j}, (-a_{1324}-a_{2134})^j]$

$$= \frac{1}{p}\sum_{j=0}^{p-2}\sum_{i=0}^{j}\sum_{k=0}^{p-2-j}(-1)^{k+1+j}C_j^{p-1}C_i^jC_k^{p-2-j}[a_{1324}, a_{4123}^{k+1}, a_{2134}^{p-2-j-k}, a_{1324}^i, a_{2134}^{j-i}].$$

Again by changing the order of summation we get

$$= \frac{1}{p}\sum_{i=0}^{p-2}\sum_{k=0}^{p-2-i}\sum_{j=i}^{p-2-k}(-1)^{k+1+j}C_j^{p-1}C_i^jC_k^{p-2-j}[a_{1324}, a_{4123}^{k+1}, a_{2134}^{p-2-j-k}, a_{1324}^i, a_{2134}^{j-i}]$$

$$= \frac{1}{p}\sum_{i=0}^{p-2}\sum_{k=0}^{p-2-i}\left(\sum_{j=i}^{p-2-k}(-1)^{k+1+j}C_j^{p-1}C_i^jC_k^{p-2-j}\right)[a_{1324}, a_{4123}^{k+1}, a_{2134}^{p-2-j-k}, a_{1324}^i, a_{2134}^{j-i}]$$

$$= -\frac{1}{p}\sum_{i=0}^{p-2}\sum_{k=0}^{p-2-i}\left(\sum_{j=i}^{p-2-k}(-1)^{k+j}C_j^{p-1}C_i^jC_k^{p-2-j}\right)[a_{1324}, a_{4123}^{k+1}, a_{2134}^{p-2-j-k}, a_{1324}^i, a_{2134}^{j-i}]$$

$$= -\frac{1}{p}\sum_{i=0}^{p-2}\sum_{k=0}^{p-2-i}((-1)^pC_i^{p-1})[a_{1324}, a_{4123}^{k+1}, a_{2134}^{p-2-j-k}, a_{1324}^i, a_{2134}^{j-i}] \text{ by Lemma 2.5.4,}$$

$$= \frac{1}{p}\sum_{i=0}^{p-2}\sum_{k=0}^{p-2-i} C_i^{p-1}[a_{1324}, a_{4123}^{k+1}, a_{2134}^{p-2-i-k}, a_{1324}^i]$$

By the Jacobi identity we get from (2) and (3) the following

$$\frac{1}{p}\sum_{i=0}^{p-2}\sum_{k=1}^{p-2-i} C_i^{p-1}[a_{2134}, a_{4123}^k, a_{2134}^{p-2-i-k}, a_{1324}^{i+1}]. \tag{3.81}$$

On the other hand when $k=0$ in (2), we get

$$-\frac{1}{p}\sum_{i=0}^{p-2} C_i^{p-1}[a_{1324}, a_{2134}^{p-1-i}, a_{1324}^i] \tag{3.82}$$

and also when $k=p-2-i$ in (3), we get

$$\frac{1}{p}\sum_{i=0}^{p-2} C_i^{p-1}[a_{1324}, a_{4123}^{p-1-i}, a_{1324}^i] \tag{3.83}$$

4. $\frac{1}{p}\sum_{j=0}^{p-2} C_j^{p-1}[a_{2134}, -a_{4123}, (a_{2134}-a_{4123})^{p-2-j}, (-a_{1324}-a_{2134})^j]$

$= \frac{1}{p}\sum_{j=0}^{p-2}\sum_{i=0}^{j}\sum_{k=0}^{p-2-j}(-1)^{k+1+j}C_j^{p-1}C_i^jC_{k-1}^{p-2-j}[a_{2134}, a_{4123}^{k+1}, a_{2134}^{p-2-j-k}, a_{1324}^i, a_{2134}^{j-i}]$

Changing the order of summation we get :

$= \frac{1}{p}\sum_{i=0}^{p-2}\sum_{k=0}^{p-2-i}(\sum_{j=i}^{p-2-k}(-1)^{k+1+j}C_j^{p-1}C_i^jC_{k-1}^{p-2-j})[a_{2134}, a_{4123}^{k+1}, a_{2134}^{p-2-j-k}, a_{1324}^i, a_{2134}^{j-i}]$

$= -\frac{1}{p}\sum_{i=0}^{p-2}\sum_{k=0}^{p-2-i}(\sum_{j=i}^{p-2-k}(-1)^{k+j}C_j^{p-1}C_i^jC_{k-1}^{p-2-j})[a_{2134}, a_{4123}^{k+1}, a_{2134}^{p-2-j-k}, a_{1324}^i, a_{2134}^{j-i}]$

$= -\frac{1}{p}\sum_{i=0}^{p-2}\sum_{k=0}^{p-2-i}((-1)^pC_i^{p-1})[a_{2134}, a_{4123}^{k+1}, a_{2134}^{p-2-j-k}, a_{1324}^i, a_{2134}^{j-i}]$ by Lemma 2.5.4,

$= \frac{1}{p}\sum_{i=0}^{p-2}\sum_{k=0}^{p-2-i} C_i^{p-1}[a_{2134}, a_{4123}^{k+1}, a_{2134}^{p-2-i-k}, a_{1324}^i]$.

5. $\frac{1}{p}\sum_{j=0}^{p-2} C_j^{p-1}[a_{4123}, -a_{1324}^{p-1-j}, (a_{4123}+a_{1324})^j]$

$= \frac{1}{p}\sum_{j=0}^{p-2}\sum_{i=0}^{j}(-1)^{p-1-j}C_j^{p-1}C_{j-i}^j[a_{4123}, a_{1324}^{p-1-j}, a_{4123}^{j-i}, a_{1324}^i]$.

Changing the order of summation we get :

$= \frac{1}{p}\sum_{i=0}^{p-2}\sum_{j=i}^{p-2}(-1)^{p-1-j}C_j^{p-1}C_{j-i}^j[a_{4123}, a_{1324}^{p-1-j}, a_{4123}^i, a_{1324}^{j-i}]$

$= \frac{1}{p}\sum_{i=0}^{p-2}(\sum_{j=i}^{p-2}(-1)^{p-1-j}C_j^{p-1}C_{j-i}^j)[a_{4123}, a_{1324}^{p-1-j}, a_{4123}^i, a_{1324}^{j-i}]$

$= \frac{1}{p}\sum_{i=0}^{p-2}(-1)^pC_i^{p-1}[a_{4123}, a_{1324}^{p-1-j}, a_{4123}^i, a_{1324}^{j-i}]$ by Lemma 2.5.4,

$= -\frac{1}{p}\sum_{i=0}^{p-2} C_i^{p-1}[a_{4123}, a_{1324}^{p-1-i}, a_{4123}^i]$.

6. $\frac{1}{p}\sum_{j=0}^{p-2} C_j^{p-1}[a_{4123}, -a_{2134}, (a_{4123}-a_{2134})^{p-2-j}, -a_{4123}^j]$

$= \frac{1}{p}\sum_{j=0}^{p-2}\sum_{k=0}^{p-2-j}(-1)^{k+1+j}C_j^{p-1}C_k^{p-2-j}[a_{4123}, a_{2134}^{k+1}, a_{4123}^{p-2-j-k}, a_{4123}^j]$

$= \frac{1}{p}\sum_{k=0}^{p-2}(\sum_{j=0}^{p-2-k}(-1)^{k+1+j}C_j^{p-1}C_k^{p-2-j})[a_{4123}, a_{2134}^{k+1}, a_{4123}^{p-2-j-k}, a_{4123}^j]$

$= -\frac{1}{p}\sum_{k=0}^{p-2}(\sum_{j=0}^{p-2-k}(-1)^{k+j}C_j^{p-1}C_k^{p-2-j})[a_{4123}, a_{2134}^{k+1}, a_{4123}^{p-2-j-k}, a_{4123}^j]$

$$= -\frac{1}{p}\sum_{k=0}^{p-2} (-1)^{p-2}[a_{4123}, a_{2134}^{k+1}, a_{4123}^{p-2-j-k}, a_{4123}^{j}]$$
$$= \frac{1}{p}\sum_{k=0}^{p-2} [a_{4123}, a_{2134}^{k+1}, a_{4123}^{p-2-k}].$$

Now from (4) and (3.81), the coefficient of

$$[a_{2134}, a_{4123}^{k+1}, a_{2134}^{p-2-k-i}, a_{1324}^{i}] \text{ is } C_{i+1}^{p} \tag{3.84}$$

On the other hand when $i = 0$ in (4) we get terms that cancel with terms of (6).

From (3.82) and (1), the coefficient of

$$[a_{2134}, a_{1324}^{p-1-j}, a_{2134}^{j}] \text{ is } C_{j+1}^{p} \tag{3.85}$$

From (3.83) and (5), the coefficient of $[a_{1324}, a_{4123}^{p-1-i}, a_{1324}^{i}]$ is

$$C_{i}^{p-1} + C_{i+1}^{p-1} = C_{i+1}^{p} \tag{3.86}$$

From (3.84), (3.85) and (3.86), we obtain

$$\begin{aligned} &\frac{1}{p}\sum_{i=1}^{p-2}\sum_{k=0}^{p-2-i} C_{i+1}^{p}[a_{2134}, a_{4123}^{k+1}, a_{1324}^{i}, a_{2134}^{p-2-k-i}] \\ &\quad +\frac{1}{p}\sum_{i=0}^{p-2} C_{i+1}^{p}[a_{2134}, a_{1324}^{p-1-i}, a_{2134}^{i}] \\ &\quad +\frac{1}{p}\sum_{i=0}^{p-2} C_{i+1}^{p}[a_{1324}, a_{4123}^{p-1-i}, a_{1324}^{i}] \end{aligned}$$

and this gives

$$\begin{aligned} &\sum_{i=0}^{p-1}\sum_{k=-1}^{p-2-i} \frac{1}{i+1}C_{i}^{p-1}[a_{2134}, a_{4123}^{k+1}, a_{1324}^{i}, a_{2134}^{p-2-k-i}] \\ &\sum_{i=0}^{p-2} \frac{1}{i+1}C_{i}^{p-1}[a_{1324}, a_{4123}^{p-1-i}, a_{1324}^{i}] \end{aligned} \tag{3.87}$$

Finally, from (3.56), (3.77) and (3.87), we get

$$\left\{\begin{array}{l}
\sum\limits_{\substack{i+j+k\leq p-1\\ i\neq 0,\ k+j\neq 0}} \gamma_1 \left\{\begin{array}{l}
[a_{2134}, a_{214}^{i}, a_{21}^{j}, a_{213}^{k}, a_{2134}^{\overline{p}-1}] + [a_{3214}, a_{324}^{i}, a_{32}^{j}, a_{321}^{k}, a_{3214}^{\overline{p}-1}] \\
+[a_{4312}, a_{432}^{i}, a_{43}^{j}, a_{431}^{k}, a_{4312}^{\overline{p}-1}] + [a_{1432}, a_{142}^{i}, a_{14}^{j}, a_{143}^{k}, a_{1432}^{\overline{p}-1}] \\
+[a_{1324}, a_{134}^{i}, a_{13}^{j}, a_{132}^{k}, a_{1324}^{\overline{p}-1}] + [a_{2413}, a_{243}^{i}, a_{24}^{j}, a_{241}^{k}, a_{2413}^{\overline{p}-1}]
\end{array}\right\} \\
\sum\limits_{\substack{i+j+k\leq p\\ k\neq 0,\ i+j\neq 0\\ j+k\neq p}} \gamma_2 \left\{\begin{array}{l}
[a_{213}, a_{2134}^{\overline{p}}, a_{214}^{i}, a_{21}^{j}, a_{213}^{k-1}] + [a_{321}, a_{3214}^{\overline{p}}, a_{324}^{i}, a_{32}^{j}, a_{321}^{k-1}] \\
+[a_{431}, a_{4312}^{\overline{p}}, a_{432}^{i}, a_{43}^{j}, a_{431}^{k-1}] + [a_{143}, a_{1432}^{\overline{p}}, a_{142}^{i}, a_{14}^{j}, a_{143}^{k-1}] \\
+[a_{132}, a_{1324}^{\overline{p}}, a_{134}^{i}, a_{132}^{k-1}, a_{13}^{j}] + [a_{241}, a_{2413}^{\overline{p}}, a_{243}^{i}, a_{24}^{j}, a_{241}^{k-1}]
\end{array}\right\} \\
\sum\limits_{\substack{i+j+k\leq p\\ i+k\neq p,\ i\neq 0}} \gamma_3 \left\{\begin{array}{l}
[a_{2134}, a_{1324}^{k}, a_{132}^{\overline{p}}, a_{2134}^{i-1}, a_{213}^{j}] - [a_{3241}, a_{2431}^{k}, a_{243}^{\overline{p}}, a_{3241}^{i-1}, a_{324}^{j}] \\
[a_{4312}, a_{3142}^{k}, a_{314}^{\overline{p}}, a_{4312}^{i-1}, a_{431}^{j}] - [a_{1423}, a_{4213}^{k}, a_{421}^{\overline{p}}, a_{1423}^{i-1}, a_{142}^{j}]
\end{array}\right\} \\
\sum\limits_{\substack{i+j+k< p\\ k\neq 0,\ i+j\neq 0}} \gamma_4 \left\{\begin{array}{l}
[a_{1324}, a_{132}^{\overline{p}}, a_{213}^{j}, a_{2134}^{i}, a_{1324}^{k-1}] - [a_{2431}, a_{243}^{\overline{p}}, a_{324}^{j}, a_{3241}^{i}, a_{2431}^{k-1}] \\
[a_{3142}, a_{314}^{\overline{p}}, a_{431}^{j}, a_{4312}^{i}, a_{3142}^{k-1}] - [a_{4213}, a_{421}^{\overline{p}}, a_{142}^{j}, a_{1423}^{i}, a_{4213}^{k-1}]
\end{array}\right\} \\
\sum\limits_{i=0}^{p-2} \sum\limits_{\substack{k=-1\\ i+k\neq -1}}^{p-2-i} \frac{1}{i+1} C_i^{p-1} \{[a_{2134}, a_{4123}^{k+1}, a_{1324}^{i}, a_{2134}^{p-2-k-i}] + [a_{1324}, a_{4123}^{p-1-i}, a_{1324}^{i}]\}
\end{array}\right\} \tag{3.88}$$

where,

$$\gamma_1 = \tfrac{1}{p-j-k} C_{k+j}^{p-1} C_k^{k+j} C_{i-1}^{p-1-k-j},\ \gamma_2 = \tfrac{1}{p-j-i} C_{i+j}^{p-1} C_i^{i+j} C_{k-1}^{p-1-i-j}$$

$$\gamma_3 = \tfrac{1}{i+k} C_{i-1}^{p-1} C_{j+k}^{p-i} C_k^{j+k},\ \gamma_4 = \tfrac{1}{p-i-j} C_{i+j+k}^{p-1} C_{k+i}^{i+j+k} C_i^{k+i}\,;\omega = \tfrac{1}{i+1} C_i^{p-1}$$

$$I_1 = \{(i,j,k) : i \neq 0\,,\ j+k \neq 0\ ,\ \ i+j+k \leq p-1\,\}$$

$$I_2 = \{(i,j,k) : k \neq 0\,,\ i+j \neq 0,\ j+k \neq p,\ \ i+j+k \leq p\,,\}$$

$$I_3 = \{(i,j,k) : i \neq 0\,,\ i+j+k \leq p\,,\ i+k \neq p\}$$

$$I_4 = \{(i,j,k) : k \neq 0,\ ,\ i+j \neq 0\,,\ \ i+j+k < p\,\}$$

$$1 \leq \tau_1 < \tau_2 < \tau_3 < \tau_4 \leq d\ ;\ \ \overline{p} = p-i-j-k$$

Finally the image of the element (3.88) in $\mathcal{M}^p M\ \otimes_G \mathbf{R}$ is the image of

$$1 \otimes e_1 \wedge e_2 \wedge e_3 \wedge e_4$$

under the connecting homomorphism $H_4(G, \mathbf{Z}_p) \longrightarrow H_0(G, \mathcal{M}^p M)$. On the other hand $tH_0(G, \mathcal{M}^p M) \cong t_p H_0(G, \mathcal{M}^p F'_{ab})$, but $tH_0(G, \mathcal{M}^p F'_{ab})$ is an elementary abelian $p$–group, this means that the torsion subgroups of both $H_0(G, \mathcal{M}^p M)$ and $H_0(G, \mathcal{M}^p F'_{ab})$ are the same. Applying the isomorphism $\mathcal{M}^p F'_{ab} \otimes_G \mathbf{R} \longrightarrow \gamma_p(F')F'''/[\gamma_p(F'), F]F'''$, we obtain

$$\left\{\begin{array}{l}
\prod\limits_{(i,j,k)\in I_1} \left\{\begin{array}{l}
[a_{2134}, a_{214}^{i}, a_{21}^{j}, a_{213}^{k}, a_{2134}^{\overline{p}-1}]\,[a_{3214}, a_{324}^{i}, a_{32}^{j}, a_{321}^{k}, a_{3214}^{\overline{p}-1}] \\
[a_{4312}, a_{432}^{i}, a_{43}^{j}, a_{431}^{k}, a_{4312}^{\overline{p}-1}]\,[a_{1432}, a_{142}^{i}, a_{14}^{j}, a_{143}^{k}, a_{1432}^{\overline{p}-1}] \\
[a_{1324}, a_{134}^{i}, a_{13}^{j}, a_{132}^{k}, a_{1324}^{\overline{p}-1}]\,[a_{2413}, a_{243}^{i}, a_{24}^{j}, a_{241}^{k}, a_{2413}^{\overline{p}-1}]
\end{array}\right\}^{\gamma_1} \\
\prod\limits_{(i,j,k)\in I_2} \left\{\begin{array}{l}
[a_{213}, a_{2134}^{\overline{p}}, a_{214}^{i}, a_{21}^{j}, a_{213}^{k-1}]\,[a_{321}, a_{3214}^{\overline{p}}, a_{324}^{i}, a_{32}^{j}, a_{321}^{k-1}] \\
[a_{431}, a_{4312}^{\overline{p}}, a_{432}^{i}, a_{43}^{j}, a_{431}^{k-1}]\,[a_{143}, a_{1432}^{\overline{p}}, a_{142}^{i}, a_{14}^{j}, a_{143}^{k-1}] \\
[a_{132}, a_{1324}^{\overline{p}}, a_{134}^{i}, a_{132}^{k-1}, a_{13}^{j}]\,[a_{241}, a_{2413}^{\overline{p}}, a_{243}^{i}, a_{24}^{j}, a_{241}^{k-1}]
\end{array}\right\}^{\gamma_2} \\
\prod\limits_{(i,j,k)\in I_3} \left\{\begin{array}{l}
[a_{2134}, a_{1324}^{k}, a_{132}^{\overline{p}}, a_{2134}^{i-1}, a_{213}^{j}]\,[a_{2341}, a_{2431}^{k}, a_{243}^{\overline{p}}, a_{3241}^{i-1}, a_{324}^{j}] \\
[a_{4312}, a_{3142}^{k}, a_{314}^{\overline{p}}, a_{4312}^{i-1}, a_{431}^{j}]\,[a_{4123}, a_{4213}^{k}, a_{421}^{\overline{p}}, a_{1423}^{i-1}, a_{142}^{j}]
\end{array}\right\}^{\gamma_3} \\
\prod\limits_{(i,j,k)\in I_4} \left\{\begin{array}{l}
[a_{1324}, a_{132}^{\overline{p}}, a_{213}^{j}, a_{2134}^{i}, a_{1324}^{k-1}]\,[a_{4231}, a_{243}^{\overline{p}}, a_{324}^{j}, a_{3241}^{i}, a_{2431}^{k-1}] \\
[a_{3142}, a_{314}^{\overline{p}}, a_{431}^{j}, a_{4312}^{i}, a_{3142}^{k-1}]\,[a_{2413}, a_{421}^{\overline{p}}, a_{142}^{j}, a_{1423}^{i}, a_{4213}^{k-1}]
\end{array}\right\}^{\gamma_4} \\
\prod\limits_{i=0}^{p-2} \prod\limits_{k=-1}^{p-2-i} \{[a_{2134}, a_{4123}^{k+1}, a_{1324}^{i}, a_{2134}^{p-2-k-i}]\,[a_{1324}, a_{4123}^{p-1-i}, a_{1324}^{i}]\}^{\omega}
\end{array}\right\} \tag{3.89}$$

The theorem follows by replacing the $a$'s in the above expression by their definitions as given on p 58.

Next we illustrate our main result by considering the following special situation.

If $p = 3$, then the generators of the torsion subgroup of $F/[\gamma_3(F'), F]F'''$ are,

$$
\{ \sum_{j+k=1} \left\{ \begin{array}{l}
[[x_{i_2}, x_{i_1}, x_{i_3}, x_{i_4}], [x_{i_2}, x_{i_1}, x_{i_4}], [x_{i_2}, x_{i_1}]^j, [x_{i_2}, x_{i_1}, x_{i_3}]^k] \\
[[x_{i_3}, x_{i_2}, x_{i_1}, x_{i_4}], [x_{i_3}, x_{i_2}, x_{i_4}], [x_{i_3}, x_{i_2}]^j, [x_{i_3}, x_{i_2}, x_{i_1}]^k] \\
[[x_{i_4}, x_{i_3}, x_{i_1}, x_{i_2}], [x_{i_4}, x_{i_3}, x_{i_2}], [x_{i_4}, x_{i_3}]^j, [x_{i_4}, x_{i_3}, x_{i_1}]^k] \\
[[x_{i_1}, x_{i_4}, x_{i_3}, x_{i_2}], [x_{i_1}, x_{i_4}, x_{i_2}], [x_{i_1}, x_{i_4}]^j, [x_{i_1}, x_{i_4}, x_{i_3}]^k] \\
[[x_{i_1}, x_{i_3}, x_{i_2}, x_{i_4}], [x_{i_1}, x_{i_3}, x_{i_4}], [x_{i_1}, x_{i_3}]^j, [x_{i_1}, x_{i_3}, x_{i_2}]^k] \\
[[x_{i_2}, x_{i_4}, x_{i_3}, x_{i_1}], [x_{i_2}, x_{i_4}, x_{i_3}], [x_{i_2}, x_{i_4}]^j, [x_{i_2}, x_{i_4}, x_{i_1}]^k]
\end{array} \right\}
$$

$$
\sum_{\substack{i+j+k=2,3 \\ k\neq 0,\, i+j\neq 0 \\ j+k\neq 3}} \beta \left\{ \begin{array}{l}
[[x_{i_2}x_{i_1}x_{i_3}], [x_{i_2}x_{i_1}x_{i_3}x_{i_4}]^{2-i-j}, [x_{i_2}x_{i_1}x_{i_4}]^i, [x_{i_2}x_{i_1}]^j, [x_{i_2}x_{i_1}x_{i_3}]^{k-1}] \\
[[x_{i_3}x_{i_2}x_{i_1}], [x_{i_3}x_{i_2}x_{i_1}x_{i_4}]^{2-i-j}, [x_{i_3}x_{i_2}x_{i_4}]^i, [x_{i_3}x_{i_2}]^j, [x_{i_3}x_{i_2}x_{i_1}]^{k-1}] \\
[[x_{i_4}x_{i_3}x_{i_1}], [x_{i_4}x_{i_3}x_{i_1}x_{i_2}]^{2-i-j}, [x_{i_4}x_{i_3}x_{i_2}]^i, [x_{i_4}x_{i_3}]^j, [x_{i_4}x_{i_3}x_{i_1}]^{k-1}] \\
[[x_{i_1}x_{i_4}x_{i_3}], [x_{i_1}x_{i_4}x_{i_3}x_{i_2}]^{2-i-j}, [x_{i_1}x_{i_4}x_{i_2}]^i, [x_{i_1}x_{i_4}]^j, [x_{i_1}x_{i_4}x_{i_3}]^{k-1}] \\
[[x_{i_1}x_{i_3}x_{i_2}], [x_{i_1}x_{i_3}x_{i_2}x_{i_4}]^{2-i-j}, [x_{i_1}x_{i_3}x_{i_4}]^i, [x_{i_1}x_{i_3}]^j, [x_{i_1}x_{i_3}x_{i_2}]^{k-1}] \\
[[x_{i_2}x_{i_4}x_{i_1}], [x_{i_2}x_{i_4}x_{i_1}x_{i_3}]^{2-i-j}, [x_{i_2}x_{i_4}x_{i_3}]^i, [x_{i_2}x_{i_4}]^j, [x_{i_2}x_{i_4}x_{i_1}]^{k-1}]
\end{array} \right\}
$$

$$
\sum_{\substack{i+j+k\leq 3 \\ i\neq 0,\, i+k\neq 3}} \gamma \left\{ \begin{array}{l}
[[x_{i_2}x_{i_1}x_{i_3}x_{i_4}], [x_{i_1}x_{i_3}x_{i_2}x_{i_4}]^k, [x_{i_1}x_{i_3}x_{i_2}]^{\overline{3}}, [x_{i_2}x_{i_1}x_{i_3}]^j, [x_{i_2}x_{i_1}x_{i_3}x_{i_4}]^{i-1}] \\
[[x_{i_3}x_{i_2}x_{i_4}x_{i_1}], [x_{i_2}x_{i_4}x_{i_3}x_{i_1}]^k, [x_{i_2}x_{i_4}x_{i_3}]^{\overline{3}}, [x_{i_3}x_{i_2}x_{i_4}]^j, [x_{i_3}x_{i_2}x_{i_4}x_{i_1}]^{i-1}] \\
[[x_{i_4}x_{i_3}x_{i_1}x_{i_2}], [x_{i_3}x_{i_1}x_{i_4}x_{i_2}]^k, [x_{i_3}x_{i_1}x_{i_4}]^{\overline{3}}, [x_{i_4}x_{i_3}x_{i_1}]^j, [x_{i_4}x_{i_3}x_{i_1}x_{i_2}]^{i-1}] \\
[[x_{i_1}x_{i_4}x_{i_2}x_{i_3}], [x_{i_4}x_{i_2}x_{i_1}x_{i_3}]^k, [x_{i_4}x_{i_2}x_{i_1}]^{\overline{3}}, [x_{i_1}x_{i_4}x_{i_2}]^j, [x_{i_1}x_{i_4}x_{i_2}x_{i_3}]^{i-1}]
\end{array} \right\}
$$

$$
\sum_{i+j=1} \left\{ \begin{array}{l}
[[x_{i_1}x_{i_3}x_{i_2}x_{i_4}], [x_{i_1}x_{i_3}x_{i_2}]^{2-i-j}, [x_{i_2}x_{i_1}x_{i_3}]^j, [x_{i_2}x_{i_1}x_{i_3}x_{i_4}]^i] \\
[[x_{i_2}x_{i_4}x_{i_3}x_{i_1}], [x_{i_2}x_{i_4}x_{i_3}]^{2-i-j}, [x_{i_3}x_{i_2}x_{i_4}]^j, [x_{i_3}x_{i_2}x_{i_4}x_{i_1}]^i] \\
[[x_{i_3}x_{i_1}x_{i_4}x_{i_2}], [x_{i_3}x_{i_1}x_{i_4}]^{2-i-j}, [x_{i_4}x_{i_3}x_{i_1}]^j, [x_{i_4}x_{i_3}x_{i_1}x_{i_2}]^i] \\
[[x_{i_4}x_{i_2}x_{i_1}x_{i_3}], [x_{i_4}x_{i_2}x_{i_1}]^{2-i-j}, [x_{i_1}x_{i_4}x_{i_2}]^j, [x_{i_1}x_{i_4}x_{i_2}x_{i_3}]^i]
\end{array} \right\}
$$

$$\sum_{\substack{k=-1 \\ i=0,1 \\ i+k\neq -1}}^{1-i} \left\{ \begin{array}{c} [[x_{i_2}x_{i_1}x_{i_3}x_{i_4}],[x_{i_4}x_{i_1}x_{i_2}x_{i_3}]^{k+1},[x_{i_1}x_{i_3}x_{i_2}x_{i_4}]^{i},[x_{i_2}x_{i_1}x_{i_3}x_{i_4}]^{1-i-k}] \\ [[x_{i_1}x_{i_3}x_{i_2}x_{i_4}],[x_{i_4}x_{i_1}x_{i_2}x_{i_3}]^{2-i},[x_{i_1}x_{i_3}x_{i_2}x_{i_4}]^{i}] \end{array} \right\},$$

where $\beta = \frac{2!}{(3-i-j)i!}$ and $\gamma = \frac{2}{(i+k)\overline{3}!j!}$.

