

SCATTERING OF ELASTIC WAVES BY CYLINDRICAL CAVITIES:
INTEGRAL-EQUATION METHODS
AND
LOW-FREQUENCY MATCHED ASYMPTOTIC EXPANSIONS

by

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A thesis submitted to
THE UNIVERSITY OF MANCHESTER
for the degree of
DOCTOR OF PHILOSOPHY
in
THE FACULTY OF SCIENCE

JUNE, 1986.

DEDICATION

A MES PARENTS, SOEURS ET FRERES.

PREFACE

The author graduated in June 1980 with the "Diplome D'etudes Superieures" in Physics from The University of Science and Technology of Algiers. He joined the Mathematics Department in October 1981 and was awarded the degree of M.Sc. in Applied Mathematics and Fluid Mechanics in October 1982.

The work reported in this thesis was carried out in the Department of Mathematics, between October 1982 and May 1986.

DECLARATION

No portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other University of institution of learning.

ACKNOWLEDGEMENTS

I wish to express my gratitude to my supervisor, Dr. P.A. Martin, for his constant guidance and invaluable help throughout the preparation of this thesis. I would also like to thank Professor F. Ursell for many helpful discussions and Mr. M. Emerson for his help in the use of computer systems.

Finally, my thanks to the Algerian Ministry of Higher Education for their financial support and to Mrs. Gell for typing this manuscript so efficiently.

ABSTRACT

This thesis is concerned with the problem of scattering of time-harmonic stress waves by an infinite cylindrical cavity of arbitrary smooth cross-section, in an otherwise unbounded, homogeneous, isotropic, linearly elastic solid. The direction of propagation is perpendicular to the axis of the cavity so that the elastic solid is in a state of plane strain. The three-dimensional problem thus reduces to a two-dimensional one.

The boundary integral equation method is used. It reduces the problem to solving a Fredholm integral equation of the second kind on the boundary. This can be done in several ways. However, the simplest integral equations fail to have a unique solution at a discrete set of values of the frequency known as the irregular frequencies (I.F.). I.F. are due to the method of solution rather than the nature of the problem as this is known to have a unique solution at all frequencies. The I.F. are also known to arise in acoustics, electromagnetics and water waves, where several methods have been devised to eliminate them.

Integral equations are derived. The solvability of these equations and the problem of I.F. are discussed. Two methods for eliminating I.F. are developed; both are based on modifying the fundamental Green's tensor. These methods are the elastodynamic analogues of methods devised by Ursell (1973) and Jones (1974) for acoustic scattering. Some results pertaining to the numerical solutions of the integral equations for a circular cavity are presented. These show the presence and elimination of I.F. Results for scattering by an elliptical cavity are also presented. These are compared with a low-frequency asymptotic solution to the boundary value problem, which is obtained by using the method of matched

asymptotic expansions, as described by Buchwald (1978); extension of Buchwald's work to cylindrical cavities with hypotrochoidal boundaries is made.

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CHAPTER I

GENERAL INTRODUCTION

The diffraction of elastic waves by inhomogeneities is an important problem in many areas of engineering and geophysics. The applications are numerous, e.g. seismology, fracture mechanics, the theory of ultrasonic inspection of materials and so on. It has been extensively studied in recent years as reviewed by Miklowitz (1960), Datta (1978) and Pao (1983). A historical review of the study of the diffraction of elastic waves is given by Pao & Mow (1973, pp.1-34). They also discuss in detail the engineering applications.

Within the context of linearised theory and for a homogeneous isotropic medium, some exact solutions for very simple geometries exist, e.g. a circular cylinder, half-space, sphere etc.... (see, e.g. Pao & Mow (1973)). For more general geometries, resort is often made to numerical methods of solution. Some of these are the finite-difference method (F.D.M.), the finite-element method (F.E.M.), the null-field method (N.F.M.) or T-matrix approach and the boundary integral equation method (B.I.E.M.).