By writing these terms explicitly and rearranging them we obtain, after applying the isomorphism $H_0(G, \mathcal{M}^3F'_{ab}) \longrightarrow \gamma_3(F')F'''/[\gamma_3(F'), F]F'''$, the following ;

$$\prod_{i+j=1}\left\{\begin{array}{c}
[[x_{i_2},x_{i_1},x_{i_3},x_{i_4}],[x_{i_2},x_{i_1},x_{i_4}],[x_{i_2},x_{i_1}]^i,[x_{i_2},x_{i_1},x_{i_3}]^j] \\
[[x_{i_3},x_{i_2},x_{i_1},x_{i_4}],[x_{i_3},x_{i_2},x_{i_4}],[x_{i_3},x_{i_2}]^i,[x_{i_3},x_{i_2},x_{i_1}]^j] \\
[[x_{i_4},x_{i_3},x_{i_1},x_{i_2}],[x_{i_4},x_{i_3},x_{i_2}],[x_{i_4},x_{i_3}]^i,[x_{i_4},x_{i_3},x_{i_1}]^j] \\
[[x_{i_1},x_{i_4},x_{i_3},x_{i_2}],[x_{i_1},x_{i_4},x_{i_2}],[x_{i_1},x_{i_4}]^i,[x_{i_1},x_{i_4},x_{i_3}]^j] \\
[[x_{i_1},x_{i_3},x_{i_2},x_{i_4}],[x_{i_1},x_{i_3},x_{i_4}],[x_{i_1},x_{i_3}]^i,[x_{i_1},x_{i_3},x_{i_2}]^j] \\
[[x_{i_2},x_{i_4},x_{i_3},x_{i_1}],[x_{i_2},x_{i_4},x_{i_3}],[x_{i_2},x_{i_4}]^i,[x_{i_2},x_{i_4},x_{i_1}]^j] \\
[[x_{i_1},x_{i_3},x_{i_2},x_{i_4}],[x_{i_2},x_{i_1},x_{i_3},x_{i_4}]^i,[x_{i_1},x_{i_3},x_{i_2}]^j,[x_{i_2},x_{i_1},x_{i_3}]] \\
[[x_{i_2},x_{i_4},x_{i_3},x_{i_1}],[x_{i_3},x_{i_2},x_{i_4},x_{i_1}]^i,[x_{i_2},x_{i_4},x_{i_3}]^j,[x_{i_3},x_{i_2},x_{i_4}]] \\
[[x_{i_3},x_{i_1},x_{i_4},x_{i_2}],[x_{i_4},x_{i_3},x_{i_1},x_{i_2}]^i,[x_{i_3},x_{i_1},x_{i_4}]^j,[x_{i_4},x_{i_3},x_{i_1}]] \\
[[x_{i_4},x_{i_2},x_{i_1},x_{i_3}],[x_{i_1},x_{i_4},x_{i_2},x_{i_3}]^i,[x_{i_4},x_{i_2},x_{i_1}]^j,[x_{i_1},x_{i_4},x_{i_2}]]
\end{array}\right\}^{(-1)^i}$$

$$\prod_{\substack{i+j+k\leq 2\\ j\neq 0}}\left\{\begin{array}{c}
[[x_{i_2}x_{i_1}x_{i_3}],[x_{i_2}x_{i_1}]^i,[x_{i_2}x_{i_1}x_{i_4}]^j,[x_{i_2}x_{i_1}x_{i_3}x_{i_4}]^k,[x_{i_2}x_{i_1}x_{i_3}]^{\overline{2}}] \\
[[x_{i_3}x_{i_2}x_{i_1}],[x_{i_3}x_{i_2}]^i,[x_{i_3}x_{i_2}x_{i_4}]^j,[x_{i_3}x_{i_2}x_{i_1}x_{i_4}]^k,[x_{i_4}x_{i_3}x_{i_1}]^{\overline{2}}] \\
[[x_{i_4}x_{i_3}x_{i_1}],[x_{i_4}x_{i_3}]^i,[x_{i_4}x_{i_3}x_{i_2}]^j,[x_{i_4}x_{i_3}x_{i_1}x_{i_2}]^k,[x_{i_4}x_{i_3}x_{i_1}]^{\overline{2}}] \\
[[x_{i_1}x_{i_4}x_{i_3}],[x_{i_1}x_{i_4}]^i,[x_{i_1}x_{i_4}x_{i_2}]^j,[x_{i_1}x_{i_4}x_{i_3}x_{i_2}]^k,[x_{i_1}x_{i_4}x_{i_3}]^{\overline{2}}] \\
[[x_{i_1}x_{i_3}x_{i_2}],[x_{i_1}x_{i_3}]^i,[x_{i_1}x_{i_3}x_{i_4}]^j,[x_{i_1}x_{i_3}x_{i_2}x_{i_4}]^k,[x_{i_1}x_{i_3}x_{i_2}]^{\overline{2}}] \\
[[x_{i_2}x_{i_4}x_{i_1}],[x_{i_2}x_{i_4}]^i,[x_{i_2}x_{i_4}x_{i_3}]^j,[x_{i_2}x_{i_4}x_{i_1}x_{i_3}]^k,[x_{i_2}x_{i_4}x_{i_1}]^{\overline{2}}] \\
[[x_{i_2}x_{i_1}x_{i_3}x_{i_4}],[x_{i_1}x_{i_3}x_{i_2}]^j,[x_{i_1}x_{i_3}x_{i_2}x_{i_4}]^k,[x_{i_2}x_{i_1}x_{i_3}]^i,[x_{i_2}x_{i_1}x_{i_3}x_{i_4}]^{\overline{2}}] \\
[[x_{i_3}x_{i_2}x_{i_4}x_{i_1}],[x_{i_2}x_{i_4}x_{i_3}]^j,[x_{i_2}x_{i_4}x_{i_3}x_{i_1}]^k,[x_{i_3}x_{i_2}x_{i_4}]^i,[x_{i_3}x_{i_2}x_{i_4}x_{i_1}]^{\overline{2}}] \\
[[x_{i_4}x_{i_3}x_{i_1}x_{i_2}],[x_{i_3}x_{i_1}x_{i_4}]^j,[x_{i_3}x_{i_1}x_{i_4}x_{i_2}]^k,[x_{i_4}x_{i_3}x_{i_1}]^i,[x_{i_4}x_{i_3}x_{i_1}x_{i_2}]^{\overline{2}}] \\
[[x_{i_1}x_{i_4}x_{i_2}x_{i_3}],[x_{i_4}x_{i_2}x_{i_1}]^j,[x_{i_4}x_{i_2}x_{i_1}x_{i_3}]^k,[x_{i_1}x_{i_4}x_{i_2}]^i,[x_{i_1}x_{i_4}x_{i_2}x_{i_3}]^{\overline{2}}]
\end{array}\right\}^{C_i^2}$$

$$[[x_{i_2}x_{i_1}x_{i_3}x_{i_4}],[x_{i_4}x_{i_3}x_{i_1}x_{i_2}],[[x_{i_2}x_{i_1}x_{i_3}x_{i_4}]]+[x_{i_2}x_{i_3}x_{i_1}x_{i_4}],[x_{i_4}x_{i_1}x_{i_2}x_{i_3}],[x_{i_4}x_{i_3}x_{i_1}x_{i_2}]]$$

Because our main result do not cover the case $p = 2$, we consider this case separately in the following subsection.

### 3.2.2 Description of $t(F/[F'', F])$ in terms of generators

In this subsection we do the computation when $p = 2$. It should be pointed out that C.K. Gupta in [1], computed generators of the torsion subgroup of $F/[F'', F]$. Using homological methods, Kuz'min in [4] obtained the same generators. However, our computation is different. Furthermore, in the end of this computation we prove that our generators are the same as Gupta's generators. The main result of this section reads as follows.

**Proposition 3.3.1.** The torsion subgroup of $F/[F'', F]$ is generated by the elements,

$$\left\{ \begin{array}{c} [[x_{i_2}, x_{i_1}, x_{i_3}, x_{i_4}], [x_{i_3}, x_{i_2}, x_{i_1}]]\ [[x_{i_4}, x_{i_3}, x_{i_2}], [x_{i_3}, x_{i_2}, x_{i_4}, x_{i_1}]] \\ [[x_{i_3}, x_{i_1}, x_{i_2}, x_{i_4}], [x_{i_4}, x_{i_3}, x_{i_1}]]\ [[x_{i_2}, x_{i_1}, x_{i_4}], [x_{i_4}, x_{i_2}, x_{i_1}, x_{i_3}]] \\ [[x_{i_2}, x_{i_1}, x_{i_3}], [x_{i_2}, x_{i_1}, x_{i_4}]]\ [[x_{i_3}, x_{i_2}, x_{i_1}], [x_{i_3}, x_{i_2}, x_{i_4}]] \\ [[x_{i_4}, x_{i_3}, x_{i_1}], [x_{i_4}, x_{i_3}, x_{i_2}]]\ [[x_{i_4}, x_{i_1}, x_{i_2}], [x_{i_4}, x_{i_1}, x_{i_3}]] \\ [[x_{i_3}, x_{i_1}, x_{i_4}], [x_{i_3}, x_{i_1}, x_{i_2}]]\ [[x_{i_4}, x_{i_2}, x_{i_3}], [x_{i_4}, x_{i_2}, x_{i_1}]] \end{array} \right\}$$

where $(1 \leq i_1 < i_2 < i_3 < i_4 \leq d\ )$.

**Proof.** The proof is by computing the connecting homomorphism,

$$H_4(F/F', \mathbf{Z}_2) \longrightarrow t(\mathcal{M}^2 M \ \otimes_G \mathbf{R})$$

where $\mathbf{R}$ is the integers localized at 2.

For computing the connecting homomorphism, we use the double complex $\mathcal{M} \otimes_G \underline{\mathcal{P}}$, with $\underline{\mathcal{P}}$ as before and the complex $\mathcal{M}$ as the following :

$$\mathcal{M}:\ 0 \longrightarrow M \Lambda M \longrightarrow P \otimes M \longrightarrow P^2 \longrightarrow \mathbf{R}G^2 \longrightarrow \mathbf{R}G \longrightarrow \mathbf{Z}_2 \longrightarrow 0.$$

Here it should be pointed out that by [[12], Lemma 3.1] the complex $\mathcal{M}$ is exact.

Consider:

$$\begin{tikzcd}
 & & & & & \mathbf{Z}_2 \otimes P_4 \\
 & & & & RG \otimes P_3 & RG \otimes P_4 \arrow[l] \arrow[u] \\
 & & & RG^2 \otimes P_2 & RG^2 \otimes P_3 \arrow[l] \arrow[u] & \\
 & & P^2 \otimes P_1 & P^2 \otimes P_2 \arrow[l] \arrow[u] & & \\
 & A \otimes RG & A \otimes P_1 \arrow[l] \arrow[u] & & & \\
 \mathcal{M}^2 M \otimes R & \mathcal{M}^2 M \otimes RG \arrow[l] \arrow[u] & & & &
\end{tikzcd}$$

where $A = P \otimes_{\mathbf{R}} M$, in this double complex we use $\otimes$ instead of $\otimes_G$.

Now we start our computation by the following element

$$1 \otimes e_1 \wedge e_2 \wedge e_3 \wedge e_4 \in \mathbf{Z}_2 \otimes P_4 \tag{3.90}$$

An inverse image of (3.90), in $\mathbf{R}G \otimes_G P_4$ is

$$1 \otimes e_1 \wedge e_2 \wedge e_3 \wedge e_4 \tag{3.91}$$

By applying the homomorphism $\mathbf{R}G \otimes P_4 \longrightarrow \mathbf{R}G \otimes P_3$ to the (3.91) we obtain

$$\sum_{i=1}^{4} (-1)^{i+1}(b_i - 1) \otimes (e_1 \wedge ....\wedge \hat{e_i} \wedge ... \wedge e_4) \tag{3.92}$$

In order to get an inverse image of (3.92), in $\mathbf{R}G^2 \otimes_G P_3$, we consider the element

$$(1 \circ 1).(b_i - 1) = b_i \circ b_i - 1 \circ 1.$$

By writing $b_i = (b_i - 1) + 1$, and expanding, we get

$$(1 \circ 1).(b_i - 1) = (b_i - 1) \circ (b_i - 1) + 2(b_i - 1) \circ 1.$$

By subtracting $(b_i - 1)^2$ from both sides, we get

$$[(1 \circ 1).(b_i - 1) - (b_i - 1)^2] = 2(b_i - 1) \circ 1\,.$$

Hence the expression

$$\frac{1}{2}[(1 \circ 1).(b_i - 1) - (b_i - 1)^2],$$

makes sense in our situation. We notice that

$$\frac{1}{2}\sum_{i=1}^{4} (-1)^{i+1}[(1 \circ 1).(b_i - 1) - (b_i - 1)^2] \otimes (e_1 \wedge ...\wedge \hat{e_i} ... \wedge e_4), \tag{3.93}$$

is an inverse image of (3.92), in $\mathbf{R}G^2 \otimes_G P_3$. This is because $(b_i - 1)^2$ is in the kernel of the map $\mathbf{R}G^2 \longrightarrow \mathbf{R}G$, and the image of $(1 \circ 1)$ under this map is 2.

After applying the homomorphism $\mathbf{R}G^2 \otimes_G P_3 \longrightarrow \mathbf{R}G^2 \otimes_G P_2$, to (3.93), we obtain

$$
\begin{aligned}
&\tfrac{1}{2}\{(b_2-1)^2.(b_1-1)-(b_1-1)^2.(b_2-1)\}\otimes e_3\wedge e_4\\
&+\tfrac{1}{2}\{(b_1-1)^2.(b_3-1)-(b_3-1)^2.(b_1-1)\}\otimes e_2\wedge e_4\\
&+\tfrac{1}{2}\{(b_4-1)^2.(b_1-1)-(b_1-1)^2.(b_4-1)\}\otimes e_2\wedge e_3\\
&+\tfrac{1}{2}\{(b_2-1)^2.(b_4-1)-(b_4-1)^2.(b_2-1)\}\otimes e_1\wedge e_3\\
&+\tfrac{1}{2}\{(b_4-1)^2.(b_3-1)-(b_3-1)^2.(b_4-1)\}\otimes e_1\wedge e_2\\
&+\tfrac{1}{2}\{(b_3-1)^2.(b_2-1)-(b_2-1)^2.(b_3-1)\}\otimes e_1\wedge e_4
\end{aligned}
\tag{3.94}
$$

In order to get an inverse image of (3.94) in $P^2\otimes_G P_2$, we consider the following element in $P^2$:

$$
e_2^2(b_1-1)-e_1^2(b_2-1)+[e_1(b_2-1)-e_2(b_1-1)]^2=\left\{\begin{array}{c}2e_2(b_1-1)\circ e_2\\ -2e_1(b_2-1)\circ e_1\\ -2e_1(b_2-1)\circ e_2(b_1-1)\end{array}\right\}
$$

Hence the expression

$$
\frac{1}{2}\{e_2^2(b_1-1)-e_1^2(b_2-1)+(e_1(b_2-1)-e_2(b_1-1))^2\}
$$

makes sense. Moreover, this term

$$
\frac{1}{2}[e_2^2(b_1-1)-e_1^2(b_2-1)+[\ ]_{21}^2]\otimes e_3\wedge e_4
$$

is an inverse image of $\frac{1}{2}\{(b_2-1)^2.(b_1-1)-(b_1-1)^2.(b_2-1)\}\otimes e_3\wedge e_4$ in $P^2\otimes_G P_2$ under the homomorphism $P^2\otimes_G P_2\longrightarrow \mathbf{R}G^2\otimes_G P_2$.

Similar considerations for the first tensor factors of the remaining terms in (3.94) imply that the following is inverse image of (3.94) in $P^2\otimes_G P_2$

$$
\begin{aligned}
&\tfrac{1}{2}\{e_2^2(b_1-1)-e_1^2(b_2-1)+[\ ]_{21}^2\}\otimes e_3\wedge e_4\\
&+\tfrac{1}{2}\{e_1^2(b_3-1)-e_3^2(b_1-1)+[\ ]_{13}^2\}\otimes e_2\wedge e_4\\
&+\tfrac{1}{2}\{e_4^2(b_1-1)-e_1^2(b_4-1)+[\ ]_{41}^2\}\otimes e_2\wedge e_3\\
&+\tfrac{1}{2}\{e_2^2(b_4-1)-e_4^2(b_2-1)+[\ ]_{24}^2\}\otimes e_1\wedge e_3\\
&+\tfrac{1}{2}\{e_4^2(b_3-1)-e_3^2(b_4-1)+[\ ]_{43}^2\}\otimes e_1\wedge e_2\\
&+\tfrac{1}{2}\{e_3^2(b_2-1)-e_2^2(b_3-1)+[\ ]_{32}^2\}\otimes e_1\wedge e_4
\end{aligned}
\tag{3.95}
$$

After applying the homomorphism $P^2\otimes_G P_2 \longrightarrow P^2\otimes_G P_1$ to (3.95), we get

$$
\begin{aligned}
&\tfrac{1}{2}\{[\ ]_{21}^2.(b_3-1)+[\ ]_{13}^2.(b_2-1)+[\ ]_{32}^2.(b_1-1)\}\otimes e_4\\
&-\tfrac{1}{2}\{[\ ]_{32}^2.(b_4-1)+[\ ]_{24}^2.(b_3-1)+[\ ]_{43}^2.(b_2-1)\}\otimes e_1\\
&+\tfrac{1}{2}\{[\ ]_{43}^2.(b_1-1)-[\ ]_{13}^2.(b_4-1)-[\ ]_{41}^2.(b_3-1)\}\otimes e_2\\
&+\tfrac{1}{2}\{[\ ]_{41}^2.(b_2-1)+[\ ]_{24}^2.(b_1-1)-[\ ]_{21}^2.(b_4-1)\}\otimes e_3
\end{aligned}
\tag{3.96}
$$

In order to get an inverse image of (3.96), in $P\otimes M\otimes_G P_1$, we consider the element

$([\ ]_{21}\otimes[\ ]_{21}).(b_3-1)=[\ ]_{21}b_3\otimes[\ ]_{21}b_3-[\ ]_{21}\otimes[\ ]_{21}.$

By writing $b_3=(b_3-1)+1$ and expanding we get

$$
([\ ]_{21}\otimes[\ ]_{21}).(b_3-1)=\left\{\begin{array}{c}[\ ]_{21}(b_3-1)\otimes[\ ]_{21}(b_3-1)\\ [\ ]_{21}(b_3-1)\otimes[\ ]_{21}+[\ ]_{21}\otimes[\ ]_{21}(b_3-1)\end{array}\right\}.
\tag{3.97}
$$

On the other hand the element

$$[\ ]_{21}(b_3-1)\otimes[\ ]_{21}-[\ ]_{21}\otimes[\ ]_{21}(b_3-1)$$

belongs to the kernel of the map $(P\otimes M\longrightarrow P^2)$. By adding this element to (3.97), we obtain

$$[\ ]_{21}(b_3-1)\otimes[\ ]_{21}(b_3-1)+2[\ ]_{21}(b_3-1)\otimes[\ ]_{21},$$

here we note that the coefficient of the second term is divisible by 2.

We do the same thing for $([\ ]_{13}\otimes[\ ]_{13}).(b_2-1)$ and $([\ ]_{32}\otimes[\ ]_{32}).(b_1-1)$. The coefficients of the first term in each one of them is not divisible by 2, so we collect them together

$$\left\{\begin{array}{c}[\ ]_{21}(b_3-1)\otimes[\ ]_{21}(b_3-1)+[\ ]_{13}(b_2-1)\otimes[\ ]_{13}(b_2-1)\\ +[\ ]_{32}(b_1-1)\otimes[\ ]_{32}(b_1-1)\end{array}\right\} \tag{3.98}$$

By the Jacobi identity, we can write $[\ ]_{32}(b_1-1)\otimes[\ ]_{32}(b_1-1)$ as follows.

$$\begin{array}{c}[\ ]_{13}(b_2-1)\otimes[\ ]_{13}(b_2-1)+[\ ]_{13}(b_2-1)\otimes[\ ]_{21}(b_3-1)\\ [\ ]_{21}(b_3-1)\otimes[\ ]_{13}(b_2-1)+[\ ]_{21}(b_3-1)\otimes[\ ]_{21}(b_3-1)\end{array} \tag{3.99}$$

By substituting (3.99) into (3.98), we obtain

$$\begin{array}{c}2[\ ]_{21}(b_3-1)\otimes[\ ]_{21}(b_3-1)+2[\ ]_{13}(b_2-1)\otimes[\ ]_{13}(b_2-1)\\ [\ ]_{13}(b_2-1)\otimes[\ ]_{21}(b_3-1)+[\ ]_{21}(b_3-1)\otimes[\ ]_{13}(b_2-1)\end{array} \tag{3.100}$$

Again the element $[\ ]_{13}(b_2-1)\otimes[\ ]_{21}(b_3-1)-[\ ]_{21}(b_3-1)\otimes[\ ]_{13}(b_2-1)$ belongs to the kernel of the map $(M\otimes M\longrightarrow M^2)$, and when we added to (3.100), we get terms with coefficients divisible by 2, namely

$$\begin{array}{c}2[\ ]_{21}(b_3-1)\otimes[\ ]_{21}(b_3-1)+2[\ ]_{13}(b_2-1)\otimes[\ ]_{21}(b_3-1)\\ +2[\ ]_{13}(b_2-1)\otimes[\ ]_{13}(b_2-1)\end{array}$$

As a result of our previous discussion the expression

$$\frac{1}{2}\left\{\begin{array}{c} ([\ ]_{21}\otimes[\ ]_{21}).(b_3-1)+([\ ]_{13}\otimes[\ ]_{13}).(b_2-1)+([\ ]_{32}\otimes[\ ]_{32}).(b_1-1)\\ [\ ]_{21}(b_3-1)\otimes[\ ]_{21}-[\ ]_{21}\otimes[\ ]_{21}(b_3-1)\\ [\ ]_{13}(b_2-1)\otimes[\ ]_{13}-[\ ]_{13}\otimes[\ ]_{13}(b_2-1)\\ [\ ]_{32}(b_1-1)\otimes[\ ]_{32}-[\ ]_{32}\otimes[\ ]_{32}(b_1-1)\\ [\ ]_{13}(b_2-1)\otimes[\ ]_{21}(b_3-1)-[\ ]_{21}(b_3-1)\otimes[\ ]_{13}(b_2-1) \end{array}\right\} \tag{3.101}$$

makes sense. Moreover, (3.101) is an inverse image of

$$\frac{1}{2}\{[\ ]_{21}^2(b_3-1)+[\ ]_{13}^2(b_2-1)+[\ ]_{32}^2(b_1-1)\}$$

under the homomorphism $P\otimes M\longrightarrow P^2$.