The F.D.M. and the F.E.M. are based on the discretization of the equations of motion in the elastic domain leading to a linear system of algebraic equations. In both cases, the corresponding matrices are sparse with a banded structure; this is an advantage that can be exploited numerically. In the case of infinite domains, this advantage is offset by the problem of truncating the infinite region and the implementation of the radiation conditions. This problem, however, can be overcome by using a hybrid method. This method is based on the

subdivision of the infinite domain into two subdomains. One finite subdomain, containing the scatterer, where the solution is sought using the F.E.M. and the remaining infinite subdomain, bounded internally by a circle, where the solution is sought using infinite series of eigenfunctions obtained by separation of variables. The two solutions in the subdomains are then completed by matching across their common circular boundary. This hybrid method has recently been used by Datta et al. (1983) for single and multiple scattering of elastic waves in two dimensions.

The B.I.E.M. and the N.F.M. are based on integral representations. These guarantee that the equations of motion and the radiation conditions are satisfied automatically. As opposed to the F.D.M. and F.E.M., the B.I.E.M. and the N.F.M. both lead to full matrices with no particular structure. This disadvantage is offset by the fact that both methods reduce the problem to finding the solution on the boundary and thus reduce the dimension of the problem by one. For harmonic time-dependence, however, the B.I.E.M. suffers from one serious drawback, which is the problem of irregular frequencies (I.F.). These are a discrete set of values of the frequency at which the resulting boundary integral equation (B.I.E.) fails to have a unique solution. This is due to the method of solution rather than the nature of the problem as this is known to have a unique solution at all frequencies. The I.F. are also known to arise in acoustic, electromagnetic and water-wave problems.

It is known that the N.F.M. in acoustics does not suffer from I.F. (see Martin (1982)); a corresponding proof for elastodynamics can probably be constructed. However, the N.F.M. has other difficulties, e.g. it involves solving an infinite system of linear equations. Nevertheless, the N.F.M. has been used extensively to solve various elastodynamic problems (see, e.g. Varadan & Varadan (1980)).

The theory of the B.I.E.M. has been known for some time and is widely developed in many fields. Comprehensive theoretical treatments of the B.I.E.M. are given by Kupradze (1965, 1979) for elastodynamics. Numerical implementations of the B.I.E.M., however, are relatively new. Some of the earliest developments in acoustics and elastodynamics were made by Banaugh & Goldsmith (1963a & b) for the steady case. Since then, many other investigations have been carried out as reviewed by Tanaka (1983) and more recently by Kitahara (1985, pp.1-7).

In this thesis, we shall use the B.I.E.M. In particular, we shall develop methods for overcoming the difficulty of I.F. In order to put our work into context, we begin by describing various methods for eliminating I.F. from the corresponding problems in acoustics. These problems are much simpler because the governing differential equation is simpler (Helmholtz equation) and the unknown function is a scalar. But first, we shall formulate three boundary value problems in acoustics which are relevant to our discussion.

Let D_+ be a bounded scatterer with a smooth surface ∂D . Determine a function u that satisfies

$$(\nabla^2 + k^2)u = 0$$

everywhere in D , the exterior of ∂D , a radiation condition at large distances from D_+ and a boundary condition on ∂D :

$$\begin{aligned} \text{either} \quad & u = f \quad (\text{Dirichlet problem}) \\ \text{or} \quad & \partial u / \partial n = f \quad (\text{Neumann problem}) \\ \text{or} \quad & \partial u / \partial n + \sigma u = f \quad (\text{Robin problem}), \end{aligned}$$

here, $k^2 > 0$ is a given constant, f and σ are given functions defined on ∂D and $\partial / \partial n$ denotes normal differentiation on ∂D away from D_+ . These are

the three basic problems in acoustics. Each problem is uniquely solvable for all real values of k^2 and for any f and σ , with $\text{Im}(\sigma) \leq 0$; see, e.g. Colton & Kress (1983, pp.65-107).

These problems can be solved using the B.I.E.M. This usually requires the derivation and solution of a Fredholm integral equation of the second kind. There are two standard approaches (both familiar from potential theory; see, e.g. Jaswon & Symm (1977)), the direct method and the indirect method. The direct method, based on the use of Green's theorem, formulates the problem in terms of variables which have definite physical meanings, while the indirect method uses variables whose physical meanings are not always clear.