Similar consideration for the first tensor factors of the remaining these terms in (3.96) then imply that

$$\frac{1}{2}\left\{\begin{array}{c} -([\ ]_{32}\otimes[\ ]_{32}).(b_4-1)-([\ ]_{24}\otimes[\ ]_{24}).(b_3-1)-([\ ]_{43}\otimes[\ ]_{43}).(b_2-1)\\ -[\ ]_{32}(b_4-1)\otimes[\ ]_{32}+[\ ]_{32}\otimes[\ ]_{32}(b_4-1)\\ [\ ]_{24}(b_3-1)\otimes[\ ]_{24}-[\ ]_{24}\otimes[\ ]_{24}(b_3-1)\\ -[\ ]_{43}(b_2-1)\otimes[\ ]_{43}+[\ ]_{43}\otimes[\ ]_{43}(b_2-1)\\ [\ ]_{42}(b_3-1)\otimes[\ ]_{32}(b_4-1)-[\ ]_{32}(b_4-1)\otimes[\ ]_{42}(b_3-1) \end{array}\right\}\otimes e_1$$

$$+\frac{1}{2}\left\{\begin{array}{c} ([\ ]_{43}\otimes[\ ]_{43}).(b_1-1)-([\ ]_{13}\otimes[\ ]_{13}).(b_4-1)-([\ ]_{41}\otimes[\ ]_{41}).(b_3-1)\\ [\ ]_{43}(b_1-1)\otimes[\ ]_{43}-[\ ]_{43}\otimes[\ ]_{43}(b_1-1)\\ [\ ]_{13}(b_4-1)\otimes[\ ]_{13}-[\ ]_{13}\otimes[\ ]_{13}(b_4-1)\\ -[\ ]_{41}(b_3-1)\otimes[\ ]_{41}+[\ ]_{41}\otimes[\ ]_{41}(b_3-1)\\ -[\ ]_{43}(b_1-1)\otimes[\ ]_{31}(b_4-1)+[\ ]_{31}(b_4-1)\otimes[\ ]_{43}(b_1-1) \end{array}\right\}\otimes e_2$$

$$+\frac{1}{2}\left\{\begin{array}{c} ([\ ]_{41}\otimes[\ ]_{41}).(b_2-1)+([\ ]_{24}\otimes[\ ]_{24}).(b_1-1)-([\ ]_{21}\otimes[\ ]_{21}).(b_4-1) \\ [\ ]_{41}(b_2-1)\otimes[\ ]_{41}-[\ ]_{41}\otimes[\ ]_{41}(b_2-1) \\ -[\ ]_{24}(b_1-1)\otimes[\ ]_{24}+[\ ]_{24}\otimes[\ ]_{24}(b_1-1) \\ -[\ ]_{21}(b_4-1)\otimes[\ ]_{21}+[\ ]_{21}\otimes[\ ]_{21}(b_4-1) \\ [\ ]_{42}(b_1-1)\otimes[\ ]_{41}(b_2-1)-[\ ]_{41}(b_2-1)\otimes[\ ]_{42}(b_1-1) \end{array}\right\}\otimes e_3$$

is a well–defined element of $P\otimes M\otimes_G P_1$ and that it is an inverse image of (3.96) under the homomorphism $P\otimes M\otimes_G P_1 \longrightarrow P^2\otimes_G P_1$.

After applying the homomorphism $P\otimes M\otimes_G P_1 \longrightarrow P\otimes M\otimes_G \mathbf{R}G$, (here we identify $P\otimes M\otimes_G \mathbf{R}G$ with $P\otimes M$), we obtain

$$
\frac{1}{2}\left\{\begin{array}{l}
\left\{\begin{array}{c}
[\ ]_{21}(b_3-1)\otimes[\ ]_{21}-[\ ]_{21}\otimes[\ ]_{21}(b_3-1)\\
-[\ ]_{13}(b_2-1)\otimes[\ ]_{13}+[\ ]_{13}\otimes[\ ]_{13}(b_2-1)\\
+[\ ]_{32}(b_1-1)\otimes[\ ]_{32}-[\ ]_{32}\otimes[\ ]_{32}(b_1-1)\\
+[\ ]_{13}(b_2-1)\otimes[\ ]_{21}(b_3-1)-[\ ]_{21}(b_3-1)\otimes[\ ]_{13}(b_2-1)
\end{array}\right\}(b_4-1)\\
+\left\{\begin{array}{c}
-[\ ]_{32}(b_4-1)\otimes[\ ]_{32}+[\ ]_{32}\otimes[\ ]_{32}(b_4-1)\\
+[\ ]_{24}(b_3-1)\otimes[\ ]_{24}-[\ ]_{24}\otimes[\ ]_{24}(b_3-1)\\
-[\ ]_{43}(b_2-1)\otimes[\ ]_{43}+[\ ]_{43}\otimes[\ ]_{43}(b_2-1)\\
-[\ ]_{24}(b_3-1)\otimes[\ ]_{32}(b_4-1)+[\ ]_{32}(b_4-1)\otimes[\ ]_{42}(b_3-1)
\end{array}\right\}(b_1-1)\\
+\left\{\begin{array}{c}
[\ ]_{43}(b_1-1)\otimes[\ ]_{43}-[\ ]_{43}\otimes[\ ]_{43}(b_1-1)\\
+[\ ]_{13}(b_4-1)\otimes[\ ]_{13}-[\ ]_{13}\otimes[\ ]_{13}(b_4-1)\\
-[\ ]_{41}(b_3-1)\otimes[\ ]_{41}+[\ ]_{41}\otimes[\ ]_{41}(b_3-1)\\
+[\ ]_{43}(b_1-1)\otimes[\ ]_{13}(b_4-1)-[\ ]_{13}(b_4-1)\otimes[\ ]_{43}(b_1-1)
\end{array}\right\}(b_2-1)\\
+\left\{\begin{array}{c}
[\ ]_{41}(b_2-1)\otimes[\ ]_{41}-[\ ]_{41}\otimes[\ ]_{41}(b_2-1)\\
-[\ ]_{24}(b_1-1)\otimes[\ ]_{24}+[\ ]_{24}\otimes[\ ]_{24}(b_1-1)\\
-[\ ]_{21}(b_4-1)\otimes[\ ]_{21}+[\ ]_{21}\otimes[\ ]_{21}(b_4-1)\\
-[\ ]_{24}(b_1-1)\otimes[\ ]_{41}(b_2-1)+[\ ]_{41}(b_2-1)\otimes[\ ]_{24}(b_1-1)
\end{array}\right\}(b_3-1)
\end{array}\right\} \tag{3.102}
$$

The inverse image of (3.102), in $\mathcal{M}^2M\otimes_G\mathbf{R}G$ (again we identify $\mathcal{M}^2M\otimes_G\mathbf{R}G$ with $\mathcal{M}^2M$) is

$$\frac{1}{2}\left\{\begin{array}{l} \left\{\begin{array}{c} [[\,]_{21}(b_3-1),[\,]_{21}\,]-[[\,]_{13}(b_2-1),[\,]_{13}\,]+[[\,]_{32}(b_1-1),[\,]_{32}\,] \\ +[[\,]_{13}(b_2-1),[\,]_{21}(b_3-1)\,] \end{array}\right\}(b_4-1) \\ +\left\{\begin{array}{c} -[[\,]_{32}(b_4-1),[\,]_{32}\,]+[[\,]_{24}(b_3-1),[\,]_{24}\,]-[[\,]_{43}(b_2-1),[\,]_{43}\,] \\ -[[\,]_{24}(b_3-1),[\,]_{32}(b_4-1)\,] \end{array}\right\}(b_1-1) \\ +\left\{\begin{array}{c} [[\,]_{43}(b_1-1),[\,]_{43}\,]+[[\,]_{13}(b_4-1),[\,]_{13}\,]-[[\,]_{41}(b_3-1),[\,]_{41}\,] \\ +[[\,]_{43}(b_1-1),[\,]_{13}(b_4-1)\,] \end{array}\right\}(b_2-1) \\ +\left\{\begin{array}{c} [[\,]_{41}(b_2-1),[\,]_{41}\,]-[[\,]_{24}(b_1-1),[\,]_{24}\,]-[[\,]_{21}(b_4-1),[\,]_{21}\,] \\ -[[\,]_{24}(b_1-1),[\,]_{41}(b_2-1)\,] \end{array}\right\}(b_3-1) \end{array}\right\} \tag{3.103}$$

Our aim is to write

$$\begin{array}{l} \left\{\begin{array}{c} [[\,]_{21}(b_3-1),[\,]_{21}\,]-[[\,]_{13}(b_2-1),[\,]_{13}\,]+[[\,]_{32}(b_1-1),[\,]_{32}\,] \\ +[[\,]_{13}(b_2-1),[\,]_{21}(b_3-1)\,] \end{array}\right\}(b_4-1) \\ +\left\{\begin{array}{c} -[[\,]_{32}(b_4-1),[\,]_{32}\,]+[[\,]_{24}(b_3-1),[\,]_{24}\,]-[[\,]_{43}(b_2-1),[\,]_{43}\,] \\ -[[\,]_{24}(b_3-1),[\,]_{32}(b_4-1)\,] \end{array}\right\}(b_1-1) \\ +\left\{\begin{array}{c} [[\,]_{43}(b_1-1),[\,]_{43}\,]+[[\,]_{13}(b_4-1),[\,]_{13}\,]-[[\,]_{41}(b_3-1),[\,]_{41}\,] \\ +[[\,]_{43}(b_1-1),[\,]_{13}(b_4-1)\,] \end{array}\right\}(b_2-1) \\ +\left\{\begin{array}{c} [[\,]_{41}(b_2-1),[\,]_{41}\,]-[[\,]_{24}(b_1-1),[\,]_{24}\,]-[[\,]_{21}(b_4-1),[\,]_{21}\,] \\ -[[\,]_{24}(b_1-1),[\,]_{41}(b_2-1)\,] \end{array}\right\}(b_3-1) \end{array}$$

as a linear combination of terms with coefficients divisible by 2.

Notice that

$$\begin{matrix} [[\,]_{21}(b_3-1),[\,]_{21}].(b_4-1) \\ -[[\,]_{21}(b_4-1),[\,]_{21}].(b_3-1) \end{matrix} = \left\{ \begin{matrix} 2[[\,]_{21}(b_3-1),[\,]_{21}(b_4-1)] \\ +[[\,]_{21}(b_3-1)(b_4-1),[\,]_{21}(b_4-1)] \\ -[[\,]_{21}(b_4-1)(b_3-1),[\,]_{21}(b_3-1)] \end{matrix} \right\} \tag{3.104}$$

By substituting this, and similar expressions for the other corresponding terms

$(-[[\,]_{13}(b_2-1),[\,]_{13}\,].(b_4-1)+[[\,]_{13}(b_4-1),[\,]_{13}].(b_2-1)\cdots\cdot$ etc) into (3.103) we get that this element is the sum of

$$\frac{1}{2}\left\{ \begin{matrix} 2[[\,]_{21}(b_3-1),[\,]_{21}(b_4-1)]+2[[\,]_{31}(b_4-1),[\,]_{31}(b_2-1)] \\ +2[[\,]_{32}(b_1-1),[\,]_{32}(b_4-1)]+2[[\,]_{42}(b_3-1),[\,]_{42}(b_1-1)] \\ +2[[\,]_{43}(b_1-1),[\,]_{43}(b_2-1)]+2[[\,]_{41}(b_2-1),[\,]_{41}(b_3-1)] \end{matrix} \right\} \tag{3.105}$$

$$\frac{1}{2}\left\{ \begin{matrix} [[\,]_{21}(b_3-1)(b_4-1),[\,]_{21}(b_4-1)]-[[\,]_{21}(b_4-1)(b_3-1),[\,]_{21}(b_3-1)] \\ +[[\,]_{31}(b_2-1)(b_4-1),[\,]_{31}(b_2-1)]-[[\,]_{31}(b_4-1)(b_2-1),[\,]_{31}(b_4-1)] \\ +[[\,]_{32}(b_1-1)(b_4-1),[\,]_{32}(b_4-1)]-[[\,]_{32}(b_4-1)(b_1-1),[\,]_{32}(b_1-1)] \\ +[[\,]_{42}(b_1-1)(b_3-1),[\,]_{42}(b_1-1)]-[[\,]_{42}(b_3-1)(b_1-1),[\,]_{42}(b_3-1)] \\ +[[\,]_{43}(b_1-1)(b_2-1),[\,]_{43}(b_2-1)]-[[\,]_{43}(b_2-1)(b_1-1),[\,]_{43}(b_1-1)] \\ +[[\,]_{41}(b_2-1)(b_3-1),[\,]_{41}(b_3-1)]-[[\,]_{41}(b_3-1)(b_2-1),[\,]_{41}(b_2-1)] \\ +[[\,]_{13}(b_2-1),[\,]_{21}(b_3-1)].(b_4-1) \\ [[\,]_{42}(b_3-1),[\,]_{32}(b_4-1)].(b_1-1) \\ -[[\,]_{43}(b_1-1),[\,]_{31}(b_4-1)].(b_2-1) \\ +[[\,]_{42}(b_1-1),[\,]_{41}(b_2-1)].(b_3-1) \end{matrix} \right\} \tag{3.106}$$

Our next goal is to write 2 times the element (3.106), as a linear combination of terms with coefficients divisible by 2. In order to do that we consider the following terms

$$
\begin{aligned}
&[[\,]_{13}(b_2-1),[\,]_{21}(b_3-1)].(b_4-1)-[[\,]_{32}(b_4-1)(b_1-1),[\,]_{32}(b_1-1)]\\
&+[[\,]_{42}(b_3-1),[\,]_{32}(b_4-1)].(b_1-1)+[[\,]_{43}(b_1-1)(b_2-1),[\,]_{43}(b_2-1)]\\
&-[[\,]_{43}(b_1-1),[\,]_{13}(b_4-1)].(b_2-1)+[[\,]_{41}(b_2-1)(b_3-1),[\,]_{41}(b_3-1)]\\
&+[[\,]_{42}(b_1-1),[\,]_{41}(b_2-1)].(b_3-1)+[[\,]_{21}(b_3-1)(b_4-1),[\,]_{21}(b_4-1)]
\end{aligned}
$$

from these terms we obtain

$$
\begin{aligned}
&2[[\,]_{13}(b_2-1),[\,]_{21}(b_3-1)(b_4-1)]+[[\,]_{13}(b_2-1)(b_4-1),[\,]_{21}(b_3-1)(b_4-1)]\\
&\quad-[[\,]_{13}(b_2-1)(b_4-1),[\,]_{13}(b_2-1)]-[[\,]_{21}(b_3-1)(b_4-1),[\,]_{21}(b_3-1)]\\
&+2[[\,]_{42}(b_3-1),[\,]_{32}(b_4-1)(b_1-1)]+[\,]_{42}(b_3-1)(b_1-1),[\,]_{32}(b_1-1)(b_4-1)]\\
&\quad+[[\,]_{42}(b_3-1)(b_1-1),[\,]_{42}(b_3-1)]+[[\,]_{32}(b_1-1)(b_4-1),[\,]_{32}(b_4-1)]\\
&+2[[\,]_{43}(b_1-1),[\,]_{31}(b_2-1)(b_4-1)]-[[\,]_{43}(b_1-1)(b_2-1),[\,]_{31}(b_2-1)(b_4-1)]\\
&\quad+[[\,]_{43}(b_2-1)(b_1-1),[\,]_{43}(b_1-1)]+[[\,]_{31}(b_2-1)(b_4-1),[\,]_{31}(b_4-1)]\\
&+2[[\,]_{42}(b_1-1),[\,]_{41}(b_2-1)(b_3-1)]+[[\,]_{42}(b_1-1)(b_3-1),[\,]_{41}(b_3-1)(b_2-1)]\\
&\quad+[[\,]_{42}(b_1-1)(b_3-1),[\,]_{42}(b_1-1)]+[[\,]_{41}(b_3-1)(b_2-1),[\,]_{41}(b_2-1)]
\end{aligned}
\tag{3.107}
$$

By substituting (3.107) into (3.106), we get

$$\frac{1}{2}\left\{\begin{array}{c} 2[[\,]_{13}(b_2-1),[\,]_{21}(b_3-1)(b_4-1)]+2[[\,]_{42}(b_3-1),[\,]_{32}(b_4-1)(b_1-1)] \\ -2[[\,]_{43}(b_1-1),[\,]_{31}(b_2-1)(b_4-1)]+2[[\,]_{42}(b_1-1),[\,]_{41}(b_2-1)(b_3-1)] \\ +2[[\,]_{21}(b_3-1),[\,]_{21}(b_3-1)(b_4-1)]+2[[\,]_{32}(b_1-1)(b_4-1),[\,]_{32}(b_4-1)] \\ +2[[\,]_{42}(b_1-1)(b_3-1),[\,]_{42}(b_1-1)] \\ +[[\,]_{13}(b_2-1)(b_4-1),[\,]_{21}(b_3-1)(b_4-1)] \\ +[[\,]_{42}(b_3-1)(b_1-1),[\,]_{32}(b_1-1)(b_4-1)] \\ -[[\,]_{43}(b_1-1)(b_2-1),[\,]_{31}(b_2-1)(b_4-1)] \\ +[[\,]_{42}(b_1-1)(b_3-1),[\,]_{41}(b_3-1)(b_2-1)] \end{array}\right\} \tag{3.108}$$

Using the Jacobi identity it is easy to see that the sum of the last four terms in (3.108), is equal to zero. Therefore (3.108) becomes

$$\begin{array}{c} [[\,]_{13}(b_2-1),[\,]_{21}(b_3-1)(b_4-1)]+[[\,]_{42}(b_3-1),[\,]_{32}(b_4-1)(b_1-1)] \\ -[[\,]_{43}(b_1-1),[\,]_{31}(b_2-1)(b_4-1)]+[[\,]_{42}(b_1-1),[\,]_{41}(b_2-1)(b_3-1)] \\ +[[\,]_{21}(b_3-1),[\,]_{21}(b_3-1)(b_4-1)]+[[\,]_{32}(b_1-1)(b_4-1),[\,]_{32}(b_4-1)] \\ +[[\,]_{42}(b_1-1)(b_3-1),[\,]_{42}(b_1-1)] \end{array}$$

and this can be rearranged as

$$\begin{array}{l} -[[\,]_{32}(b_1-1),[\,]_{21}(b_3-1)(b_4-1)+[[\,]_{43}(b_2-1),[\,]_{32}(b_1-1)(b_4-1)]] \\ +[[\,]_{42}(b_1-1),[\,]_{21}(b_3-1)(b_4-1)]-[[\,]_{43}(b_1-1),[\,]_{31}(b_2-1)(b_4-1)] \end{array} \tag{3.109}$$

Finally from (3.105) and (3.109), we obtain

$$\left\{\begin{array}{c} [[\,]_{21}(b_3-1),[\,]_{21}(b_4-1)]+[[\,]_{31}(b_4-1),[\,]_{31}(b_2-1)] \\ +[[\,]_{32}(b_1-1),[\,]_{32}(b_4-1)]+[[\,]_{42}(b_3-1),[\,]_{42}(b_1-1)] \\ +[[\,]_{43}(b_1-1),[\,]_{43}(b_2-1)]+[[\,]_{41}(b_2-1),[\,]_{41}(b_3-1)] \\ -[[\,]_{32}(b_1-1),[\,]_{21}(b_3-1)(b_4-1)+[[\,]_{43}(b_2-1),[\,]_{32}(b_1-1)(b_4-1)] \\ +[[\,]_{42}(b_1-1),[\,]_{21}(b_3-1)(b_4-1)]-[[]_{43}(b_1-1),[]_{31}(b_2-1)(b_4-1)] \end{array}\right\}. \tag{3.110}$$

and the image of this element in $M\Lambda M \otimes_G \mathbf{R}$ is the image of $1\otimes e_1\wedge e_2\wedge e_3\wedge e_4$ under the connecting homomorphism $H_4(G,\mathbf{Z}_2)\longrightarrow H_0(G,M\Lambda M)$. On the other hand $tH_0(G,M\Lambda M)\cong t_pH_0(G,F'_{ab}\Lambda F'_{ab})$, but $tH_0(G,F'_{ab}\Lambda F'_{ab})$ is an elementary abelian $p$-group, this means that the torsion subgroups of both $H_0(G,M\Lambda M)$ and $H_0(G,F'_{ab}\Lambda F'_{ab})$ are the same. Therefore the theorem follows by applying the isomorphism $H_0(G,F'_{ab}\Lambda F'_{ab})\longrightarrow F''/[F'',F]$.

**We close this subsection by proving that our results for $p=2$ are consistent with Gupta's result.**

Now, we start with Gupta's element, and we try to convert it to our element, for the case $p=2$.

Gupta's element is

$$\begin{gathered}\Big[[x_{i_1},x_{i_2}],[x_{i_3}^{-1},x_{i_4}^{-1}]\Big]\Big[[x_{i_3},x_{i_4}],[x_{i_1}^{-1},x_{i_2}^{-1}]\Big]\\ \Big[[x_{i_1},x_{i_3}],[x_{i_4}^{-1},x_{i_2}^{-1}]\Big]\Big[[x_{i_4},x_{i_2}],[x_{i_1}^{-1},x_{i_3}^{-1}]\Big]\\ \Big[[x_{i_1},x_{i_4}],[x_{i_2}^{-1},x_{i_3}^{-1}]\Big]\Big[[x_{i_2},x_{i_3}],[x_{i_1}^{-1},x_{i_4}^{-1}]\Big]\end{gathered}$$

For simplicity, we consider

$$\begin{array}{c}\Big[[x_1, x_2], [x_3^{-1}, x_4^{-1}]\Big]\ \Big[[x_3, x_4], [x_1^{-1}, x_2^{-1}]\Big] \\ \Big[[x_1, x_3], [x_4^{-1}, x_2^{-1}]\Big]\ \Big[[x_4, x_2], [x_1^{-1}, x_3^{-1}]\Big] \\ \Big[[x_1, x_4], [x_2^{-1}, x_3^{-1}]\Big]\ \Big[[x_2, x_3], [x_1^{-1}, x_4^{-1}]\Big]\end{array} \tag{3.111}$$

For $b_i, b_j \in G$, we have

$$\begin{aligned}[x_i, x_j].(b_ib_j)^{-1} = [x_i, x_j]^{(x_ix_j)^{-1}} &= x_i x_j[x_i, x_j](x_i x_j)^{-1} \\ &= x_i x_j x_i^{-1} x_j^{-1} x_i x_j x_j^{-1} x_i^{-1} \\ &= [x_i^{-1}, x_j^{-1}]\end{aligned}$$

this means that (3.111) is equal the following

$$\begin{array}{c}\Big[[x_1, x_2], [x_3, x_4]^{(x_3x_4)^{-1}}\Big]\ \Big[[x_3, x_4], [x_1, x_2]^{(x_1x_2)^{-1}}\Big] \\ \Big[[x_1, x_3], [x_4, x_2]^{(x_4x_2)^{-1}}\Big]\ \Big[[x_4, x_2], [x_1, x_3]^{(x_1x_3)^{-1}}\Big] \\ \Big[[x_1, x_4], [x_2, x_3]^{(x_2x_3)^{-1}}\Big]\ \Big[[x_2, x_3], [x_1, x_4]^{(x_1x_4)^{-1}}\Big]\end{array}$$

Using the fact that $\Big[[\,,\,], [\,,\,]^{x^{-1}}\Big] = [[\,,\,]^x, [\,,\,]]$ modulo $[F'', F]$, and applying the isomorphism

$$F''/[F'', F] \longrightarrow F'_{ab} \Lambda F'_{ab} \otimes_G \mathbf{Z}$$

we obtain the following

$$\begin{array}{c}[[\,]_{12} b_3 b_4, [\,]_{34}] + [[\,]_{34} b_1 b_2, [\,]_{12}] \\ + [[\,]_{13} b_4 b_2, [\,]_{42}] + [[\,]_{42} b_1 b_3, [\,]_{13}] \\ + [[\,]_{14} b_2 b_3, [\,]_{23}] + [[\,]_{23} b_1 b_4, [\,]_{14}]\end{array}$$

By writing $b_i = (b_i - 1) + 1$ $(i = 1, 2, 3, 4)$, Gupta's element becomes as the following

$$\begin{aligned}
&[[\,]_{43}, [\,]_{12}(b_3 - 1)(b_4 - 1)] + [[\,]_{43}, [\,]_{12}(b_3 - 1)] + [[\,]_{43}, [\,]_{12}(b_4 - 1)] \\
&+[[\,]_{21}, [\,]_{34}(b_1 - 1)(b_2 - 1)] + [[\,]_{21}, [\,]_{34}(b_1 - 1)] + [[\,]_{21}, [\,]_{34}(b_2 - 1)] \\
&+[[\,]_{24}, [\,]_{13}(b_2 - 1)(b_4 - 1)] + [[\,]_{24}, [\,]_{13}(b_2 - 1)] + [[\,]_{24}, [\,]_{13}(b_4 - 1)] \\
&+[[\,]_{13}, [\,]_{24}(b_1 - 1)(b_3 - 1)] + [[\,]_{13}, [\,]_{24}(b_1 - 1)] + [[\,]_{13}, [\,]_{24}(b_3 - 1)] \\
&+[[\,]_{32}, [\,]_{14}(b_2 - 1)(b_3 - 1)] + [[\,]_{32}, [\,]_{14}(b_2 - 1)] + [[\,]_{32}, [\,]_{14}(b_3 - 1)] \\
&+[[\,]_{14}, [\,]_{32}(b_4 - 1)(b_1 - 1)] + [[\,]_{14}, [\,]_{32}(b_4 - 1)] + [[\,]_{14}, [\,]_{32}(b_1 - 1)]
\end{aligned} \tag{3.112}$$

Using the Jacobi identity it is easy to see that (3.112), becomes as follows

$$\left.\begin{array}{l}
\sum_{\tau \in J} (-1)^{\tau} \left\{ \begin{array}{c}
[[\,]_{2\tau 1\tau}, [\,]_{1\tau 4\tau}(b_{2\tau} - 1)(b_{3\tau} - 1)] + [[\,]_{2\tau 1\tau}, [\,]_{3\tau 1\tau}(b_{2\tau} - 1)(b_{4\tau} - 1)] \\
+[[\,]_{2\tau 1\tau}, [\,]_{1\tau 4\tau}(b_{3\tau} - 1)] + [[\,]_{2\tau 1\tau}, [\,]_{3\tau 1\tau}(b_{4\tau} - 1)] \\
+[[\,]_{2\tau 1\tau}, [\,]_{3\tau 2\tau}(b_{4\tau} - 1)] + [[\,]_{2\tau 1\tau}, [\,]_{2\tau 4\tau}(b_{3\tau} - 1)]
\end{array} \right\} \\
+[[\,]_{13}, [\,]_{21}(b_4 - 1)(b_3 - 1)] + [[\,]_{13}, [\,]_{14}(b_2 - 1)(b_3 - 1)] + [[\,]_{13}, [\,]_{21}(b_4 - 1)] \\
\qquad +[[\,]_{13}, [\,]_{14}(b_2 - 1)] + [[\,]_{13}, [\,]_{23}(b_4 - 1)] + [[\,]_{13}, [\,]_{34}(b_2 - 1)] \\
+[[\,]_{24}, [\,]_{12}(b_3 - 1)(b_4 - 1)] + [[\,]_{24}, [\,]_{23}(b_1 - 1)(b_4 - 1)] + [[\,]_{24}, [\,]_{12}(b_3 - 1)] \\
\qquad +[[\,]_{24}, [\,]_{23}(b_1 - 1)] + [[\,]_{24}, [\,]_{14}(b_3 - 1)] + [[\,]_{24}, [\,]_{43}(b_1 - 1)]
\end{array}\right\} \tag{3.113}$$

where $J$ is the subgroup of $S_4$ generated by $\langle(1234)\rangle$.

For the absence of the space we put

$$[[\,]_{21}(b_3 - 1), [\,]_{41}(b_2 - 1)(b_3 - 1)]\tau = [[\,]_{2\tau 1\tau}(b_{3\tau} - 1), [\,]_{4\tau 1\tau}(b_{2\tau} - 1)(b_{3\tau} - 1)]$$

and we mean the same thing for the similar cases. Using the fact that the action is trivial, we can rewrite (3.113) as follows

$$
\left.
\begin{array}{c}
\sum_{\tau\in J}(-1)^{\tau}\left\{
\begin{array}{c}
[[\,]_{21}(b_3-1),[\,]_{41}(b_2-1)(b_3-1)]\tau+[[\,]_{21}(b_3-1),[\,]_{41}(b_2-1)]\tau\\
+[[\,]_{21}(b_4-1),[\,]_{13}(b_2-1)(b_4-1)]\tau+[[\,]_{21}(b_4-1),[\,]_{13}(b_2-1)]\tau\\
+[[\,]_{21}(b_3-1),[\,]_{41}(b_{3\tau}-1)]\tau+[[\,]_{21}(b_3-1),[\,]_{41}]\tau\\
+[[\,]_{21}(b_4-1),[\,]_{13}(b_4-1)]\tau+[[\,]_{21}(b_4-1),[\,]_{13}]\tau\\
+[[\,]_{21}(b_4-1),[\,]_{23}(b_4-1)]\tau+[[\,]_{21}(b_4-1),[\,]_{23}]\tau\\
+[[\,]_{21}(b_3-1),[\,]_{42}(b_3-1)]\tau+[[\,]_{21}(b_3-1),[\,]_{42}]\tau
\end{array}
\right\}\\
+[[\,]_{13}(b_4-1),[\,]_{12}(b_3-1)(b_4-1)]+[[\,]_{13}(b_4-1),[\,]_{12}(b_3-1)]\\
+[[\,]_{13}(b_2-1),[\,]_{41}(b_3-1)(b_2-1)]+[[\,]_{13}(b_2-1),[\,]_{41}(b_3-1)]\\
+[[]_{13}(b_4-1),[\,]_{12}(b_4-1)]+[[\,]_{13}(b_4-1),[\,]_{12}]+[[\,]_{13}(b_2-1),[\,]_{41}(b_2-1)]\\
+[[\,]_{13}(b_2-1),[\,]_{41}]+[[\,]_{13}(b_4-1),[\,]_{32}(b_4-1)]+[[\,]_{13}(b_4-1),[\,]_{32}]\\
+[[\,]_{13}(b_2-1),[\,]_{43}(b_2-1)]+[[\,]_{13}(b_2-1),[\,]_{43}]\\
+[[\,]_{24}(b_3-1),[\,]_{21}(b_4-1)(b_3-1)]+[[\,]_{24}(b_3-1),[\,]_{21}(b_4-1)]\\
+[[\,]_{24}(b_1-1),[\,]_{32}(b_4-1)(b_1-1)]+[[\,]_{24}(b_1-1),[\,]_{32}(b_4-1)]\\
+[[]_{24}(b_3-1),[\,]_{21}(b_3-1)]+[[\,]_{24}(b_3-1),[\,]_{21}]+[[\,]_{24}(b_1-1),[\,]_{32}(b_1-1)]\\
+[[\,]_{24}(b_1-1),[\,]_{32}]+[[\,]_{24}(b_3-1),[\,]_{41}(b_3-1)]+[[\,]_{24}(b_3-1),[\,]_{41}]\\
+[[\,]_{24}(b_1-1),[\,]_{34}(b_1-1)]+[[\,]_{24}(b_1-1),[\,]_{34}]
\end{array}
\right\}
\tag{3.114}
$$

By rewriting (3.114) we obtain the following decomposition,

$$
\left\{
\begin{array}{l}
[[\,]_{21}(b_3-1),[\,]_{41}(b_2-1)(b_3-1)]+[[\,]_{13}(b_2-1),[\,]_{41}(b_3-1)(b_2-1)] \\
+[[\,]_{21}(b_4-1),[\,]_{13}(b_2-1)(b_4-1)]+[[\,]_{41}(b_2-1),[\,]_{31}(b_2-1)(b_4-1)] \\
+[[\,]_{32}(b_4-1),[\,]_{21}(b_3-1)(b_4-1)]+[[\,]_{24}(b_3-1),[\,]_{21}(b_4-1)(b_3-1)] \\
+[[\,]_{41}(b_3-1),[\,]_{12}(b_4-1)(b_3-1)]+[[\,]_{13}(b_4-1),[\,]_{12}(b_3-1)(b_4-1)] \\
+[[\,]_{43}(b_1-1),[\,]_{23}(b_4-1)(b_1-1)]+[[\,]_{43}(b_2-1),[\,]_{31}(b_4-1)(b_2-1)] \\
+[[\,]_{32}(b_1-1),[\,]_{43}(b_2-1)(b_1-1)]+[[\,]_{24}(b_1-1),[\,]_{32}(b_4-1)(b_1-1)]
\end{array}
\right\}
$$

$$
+\left\{
\begin{array}{l}
[[\,]_{21}(b_3-1),[\,]_{41}(b_3-1)]+[[\,]_{21}(b_3-1),[\,]_{41}]+[[\,]_{41}(b_3-1),[\,]_{12}] \\
+[[\,]_{21}(b_4-1),[\,]_{13}(b_4-1)]+[[\,]_{21}(b_4-1),[\,]_{13}]+[[\,]_{13}(b_4-1),[\,]_{12}] \\
+[[\,]_{21}(b_3-1),[\,]_{42}(b_3-1)]+[[\,]_{21}(b_3-1),[\,]_{42}]+[[\,]_{42}(b_3-1),[\,]_{21}] \\
+[[\,]_{21}(b_4-1),[\,]_{23}(b_4-1)]+[[\,]_{21}(b_4-1),[\,]_{23}]+[[\,]_{32}(b_4-1),[\,]_{21}] \\
+[[\,]_{43}(b_1-1),[\,]_{23}(b_1-1)]+[[\,]_{43}(b_1-1),[\,]_{23}]+[[\,]_{32}(b_1-1),[\,]_{43}] \\
+[[\,]_{43}(b_2-1),[\,]_{31}(b_2-1)]+[[\,]_{43}(b_2-1),[\,]_{31}]+[[\,]_{13}(b_2-1),[\,]_{43}] \\
+[[\,]_{43}(b_1-1),[\,]_{24}(b_1-1)]+[[\,]_{43}(b_1-1),[\,]_{24}]+[[\,]_{24}(b_1-1),[\,]_{34}] \\
+[[\,]_{43}(b_2-1),[\,]_{41}(b_2-1)]+[[\,]_{43}(b_2-1),[\,]_{41}]+[[\,]_{41}(b_2-1),[\,]_{34}] \\
+[[\,]_{32}(b_1-1),[\,]_{42}(b_1-1)]+[[\,]_{32}(b_1-1),[\,]_{42}]+[[\,]_{24}(b_1-1),[\,]_{32}] \\
+[[\,]_{32}(b_4-1),[\,]_{31}(b_4-1)]+[[\,]_{32}(b_4-1),[\,]_{31}]+[[\,]_{13}(b_4-1),[\,]_{32}] \\
+[[\,]_{41}(b_3-1),[\,]_{42}(b_3-1)]+[[\,]_{41}(b_3-1),[\,]_{42}]+[[\,]_{24}(b_3-1),[\,]_{41}] \\
+[[\,]_{13}(b_2-1),[\,]_{41}(b_2-1)]+[[\,]_{13}(b_2-1),[\,]_{41}]+[[\,]_{41}(b_2-1),[\,]_{31}]
\end{array}
\right\}
$$

$$
+\left\{\begin{array}{l}
[[\,]_{21}(b_3-1),[\,]_{41}(b_2-1)]+[[\,]_{41}(b_2-1),[\,]_{31}(b_2-1)] \\
+[[\,]_{43}(b_1-1),[\,]_{23}(b_4-1)]+[[\,]_{13}(b_4-1),[\,]_{32}(b_4-1)] \\
+[[\,]_{32}(b_1-1),[\,]_{42}(b_3-1)]+[[\,]_{24}(b_3-1),[\,]_{21}(b_3-1)] \\
+[[\,]_{32}(b_4-1),[\,]_{21}(b_4-1)]+[[\,]_{24}(b_1-1),[\,]_{32}(b_4-1)] \\
+[[\,]_{21}(b_4-1),[\,]_{13}(b_2-1)]+[[\,]_{43}(b_2-1),[\,]_{31}(b_4-1)] \\
+[[\,]_{32}(b_4-1),[\,]_{21}(b_3-1)]+[[\,]_{32}(b_1-1),[\,]_{43}(b_1-1)] \\
+[[\,]_{41}(b_2-1),[\,]_{31}(b_4-1)]+[[\,]_{41}(b_3-1),[\,]_{12}(b_4-1)] \\
+[[\,]_{41}(b_3-1),[\,]_{12}(b_3-1)]+[[\,]_{41}(b_2-1),[\,]_{34}(b_2-1)] \\
+[[\,]_{13}(b_4-1),[\,]_{12}(b_3-1)]+[[\,]_{13}(b_2-1),[\,]_{41}(b_3-1)] \\
+[[\,]_{13}(b_4-1),[\,]_{12}(b_4-1)]+[[\,]_{13}(b_2-1),[\,]_{43}(b_2-1)] \\
+[[\,]_{24}(b_3-1),[\,]_{21}(b_4-1)]+[[\,]_{24}(b_1-1),[\,]_{32}(b_1-1)] \\
+[[\,]_{24}(b_3-1),[\,]_{41}(b_3-1)]+[[\,]_{24}(b_1-1),[\,]_{34}(b_1-1)]
\end{array}\right\}.
$$

Again, using the fact that the action is trivial, we can easily see that the middle summand is zero. On the other hand from the first and the last summand we obtain the follows

$$
\left.
\begin{aligned}
&[[\,]_{23}(b_1 - 1), [\,]_{41}(b_2 - 1)(b_3 - 1)] + [[\,]_{32}(b_1 - 1), [\,]_{42}(b_1 - 1)(b_3 - 1)] \\
&+[[\,]_{24}(b_1 - 1), [\,]_{13}(b_2 - 1)(b_4 - 1)] + [[\,]_{24}(b_1 - 1), [\,]_{32}(b_1 - 1)(b_4 - 1)] \\
&+[[\,]_{43}(b_1 - 1), [\,]_{23}(b_4 - 1)(b_1 - 1)] + [[\,]_{43}(b_1 - 1), [\,]_{12}(b_3 - 1)(b_4 - 1)] \\
&+[[\,]_{43}(b_2 - 1), [\,]_{31}(b_4 - 1)(b_2 - 1)] + [[\,]_{43}(b_2 - 1), [\,]_{21}(b_4 - 1)(b_3 - 1)] \\
&\qquad +[[\,]_{23}(b_1 - 1), [\,]_{41}(b_2 - 1)] + [[\,]_{24}(b_1 - 1), [\,]_{32}(b_1 - 1)] \\
&\qquad +[[\,]_{41}(b_3 - 1), [\,]_{23}(b_4 - 1)] + [[\,]_{24}(b_3 - 1), [\,]_{41}(b_3 - 1)] \\
&\qquad +[[\,]_{32}(b_4 - 1), [\,]_{41}(b_2 - 1)] + [[\,]_{41}(b_2 - 1), [\,]_{34}(b_2 - 1)] \\
&\qquad +[[\,]_{41}(b_2 - 1), [\,]_{31}(b_4 - 1)] + [[\,]_{13}(b_4 - 1), [\,]_{12}(b_4 - 1)] \\
&\qquad +[[\,]_{41}(b_3 - 1), [\,]_{12}(b_3 - 1)] + [[\,]_{13}(b_4 - 1), [\,]_{12}(b_3 - 1)] \\
&\qquad +[[\,]_{31}(b_2 - 1), [\,]_{42}(b_3 - 1)] + [[\,]_{13}(b_2 - 1), [\,]_{43}(b_2 - 1)] \\
&\qquad +[[\,]_{21}(b_4 - 1), [\,]_{13}(b_2 - 1)] + [[\,]_{43}(b_2 - 1), [\,]_{31}(b_4 - 1)] \\
&\qquad +[[\,]_{32}(b_4 - 1), [\,]_{21}(b_3 - 1)] + [[\,]_{32}(b_1 - 1), [\,]_{43}(b_1 - 1)] \\
&\qquad +[[\,]_{41}(b_3 - 1), [\,]_{12}(b_4 - 1)] + [[\,]_{13}(b_2 - 1), [\,]_{41}(b_3 - 1)] \\
&\qquad +[[\,]_{24}(b_3 - 1), [\,]_{21}(b_4 - 1)] + [[\,]_{24}(b_1 - 1), [\,]_{34}(b_1 - 1)]
\end{aligned}
\right\} \qquad (3.115)
$$

From (3.115), we obtain

$$\left\{\begin{array}{c}
[[\,]_{23}(b_1-1),[\,]_{12}(b_3-1)(b_4-1)]+[[\,]_{24}(b_1-1),[\,]_{12}(b_3-1)(b_4-1)] \\
+[[\,]_{43}(b_1-1),[\,]_{13}(b_2-1)(b_4-1)]+[[\,]_{43}(b_2-1),[\,]_{32}(b_1-1)(b_4-1)] \\
+[[\,]_{23}(b_1-1),[\,]_{21}(b_4-1)]+[[\,]_{21}(b_4-1),[\,]_{13}(b_2-1)] \\
+[[\,]_{41}(b_3-1),[\,]_{43}(b_2-1)]+[[\,]_{43}(b_2-1),[\,]_{31}(b_4-1)] \\
+[[\,]_{32}(b_1-1),[\,]_{43}(b_1-1)]+[[\,]_{43}(b_1-1),[\,]_{12}(b_3-1)] \\
+[[\,]_{42}(b_3-1),[\,]_{41}(b_2-1)]+[[\,]_{24}(b_3-1),[\,]_{21}(b_4-1)] \\
+[[\,]_{42}(b_1-1),[\,]_{31}(b_4-1)]+[[\,]_{24}(b_1-1),[\,]_{34}(b_1-1)] \\
+[[\,]_{31}(b_2-1),[\,]_{32}(b_4-1)]+[[\,]_{32}(b_4-1),[\,]_{21}(b_3-1)] \\
+[[\,]_{41}(b_3-1),[\,]_{12}(b_4-1)]+[[\,]_{13}(b_2-1),[\,]_{41}(b_3-1)]
\end{array}\right\} \tag{3.116}$$

We obtain from (3.116), the following ;

$$\left\{\begin{array}{c}
[[\,]_{23}(b_1-1),[\,]_{12}(b_3-1)(b_4-1)]+[[\,]_{24}(b_1-1),[\,]_{12}(b_3-1)(b_4-1)] \\
+[[\,]_{43}(b_1-1),[\,]_{13}(b_2-1)(b_4-1)]+[[\,]_{43}(b_2-1),[\,]_{32}(b_1-1)(b_4-1)] \\
+[[\,]_{21}(b_3-1),[\,]_{21}(b_4-1)]+[[\,]_{43}(b_1-1),[\,]_{43}(b_2-1)] \\
+[[\,]_{32}(b_1-1),[\,]_{32}(b_4-1)]+[[\,]_{42}(b_3-1),[\,]_{42}(b_1-1)] \\
+[[\,]_{31}(b_2-1),[\,]_{43}(b_1-1)]+[[\,]_{13}(b_2-1),[\,]_{41}(b_3-1)] \\
+[[\,]_{42}(b_1-1),[\,]_{41}(b_3-1)]+[[\,]_{41}(b_3-1),[\,]_{12}(b_4-1)]
\end{array}\right\}$$

and this gives

$$\begin{array}{c}
[[\,]_{23}(b_1-1),[\,]_{12}(b_3-1)(b_4-1)]+[[\,]_{24}(b_1-1),[\,]_{12}(b_3-1)(b_4-1)] \\
+[[\,]_{43}(b_1-1),[\,]_{13}(b_2-1)(b_4-1)]+[[\,]_{43}(b_2-1),[\,]_{32}(b_1-1)(b_4-1)] \\
+[[\,]_{21}(b_3-1),[\,]_{21}(b_4-1)]+[[\,]_{43}(b_1-1),[\,]_{43}(b_2-1)] \\
+[[\,]_{32}(b_1-1),[\,]_{32}(b_4-1)]+[[\,]_{42}(b_3-1),[\,]_{42}(b_1-1)] \\
+[[\,]_{31}(b_4-1),[\,]_{31}(b_2-1)]+[[\,]_{41}(b_2-1),[\,]_{41}(b_3-1)]
\end{array}$$

which is exactly (3.110), [i.e. our element when $p=2$].

### 3.2.3 Applications

We conclude this section with an interesting application of the main result.

First we consider the following quotient

$$F/[\gamma_c(F'),F] \tag{3.117}$$

where $F$ as before and $c \geq 2$ is a positive integer. For $c = 2$ and $c = 3$, (3.117) coincides with $F/[\gamma_c(F'),F]F'''$, the object of study in this thesis, and for $c \geq 4$ the latter is a homomorphic image of the former. In fact, (3.117) is the free–by–(nilpotent of class $c-1$)–by–abelian group, and we have an exact sequence

$$1 \longrightarrow \gamma_c(F')/[\gamma_c(F'),F] \longrightarrow F/[\gamma_c(F'),F] \longrightarrow F/\gamma_c(F') \longrightarrow 1. \tag{3.118}$$

While $F/\gamma_c(F')$ is torsion–free, elements of finite order do occur in $\gamma_c(F')/[\gamma_c(F'),F]$, see [11], [16], [3]. In the case where $c = p$, $p$ a prime, Stöhr proved in [11] that there is an isomorphism

$$t\left(\gamma_p(F')/[\gamma_p(F'),F]\right) \cong H_4(F/F',\mathbf{Z}_p),$$

i.e. the elements of finite order form an elementary abelain $p$–subgroup of rank $C_4^d$ of the centre of (3.118) where $c = p$. It turns out that our main theorem can be exploited to obtain generators for this torsion subgroup in terms of generators of $F$. Recall that by Proposition 2.4.9 there is an isomorphism

$$\gamma_p(F')F'''/[\gamma_p(F'),F]F''' \cong \mathcal{M}^p(F'_{ab}) \otimes_G \mathbf{Z},$$

and, by Proposition 2.4.8

$$\gamma_p(F')/[\gamma_p(F'), F] \cong \mathcal{L}^p(F'_{ab}) \otimes_G \mathbf{Z}.$$

We shall exhibit a homomorphism

$$\mathcal{M}^p(F'_{ab}) \otimes_G \mathbf{Z} \longrightarrow \mathcal{L}^p(F'_{ab}) \otimes_G \mathbf{Z}$$

which maps the torsion subgroup of $\mathcal{M}^p(F'_{ab}) \otimes_G \mathbf{Z}$ isomorphically onto the torsion subgroup of $\mathcal{L}^p(F'_{ab}) \otimes_G \mathbf{Z}$. The key ingredients for this construction are recent results from [29], which we explain now.

Let $A$ be a $\mathbf{Z}$–free $G$–module, and let $p$ be a prime.

On the one hand, we have the natural projection homomorphism,

$$\nu_p : \mathcal{L}^p(A) \longrightarrow \mathcal{M}^p(A)$$

of the $p$–th free Lie power of $A$ onto the $p$–th free metabelian Lie power of $A$. On the other hand, Bryant and Stöhr proved in [29] that the map

$$[a_1, a_2, \cdots a_p] \longrightarrow \frac{1}{p}\left(\sum_{\tau} [a_1, a_{2\tau}, \cdots a_{p\tau}] - \sum_{\pi} [a_2, a_{1\pi}, \cdots a_{p\pi}]\right) \tag{3.119}$$

where $a_1, \cdots a_p \in A$, and $\tau$ and $\pi$ range over all permutations of $\{2, \cdots, p\}$ and $\{1, 3, \cdots, p\}$, extends to a $G$–module homomorphism

$$\psi_p : \mathcal{M}^p(A) \longrightarrow \mathcal{L}^p(A)$$

and the composite

$$\mathcal{M}^p(A) \xrightarrow{\psi_p} \mathcal{L}^p(A) \xrightarrow{\nu_p} \mathcal{M}^p(A) \tag{3.120}$$

amounts to multiplication by $(p-2)!$ in $\mathcal{M}^p(A)$ (i.e. $\psi_p\ \nu_p = (p-2)!$).

It should be pointed out that it is by no means obvious that (3.119) is a correctly defined map, as it involves the coefficient $\frac{1}{p}$. The correctness of (3.119), however, follows easily from a result of Wall, [Lemma 1 on page 677 in [30]], and in [29] Bryant and Stöhr exhibit an explicit expression of the element on the right hand side of (3.119) as a liner combination of Lie elements with integer coefficient.

Now let $A = F'_{ab}$. Then tensoring (3.120) with $\mathbf{Z}$ gives over $G$,

$$\mathcal{M}^p(F'_{ab}) \otimes_G \mathbf{Z} \xrightarrow{\psi_p \otimes 1} \mathcal{L}^p(F'_{ab}) \otimes_G \mathbf{Z} \xrightarrow{\nu_p \otimes 1} \mathcal{M}^p(F'_{ab}) \otimes_G \mathbf{Z},$$

where the composite $\psi_p\ \nu_p \otimes 1$ is also multiplication by $(p-2)!$. Since the torsion subgroups of both $\mathcal{M}^p(F'_{ab}) \otimes_G \mathbf{Z}$ and $\mathcal{L}^p(F'_{ab}) \otimes_G \mathbf{Z}$ are finitely generated elementary abelian $p$-groups, it follows that the restriction of $\psi_p \otimes 1$ to $t(\mathcal{M}^p(F'_{ab}) \otimes_G \mathbf{Z}\ )$ maps this torsion subgroup isomorphically onto the torsion subgroup of $\mathcal{L}^p(F'_{ab}) \otimes_G \mathbf{Z}$, as required. Finally, in view of (3.119) we can summarize our discussion as follows

**Theorem 3.2.3.** Let $p$ be any odd prime. Then the torsion subgroup of $F/[\gamma_p(F'), F]$ is generated by the elements $(W_p(x_{\tau_1}, x_{\tau_2}, x_{\tau_3}, x_{\tau_4}))\overline{\psi}$. Where $(W_p(x_{\tau_1}, x_{\tau_2}, x_{\tau_3}, x_{\tau_4}))$ as in Theorem 3.2.1, and $\overline{\psi}$ is the composite of the isomorphism

$$\gamma_p(F')F'''/[\gamma_p(F'), F]F''' \longrightarrow \mathcal{M}^p(F'_{ab}) \otimes_G \mathbf{Z},$$

the homomorphism $\psi_p \otimes 1$ and the isomorphism

$$\mathcal{L}^p(F'_{ab}) \otimes_G \mathbf{Z} \longrightarrow \gamma_p(F')/[\gamma_p(F'), F].$$

# Chapter 4

# Further investigation of torsion in free central extensions

Let $G$ be a group that is given by a free presentation

$$1 \longrightarrow N \longrightarrow F \longrightarrow G \longrightarrow 1$$

where $F$ is a free group on $x_1, \cdots, x_d$, and let $\gamma_{p^n}N$ denote the $p^n$–term of the lower central series of $N$. As was already mentioned in chapter one, R. Stöhr in [14] has shown that, if $G$ has no elements of order $p$ and $H_s(G, \mathbf{Z}_p) = 0$ for all $s \geq 5$, then the torsion subgroup of the central extension $F/[\gamma_{p^n}(N), F]N''$ can be identified with the fourth homology group of $G$ with coefficients in $\mathbf{Z}_p$. Furthermore, for $p = 2$ and $n = 2$, R. Stöhr in [13] obtained a complete description to the torsion subgroup of $F/[\gamma_4(N), F]N''$ in the case where $G$ is any group without elements of order 2. He obtained the following isomorphism:

$$t(F/[\gamma_4(N), F]N'') \cong H_7(G, \mathbf{Z}_2) \oplus H_6(G, \mathbf{Z}_4) \oplus H_4(G, \mathbf{Z}_2).$$

We start this a chapter with an alternative proof to the first result, which gives a homological description of the torsion subgroup of $F/[\gamma_{p^n}(N), F]N''$, under the homological finiteness condition on $G$ that $H_s(G, \mathbf{Z}_p) = 0$ for all $s \geq 5$.