In the indirect method, u is represented as a single layer,

$$u(P) = \int_{\partial D} \mu(q) G(P;q) ds_q \quad (1.1)$$

for the Neumann problem, and as a double layer,

$$u(P) = \int_{\partial D} v(q) \partial G(P;q) / \partial n_q ds_q \quad (1.2)$$

for the Dirichlet problem. Here, the functions μ and v are densities,

$G(P;Q) = \frac{i\pi}{2} H_0^{(1)}(kR)$ and R is the distance between the two points P and Q .

The direct method begins with an application of Green's theorem in D to u and G to give

$$2\pi u(P) = \int_{\partial D} \{u(q) \partial G(P;q) / \partial n_q - G(P;q) \partial u(q) / \partial n_q\} ds_q. \quad (1.3)$$

In both cases, u satisfies the differential equation and the radiation

condition. It remains to satisfy the boundary condition on ∂D . To fix ideas, consider just the Neumann problem. Applying the boundary condition, to (1.1) and (1.3), gives

$$-\pi\mu(p) + \int_{\partial D} \mu(q) \partial G(p;q) / \partial n_p ds_q = f(p), \quad (1.4)$$

$$-\pi u(p) + \int_{\partial D} u(q) \partial G(p;q) / \partial n_q ds_q = \int_{\partial D} G(p;q) f(q) ds_q. \quad (1.5)$$

It is known that both of these integral equations fail to have a unique solution at a discrete set of values of k (see, e.g., Colton & Kress (1983, pp.65-107)). It is these that are the I.F. Similar considerations apply to the Dirichlet and Robin problems.

Various methods for eliminating I.F. have been considered. For the Dirichlet problem, Leis (1965), Brakhage and Werner (1965), and Panich (1965) independently suggested representing u as a linear combination of a single layer and a double layer. With a suitable condition on the coupling constant, this, indeed, led to a uniquely solvable integral equation. When the same approach is used for the Neumann problem, it also yields a uniquely solvable integral equation, but this equation has a highly singular kernel. This is due to the presence of the normal derivative of the double layer which in general does not exist on the boundary. Various techniques have been devised in an attempt to obviate this difficulty. Panich (1965) proposed a method based on the replacement of the double layer distribution by another distribution which involves the static fundamental solution. This, he showed, resulted in a kernel which is only weakly singular. Kussmaul (1969) employs the combined single and double layer representation and then uses a regularization technique.

Another approach, for eliminating I.F., proposed by Burton & Miller (1971) for the Neumann problem, is based on the use of (1.3). They prove that a suitable linear combination of (1.5) and the normal derivative of (1.3) on ∂D , is free from the I.F. This also leads to a highly singular kernel, similar to that of the previous approach. Burton & Miller presented a procedure for weakening this singularity. Later Burton (1973) extended this approach to Dirichlet and Robin problems. Meyer et al. (1978) and Terai (1980) used Burton & Miller's (1973) approach and proposed other ways of dealing with the highly singular kernel.

Schenck (1968) suggested, in addition to (1.5), applying the corresponding interior formula at a finite but selected number of interior points. This leads to an overdetermined but consistent system of linear equations. No satisfactory criterion for choosing the interior points was proposed.

Finally, Ursell (1973) suggested an approach based on the modification of the fundamental solution G . He added to G an infinite series of multipoles such that the resulting modified fundamental solution satisfies a dissipative type of condition on a boundary inside the scatterer. This, indeed, frees the boundary integral equation from all the I.F. and moreover does not involve a strongly singular kernel. It presents, however, the disadvantage of adding an infinite number of terms involving calculations of complicated expressions for the coefficients of the multipoles. Jones (1974) proposed a modification to Ursell's approach by replacing the infinite series with a finite one whose coefficients are required to satisfy only some mild conditions. However, uniqueness of the solution of the boundary integral equation is guaranteed only for a restricted range of the frequency. Later Ursell (1978) simplified the proof of a key theorem in Jones' (1974) approach. Kleinman & Roach (1982)

extended this method to three dimensions and suggested various criteria for choosing the coefficients of the multipoles. Based on this approach, Sevat (1982) carried out numerical calculations in two dimensions and demonstrated the effect of the position of the multipoles inside the scatterer. A more detailed discussion of these methods along with some references to their numerical applications are given in Colton & Kress (1983, pp.90-97) and Rizzo et al. (1985).