## 4.1 Description of $t(F/[\gamma_{p^n}(N), F]N'')$ in homological terms

In this section we will assume that the group $G$ has no elements of order $p$. This enables us to appeal to Lemma 2.2.5.

We begin this section by one of the main results of [14], which provides us with information about torsion elements in $F/[\gamma_{p^n}(N), F]N''$. However, our proof of this result differs from that in [14].

**Theorem 4.1.1,** ([14], Theorem 2). Let $G$ be a $p$-torsion-free group such that $H_s(G, \mathbf{Z}_p) = 0$ for all $s \geq 5$. Then $tH_0(G, \mathcal{M}^{p^n} M) \cong H_4(G, \mathbf{Z}_p)$.

In order to give an alternative proof to this result we need some technical results. For that we need to introduce the following notation.

Let $f(x) = \sum_i m_i x^i$ be a polynomial with non-negative integer coefficients. For any $G$-module $B$, we set

$$fH_k(G, B) = \bigoplus_i H_{k+i}(G, B)^{\oplus m_i}$$

where $H_{k+i}(G, B)^{\oplus m_i}$ is the direct sum of $m_i$ isomorphic copes of $H_{k+i}(G, B)$.

For any integer $c \geq 2$ we define the Kuz'min polynomials $f_c^{(p)}$ by the following recursion :

$$f_c^{(p)} = \begin{cases} 0 & \text{if} \quad c \not\equiv 0,1 \bmod p, \\ x^2 & \text{if} \quad c = p, \\ x f_{c-1}^{(p)} & \text{if} \quad c \equiv 1 \bmod p, \\ x^2 f_{c-p}^{(p)} + f_{c/p} & \text{if} \quad c \equiv 0 \bmod p \text{ with } c > p. \end{cases}$$

**Proposition 4.1.2**, [Modification of the main result of [20]]. Let $K$ be an $\mathbf{R}G$–module whose underlying abelian group is a free $\mathbf{R}$–module. Then, for any $c \geq 2$ and $k \geq 1$, there is an isomorphism

$$H_k(G, K \otimes \Delta^c) \cong f_c^{(p)} H_k(G, K \otimes \mathbf{Z}_p).$$

The following Lemma shows that the non–zero Kuz'min polynomials $f_c^{(p)}$ have the property that the term of lowest degree occurs with coefficient 1. In order to state this Lemma, we let $\sigma_p(n)$ denote the sum of the base $p$ digits of the natural number $n$, $\varepsilon_p(n)$ the number of non–zero base $p$ digits of $n$, and we put $\delta_p(n) = 2\sigma_p(n) - \varepsilon_p(n) + 1$.

**Lemma 4.1.3.** ([20], section 7). For any $n \geq 2$, the term of lowest degree in the Kuz'min polynomial $f_n^{(p)}$ is $x^{\delta_p(n)}$.

In particular, by Proposition 4.1.2 we have that $H_k(G, K \otimes \Delta^{p^n})$ is a direct sum of $H_{k+2}(G, K \otimes \mathbf{Z}_p)$ (since $\delta_p(p^n) = 2\sigma_p(p^n) - \varepsilon_p(p^n) + 1 = 2 \cdot 1 - 1 + 1 = 2$) and possibly some higher dimensional homology groups $H_{k+s}(G, K \otimes \mathbf{Z}_p)$ with $s > 2$.

We also need the following result from [14].

**Lemma 4.1.4**, [[14], Lemma 3.5]. Let $K$ be an $\mathbf{R}$–free $G$–module such that each of the homology groups $H_k(G, K)$, $k \geq 1$, has a finite filtration whose quotients are isomorphic to sections of some homology groups $H_{k+s}(G, \mathbf{Z}_p)$ where $s \geq s_0$ for some fixed integer $s_0 \geq 2$. Then $H_k(G, K \otimes \mathbf{Z}_p)$ has a finite filtration whose quotients are isomorphic to sections of some homology groups $H_{k+s}(G, \mathbf{Z}_p)$ where $s \geq s_0 - 1$.

Now we can prove the following observation.

**Lemma 4.1.5.** Let $A$ be an $\mathbf{R}$–free $G$–module such that each of the homology groups $H_k(G, A)$, $k \geq 1$, has a finite filtration whose quotients are isomorphic to sections of some homology groups $H_{k+s}(G, \mathbf{Z}_p)$ where $s \geq 2$ . Then $H_k(G, M^n \otimes A)$ has a finite filtration whose quotients are isomorphic to sections of some homology groups $H_{k+t}(G, \mathbf{Z}_p)$ where $t \geq 4$.

**Proof.** The proof is by induction on $n$. If $n = 1$, by theorem 2.2.3, we obtain

$$H_k(G, M \otimes A) \cong H_{k+2}(G, A)$$

and by our assumption $H_{k+2}(G, A)$ has a finite filtration whose quotients are isomorphic to sections of some homology groups $H_{k+t}(G, \mathbf{Z}_p)$ where $t \geq 4$. Thus $H_k(G, M \otimes A)$ has a finite filtration whose quotients are isomorphic to sections of some homology groups $H_{k+t}(G, \mathbf{Z}_p)$ where $t \geq 4$.

For the induction step we consider the following exact sequence, where $A$ is an $\mathbf{R}$–free $G$–module

$$0 \longrightarrow M^n \otimes A \longrightarrow P^n \otimes A \longrightarrow P^n / M^n \otimes A \longrightarrow 0.$$

From that we get a long exact homology sequence,

$$\to H_{k+1}(G, P^n \otimes A) \to H_{k+1}(G, P^n / M^n \otimes A) \to H_k(G, M^n \otimes A) \to H_k(G, P^n \otimes A) \to$$

By Lemma 2.2.5, the outside terms are zero for $k \geq 1$. Then we get

$$H_k(G, M^n \otimes A) \cong H_{k+1}(G, P^n/M^n \otimes A), \quad \forall k \geq 1.$$

On the other hand $P^n/M^n \otimes A$ has a finite filtration whose quotients are isomorphic to $(M^i \otimes \Delta^{n-i}) \otimes A$ $(i = 0, 1 \cdots, n-1)$. Here we have two cases:

(1) If $n - i \geq p$, then by Proposition 4.1.2 we get the following isomorphisms.

$$\begin{aligned} H_{k+1}(G, M^i \otimes \Delta^{n-i} \otimes A) &\cong \quad f_{n-i}^{(p)} H_{k+1}(G, M^i \otimes A \otimes \mathbf{Z}_p) \\ &\cong \quad \oplus H_{k+1+s}(G, M^i \otimes A \otimes \mathbf{Z}_p), \;\; s \geq 2. \end{aligned}$$

By induction $H_{k+1+s}(G, M^i \otimes A)$ has a finite filtration whose quotients are isomorphic to sections of some homology groups $H_{k+1+s+t}(G, \mathbf{Z}_p)$, where $t \geq 2$. Hence by Lemma 4.1.4, $H_{k+1+s}(G, M^i \otimes A \otimes \mathbf{Z}_p)$ has a finite filtration whose quotients are isomorphic to sections of some homology groups $H_{k+1+s+t}(G, \mathbf{Z}_p)$, where $t \geq 1$.

(2) If $n - i < p$, then by definition of the Kuz'min polynomials it is enough to consider the case $n - i = 1$, because if $2 \leq n - i < p$, then the Kuz'min polynomials become zero polynomials and this shows that $H_{k+1}(G, M^i \otimes \Delta^{n-i} \otimes A) = 0$.

Now, if $n - i = 1$, then by the Reduction Theorem 2.2.2, we obtain

$$H_{k+1}(G, M^{n-1} \otimes \Delta \otimes A) \cong H_{k+2}(G, M^{n-1} \otimes A).$$

Again, by induction, $H_{k+2}(G, M^{n-1} \otimes A)$ has a finite filtration whose quotients are isomorphic to sections of some homology groups $H_{k+2+s}(G, \mathbf{Z}_p)$, where $s \geq 2$.

In either case we can conclude that $H_k(G, M^n \otimes A)$ has a finite filtration whose quotients are isomorphic to sections of some homology groups $H_{k+t_0}(G, \mathbf{Z}_p)$, where $t_0 \geq 4$, and this completes the proof of the Lemma.

Towards achieving our goal in this section, we shall need the following Lemma.

**Lemma 4.1.6.** The chain complex

$$0 \longrightarrow \mathcal{M}^{p^n}M \xrightarrow{\delta_5} P \otimes M^{p^n-1} \xrightarrow{\delta_4} P^{p^n} \xrightarrow{\delta_3} \triangle^{p^n} \longrightarrow 0$$

satisfies the hypothesis of the Lemma 3.1.4.

**Proof.** Lemma 2.2.5, implies That $H_k(G, P^{p^n}) = H_k(G, P \otimes M^{p^n-1}) = 0$. On the other hand Lemma 6.1 in [2], tells us that $\mathcal{M}^{p^n}M \xrightarrow{\delta_5} P \otimes M^{p^n-1} \xrightarrow{\delta_4} P^{p^n}$ is exact (i.e. $\ker \delta_4 = im\delta_5$). For the other conditions, it is sufficient to prove $H_k(G, \ker \delta_3 / im\delta_4) = 0$ for all $k \geq 1$.

Now the free $G$-module $P^{p^n}$ has a finite filtration

$$0 < M^{p^n} = K^{p^n}_{p^n-1} < K^{p^n}_{p^n-2} < .... < K^{p^n}_0 < K^{p^n}_{-1} = P^{p^n}$$

with $K^{p^n}_{i-1}/K^{p^n}_i \cong M^i \otimes \triangle^{p^n-i}$. Therefore to show $H_k(G, \ker \delta_3 / im\delta_4) = 0$, it is enough to show

$$H_k(G, M^m \otimes \triangle^{p^n-m}) = 0 \ \ \forall k \geq 1, \, m \leq p^n - 2,$$

On the other hand, by Proposition 4.1.3, we get

$$\begin{aligned} H_k(G, M^m \otimes \triangle^{p^n-m}) &\cong f^{(p)}_{p^n-m} H_k(G, M^m \otimes \mathbf{Z}_p) \\ &= \oplus H_{k+s}(G, M^m \otimes \mathbf{Z}_p), \ s \geq 2 \end{aligned}$$

By Lemma 4.1.5, $H_{k+s}(G, M^m \otimes \mathbf{Z}_p)$ has a finite filtration whose quotients are isomorphic to sections of some homology groups $H_{k+t}(G, \mathbf{Z}_p)$, where $t \geq 4$. By assumption, the homology groups $H_{k+t}(G, \mathbf{Z}_p)$ are trivial for all $k + t \geq 5$. Hence $H_k(G, M^m \otimes \Delta^{p^n - m}) = 0$ for all $k \geq 1$.

Now we proceed to the our actual concern in this section, namely, to give an alternative proof to the main result of [14].

<u>**Proof of the Theorem 4.1.1.**</u> In view of Corollary 3.1.4, and Lemma 4.1.6, the complex

$$0 \longrightarrow \mathcal{M}^{p^n} M \xrightarrow{\delta_5} P \otimes M^{p^n - 1} \xrightarrow{\delta_4} P^{p^n} \xrightarrow{\delta_3} \Delta^{p^n} \longrightarrow 0,$$

gives the following exact sequence

$$0 \longrightarrow H_2(G, \Delta^{p^n}) \longrightarrow \mathcal{M}^{p^n} M \otimes_G \mathbf{R} \longrightarrow P \otimes M^{p^n - 1} \otimes_G \mathbf{R}$$

On the other hand $H_2(G, \Delta^{p^n}) \cong f^{(p)}_{p^n} H_2(G, \mathbf{Z}_p) = \bigoplus_s H_{2+s}(G, \mathbf{Z}_p)$, $s \geq 2$. But by assumption $H_k(G, \mathbf{Z}_p) = 0$ for all $k \geq 5$. Thus $H_2(G, \Delta^{p^n}) \cong H_4(G, \mathbf{Z}_p)$, and the exact sequence turns into

$$0 \longrightarrow H_4(G, \mathbf{Z}_p) \longrightarrow \mathcal{M}^{p^n} M \otimes_G \mathbf{R} \longrightarrow P \otimes M^{p^n - 1} \otimes_G \mathbf{R}.$$

Since $P \otimes M^{p^n - 1} \otimes_G \mathbf{R}$ is a free $\mathbf{R}$-module, $tH_0(G, \mathcal{M}^{p^n} M) \cong H_4(G, \mathbf{Z}_p)$.

This completes the proof of the theorem.

## 4.2 Torsion subgroups of $(F/[\gamma_{p^n}(F'), F]F''')$ and their description in terms of generators

Our principal objective in this section will be to obtain results similar to those in section 2 of chapter 3.

Consider the quotient $F/[\gamma_{p^n}(F'), F]F'''$, where $F$ is a free group on $\{x_1, \cdots, x_d\}$. This quotient turns out to be torsion–free when $d \leq 3$, and if the rank of $F$ is greater than 4, then any four of the free generators $x_1, \cdots, x_d$ generate a rank 4 subgroup of $F/[\gamma_{p^n}(F'), F]F'''$, and hence the rank 4 torsion elements will occur in all higher ranks.

To be more explicit, we illustrate the following case: if $p = 2$, and $n = 2$, then as a consequence of the main result of [13] we have

$$t(F/[\gamma_4(F'), F]F''') \cong H_7(F/F', \mathbf{Z}_2) \oplus H_6(F/F', \mathbf{Z}_4) \oplus H_4(F/F', \mathbf{Z}_2).$$

From that we do at least have the following

1. $F/[\gamma_{2^2}(F'), F]F'''$ is torsion–free for $d \leq 3$.
2. If $d = 4$ or 5, then $t(F/[\gamma_{2^2}(F'), F]F''') \cong H_4(F/F', \mathbf{Z}_2)$.
3. If $d > 5$, then any 4 of the free generators $x_1, \cdots, x_d$ generate a rank 4 subgroup of $F/[\gamma_4(F'), F]F'''$, hence the rank 4 torsion elements appear in all higher ranks.

We return to the general case. As an obvious consequence of the proof of theorem 4.1.1, one can get torsion elements in $\mathcal{M}^{p^n}M \otimes_G \mathbf{R} \cong \gamma_{p^n}(F')F'''/[\gamma_{p^n}(F'), F]F'''$, where the second term is the kernel of $[F/[\gamma_{p^n}(F'),F]F''' \to F/\gamma_{p^n}(F')F''']$. In this section we describe these torsion elements in group theoretic terms, where $p$ is any prime and $n \geq 1$. Moreover, if $d = 4$, then we give a complete description of all torsion elements in $F/[\gamma_{p^n}(F'),F]F'''$, otherwise we describe just rank 4 torsion elements in this subgroup. The motivation of this investigation came from [14].

In order to state the main result of this section we need to introduce the following elements $W_{p^n}(x_{\tau_1},x_{\tau_2},x_{\tau_3},x_{\tau_4})$ where these elements are obtained from $W_p(x_{\tau_1},x_{\tau_2},x_{\tau_3},x_{\tau_4})$ by replacing $p$ by $p^n$ (in commutators and coefficients), $p$ is odd prime. Also we need $W_{2^n}(x_{\tau_1},x_{\tau_2},x_{\tau_3},x_{\tau_4})$, where $W_{2^n}(x_{\tau_1},x_{\tau_2},x_{\tau_3},x_{\tau_4}) =$

$$\prod_{(i,j,k)\in\hat{I}_1}\left\{\begin{array}{l}
[[x_{\tau_2}x_{\tau_1}x_{\tau_3}x_{\tau_4}],[x_{\tau_2}x_{\tau_1}x_{\tau_4}]^i,[x_{\tau_2}x_{\tau_1}]^j,[x_{\tau_2}x_{\tau_1}x_{\tau_3}]^k],[x_{\tau_2}x_{\tau_1}x_{\tau_3}x_{\tau_4}]^{\bar{\bar{2}}-1}]\\
[[x_{\tau_3}x_{\tau_2}x_{\tau_1}x_{\tau_4}],[x_{\tau_3}x_{\tau_2}x_{\tau_4}]^i,[x_{\tau_3}x_{\tau_2}]^j,[x_{\tau_3}x_{\tau_2}x_{\tau_1}]^k,[x_{\tau_3}x_{\tau_2}x_{\tau_1}x_{\tau_4}]^{\bar{\bar{2}}-1}]\\
[[x_{\tau_4}x_{\tau_3}x_{\tau_1}x_{\tau_2}],[x_{\tau_4}x_{\tau_3}x_{\tau_2}]^i,[x_{\tau_4}x_{\tau_3}]^j,[x_{\tau_4}x_{\tau_3}x_{\tau_1}]^k,[x_{\tau_4}x_{\tau_3}x_{\tau_1}x_{\tau_2}]^{\bar{\bar{2}}-1}]\\
[[x_{\tau_4}x_{\tau_1}x_{\tau_3}x_{\tau_2}],[x_{\tau_4}x_{\tau_1}x_{\tau_3}]^i,[x_{\tau_4}x_{\tau_1}]^j,[x_{\tau_4}x_{\tau_1}x_{\tau_2}]^k,[x_{\tau_4}x_{\tau_1}x_{\tau_3}x_{\tau_2}]^{\bar{\bar{2}}-1}]\\
[[x_{\tau_1}x_{\tau_3}x_{\tau_2}x_{\tau_4}],[x_{\tau_1}x_{\tau_3}x_{\tau_4}]^i,[x_{\tau_1}x_{\tau_3}]^j,[x_{\tau_1}x_{\tau_3}x_{\tau_2}]^k,[x_{\tau_1}x_{\tau_3}x_{\tau_2}x_{\tau_4}]^{\bar{\bar{2}}-1}]\\
[[x_{\tau_2}x_{\tau_4}x_{\tau_3}x_{\tau_1}],[x_{\tau_2}x_{\tau_4}x_{\tau_3}]^i,[x_{\tau_2}x_{\tau_4}]^j,[x_{\tau_2}x_{\tau_4}x_{\tau_1}]^k,[x_{\tau_2}x_{\tau_4}x_{\tau_3}x_{\tau_1}]^{\bar{\bar{2}}-1}]
\end{array}\right\}^{\delta_1}$$

$$\prod_{(i,j,k)\in\hat{I}_2}\left\{\begin{array}{l}
[[x_{\tau_2}x_{\tau_1}x_{\tau_3}],[x_{\tau_2}x_{\tau_1}x_{\tau_3}x_{\tau_4}]^{\bar{\bar{2}}},[x_{\tau_2}x_{\tau_1}x_{\tau_4}]^i,[x_{\tau_2}x_{\tau_1}]^j,[x_{\tau_2}x_{\tau_1}x_{\tau_3}]^{k-1}]\\
[[x_{\tau_3}x_{\tau_2}x_{\tau_1}],[x_{\tau_3}x_{\tau_2}x_{\tau_1}x_{\tau_4}]^{\bar{\bar{2}}},[x_{\tau_3}x_{\tau_2}x_{\tau_4}]^i,[x_{\tau_3}x_{\tau_2}]^j,[x_{\tau_3}x_{\tau_2}x_{\tau_1}]^{k-1}]\\
[[x_{\tau_4}x_{\tau_3}x_{\tau_1}],[x_{\tau_4}x_{\tau_3}x_{\tau_1}x_{\tau_2}]^{\bar{\bar{2}}},[x_{\tau_4}x_{\tau_3}x_{\tau_2}]^i,[x_{\tau_4}x_{\tau_3}]^j,[x_{\tau_4}x_{\tau_3}x_{\tau_1}]^{k-1}]\\
[[x_{\tau_4}x_{\tau_1}x_{\tau_2}],[x_{\tau_4}x_{\tau_1}x_{\tau_3}x_{\tau_2}]^{\bar{\bar{2}}},[x_{\tau_4}x_{\tau_1}x_{\tau_3}]^i,[x_{\tau_4}x_{\tau_1}]^j,[x_{\tau_4}x_{\tau_1}x_{\tau_2}]^{k-1}]\\
[[x_{\tau_1}x_{\tau_3}x_{\tau_2}],[x_{\tau_1}x_{\tau_3}x_{\tau_2}x_{\tau_4}]^{\bar{\bar{2}}},[x_{\tau_1}x_{\tau_3}x_{\tau_4}]^i,[x_{\tau_1}x_{\tau_3}]^j,[x_{\tau_1}x_{\tau_3}x_{\tau_2}]^{k-1}]\\
[[x_{\tau_2}x_{\tau_4}x_{\tau_1}],[x_{\tau_2}x_{\tau_4}x_{\tau_3}x_{\tau_1}]^{\bar{\bar{2}}},[x_{\tau_2}x_{\tau_4}x_{\tau_3}]^i,[x_{\tau_2}x_{\tau_4}]^j,[x_{\tau_2}x_{\tau_4}x_{\tau_1}]^{k-1}]
\end{array}\right\}^{\delta_2}$$

$$\prod_{(i,j,k)\in\hat{I}_3}\left\{\begin{array}{l}[[x_{\tau_1}x_{\tau_2}x_{\tau_3}x_{\tau_4}],[x_{\tau_1}x_{\tau_3}x_{\tau_2}x_{\tau_4}]^k,[x_{\tau_1}x_{\tau_3}x_{\tau_2}]^{\overline{\overline{2}}},[x_{\tau_2}x_{\tau_1}x_{\tau_3}x_{\tau_4}]^{i-1},[x_{\tau_2}x_{\tau_1}x_{\tau_3}]^j]\\ [[x_{\tau_3}x_{\tau_2}x_{\tau_4}x_{\tau_1}],[x_{\tau_2}x_{\tau_4}x_{\tau_3}x_{\tau_1}]^k,[x_{\tau_2}x_{\tau_4}x_{\tau_3}]^{\overline{\overline{2}}},[x_{\tau_3}x_{\tau_2}x_{\tau_4}x_{\tau_1}]^{i-1},[x_{\tau_3}x_{\tau_2}x_{\tau_4}]^j]\\ [[x_{\tau_3}x_{\tau_1}x_{\tau_4}x_{\tau_2}],[x_{\tau_4}x_{\tau_3}x_{\tau_1}x_{\tau_2}]^k,[x_{\tau_4}x_{\tau_3}x_{\tau_1}]^{\overline{\overline{2}}},[x_{\tau_2}x_{\tau_1}x_{\tau_3}x_{\tau_4}]^{i-1},[x_{\tau_3}x_{\tau_1}x_{\tau_4}]^j]\\ [[x_{\tau_4}x_{\tau_2}x_{\tau_1}x_{\tau_3}],[x_{\tau_2}x_{\tau_4}x_{\tau_3}x_{\tau_1}]^k,[x_{\tau_2}x_{\tau_4}x_{\tau_1}]^{\overline{\overline{2}}},[x_{\tau_4}x_{\tau_1}x_{\tau_3}x_{\tau_2}]^{i-1},[x_{\tau_4}x_{\tau_2}x_{\tau_1}]^j]\end{array}\right\}^{\delta_3}$$

$$\prod_{(i,j,k)\in\hat{I}_4}\left\{\begin{array}{l}[[x_{\tau_1}x_{\tau_3}x_{\tau_2}],[x_{\tau_1}x_{\tau_3}x_{\tau_2}x_{\tau_4}]^k,[x_{\tau_2}x_{\tau_1}x_{\tau_3}]^j,[x_{\tau_2}x_{\tau_1}x_{\tau_3}x_{\tau_4}]^i,[x_{\tau_1}x_{\tau_3}x_{\tau_2}]^{\overline{\overline{2}}-1}]\\ [[x_{\tau_4}x_{\tau_2}x_{\tau_3}],[x_{\tau_2}x_{\tau_4}x_{\tau_3}x_{\tau_1}]^k,[x_{\tau_3}x_{\tau_2}x_{\tau_4}]^j,[x_{\tau_3}x_{\tau_2}x_{\tau_4}x_{\tau_1}]^i,[x_{\tau_4}x_{\tau_2}x_{\tau_3}]^{\overline{\overline{2}}-1}]\\ [[x_{\tau_4}x_{\tau_3}x_{\tau_1}],[x_{\tau_4}x_{\tau_3}x_{\tau_1}x_{\tau_2}]^k,[x_{\tau_3}x_{\tau_1}x_{\tau_4}]^j,[x_{\tau_3}x_{\tau_1}x_{\tau_4}x_{\tau_2}]^i,[x_{\tau_4}x_{\tau_3}x_{\tau_1}]^{\overline{\overline{2}}-1}]\\ [[x_{\tau_4}x_{\tau_2}x_{\tau_1}],[x_{\tau_2}x_{\tau_4}x_{\tau_3}x_{\tau_1}]^k,[x_{\tau_4}x_{\tau_1}x_{\tau_2}]^j,[x_{\tau_1}x_{\tau_4}x_{\tau_2}x_{\tau_3}]^i,[x_{\tau_2}x_{\tau_4}x_{\tau_1}]^{\overline{\overline{2}}-1}]\end{array}\right\}^{\delta_4}$$

$$\prod_{i=1}^{2^n-2}\prod_{k=0}^{2^n-2-i}[[x_{\tau_2}x_{\tau_1}x_{\tau_3}x_{\tau_4}],[x_{\tau_4}x_{\tau_1}x_{\tau_3}x_{\tau_2}]^{k+1},[x_{\tau_1}x_{\tau_3}x_{\tau_2}x_{\tau_4}]^i,[x_{\tau_2}x_{\tau_1}x_{\tau_3}x_{\tau_4}]^{2^n-2-k-i}]^{\omega_1}$$

$$\prod_{i=1}^{2^n-2}[[x_{\tau_2}x_{\tau_1}x_{\tau_3}x_{\tau_4}],[x_{\tau_1}x_{\tau_3}x_{\tau_2}x_{\tau_4}]^{2^n-1-i},[x_{\tau_2}x_{\tau_1}x_{\tau_3}x_{\tau_4}]^i]^{\omega_2}$$

$$\prod_{i=1}^{2^n-2}[[x_{\tau_1}x_{\tau_3}x_{\tau_2}x_{\tau_4}],[x_{\tau_4}x_{\tau_1}x_{\tau_3}x_{\tau_2}]^{2^n-1-i},[x_{\tau_1}x_{\tau_3}x_{\tau_2}x_{\tau_4}]^i]^{\alpha_i}\};$$

and,

$$\delta_1=\frac{i[(2^n-1)!]2^{n-1}}{k!j!i!(2^n-i-j-k)!(2^n-j-k)},\quad \delta_2=\frac{k[(2^n-1)!]2^{n-1}}{i!j!k!(2^n-i-j-k)!(2^n-j-i)}$$

$$\delta_3=\frac{(2^n-1)!(2^{n-1})}{k!j!(i-1)!(\overline{2^n})!(i+k)},\quad \delta_4=\frac{(2^n-1)!(2^{n-1})}{i!j!k!(\overline{2^n}-1)!(2^n-i-j)}$$

$$\omega_1=\frac{2^{n-1}}{2^n-i}C_i^{2^n-1},\quad \omega_2=\frac{(2^{n-1}-1-i)}{i+1}C_i^{2^n-1},\quad \alpha_i=\frac{C^{2^n-1}-(-1)^i}{2}$$

$$\hat{I}_1=\{(i,j,k):i\neq 0\ ,\ i+j+k<2^n\ ,\ k+j\neq 0\}$$

$$\hat{I}_2=\{(i,j,k):k\neq 0\ ,\ j+k\neq 2^n\ ,\ i+j+k\leq 2^n\ ,\ i+j\neq 0\}$$

$$\hat{I}_3=\{(i,j,k):i\neq 0\ ,\ i+j+k\leq 2^n\ ,\ i+k\neq 2^n\}$$

$$\hat{I}_4=\{(i,j,k):i+k\neq 0\ ,\ i+j+k<2^n\ ,\ i+j\neq 0\}$$

$$1\leq\tau_1<\tau_2<\tau_3<\tau_4\leq d\ ;\ \overline{\overline{2}}=2^n-i-j-k\ .$$

**Theorem 4.2.1.** Let $F$ be a free group on $x_1,\cdots,x_d$ $(d\geq 4)$, then the following statements hold.

1. If $p$ is an odd prime, then the elements $W_{p^n}(x_{\tau_1}, x_{\tau_2}, x_{\tau_3}, x_{\tau_4})$ generate an elementary abelian $p$–group of rank $C_4^d$ in $F/[\gamma_{p^n}(F'), F]F'''$.

2. The elements $W_{2^n}(x_{\tau_1}, x_{\tau_2}, x_{\tau_3}, x_{\tau_4})$ generate an elementary abelian 2–group of rank $C_4^d$ in $F/[\gamma_{2^n}(F'), F]F'''$.

**Proof.** The proof is by computing the connecting homomorphisms

$$H_4(G, \mathbf{Z}_p) \longrightarrow t(\mathcal{M}^{p^n} M \otimes_G \mathbf{R}),$$

where $G = F/F'$ [i.e. free abelian group], this computation enables us to describe these torsion elements in terms of generators. In order to compute these connecting homomorphisms, we need to recall the following homomorphism from [19],

$$\pi_{p^n-1}^{p^n} : \mathbf{R}G^{p^n} \longrightarrow \mathbf{R}G^{p^n-1}$$

where $p$ is any prime number, and $\pi_{p^n-1}^{p^n}$ is defined by

$$(\alpha_1 \circ \alpha_2 \circ \dots. \circ \alpha_{p^n})\, \pi_{p^n-1}^{p^n} = \sum_{i=1}^{p^n} (\alpha_i \varepsilon)\alpha_1 \circ \dots. \circ \hat{\alpha_i} \circ \dots \circ \alpha_{p^n}$$

where $\alpha_1, \cdots \alpha_{p^n} \in \mathbf{R}G$, the circumflex denotes that $\alpha_i$ is omitted, and $\varepsilon$ is the augmentation map $\mathbf{R}G \longrightarrow \mathbf{R}$. In particular,

$$\underbrace{(1 \circ 1 \circ \cdots \circ 1)}_{p^n}\, \pi_{p^n-1}^{p^n} \rightarrow p^n\, \underbrace{(1 \circ 1 \circ \cdots \circ 1)}_{p^n-1}$$

$$((g_1 - 1) \circ \cdots \circ (g_l - 1) \circ 1 \circ \cdots \circ 1)\pi_{p^n-1}^{p^n} \rightarrow (p^n - l)(g_1 - 1) \circ \cdots \circ (g_l - 1) \circ 1 \circ \cdots \circ 1$$

for $l < p^n$, and for $l = p^n$ we get

$$((g_1 - 1) \circ \cdots \circ (g_{p^n} - 1))\pi^{p^n}_{p^n-1} = 0.$$

On the other hand, it follows from ([19], Lemma 3.1) that $\mathrm{coker}\pi^{p^n}_{p^n-1}$ is a direct sum of cyclic $p$-groups. The order of the element

$$\underbrace{(1 \circ 1 \circ \cdots \circ 1)}_{p^n-1} + \mathrm{Im}\pi^{p^n}_{p^n-1}$$

in $\mathrm{coker}\,\pi^{p^n}_{p^n-1}$ is $p^n$, whereas the orders of the remaining generators are $< p^n$. Hence we have an exact sequence

$$0 \longrightarrow \mathbf{Z}_p \longrightarrow \mathrm{coker}\pi^{p^n}_{p^n-1} \longrightarrow \mathrm{coker}\pi^{p^n}_{p^n-1} \otimes \mathbf{Z}_{p^{n-1}} \longrightarrow 0 \tag{4.1}$$

where the embedding $\mathbf{Z}_p \longrightarrow \mathrm{coker}\pi^{p^n}_{p^n-1}$ is given by

$$1 \longmapsto p^{n-1} \underbrace{(1 \circ 1 \circ \cdots \circ 1)}_{p^n-1} + \mathrm{Im}\pi^{p^n}_{p^n-1}.$$

Furthermore, in view of the proof of ([19], theorem 2) the map $\mathbf{Z}_p \longrightarrow \mathrm{coker}\pi^{p^n}_{p^n-1}$ induces an injective map in homology (i.e. the map $H_k(G, \mathbf{Z}_p) \longrightarrow H_k(G, \mathrm{coker}\pi^{p^n}_{p^n-1})$ is injective).

Therefore the image of $\mathbf{Z}_p \otimes_G P_4$ in $\mathrm{coker}\pi^{p^n}_{p^n-1} \otimes_G P_4$ gives us a non-trivial cycle.

Now for this computation, we can use the double complex $\overline{\mathcal{M}} \otimes_G \underline{\mathbf{P}}$, where $\underline{\mathbf{P}}$ is the Koszul complex (projective resolution of the trivial $G$-module $\mathbf{Z}$), and $\overline{\mathcal{M}}$ was as in chapter 2, i.e.

$$\overline{\mathcal{M}} : 0 \to \mathcal{M}^{p^n} M \to P \otimes M^{p^n-1} \to P^{p^n} \to \mathbf{R}G^{p^n} \xrightarrow{\pi^{p^n}_{p^n-1}} \mathbf{R}G^{p^n-1} \to \mathrm{coker}\pi^{p^n}_{p^n-1} \to 0.$$

All tools are now at hand, and we can now start our computation, first we consider the double complex $\overline{\mathcal{M}} \otimes_G \underline{\mathbf{P}}$, and we follow a method analogous to the one described in Remark 3.1.5. Our starting point is the abelian group $\mathrm{coker}\pi^{p^n}_{p^n-1} \otimes_G P_4$, where we choose a cycle which is the image of the given generator of the abelian group $\mathbf{Z}_p \otimes_G P_4$, and we go along down to $\mathcal{M}^{p^n}M \otimes_G \mathbf{R}$ .

Here we consider two cases, one when $p$ is any odd prime, and the other one when $p = 2$.

### 4.2.1 The computation when $p$ is any odd prime and $n > 1$.

First we do the computation when $p$ is any odd prime and $n$ is any natural number greater than 1. In the double complex $\overline{\mathcal{M}} \otimes_G \underline{\mathbf{P}}$, we put coker $= \mathrm{coker}\pi^{p^n}_{p^n-1}$, $\mathbf{R}G^{p^n-1} = B$, $P \otimes M^{p^n-1} = A$ and $\mathcal{M}^{p^n}M = C$. Now, we consider the following diagram:

$$\begin{tikzcd}
 & & & & & \mathrm{coker} \otimes P_4 \\
 & & & & B \otimes P_3 & B \otimes P_4 \arrow[l] \arrow[u] \\
 & & & \mathbf{R}G^{p^n} \otimes P_2 & \mathbf{R}G^{p^n} \otimes P_3 \arrow[l] \arrow[u] & \\
 & & P^{p^n} \otimes P_1 & P^{p^n} \otimes P_2 \arrow[l] \arrow[u] & & \\
 & A \otimes \mathbf{R}G & A \otimes P_1 \arrow[l] \arrow[u] & & & \\
C \otimes \mathbf{R} & C \otimes \mathbf{R}G \arrow[l] \arrow[u] & & & &
\end{tikzcd}$$

The element

$$[p^{n-1}\underbrace{(1 \circ 1 \circ \cdots \circ 1)}_{p^n-1} + \mathrm{Im}\,\pi^{p^n}_{p^n-1}] \otimes e_1 \wedge e_2 \wedge e_3 \wedge e_4 \in p^{n-1}\mathrm{coker} \otimes_G P_4.$$

An inverse image of this in $\mathbf{R}G^{p^n-1} \otimes_G P_4$ is

$$p^{n-1}\underbrace{(1 \circ 1 \circ \cdots \circ 1)}_{p^n-1} \otimes e_1 \wedge e_2 \wedge e_3 \wedge e_4. \tag{4.2}$$

By applying the homomorphism $\mathbf{R}G^{p^n-1} \otimes_G P_4 \longrightarrow \mathbf{R}G^{p^n-1} \otimes_G P_3$ to (4.2), we obtain

$$p^{n-1}\sum_{i=1}^{4}(-1)^{i+1}(1 \circ 1 \circ \cdots \circ 1).(b_i - 1) \otimes (e_1 \wedge \cdots \hat{e_i} \wedge \cdots \wedge e_4) \tag{4.3}$$

By computation analogous to the computation after step (3.13) in chapter 3, we obtain an inverse image of (4.3) in $\mathbf{R}G^{p^n} \otimes_G P_3$, and this inverse is

$$\frac{1}{p}\sum_{i=1}^{4}(-1)^{i+1}[(1\circ 1\circ\cdots\circ 1).(b_i-1)-(b_i-1)^{p^n}]\otimes(e_1\wedge\cdots\hat{e_i}\wedge\cdots\wedge e_4) \qquad (4.4)$$

After applying the homomorphism $\mathbf{R}G^{p^n} \otimes_G P_3 \longrightarrow \mathbf{R}G^{p^n} \otimes_G P_2$ to (4.4), and rearranging the result, we get

$$\frac{1}{p}\sum_{\tau}(-1)^{\tau}\{(b_{2\tau}-1)^{p^n}\cdot(b_{1\tau}-1)-(b_{1\tau}-1)^{p^n}\cdot(b_{2\tau}-1)\}\otimes e_{3\tau}\wedge e_{4\tau} \qquad (4.5)$$

where $\tau$ ranges over all permutations of $\{1,2,3,4\}$ with $1\tau < 2\tau$ and $3\tau < 4\tau$.

An inverse image of (4.5) in $P^{p^n} \otimes_G P_2$ is

$$\frac{1}{p}\sum_{\tau}(-1)^{\tau}[e_{2\tau}^{p^n}(b_{1\tau}-1)-e_{1\tau}^{p^n}(b_{2\tau}-1)+[\ ]_{2\tau 1\tau}^{p^n}]\otimes e_{3\tau}\wedge e_{4\tau} \qquad (4.6)$$

After applying the homomorphism $P^{p^n} \otimes_G P_2 \longrightarrow P^{p^n} \otimes_G P_1$ to (4.6), we get

$$\begin{array}{l}
\frac{1}{p}\{[\ ]_{21}^{p^n}.(b_3-1)+[\ ]_{13}^{p^n}.(b_2-1)+[\ ]_{32}^{p^n}.(b_1-1)\}\otimes e_4\\
-\frac{1}{p}\{[\ ]_{32}^{p^n}.(b_4-1)+[\ ]_{24}^{p^n}.(b_3-1)+[\ ]_{43}^{p^n}.(b_2-1)\}\otimes e_1\\
\frac{1}{p}\{[\ ]_{43}^{p^n}.(b_1-1)+[\ ]_{31}^{p^n}.(b_4-1)+[\ ]_{14}^{p^n}.(b_3-1)\}\otimes e_2\\
-\frac{1}{p}\{[\ ]_{14}^{p^n}.(b_2-1)+[\ ]_{42}^{p^n}.(b_1-1)+[\ ]_{21}^{p^n}.(b_4-1)\}\otimes e_3
\end{array} \qquad (4.7)$$

We notice that (4.7) is very similar to (3.17) (just replacing the power $p$ in the commutators by $p^n$). Thus our final inverse image in $\mathcal{M}^{p^n}M \otimes_G \mathbf{R}G$ can be obtained from (3.84), by replacing the $p$'s in the commutators and in coefficients by $p^n$ as follows

$$
\begin{aligned}
&\sum_{(i,j,k)\in I_1^n}{}^{\beta_1} \left\{ \begin{array}{l} [a_{2134}, a_{214}^{i}, a_{21}^{j}, a_{213}^{k}, a_{2134}^{\overline{\overline{p}}-1}] + [a_{3214}, a_{324}^{i}, a_{32}^{j}, a_{321}^{k}, a_{3214}^{\overline{\overline{p}}-1}] \\ +[a_{4312}, a_{432}^{i}, a_{43}^{j}, a_{431}^{k}, a_{4312}^{\overline{\overline{p}}-1}] + [a_{1432}, a_{142}^{i}, a_{14}^{j}, a_{143}^{k}, a_{1432}^{\overline{\overline{p}}-1}] \\ +[a_{1324}, a_{134}^{i}, a_{13}^{j}, a_{132}^{k}, a_{1324}^{\overline{\overline{p}}-1}] + [a_{2413}, a_{243}^{i}, a_{24}^{j}, a_{241}^{k}, a_{2413}^{\overline{\overline{p}}-1}] \end{array} \right\} \\
&+ \sum_{(i,j,k)\in I_2^n}{}^{\beta_2} \left\{ \begin{array}{l} [a_{213}, a_{2134}^{\overline{\overline{p}}}, a_{214}^{i}, a_{21}^{j}, a_{213}^{k-1}] + [a_{321}, a_{3214}^{\overline{\overline{p}}}, a_{324}^{i}, a_{32}^{j}, a_{321}^{k-1}] \\ +[a_{431}, a_{4312}^{\overline{\overline{p}}}, a_{432}^{i}, a_{43}^{j}, a_{431}^{k-1}] + [a_{143}, a_{1432}^{\overline{\overline{p}}}, a_{142}^{i}, a_{14}^{j}, a_{143}^{k-1}] \\ +[a_{132}, a_{1324}^{\overline{\overline{p}}}, a_{134}^{i}, a_{132}^{k-1}, a_{13}^{j}] + [a_{241}, a_{2413}^{\overline{\overline{p}}}, a_{243}^{i}, a_{24}^{j}, a_{241}^{k-1}] \end{array} \right\} \\
&+ \sum_{(i,j,k)\in I_3^n}{}^{\beta_3} \left\{ \begin{array}{l} [a_{2134}, a_{1324}^{k}, a_{132}^{\overline{\overline{p}}}, a_{2134}^{i-1}, a_{213}^{j}] + [a_{2341}, a_{2431}^{k}, a_{243}^{\overline{\overline{p}}}, a_{3241}^{i-1}, a_{324}^{j}] \\ [a_{4312}, a_{3142}^{k}, a_{314}^{\overline{\overline{p}}}, a_{4312}^{i-1}, a_{431}^{j}] + [a_{4123}, a_{4213}^{k}, a_{421}^{\overline{\overline{p}}}, a_{1423}^{i-1}, a_{142}^{j}] \end{array} \right\} \\
&+ \sum_{(i,j,k)\in I_4^n}{}^{\beta_4} \left\{ \begin{array}{l} [a_{1324}, a_{132}^{\overline{\overline{p}}}, a_{213}^{j}, a_{2134}^{i}, a_{1324}^{k-1}] + [a_{4231}, a_{243}^{\overline{\overline{p}}}, a_{324}^{j}, a_{3241}^{i}, a_{2431}^{k-1}] \\ [a_{3142}, a_{314}^{\overline{\overline{p}}}, a_{431}^{j}, a_{4312}^{i}, a_{3142}^{k-1}] + [a_{2413}, a_{421}^{\overline{\overline{p}}}, a_{142}^{j}, a_{1423}^{i}, a_{4213}^{k-1}] \end{array} \right\} \\
&+ \sum_{i=0}^{p^n-1} \sum_{k=-1}^{p^n-1-i}{}^{\omega_n} \left\{ [a_{2134}, a_{4123}^{k+1}, a_{1324}^{i}, a_{2134}^{p^n-2-k-i}] + [a_{1324}, a_{4123}^{p^n-1-i}, a_{1324}^{i}] \right\}
\end{aligned} \tag{4.8}
$$

where,

$$\beta_1 = \tfrac{p^{n-1}}{(p^n-k-j)} C_{j+k}^{p^n-1} C_k^{j+k} C_{i-1}^{p^n-1-k-j}, \quad \beta_2 = \tfrac{p^{n-1}}{(p^n-i-j)} C_{j+i}^{p^n-1} C_i^{j+i} C_{k-1}^{p^n-1-i-j}$$

$$\beta_3 = \tfrac{p^{n-1}}{i+k} C_{i-1}^{p^n-1} C_{j+k}^{p^n-i} C_k^{j+k}, \quad \beta_4 = \tfrac{p^{n-1}}{(p^n-i-j)} C_{i+j+k}^{p^n-1} C_{k+i}^{i+j+k} C_i^{k+i}$$

$$\omega_n = \tfrac{1}{i+1} C_i^{p^n-1}, \quad \overline{\overline{p}} = p^n - i - j - k$$

$$I_1^n = \{(i,j,k) : i \neq 0,\ j+k \neq 0, i+j+k < p^n\}$$

$$I_2^n = \{(i,j,k) : k \neq 0,\ j+i \neq 0,\ j+k \neq p^n,\ i+j+k \leq p^n\}$$

$$I_3^n = \{(i,j,k) : i \neq 0,\ i+k \neq p^n,\ i+j+k \leq p^n\}$$

$$I_4^n = \{(i,j,k) : k \neq 0,\ j+i \neq 0,\ i+j+k < p^n\}.$$

Finally the image of the element (4.8) in $\mathcal{M}^{p^n} M \otimes_G \mathbf{R}$ is the image of

$$[p^{n-1} \underbrace{(1 \circ 1 \circ \cdots \circ 1)}_{p^n-1} + \mathrm{Im}\pi_{p^n-1}^{p^n}] \otimes e_1 \wedge e_2 \wedge e_3 \wedge e_4$$

under the connecting homomorphism $H_4(G,\mathrm{coker}\pi^{p^n}_{p^n-1}) \longrightarrow H_0(G,\mathcal{M}^{p^n}M)$. As before going from $H_0(G,\mathcal{M}^{p^n}M)$ to $H_0(G,\mathcal{M}^{p^n}F'_{ab})$ does not effect the result, then applying the isomorphism $H_0(G,\mathcal{M}^{p^n}F'_{ab}) \longrightarrow \gamma_{p^n}(F')F'''/[\gamma_{p^n}(F'),F]F'''$, and then replacing the $a$'s by their definitions as given on p58 we obtain our desired element, and this finishes the proof of statement 1.

### 4.2.2 The computation when $p = 2$ and $n$ is any natural number.

Due to the even powers, the computation turns out to be slightly different from the other cases, so we do it in some detail. Consider the element

$$2^{n-1}\,\underbrace{(1\circ 1\circ\cdots\circ 1)}_{2^n-1}+Im\pi^{2^n}_{2^n-1}]\otimes e_1\wedge e_2\wedge e_3\wedge e_4\in 2^{n-1}\mathrm{coker}\pi^{2^n}_{2^n-1}\otimes_G P_4.$$

An inverse image of this in $\mathbf{R}G^{2^n-1}\otimes_G P_4$ is

$$2^{n-1}(1\circ 1\circ\cdots\circ 1)\otimes e_1\wedge e_2\wedge e_3\wedge e_4.$$

Applying the homomorphism $\mathbf{R}G^{2^n-1}\otimes_G P_4 \longrightarrow \mathbf{R}G^{2^n-1}\otimes_G P_3$ we obtain

$$2^{n-1}\sum_{i=1}^{4}(-1)^{i+1}(1\circ 1\circ\cdots\circ 1).(b_i-1)\otimes(e_1\wedge\cdots\hat{e_i}\wedge\cdots\wedge e_4) \tag{4.9}$$

Exactly as before, we can write an inverse image of (4.9) in $\mathbf{R}G^{2^n}\otimes_G P_3$, and this inverse image is:

$$\frac{1}{2}\sum_{i=1}^{4}(-1)^{i+1}[(1\circ\cdots\circ 1).(b_i-1)-(b_i-1)^{2^n}]\otimes e_1\wedge\cdots\hat{e_i}\wedge\cdots\wedge e_4 \tag{4.10}$$

After applying the homomorphism $\mathbf{R}G^{2^n} \otimes_G P_3 \longrightarrow \mathbf{R}G^{2^n} \otimes_G P_2$ to (4.10), we obtain

$$\begin{aligned}
&\tfrac{1}{2}\{(b_2-1)^{2^n}.(b_1-1)-(b_1-1)^{2^n}.(b_2-1)\}\otimes e_3\wedge e_4\\
&+\tfrac{1}{2}\{(b_1-1)^{2^n}.(b_3-1)-(b_3-1)^{2^n}.(b_1-1)\}\otimes e_2\wedge e_4\\
&+\tfrac{1}{2}\{(b_4-1)^{2^n}.(b_1-1)-(b_1-1)^{2^n}.(b_4-1)\}\otimes e_2\wedge e_3\\
&+\tfrac{1}{2}\{(b_2-1)^{2^n}.(b_4-1)-(b_4-1)^{2^n}.(b_2-1)\}\otimes e_1\wedge e_3\\
&+\tfrac{1}{2}\{(b_4-1)^{2^n}.(b_3-1)-(b_3-1)^{2^n}.(b_4-1)\}\otimes e_1\wedge e_2\\
&+\tfrac{1}{2}\{(b_3-1)^{2^n}.(b_2-1)-(b_2-1)^{2^n}.(b_3-1)\}\otimes e_1\wedge e_4
\end{aligned} \tag{4.11}$$

An inverse image of (4.11) in $P^{2^n} \otimes_G P_2$ is:

$$\begin{aligned}
&\tfrac{1}{2}\{e_2^{2^n}(b_1-1)-e_1^{2^n}(b_2-1)+[\ ]_{21}^{2^n}\}\otimes e_3\wedge e_4\\
&+\tfrac{1}{2}\{e_1^{2^n}(b_3-1)-e_3^{2^n}(b_1-1)+[\ ]_{13}^{2^n}\}\otimes e_2\wedge e_4\\
&+\tfrac{1}{2}\{e_4^{2^n}(b_1-1)-e_1^{2^n}(b_4-1)+[\ ]_{41}^{2^n}\}\otimes e_2\wedge e_3\\
&+\tfrac{1}{2}\{e_2^{2^n}(b_4-1)-e_4^{2^n}(b_2-1)+[\ ]_{24}^{2^n}\}\otimes e_1\wedge e_3\\
&+\tfrac{1}{2}\{e_4^{2^n}(b_3-1)-e_3^{2^n}(b_4-1)+[\ ]_{43}^{2^n}\}\otimes e_1\wedge e_2\\
&+\tfrac{1}{2}\{e_3^{2^n}(b_2-1)-e_2^{2^n}(b_3-1)+[\ ]_{32}^{2^n}\}\otimes e_1\wedge e_4
\end{aligned} \tag{4.12}$$

After applying the homomorphism $P^{2^n} \otimes_G P_2 \longrightarrow P^{2^n} \otimes_G P_1$ to (4.12), we get:

$$\begin{aligned}
&\tfrac{1}{2}\{[\ ]_{21}^{2^n}.(b_3-1)+[\ ]_{13}^{2^n}.(b_2-1)+[\ ]_{32}^{2^n}.(b_1-1)\}\otimes e_4\\
&+\tfrac{1}{2}\{-[\ ]_{32}^{2^n}.(b_4-1)-[\ ]_{24}^{2^n}.(b_3-1)-[\ ]_{43}^{2^n}.(b_2-1)\}\otimes e_1\\
&+\tfrac{1}{2}\{[\ ]_{43}^{2^n}.(b_1-1)-[\ ]_{13}^{2^n}.(b_4-1)-[\ ]_{41}^{2^n}.(b_3-1)\}\otimes e_2\\
&+\tfrac{1}{2}\{[\ ]_{41}^{2^n}.(b_2-1)+[\ ]_{24}^{2^n}.(b_1-1)-[\ ]_{21}^{2^n}.(b_4-1)\}\otimes e_3
\end{aligned} \tag{4.13}$$