Similar techniques to overcome the problem of I.F. in electromagnetics are also available. The combined single and double layer approach is discussed in Colton & Kress (1983, pp.140-146), while the Jones-Ursell approach has recently been considered by Brandt et al. (1985).

In water waves, several methods have also been devised for overcoming the problem of I.F. Some of these are similar to those in acoustics and others are not. For a review of these methods see Mei (1978), Ursell (1981) and Martin (1985).

In elastodynamics, Kobayashi & Nishimura (1982) used Jones' (1974) method for two-dimensional problems. Later Jones (1984) extended the Jones-Ursell method to three-dimensional problems. He considered, however, an infinite series of multipoles. Takakuda (1984) adopted the analogue of Burton & Miller's (1971) approach and dealt with the resulting highly singular kernel in the same manner as Burton & Miller. Independently, Jones (1985) also used Burton & Miller's approach and proposed a method for dealing with the singularity which, however, requires some assumptions on the behaviour of the unknown over the boundary. Finally Rizzo et al. (1985) used the analogue of Schenck's (1968) approach but no satisfactory criterion for choosing the interior points was given.

In this thesis we consider the problem of scattering of an incident wave by an infinite cylindrical cavity of arbitrary smooth cross-section,

in an otherwise unbounded, homogeneous, isotropic, linearly elastic solid. The incident wave is supposed to be a time-harmonic stress-wave of frequency ω , propagating perpendicular to the axis of the cylinder so that the elastic solid is in a state of plane strain. The three-dimensional problem is thus reduced to a two-dimensional one.

In Chapter II, the boundary value problem is formulated. In Chapter III, integral representations for the elastic displacement vector are derived, and B.I.E.s are obtained. The solvability of these B.I.E.s as well as the problem of I.F. are discussed. In Chapter IV, a method for overcoming the problem of I.F. is suggested. This method, based on the modification of the fundamental solution, is the analogue of Ursell's (1973) method in acoustics. It involves the addition of an infinite series of multipoles to the fundamental solution such that the modified fundamental solution satisfies a dissipative type of condition on a circular boundary inside the scatterer. This, we show, results in a modified B.I.E. that is uniquely solvable at all frequencies. In Chapter V, another method for eliminating the I.F. is proposed. This method, also based on the modification of the fundamental solution, is the analogue of the Jones-Ursell method in acoustics and Ursell's (1981) method in water waves. It involves the addition of a finite series of multipoles. With some suitable mild conditions on the coefficients of the multipoles, we show that a restricted range of frequencies is freed from I.F. Independently, Jones (1984) adopted this method for three-dimensional elastodynamic problems, while Kobayashi & Nishimura (1982) followed Jones' (1974) approach for two-dimensional problems. Jones (1984), however, uses an infinite series of multipoles and shows that all I.F. are eliminated under some mild restrictions on the coefficients of the multipoles. Kobayashi & Nishimura (1982) use a finite series but assume the coefficients of the multipoles to be equal and independent of

the summation index. With a suitable condition on these coefficients, they show that a restricted range of frequencies is freed from I.F. Their argument, however, is difficult to follow. Our approach involves a more general form for the additional finite series of multipoles. We recover the results of Jones (1984) (when adapted to two-dimensional problems) and Kobayashi & Nishimura (1982) as special cases. Chapter VI is devoted to some numerical results pertaining to the solutions of the modified and non-modified B.I.E.s for the special case of a circular cavity. Two types of incident waves are considered, a P-wave and an S-wave. The numerical results are compared with the exact solution obtained by separation of variables. The existence and the elimination of the I.F. are demonstrated. Some discussions regarding the conditions on the coefficients of the multipoles and the effect of their position inside the scatterer are given. The chapter is concluded with some numerical results for the scattering of a P-wave and an S-wave by an elliptic cavity.