At this point, we could not get an inverse image of (4.13), just by replacing the $p$'s in (3.31), (3.32), (3.33), (3.34) (in the commutators and in the binomial coefficients) by $2^n$, as we did in the last computation. However, the steps in the computation of inverse image of (3.26) may be followed to produce an inverse image of (4.13), and this inverse image in $P \otimes M^{2^n-1}$ is the sum of the following four terms:

$$\frac{1}{2}\left\{\begin{array}{c} ([\,]_{21} \otimes [\,]_{21}^{2^n-1})(b_3-1) + ([\,]_{13} \otimes [\,]_{13}^{2^n-1})(b_2-1) + ([\,]_{32} \otimes [\,]_{32}^{2^n-1})(b_1-1) \\ + \sum_{i=0}^{2^n-2} C_i^{2^n-1} \left\{\begin{array}{c} [\,]_{21}(b_3-1) \otimes [\,]_{21}(b_3-1)^{2^n-2-i} \circ [\,]_{21}^{i+1} \\ -[\,]_{21} \otimes [\,]_{21}(b_3-1)^{2^n-1-i} \circ [\,]_{21}^{i} \\ -[\,]_{31}(b_2-1) \otimes [\,]_{31}(b_2-1)^{2^n-2-i} \circ [\,]_{31}^{i+1} \\ +[\,]_{31} \otimes [\,]_{31}(b_2-1)^{2^n-1-i} \circ [\,]_{31}^{i} \\ +[\,]_{32}(b_1-1) \otimes [\,]_{32}(b_1-1)^{2^n-2-i} \circ [\,]_{32}^{i+1} \\ -[\,]_{32} \otimes [\,]_{32}(b_1-1)^{2^n-1-i} \circ [\,]_{32}^{i} \\ +[\,]_{13}(b_2-1) \otimes [\,]_{13}(b_2-1)^{2^n-2-i} \circ [\,]_{21}(b_3-1)^{i+1} \\ -[\,]_{21}(b_3-1) \otimes [\,]_{13}(b_2-1)^{2^n-1-i} \circ [\,]_{21}(b_3-1)^{i} \end{array}\right\} \end{array}\right\} \otimes e_4$$

$$+\frac{1}{2}\left\{\begin{array}{c} -([\,]_{32} \otimes [\,]_{32}^{2^n-1})(b_4-1) - ([\,]_{24} \otimes [\,]_{24}^{2^n-1})(b_3-1) - ([\,]_{43} \otimes [\,]_{43}^{2^n-1})(b_2-1) \\ + \sum_{i=0}^{2^n-2} C_i^{2^n-1} \left\{\begin{array}{c} -[\,]_{32}(b_4-1) \otimes [\,]_{32}(b_4-1)^{2^n-2-i} \circ [\,]_{32}^{i+1} \\ +[\,]_{32} \otimes [\,]_{32}(b_4-1)^{2^n-1-i} \circ [\,]_{32}^{i} \\ [\,]_{42}(b_3-1) \otimes [\,]_{42}(b_3-1)^{2^n-2-i} \circ [\,]_{42}^{i+1} \\ -[\,]_{42} \otimes [\,]_{42}(b_3-1)^{2^n-1-i} \circ [\,]_{42}^{i} \\ -[\,]_{43}(b_2-1) \otimes [\,]_{43}(b_2-1)^{2^n-2-i} \circ [\,]_{43}^{i+1} \\ +[\,]_{43} \otimes [\,]_{43}(b_2-1)^{2^n-1-i} \circ [\,]_{43}^{i} \\ +[\,]_{42}(b_3-1) \otimes [\,]_{42}(b_3-1)^{2^n-2-i} \circ [\,]_{32}(b_4-1)^{i+1} \\ -[\,]_{32}(b_4-1) \otimes [\,]_{42}(b_3-1)^{2^n-1-i} \circ [\,]_{32}(b_4-1)^{i} \end{array}\right\} \end{array}\right\} \otimes e_1$$

$$
+\frac{1}{2}\left\{\begin{array}{c}
([\,]_{43}\otimes[\,]_{43}^{2^n-1})(b_1-1)-([\,]_{13}\otimes[\,]_{13}^{2^n-1})(b_4-1)-([\,]_{41}\otimes[\,]_{41}^{2^n-1})(b_3-1)\\
+\sum_{i=0}^{2^n-2} C_i^{2^n-1}\left\{\begin{array}{c}
[\,]_{43}(b_1-1)\otimes[\,]_{43}(b_1-1)^{2^n-2-i}\circ[\,]_{43}^{i+1}\\
-[\,]_{43}\otimes[\,]_{43}(b_1-1)^{2^n-1-i}\circ[\,]_{43}^{i}\\
-[\,]_{41}(b_3-1)\otimes[\,]_{41}(b_3-1)^{2^n-2-i}\circ[\,]_{41}^{i+1}\\
+[\,]_{41}\otimes[\,]_{41}(b_3-1)^{2^n-1-i}\circ[\,]_{41}^{i}\\
+[\,]_{31}(b_4-1)\otimes[\,]_{31}(b_4-1)^{2^n-2-i}\circ[\,]_{31}^{i+1}\\
-[\,]_{31}\otimes[\,]_{31}(b_4-1)^{2^n-1-i}\circ[\,]_{31}^{i}\\
-[\,]_{43}(b_1-1)\otimes[\,]_{43}(b_1-1)^{2^n-2-i}\circ[\,]_{31}(b_4-1)^{i+1}\\
+[\,]_{31}(b_4-1)\otimes[\,]_{43}(b_1-1)^{2^n-1-i}\circ[\,]_{31}(b_4-1)^{i}
\end{array}\right\}
\end{array}\right\}\otimes e_2
$$

$$
+\frac{1}{2}\left\{\begin{array}{c}
([\,]_{41}\otimes[\,]_{41}^{2^n-1})(b_2-1)+([\,]_{24}\otimes[\,]_{24}^{2^n-1})(b_1-1)-([\,]_{21}\otimes[\,]_{21}^{2^n-1})(b_4-1)\\
+\sum_{i=0}^{2^n-2} C_i^{2^n-1}\left\{\begin{array}{c}
[\,]_{41}(b_2-1)\otimes[\,]_{41}(b_2-1)^{2^n-2-i}\circ[\,]_{41}^{i+1}\\
-[\,]_{41}\otimes[\,]_{41}(b_2-1)^{2^n-1-i}\circ[\,]_{41}^{i}\\
-[\,]_{42}(b_1-1)\otimes[\,]_{42}(b_1-1)^{2^n-2-i}\circ[\,]_{42}^{i+1}\\
+[\,]_{42}\otimes[\,]_{42}(b_2-1)^{2^n-1-i}\circ[\,]_{42}^{i}\\
-[\,]_{21}(b_4-1)\otimes[\,]_{21}(b_4-1)^{2^n-2-i}\circ[\,]_{21}^{i+1}\\
+[\,]_{21}\otimes[\,]_{21}(b_4-1)^{2^n-1-i}\circ[\,]_{21}^{i}\\
-[\,]_{24}(b_1-1)\otimes[\,]_{24}(b_1-1)^{2^n-2-i}\circ[\,]_{41}(b_2-1)^{i+1}\\
+[\,]_{41}(b_2-1)\otimes[\,]_{24}(b_1-1)^{2^n-1-i}\circ[\,]_{41}(b_1-1)^{i}
\end{array}\right\}
\end{array}\right\}\otimes e_3
$$

After applying the homomorphism $P\otimes M^{2^n-1}\otimes_G P_1 \longrightarrow P\otimes M^{2^n-1}\otimes_G \mathbf{R}G$, we get

$$\frac{1}{2}\sum_{i=0}^{2^n-2} C_i^{2^n-1}\left\{\begin{array}{c} [\,]_{21}(b_3-1)\otimes[\,]_{21}(b_3-1)^{2^n-2-i}\circ[\,]_{21}^{i+1} \\ -[\,]_{21}\otimes[\,]_{21}(b_3-1)^{2^n-1-i}\circ[\,]_{21}^{i} \\ -[\,]_{31}(b_2-1)\otimes[\,]_{31}(b_2-1)^{2^n-2-i}\circ[\,]_{31}^{i+1} \\ +[\,]_{31}\otimes[\,]_{31}(b_2-1)^{2^n-1-i}\circ[\,]_{31}^{i} \\ +[\,]_{32}(b_1-1)\otimes[\,]_{32}(b_1-1)^{2^n-2-i}\circ[\,]_{32}^{i+1} \\ -[\,]_{32}\otimes[\,]_{32}(b_1-1)^{2^n-1-i}\circ[\,]_{32}^{i} \\ +[\,]_{13}(b_2-1)\otimes[\,]_{13}(b_2-1)^{2^n-2-i}\circ[\,]_{21}(b_3-1)^{i+1} \\ -[\,]_{21}(b_3-1)\otimes[\,]_{13}(b_2-1)^{2^n-1-i}\circ[\,]_{21}(b_3-1)^{i} \end{array}\right\}(b_4-1)$$

$$\frac{1}{2}\sum_{i=0}^{2^n-2} C_i^{2^n-1}\left\{\begin{array}{c} -[\,]_{32}(b_4-1)\otimes[\,]_{32}(b_4-1)^{2^n-2-i}\circ[\,]_{32}^{i+1} \\ +[\,]_{32}\otimes[\,]_{32}(b_4-1)^{2^n-1-i}\circ[\,]_{32}^{i} \\ +[\,]_{42}(b_3-1)\otimes[\,]_{42}(b_3-1)^{2^n-2-i}\circ[\,]_{42}^{i+1} \\ -[\,]_{42}\otimes[\,]_{42}(b_3-1)^{2^n-1-i}\circ[\,]_{42}^{i} \\ -[\,]_{43}(b_2-1)\otimes[\,]_{43}(b_2-1)^{2^n-2-i}\circ[\,]_{43}^{i+1} \\ +[\,]_{43}\otimes[\,]_{43}(b_2-1)^{2^n-1-i}\circ[\,]_{43}^{i} \\ -[\,]_{24}(b_3-1)\otimes[\,]_{24}(b_3-1)^{2^n-2-i}\circ[\,]_{32}(b_4-1)^{i+1} \\ +[\,]_{32}(b_4-1)\otimes[\,]_{24}(b_3-1)^{2^n-1-i}\circ[\,]_{32}(b_4-1)^{i} \end{array}\right\}(b_1-1)$$

$$\frac{1}{2}\sum_{i=0}^{2^n-2} C_i^{2^n-1}\left\{\begin{array}{c} [\,]_{43}(b_1-1)\otimes[\,]_{43}(b_1-1)^{2^n-2-i}\circ[\,]_{43}^{i+1} \\ -[\,]_{43}\otimes[\,]_{43}(b_1-1)^{2^n-1-i}\circ[\,]_{43}^{i} \\ -[\,]_{41}(b_3-1)\otimes[\,]_{41}(b_3-1)^{2^n-2-i}\circ[\,]_{41}^{i+1} \\ +[\,]_{41}\otimes[\,]_{41}(b_3-1)^{2^n-1-i}\circ[\,]_{41}^{i} \\ +[\,]_{31}(b_4-1)\otimes[\,]_{31}(b_4-1)^{2^n-2-i}\circ[\,]_{31}^{i+1} \\ -[\,]_{31}\otimes[\,]_{31}(b_4-1)^{2^n-1-i}\circ[\,]_{31}^{i} \\ -[\,]_{43}(b_1-1)\otimes[\,]_{43}(b_1-1)^{2^n-2-i}\circ[\,]_{31}(b_4-1)^{i+1} \\ +[\,]_{31}(b_4-1)\otimes[\,]_{43}(b_1-1)^{2^n-1-i}\circ[\,]_{31}(b_4-1)^{i} \end{array}\right\}(b_2-1)$$

$$\frac{1}{2}\sum_{i=0}^{2^n-2} C_i^{2^n-1}\left\{\begin{array}{c} [\,]_{41}(b_2-1)\otimes[\,]_{41}(b_2-1)^{2^n-2-i}\circ[\,]_{41}^{i+1} \\ -[\,]_{41}\otimes[\,]_{41}(b_2-1)^{2^n-1-i}\circ[\,]_{41}^{i} \\ -[\,]_{42}(b_1-1)\otimes[\,]_{42}(b_1-1)^{2^n-2-i}\circ[\,]_{42}^{i+1} \\ +[\,]_{42}\otimes[\,]_{42}(b_2-1)^{2^n-1-i}\circ[\,]_{42}^{i} \\ -[\,]_{21}(b_4-1)\otimes[\,]_{21}(b_4-1)^{2^n-2-i}\circ[\,]_{21}^{i+1} \\ +[\,]_{21}\otimes[\,]_{21}(b_4-1)^{2^n-1-i}\circ[\,]_{21}^{i} \\ -[\,]_{24}(b_1-1)\otimes[\,]_{24}(b_1-1)^{2^n-2-i}\circ[\,]_{41}(b_2-1)^{i+1} \\ +[\,]_{41}(b_2-1)\otimes[\,]_{24}(b_1-1)^{2^n-1-i}\circ[\,]_{41}(b_1-1)^{i} \end{array}\right\}(b_3-1)$$

Therefore an inverse image of this element in $\mathcal{M}^{2^n}M \otimes \mathbf{R}G$ is

$$\frac{1}{2}\sum_{i=0}^{2^n-2} C_i^{2^n-1}\left\{\begin{array}{c}
[[\,]_{21}(b_3-1),[\,]_{21}^{i+1},[\,]_{21}(b_3-1)^{2^n-2-i}](b_4-1)\\
-[[\,]_{31}(b_2-1),[\,]_{31}^{i+1},[\,]_{31}(b_2-1)^{2^n-2-i}](b_4-1)\\
+[[\,]_{32}(b_1-1),[\,]_{32}^{i+1},[\,]_{32}(b_1-1)^{2^n-2-i}](b_4-1)\\
+[[\,]_{13}(b_2-1),[\,]_{21}(b_3-1)^{i+1},[\,]_{13}(b_2-1)^{2^n-2-i}](b_4-1)\\
-[[\,]_{32}(b_4-1),[\,]_{32}^{i+1},[\,]_{32}(b_4-1)^{2^n-2-i}](b_1-1)\\
+[[\,]_{42}(b_3-1),[\,]_{42}^{i+1},[\,]_{42}(b_3-1)^{2^n-2-i}](b_1-1)\\
-[[\,]_{43}(b_2-1),[\,]_{43}^{i+1},[\,]_{43}(b_2-1)^{2^n-2-i}](b_1-1)\\
-[[\,]_{24}(b_3-1),[\,]_{32}(b_4-1)^{i+1},[\,]_{24}(b_3-1)^{2^n-2-i}](b_1-1)\\
+[[\,]_{43}(b_1-1),[\,]_{43}^{i+1},[\,]_{43}(b_1-1)^{2^n-2-i}](b_2-1)\\
-[[\,]_{41}(b_3-1),[\,]_{41}^{i+1},[\,]_{41}(b_3-1)^{2^n-2-i}](b_2-1)\\
+[[\,]_{31}(b_4-1),[\,]_{31}^{i+1},[\,]_{31}(b_4-1)^{2^n-2-i}](b_2-1)\\
-[[\,]_{43}(b_1-1),[\,]_{31}(b_4-1)^{i+1},[\,]_{43}(b_1-1)^{2^n-2-i}](b_2-1)\\
+[[\,]_{41}(b_2-1),[\,]_{41}^{i+1},[\,]_{41}(b_2-1)^{2^n-2-i}](b_3-1)\\
-[[\,]_{42}(b_1-1),[\,]_{42}^{i+1},[\,]_{42}(b_1-1)^{2^n-2-i}](b_3-1)\\
-[[\,]_{21}(b_4-1),[\,]_{21}^{i+1},[\,]_{21}(b_4-1)^{2^n-2-i}](b_3-1)\\
-[[\,]_{24}(b_1-1),[\,]_{41}(b_2-1)^{i+1},[\,]_{24}(b_1-1)^{2^n-2-i}](b_3-1)
\end{array}\right\} \tag{4.14}$$

As before, our first aim is to write 2 times the element (4.14) as a linear combination of terms with coefficients multiple of 2. By computation similar to the one in chapter 3, we obtain from (4.14) (using notation already introduced in chapter 3) the following:

$$\frac{1}{2}\left\{\begin{array}{l}
\sum\limits_{\substack{i+j+k<2^n\\ i\neq 0}} \overline{\delta}_1 \left\{\begin{array}{l}
[a_{2134}, a_{214}^{i}, a_{21}^{j}, a_{213}^{k}, a_{2134}^{\bar{\bar{2}}-1}] + [a_{3142}, a_{312}^{i}, a_{31}^{j}, a_{314}^{k}, a_{3142}^{\bar{\bar{2}}-1}] \\
[a_{3214}, a_{324}^{i}, a_{32}^{j}, a_{321}^{k}, a_{3214}^{\bar{\bar{2}}-1}] + [a_{4231}, a_{421}^{i}, a_{42}^{j}, a_{423}^{k}, a_{4231}^{\bar{\bar{2}}-1}] \\
[a_{4312}, a_{432}^{i}, a_{43}^{j}, a_{431}^{k}, a_{4312}^{\bar{\bar{2}}-1}] + [a_{4132}, a_{413}^{i}, a_{41}^{j}, a_{412}^{k}, a_{4132}^{\bar{\bar{2}}-1}]
\end{array}\right\} \\
\sum\limits_{\substack{i+j+k\leq 2^n\\ k\neq 0,\, j+k\neq 2^n}} \overline{\delta}_2 \left\{\begin{array}{l}
[a_{213}, a_{2134}^{\bar{\bar{2}}}, a_{214}^{i}, a_{21}^{j}, a_{213}^{k-1}] + [a_{314}, a_{3142}^{\bar{\bar{2}}}, a_{312}^{i}, a_{31}^{j}, a_{314}^{k-1}] \\
[a_{321}, a_{3214}^{\bar{\bar{2}}}, a_{324}^{i}, a_{32}^{j}, a_{321}^{k-1}] + [a_{423}, a_{4231}^{\bar{\bar{2}}}, a_{421}^{i}, a_{42}^{j}, a_{423}^{k-1}] \\
[a_{431}, a_{4312}^{\bar{\bar{2}}}, a_{432}^{i}, a_{43}^{j}, a_{431}^{k-1}] + [a_{412}, a_{4123}^{\bar{\bar{2}}}, a_{413}^{i}, a_{41}^{j}, a_{412}^{k-1}]
\end{array}\right\} \\
\sum\limits_{i=0}^{2^n-2} C_i^{2^n-1} \left\{\begin{array}{l}
[[\,]_{13}(b_2-1), [\,]_{21}(b_3-1)^{i+1}, [\,]_{13}(b_2-1)^{2^n-2-i}].(b_4-1) \\
-[[\,]_{24}(b_3-1), [\,]_{32}(b_4-1)^{i+1}, [\,]_{24}(b_{3-1})^{2^n-2-i}].(b_1-1) \\
-[[\,]_{43}(b_1-1), [\,]_{31}(b_4-1)^{i+1}, [\,]_{43}(b_1-1)^{2^n-2-i}](b_2-1) \\
-[[\,]_{24}(b_1-1), [\,]_{41}(b_2-1)^{i+1}, [\,]_{41}(b_2-1)^{2^n-2-i}].(b_3-1)
\end{array}\right\}
\end{array}\right\} \tag{4.15}$$

where $\overline{\delta}_1 = \frac{i[(2^n-1)!]2^n}{k!j!i!(2^n-i-j-k)!(2^n-j-k)}$ and $\overline{\delta}_2 = \frac{k[(2^n-1)!]2^n}{i!j!k!(2^n-i-j-k)!(2^n-j-i)}$

In (4.15) when $j = k = 0$, $\overline{\delta}_1$ is not divisible by 2, moreover, when $j = i = 0$ the integer $\overline{\delta}_2$ is also not divisible by 2. By rewriting (4.15) we get the following summands:

$$
\left\{
\begin{array}{l}
\displaystyle\sum_{\substack{i+j+k<2^n \\ i\neq 0,\, j+k\neq 0}} \delta_1
\left\{
\begin{array}{l}
[a_{2134}, a_{214}^{i}, a_{21}^{j}, a_{213}^{k}, a_{2134}^{\bar{\bar{2}}-1}] + [a_{3142}, a_{312}^{i}, a_{31}^{j}, a_{314}^{k}, a_{3142}^{\bar{\bar{2}}-1}] \\
{[a_{3214}, a_{324}^{i}, a_{32}^{j}, a_{321}^{k}, a_{3214}^{\bar{\bar{2}}-1}]} + [a_{4231}, a_{421}^{i}, a_{42}^{j}, a_{423}^{k}, a_{4231}^{\bar{\bar{2}}-1}] \\
{[a_{4312}, a_{432}^{i}, a_{43}^{j}, a_{431}^{k}, a_{4312}^{\bar{\bar{2}}-1}]} + [a_{4132}, a_{413}^{i}, a_{41}^{j}, a_{412}^{k}, a_{4132}^{\bar{\bar{2}}-1}]
\end{array}
\right\} \\
\displaystyle\sum_{\substack{i+j+k\leq 2^n \\ k\neq 0,\, j+k\neq 2^n \\ j+i\neq 0}} \delta_2
\left\{
\begin{array}{l}
[a_{213}, a_{2134}^{\bar{\bar{2}}}, a_{214}^{i}, a_{21}^{j}, a_{213}^{k-1}] + [a_{314}, a_{3142}^{\bar{\bar{2}}}, a_{312}^{i}, a_{31}^{j}, a_{314}^{k-1}] \\
+[a_{321}, a_{3214}^{\bar{\bar{2}}}, a_{324}^{i}, a_{32}^{j}, a_{321}^{k-1}] + [a_{423}, a_{4231}^{\bar{\bar{2}}}, a_{421}^{i}, a_{42}^{j}, a_{423}^{k-1}] \\
+[a_{431}, a_{4312}^{\bar{\bar{2}}}, a_{432}^{i}, a_{43}^{j}, a_{431}^{k-1}] + [a_{412}, a_{4123}^{\bar{\bar{2}}}, a_{413}^{i}, a_{41}^{j}, a_{412}^{k-1}]
\end{array}
\right\}
\end{array}
\right\}
\tag{4.16}
$$

and

$$
\frac{1}{2}
\left\{
\begin{array}{l}
\displaystyle\sum_{i=1}^{2^n-1} C_{i-1}^{2^n-1}
\left\{
\begin{array}{l}
[a_{2134}, a_{214}^{i}, a_{2134}^{2^n-1-i}] + [a_{3142}, a_{312}^{i}, a_{3142}^{2^n-1-i}] \\
+[a_{3214}, a_{324}^{i}, a_{3214}^{2^n-1-i}] + [a_{4231}, a_{421}^{i}, a_{4231}^{2^n-1-i}] \\
+[a_{4312}, a_{432}^{i}, a_{4312}^{2^n-1-i}] + [a_{4132}, a_{413}^{i}, a_{4132}^{2^n-1-i}]
\end{array}
\right\} \\
\displaystyle -\sum_{k=1}^{2^n-1} C_{k-1}^{2^n-1}
\left\{
\begin{array}{l}
[a_{2134}, a_{213}^{k}, a_{2134}^{2^n-1-k}] + [a_{3142}, a_{314}^{k}, a_{3142}^{2^n-1-k}] \\
{[a_{3214}, a_{321}^{k}, a_{3214}^{2^n-1-k}]} + [a_{4231}, a_{423}^{k}, a_{4231}^{2^n-1-k}] \\
{[a_{4312}, a_{431}^{k}, a_{4312}^{2^n-1-k}]} + [a_{4123}, a_{412}^{k}, a_{4123}^{2^n-1-k}]
\end{array}
\right\} \\
\displaystyle\sum_{i=0}^{2^n-2} C_{i}^{2^n-1}
\left\{
\begin{array}{l}
[[\,]_{13}(b_2-1), [\,]_{21}(b_3-1)^{i+1}, [\,]_{13}(b_2-1)^{2^n-2-i}].(b_4-1) \\
-[[\,]_{24}(b_3-1), [\,]_{32}(b_4-1)^{i+1}, [\,]_{24}(b_{3-1})^{2^n-2-i}].(b_1-1) \\
-[[\,]_{43}(b_1-1), [\,]_{31}(b_4-1)^{i+1}, [\,]_{43}(b_1-1)^{2^n-2-i}].(b_2-1) \\
-[[\,]_{24}(b_1-1), [\,]_{41}(b_2-1)^{i+1}, [\,]_{24}(b_1-1)^{2^n-2-i}].(b_3-1)
\end{array}
\right\}
\end{array}
\right\}
\tag{4.17}
$$

where $\delta_1 = \frac{i[(2^n-1)!]2^{n-1}}{k!j!i!(2^n-i-j-k)!(2^n-j-k)}$ and $\delta_2 = \frac{k[(2^n-1)!]2^{n-1}}{i!j!k!(2^n-i-j-k)!(2^n-j-i)}$.

Our next goal is to write 2 times the element (4.17) as a linear combination of terms with coefficients multiple of 2. Again by computation similar to the one in chapter 3, we obtain from (4.17) the following:

$$\frac{1}{2}\left\{\begin{array}{l}
\displaystyle\sum_{\substack{i+j+k\leq 2^n\\ i\neq 0,\, k\neq 0}} \overline{\delta}_3 \left\{\begin{array}{c}
[a_{1324}, a_{2134}^{i}, a_{213}^{j}, a_{1324}^{k-1}, a_{132}^{\bar{\bar{2}}}] - [a_{2431}, a_{3241}^{i}, a_{324}^{j}, a_{2431}^{k-1}, a_{243}^{\bar{\bar{2}}}] \\
-[a_{4312}, a_{3142}^{i}, a_{314}^{j}, a_{4312}^{k-1}, a_{431}^{\bar{\bar{2}}}] - [a_{2413}, a_{4123}^{i}, a_{412}^{j}, a_{2413}^{k-1}, a_{241}^{\bar{\bar{2}}}]
\end{array}\right\} \\
+\displaystyle\sum_{\substack{i+j+k<2^n\\ i+k\neq 0}} \overline{\delta}_4 \left\{\begin{array}{c}
[a_{132}, a_{1324}^{k}, a_{2134}^{i}, a_{213}^{j}, a_{132}^{\bar{\bar{2}}-1}] - [a_{243}, a_{2431}^{k}, a_{3241}^{i}, a_{324}^{j}, a_{243}^{\bar{\bar{2}}-1}] \\
-[a_{431}, a_{4312}^{k}, a_{3142}^{i}, a_{314}^{j}, a_{431}^{\bar{\bar{2}}-1}] - [a_{241}, a_{2413}^{k}, a_{4123}^{i}, a_{412}^{j}, a_{241}^{\bar{\bar{2}}-1}]
\end{array}\right\} \\
\qquad +2\displaystyle\sum_{k=1}^{2^n-1} C_{k-1}^{2^n-1} \left\{\begin{array}{c}
-[a_{2134}, a_{213}^{k}, a_{2134}^{2^n-1-k}] + [a_{3241}, a_{324}^{k}, a_{3241}^{2^n-1-k}] \\
+[a_{1324}, a_{134}^{k}, a_{1324}^{2^n-1-k}]
\end{array}\right\} \\
\qquad +\displaystyle\sum_{k=1}^{2^n-1} C_{k-1}^{2^n-1} \left\{\begin{array}{c}
+[a_{2413}, a_{243}^{k}, a_{2413}^{2^n-1-k}] - [a_{1324}, a_{132}^{k}, a_{1324}^{2^n-1-k}] \\
-[a_{2413}, a_{241}^{k}, a_{2413}^{2^n-1-k}] - [a_{4312}, a_{431}^{k}, a_{4312}^{2^n-1-k}]
\end{array}\right\}
\end{array}\right\} \tag{4.18}$$

where,

$$\overline{\delta}_3 = \frac{(2^n-1)!(2^n)}{k!j!(i-1)!(\bar{\bar{2}})!(i+k)}, \overline{\delta}_4 = \frac{(2^n-1)!(2^n)}{i!j!k!(\bar{\bar{2}}-1)!(2^n-i-j)}.$$

We notice that $\overline{\delta}_3$ is not divisible by 2, when $i+k=2^n$. Moreover, $\overline{\delta}_4$ is not divisible by 2, when $i=j=0$. So we decompose (4.18), into two summands

$$\left\{\begin{array}{l}
\displaystyle\sum_{\substack{i\neq 0, i+j+k\leq 2^n\\ i+k\neq 2^n,\, k\neq 0}} \delta_3 \left\{\begin{array}{c}
[a_{1324}, a_{2134}^{i}, a_{213}^{j}, a_{1324}^{k-1}, a_{132}^{\bar{\bar{2}}}] - [a_{2431}, a_{3241}^{i}, a_{324}^{j}, a_{2431}^{k-1}, a_{243}^{\bar{\bar{2}}}] \\
-[a_{4312}, a_{3142}^{i}, a_{314}^{j}, a_{4312}^{k-1}, a_{431}^{\bar{\bar{2}}}] - [a_{2413}, a_{4123}^{i}, a_{412}^{j}, a_{2413}^{k-1}, a_{241}^{\bar{\bar{2}}}]
\end{array}\right\} \\
+\displaystyle\sum_{\substack{i+j+k<2^n\\ i+k\neq 0,\, i+j\neq 0}} \delta_4 \left\{\begin{array}{c}
[a_{132}, a_{1324}^{k}, a_{2134}^{i}, a_{213}^{j}, a_{132}^{\bar{\bar{2}}-1}] - [a_{243}, a_{2431}^{k}, a_{3241}^{i}, a_{324}^{j}, a_{243}^{\bar{\bar{2}}-1}] \\
-[a_{431}, a_{4312}^{k}, a_{3142}^{i}, a_{314}^{j}, a_{431}^{\bar{\bar{2}}-1}] - [a_{241}, a_{2413}^{k}, a_{4123}^{i}, a_{412}^{j}, a_{241}^{\bar{\bar{2}}-1}]
\end{array}\right\} \\
\qquad +\displaystyle\sum_{k=1}^{2^n-1} C_{k-1}^{2^n-1} \left\{-[a_{2134}, a_{213}^{k}, a_{2134}^{2^n-1-k}] + [a_{3241}, a_{324}^{k}, a_{3241}^{2^n-1-k}]\right\}
\end{array}\right\} \tag{4.19}$$

and

$$
\begin{aligned}
&\tfrac{1}{2}\sum_{i=1}^{2^n-1} C_{i-1}^{2^n-1}\left\{\begin{array}{c}[a_{1324}, a_{2134}^{i}, a_{1324}^{2^n-1-i}] - [a_{2431}, a_{3241}^{i}, a_{2431}^{2^n-1-i}] \\ -[a_{4312}, a_{3142}^{i}, a_{4312}^{2^n-1-i}] - [a_{2413}, a_{4123}^{i}, a_{2413}^{2^n-1-i}]\end{array}\right\} \\
&+\tfrac{1}{2}\sum_{k=1}^{2^n-1} C_{k}^{2^n-1}\left\{\begin{array}{c}[a_{132}, a_{1324}^{k}, a_{132}^{2^n-1-k}] - [a_{243}, a_{2431}^{k}, a_{234}^{2^n-1-k}] \\ -[a_{431}, a_{4312}^{k}, a_{431}^{2^n-1-k}] - [a_{241}, a_{2413}^{k}, a_{241}^{2^n-1-k}\end{array}\right\} \\
&+\tfrac{1}{2}\sum_{k=1}^{2^n-1} C_{k-1}^{2^n-1}\left\{\begin{array}{c}+[a_{2413}, a_{243}^{k}, a_{2413}^{2^n-1-k}] - [a_{1324}, a_{132}^{k}, a_{1324}^{2^n-1-k}] \\ -[a_{2413}, a_{241}^{k}, a_{2413}^{2^n-1-k}] - [a_{4312}, a_{431}^{k}, a_{4312}^{2^n-1-k}]\end{array}\right\}
\end{aligned}
\tag{4.20}
$$

where $\delta_3 = \frac{(2^n-1)!(2^{n-1})}{k!j!(i-1)!(\bar{2})!(i+k)}$ and $\delta_4 = \frac{(2^n-1)!(2^{n-1})}{i!j!k!(\bar{2}-1)!(2^n-i-j)}$.

From (4.20), we obtain

$$
\begin{aligned}
&\tfrac{1}{2}\sum_{i=1}^{2^n-1} C_{i-1}^{2^n-1}\left\{\begin{array}{c}[a_{1324}, a_{2134}^{i}, a_{1324}^{2^n-1-i}] - [a_{2431}, a_{3241}^{i}, a_{2431}^{2^n-1-i}] \\ -[a_{4312}, a_{3142}^{i}, a_{4312}^{2^n-1-i}] - [a_{2413}, a_{4123}^{i}, a_{2413}^{2^n-1-i}]\end{array}\right\} \\
&+\tfrac{1}{2}\sum_{k=1}^{2^n-1} C_{k-1}^{2^n-1}\{2[a_{2413}, a_{243}^{k}, a_{2413}^{2^n-1-k}] - 2[a_{1324}, a_{132}^{k}, a_{1324}^{2^n-1-k}]\}
\end{aligned}
\tag{4.21}
$$

From (4.19) and (4.21), we obtain the following two summands:

$$
\frac{1}{2}\sum_{i=1}^{2^n-1} C_{i-1}^{2^n-1}\left\{\begin{array}{c}[a_{1324}, a_{2134}^{i}, a_{1324}^{2^n-1-i}] - [a_{2431}, a_{3241}^{i}, a_{2431}^{2^n-1-i}] \\ -[a_{4312}, a_{3142}^{i}, a_{4312}^{2^n-1-i}] - [a_{2413}, a_{4123}^{i}, a_{2413}^{2^n-1-i}]\end{array}\right\}
\tag{4.22}
$$

and

$$
\left\{\begin{array}{l}
\displaystyle\sum_{\substack{i\neq 0, i+j+k\leq 2^n \\ i+k\neq 2^n, k\neq 0}} \delta_3\left\{\begin{array}{c}[a_{1324}, a_{2134}^{i}, a_{213}^{j}, a_{1324}^{k-1}, a_{132}^{\bar{\bar{2}}}] - [a_{2431}, a_{3241}^{i}, a_{324}^{j}, a_{2431}^{k-1}, a_{243}^{\bar{\bar{2}}}] \\ -[a_{4312}, a_{3142}^{i}, a_{314}^{j}, a_{4312}^{k-1}, a_{431}^{\bar{\bar{2}}}] - [a_{2413}, a_{4123}^{i}, a_{412}^{j}, a_{2413}^{k-1}, a_{241}^{\bar{\bar{2}}}]\end{array}\right\} \\
\displaystyle+\sum_{\substack{i+j+k<2^n \\ i+k\neq 0, i+j\neq 0}} \delta_4\left\{\begin{array}{c}[a_{132}, a_{1324}^{k}, a_{2134}^{i}, a_{213}^{j}, a_{132}^{\bar{\bar{2}}-1}] - [a_{243}, a_{2431}^{k}, a_{3241}^{i}, a_{324}^{j}, a_{243}^{\bar{\bar{2}}-1}] \\ -[a_{431}, a_{4312}^{k}, a_{3142}^{i}, a_{314}^{j}, a_{431}^{\bar{\bar{2}}-1}] - [a_{241}, a_{2413}^{k}, a_{4123}^{i}, a_{412}^{j}, a_{241}^{\bar{\bar{2}}-1}]\end{array}\right\} \\
\displaystyle\qquad+\sum_{k=1}^{2^n-1} C_{k-1}^{2^n-1}\left\{\begin{array}{c}-[a_{2134}, a_{213}^{k}, a_{2134}^{2^n-1-k}] + [a_{3241}, a_{324}^{k}, a_{3241}^{2^n-1-k}] \\ [a_{2413}, a_{243}^{k}, a_{2413}^{2^n-1-k}] - [a_{1324}, a_{132}^{k}, a_{1324}^{2^n-1-k}]\end{array}\right\}
\end{array}\right\}
\tag{4.23}
$$

by rewriting (4.23), we obtain

$$\left\{ \begin{array}{l} \displaystyle\sum_{\substack{i+j+k\leq 2^n \\ i+k\neq 2^n, i\neq 0}} \delta_3 \left\{ \begin{array}{c} -[a_{2134}, a_{1324}^{k}, a_{213}^{j}, a_{2134}^{i-1}, a_{132}^{\bar{\bar{2}}}] + [a_{3241}, a_{2431}^{k}, a_{324}^{j}, a_{3241}^{i-1}, a_{243}^{\bar{\bar{2}}}] \\ +[a_{3142}, a_{4312}^{k}, a_{314}^{j}, a_{2134}^{i-1}, a_{431}^{\bar{\bar{2}}}] + [a_{4123}, a_{2413}^{k}, a_{412}^{j}, a_{4123}^{i-1}, a_{241}^{\bar{\bar{2}}}] \end{array} \right\} \\ \displaystyle +\sum_{\substack{i+j+k<2^n \\ i+k\neq 0, i+j\neq 0}} \delta_4 \left\{ \begin{array}{c} [a_{132}, a_{1324}^{k}, a_{2134}^{i}, a_{213}^{j}, a_{132}^{\bar{\bar{2}}-1}] - [a_{243}, a_{2431}^{k}, a_{3241}^{i}, a_{324}^{j}, a_{243}^{\bar{\bar{2}}-1}] \\ -[a_{431}, a_{4312}^{k}, a_{3142}^{i}, a_{314}^{j}, a_{431}^{\bar{\bar{2}}-1}] - [a_{241}, a_{2413}^{k}, a_{4123}^{i}, a_{412}^{j}, a_{241}^{\bar{\bar{2}}-1}] \end{array} \right\} \end{array} \right\} \tag{4.24}$$

From (4.22), we get

$$\frac{1}{2}\sum_{j=0}^{2^n-2} C_j^{2^n-1} \left\{ \begin{array}{c} -[a_{2134}, a_{1324}^{2^n-1-j}, a_{2134}^{j}] + [a_{3241}, a_{2431}^{2^n-1-j}, a_{3241}^{j}] \\ +[a_{3142}, a_{4312}^{2^n-1-j}, a_{3142}^{j}] + [a_{4123}, a_{2413}^{2^n-1-j}, a_{4123}^{j}] \end{array} \right\} \tag{4.25}$$

Our final goal is to write

$$\sum_{j=0}^{2^n-2} C_j^{2^n-1} \left\{ \begin{array}{c} -[a_{2134}, a_{1324}^{2^n-1-j}, a_{2134}^{j}] + [a_{3241}, a_{2431}^{2^n-1-j}, a_{3241}^{j}] \\ +[a_{3142}, a_{4312}^{2^n-1-j}, a_{3142}^{j}] + [a_{4123}, a_{2413}^{2^n-1-j}, a_{4123}^{j}] \end{array} \right\}$$

as a linear combination of terms divisible by 2.

Now, the computation in step 3 in our main result, can be applied to the following terms:

$$\sum_{j=0}^{2^n-2} C_j^{2^n-1} \left\{ \begin{array}{c} -[a_{2134}, a_{1324}^{2^n-1-j}, a_{2134}^{j}] + [a_{3241}, a_{2431}^{2^n-1-j}, a_{3241}^{j}] \\ +[a_{3142}, a_{4312}^{2^n-1-j}, a_{3142}^{j}] + [a_{4123}, a_{2413}^{2^n-1-j}, a_{4123}^{j}] \end{array} \right\}$$

to produce the following

$$\sum_{i=0}^{2^n-2} \frac{2(2^{n-1}-1-i)}{i+1} C_i^{2^n-1}[a_{2134}, a_{1324}^{2^n-1-i}, a_{2134}^i]$$
$$\sum_{i=0}^{2^n-2} (C_i^{2^n-1} - (-1)^i)[a_{1324}, a_{4123}^{2^n-1-i}, a_{1324}^i]$$
$$\sum_{i=1}^{2^n-2}\sum_{k=0}^{2^n-2-i} C_i^{2^n}[a_{2134}, a_{4123}^{k+1}, a_{2134}^{2^n-2-i-k}, a_{1324}^i]$$

Hence (4.25) becomes as the following:

$$\begin{gathered}\sum_{i=0}^{2^n-2} \frac{(2^{n-1}-1-i)}{i+1} C_i^{2^n-1}[a_{2134}, a_{1324}^{2^n-1-i}, a_{2134}^i]\\ \frac{1}{2}\sum_{i=0}^{2^n-2} (C_i^{2^n-1} - (-1)^i)[a_{1324}, a_{4123}^{2^n-1-i}, a_{1324}^i]\\ \sum_{i=1}^{2^n-2}\sum_{k=0}^{2^n-2-i} \frac{2^n-1}{2^n-i} C_i^{2^n-1}[a_{2134}, a_{4123}^{k+1}, a_{2134}^{2^n-2-i-k}, a_{1324}^i]\end{gathered} \tag{4.26}$$

However, by Lemma 2.5.5, the integer $C_i^{2^n-1}$ is odd $(0 \le i \le 2^n-1)$, thus the coefficient of $[a_{1324}, a_{4123}^{2^n-1-i}, a_{1324}^i]$ is even, and we can write the coefficient as

$$C_i^{2^n-1} - (-1)^i = 2\alpha_i \tag{4.27}$$

for some integer $\alpha_i$. From (4.26) and (4.27) we obtain

$$\begin{gathered}\sum_{i=0}^{2^n-2} \frac{(2^{n-1}-1-i)}{i+1} C_i^{2^n-1}[a_{2134}, a_{1324}^{2^n-1-i}, a_{2134}^i]\\ \sum_{i=0}^{2^n-2} \alpha_i[a_{1324}, a_{4123}^{2^n-1-i}, a_{1324}^i]\\ \sum_{i=1}^{2^n-2}\sum_{k=0}^{2^n-2-i} \frac{2^n-1}{2^n-i} C_i^{2^n-1}[a_{2134}, a_{4123}^{k+1}, a_{2134}^{2^n-2-i-k}, a_{1324}^i]\end{gathered} \tag{4.28}$$

Thus from (4.16), (4.24) and (4.28), we obtain the following:

$$
\left\{
\begin{array}{l}
\displaystyle\sum_{\substack{i+j+k<2^n \\ j+k\neq 0,\, i\neq 0}} \delta_1 \left\{
\begin{array}{l}
[a_{2134}, a^{i}_{214}, a^{j}_{21}, a^{k}_{213}, a^{\bar{\bar{2}}-1}_{2134}] + [a_{1324}, a^{i}_{134}, a^{j}_{13}, a^{k}_{132}, a^{\bar{\bar{2}}-1}_{1324}] \\
[a_{3214}, a^{i}_{324}, a^{j}_{32}, a^{k}_{321}, a^{\bar{\bar{2}}-1}_{3214}] + [a_{2413}, a^{i}_{243}, a^{j}_{24}, a^{k}_{241}, a^{\bar{\bar{2}}-1}_{2413}] \\
[a_{4312}, a^{i}_{432}, a^{j}_{43}, a^{k}_{431}, a^{\bar{\bar{2}}-1}_{4312}] + [a_{4132}, a^{i}_{413}, a^{j}_{41}, a^{k}_{412}, a^{\bar{\bar{2}}-1}_{4132}]
\end{array}
\right\} \\
+ \displaystyle\sum_{\substack{i+j+k\leq 2^n \\ i+j\neq 0,\, k\neq 0 \\ j+k\neq 2^n}} \delta_2 \left\{
\begin{array}{l}
[a_{213}, a^{\bar{\bar{2}}}_{2134}, a^{i}_{214}, a^{j}_{21}, a^{k-1}_{213}] + [a_{132}, a^{\bar{\bar{2}}}_{1324}, a^{i}_{134}, a^{j}_{13}, a^{k-1}_{132}] \\
[a_{321}, a^{\bar{\bar{2}}}_{3214}, a^{i}_{324}, a^{j}_{32}, a^{k-1}_{321}] + [a_{241}, a^{\bar{\bar{2}}}_{2413}, a^{i}_{243}, a^{j}_{24}, a^{k-1}_{241}] \\
[a_{431}, a^{\bar{\bar{2}}}_{4312}, a^{i}_{432}, a^{j}_{43}, a^{k-1}_{431}] + [a_{412}, a^{\bar{\bar{2}}}_{4123}, a^{i}_{413}, a^{j}_{41}, a^{k-1}_{412}]
\end{array}
\right\} \\
+ \displaystyle\sum_{\substack{i+j+k\leq 2^n \\ i+k\neq 2^n,\, i\neq 0}} \delta_4 \left\{
\begin{array}{l}
-[a_{2134}, a^{k}_{1324}, a^{j}_{213}, a^{i-1}_{2134}, a^{\bar{\bar{2}}}_{132}] + [a_{3241}, a^{k}_{2431}, a^{j}_{324}, a^{i-1}_{3241}, a^{\bar{\bar{2}}}_{243}] \\
+[a_{3142}, a^{k}_{4312}, a^{j}_{314}, a^{i-1}_{2134}, a^{\bar{\bar{2}}}_{431}] + [a_{4123}, a^{k}_{2413}, a^{j}_{412}, a^{i-1}_{4123}, a^{\bar{\bar{2}}}_{241}]
\end{array}
\right\} \\
+ \displaystyle\sum_{\substack{i+j+k<2^n \\ i+k\neq 0,\, i+j\neq 0}} \delta_4 \left\{
\begin{array}{l}
[a_{132}, a^{k}_{1324}, a^{i}_{2134}, a^{j}_{213}, a^{\bar{\bar{2}}-1}_{132}] - [a_{243}, a^{k}_{2431}, a^{i}_{3241}, a^{j}_{324}, a^{\bar{\bar{2}}-1}_{243}] \\
-[a_{431}, a^{k}_{4312}, a^{i}_{3142}, a^{j}_{314}, a^{\bar{\bar{2}}-1}_{431}] - [a_{241}, a^{k}_{2413}, a^{i}_{4123}, a^{j}_{412}, a^{\bar{\bar{2}}-1}_{241}]
\end{array}
\right\} \\
\qquad + \displaystyle\sum_{i=1}^{2^n-2}\sum_{k=0}^{2^n-2-i} \tfrac{2^{n-1}}{2^n-i} C_i^{2^n-1}[a_{2134}, a^{k+1}_{4123}, a^{2^n-2-i-k}_{2134}, a^{i}_{1324}] \\
+ \displaystyle\sum_{i=0}^{2^n-2} \tfrac{(2^{n-1}-1-i)}{i+1} C_i^{2^n-1}[a_{2134}, a^{2^n-1-i}_{1324}, a^{i}_{2134}] + \sum_{i=0}^{2^n-2} \alpha_i [a_{1324}, a^{2^n-1-i}_{4123}, a^{i}_{1324}]
\end{array}
\right\}
\tag{4.29}
$$

Finally, the image of the element (4.29) in $\mathcal{M}^{2^n}M \otimes_G \mathbf{R} \cong H_0(G, \mathcal{M}^{2^n}M)$ is the image of

$$
2^{n-1}\, \underbrace{(1 \circ 1 \circ \cdots \circ 1)}_{2^{n-1}} + Im\pi^{2^n}_{2^n-1}] \otimes e_1 \wedge e_2 \wedge e_3 \wedge e_4
$$

under the connecting homomorphism $H_4(G, \mathrm{coker}\pi^{2^n}_{2^n-1}) \longrightarrow H_0(G, \mathcal{M}^{2^n}M)$. Applying the isomorphism $H_0(G, \mathcal{M}^{2^n}F'_{ab}) \longrightarrow \gamma_{2^n}(F')F'''/[\gamma_{2^n}(F'), F]F'''$, we get

$$
\left\{
\begin{array}{l}
\displaystyle\prod_{(i,j,k)\in\hat{I}_1}
\left\{
\begin{array}{l}
[a_{2134}, a^{i}_{214}, a^{j}_{21}, a^{k}_{213}, a^{\bar{\bar{2}}-1}_{2134}]\,[a_{1324}, a^{i}_{134}, a^{j}_{13}, a^{k}_{132}, a^{\bar{\bar{2}}-1}_{1324}] \\
[a_{3214}, a^{i}_{324}, a^{j}_{32}, a^{k}_{321}, a^{\bar{\bar{2}}-1}_{3214}]\,[a_{2413}, a^{i}_{243}, a^{j}_{24}, a^{k}_{241}, a^{\bar{\bar{2}}-1}_{2413}] \\
[a_{4312}, a^{i}_{432}, a^{j}_{43}, a^{k}_{431}, a^{\bar{\bar{2}}-1}_{4312}]\,[a_{4132}, a^{i}_{413}, a^{j}_{41}, a^{k}_{412}, a^{\bar{\bar{2}}-1}_{4132}]
\end{array}
\right\}^{\delta_1} \\
\displaystyle\prod_{(i,j,k)\in\hat{I}_2}
\left\{
\begin{array}{l}
[a_{213}, a^{\bar{\bar{2}}}_{2134}, a^{i}_{214}, a^{j}_{21}, a^{k-1}_{213}]\,[a_{132}, a^{\bar{\bar{2}}}_{1324}, a^{i}_{134}, a^{j}_{13}, a^{k-1}_{132}] \\
[a_{321}, a^{\bar{\bar{2}}}_{3214}, a^{i}_{324}, a^{j}_{32}, a^{k-1}_{321}]\,[a_{241}, a^{\bar{\bar{2}}}_{2413}, a^{i}_{243}, a^{j}_{24}, a^{k-1}_{241}] \\
[a_{431}, a^{\bar{\bar{2}}}_{4312}, a^{i}_{432}, a^{j}_{43}, a^{k-1}_{431}]\,[a_{412}, a^{\bar{\bar{2}}}_{4123}, a^{i}_{413}, a^{j}_{41}, a^{k-1}_{412}]
\end{array}
\right\}^{\delta_2} \\
\displaystyle\prod_{(i,j,k)\in\hat{I}_3}
\left\{
\begin{array}{l}
[a_{1234}, a^{k}_{1324}, a^{j}_{213}, a^{i-1}_{2134}, a^{\bar{\bar{2}}}_{132}]\,[a_{3241}, a^{k}_{2431}, a^{j}_{324}, a^{i-1}_{3241}, a^{\bar{\bar{2}}}_{243}] \\
[a_{3142}, a^{k}_{4312}, a^{j}_{314}, a^{i-1}_{2134}, a^{\bar{\bar{2}}}_{431}]\,[a_{4123}, a^{k}_{2413}, a^{j}_{412}, a^{i-1}_{4123}, a^{\bar{\bar{2}}}_{241}]
\end{array}
\right\}^{\delta_3} \\
\displaystyle\prod_{(i,j,k)\in\hat{I}_4}
\left\{
\begin{array}{l}
[a_{132}, a^{k}_{1324}, a^{i}_{2134}, a^{j}_{213}, a^{\bar{\bar{2}}-1}_{132}]\,[a_{423}, a^{k}_{2431}, a^{i}_{3241}, a^{j}_{324}, a^{\bar{\bar{2}}-1}_{243}] \\
[a_{431}, a^{k}_{4312}, a^{i}_{3142}, a^{j}_{314}, a^{\bar{\bar{2}}-1}_{431}]\,[a_{421}, a^{k}_{2413}, a^{i}_{4123}, a^{j}_{412}, a^{\bar{\bar{2}}-1}_{241}]
\end{array}
\right\}^{\delta_4} \\
\displaystyle\prod_{i=1}^{2^n-2}\prod_{k=0}^{2^n-2-i} [a_{2134}, a^{k+1}_{4123}, a^{2^n-2-i-k}_{2134}, a^{i}_{1324}]^{\frac{2^{n-1}}{2^n-i}C^{2^n-1}_i} \\
\displaystyle\prod_{i=0}^{2^n-2} \left\{[a_{2134}, a^{2^n-1-i}_{1324}, a^{i}_{2134}]^{\frac{(2^{n-1}-1-i)}{i+1}C^{2^n-1}_i}\,[a_{1324}, a^{2^n-1-i}_{4123}, a^{i}_{1324}]^{\alpha_i}\right\}
\end{array}
\right)
\tag{4.30}
$$

Hence the last part of the theorem follows by replacing the $a$'s in the above expression by their definitions as given on p 58.

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