Due to the lack of published numerical solutions with which we could compare our results, we set out to look at low-frequency asymptotic solutions using the method of matched asymptotic expansions. This method, widely used in fluid mechanics, has only recently demonstrated its usefulness in elastodynamics as reviewed by Datta (1978). Buchwald (1978) developed a method for studying the diffraction of elastic waves by a small circular cylindrical cavity in an otherwise unbounded domain. His method involves the derivation of a relationship between the equations of plane elastodynamics and elastostatics. Later Buchwald and Tran Cong (1984) extended this method, using Muskhelishvili (1963) conformal mapping technique, to the diffraction of elastic waves by a small elliptic cylindrical cavity. In Chapter VII, we clarify and systematize some

aspects of Buchwald & Tran Cong's work, and extend this method to the diffraction of elastic waves by a small cylindrical cavity whose arbitrary cross-section can be mapped onto a circle by one of a certain class of mappings. We consider only incident P-waves, although other types of incident waves could be used. For elliptic cavities, we compare the asymptotic formula with our numerical solutions obtained using the B.I.E.M. in Chapter VI; good agreement is found.

CHAPTER II

FORMULATION OF THE PROBLEM

II.1 The Equations of Motion

Consider a linearly elastic solid, of mass density ρ , occupying three-dimensional space. The motion of this body is described by the displacement vector \underline{u} and the stress tensor $\underline{\tau}$ which satisfy two systems of equations, the stress equations of motion

$$\text{div } \underline{\tau} + \rho \underline{f} = \rho \frac{\partial^2 \underline{u}}{\partial t^2} \quad (2.1.1)$$

and the stress-displacement relations, derived through Hooke's law

$$\underline{\tau} = \underline{C} : \text{grad } \underline{u} \quad (2.1.2)$$

where

$\underline{f}(\underline{r})$ is the body force vector,

$\underline{C}(\underline{r})$ is the fourth-rank stiffness tensor of the material

and

$$\underline{r} = (x_1, x_2, x_3) = (x, y, z).$$

In Cartesian components, (2.1.2) reads

$$\tau_{ij} = C_{ijkl} \frac{\partial u_l}{\partial x_k} \quad (2.1.3)$$

where repeated indices imply the Einstein summation convention. Substituting for $\underline{\tau}$ from (2.1.2) into (2.1.1) and assuming, hereafter, that $\underline{f} \equiv 0$ and that the time dependence is $e^{-i\omega t}$, we obtain the equations of Navier

$$\text{div}(\underline{\underline{C}}:\text{grad } \underline{u}) + \rho \omega^2 \underline{u} = \underline{0}. \quad (2.1.4)$$

The elastic stiffness coefficients satisfy the relations

$$C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij}, \quad (2.1.5)$$

which imply

$$\tau_{ij} = \tau_{ji}. \quad (2.1.6)$$

This reduces the number of independent elastic coefficients to 21; if the body is isotropic, just two of them will actually be independent. For an isotropic body the elastic coefficients are usually expressed in terms of the Lamé coefficients λ and μ by the relations

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \quad (2.1.7)$$

and equation (2.1.3) assumes the form

$$\tau_{ij} = \lambda \delta_{ij} \text{div } \underline{u} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right). \quad (2.1.8)$$

If the body is homogeneous ρ and C_{ijkl} are independent of position. Thus, for an isotropic homogeneous body, the Navier equations of motion (2.1.4) become

$$(\lambda + 2\mu) \text{grad}(\text{div } \underline{u}) - \mu \cdot \text{curl}(\text{curl } \underline{u}) + \rho \omega^2 \underline{u} = \underline{0}. \quad (2.1.9)$$

Introducing the wave speeds C_p and C_s , where C_p is the speed of compressional waves in the solid and C_s that of shear waves, we have

$$C_p^2 = (\lambda + 2\mu)/\rho \quad \text{and} \quad C_s^2 = \mu/\rho.$$

Defining the two characteristic wave-numbers, k and K , by

$$k = \omega/C_p \quad \text{and} \quad K = \omega/C_s \quad (2.1.10)$$

equation (2.1.9) takes the form

$$\frac{1}{k^2} \text{grad}(\text{div } \underline{u}) - \frac{1}{K^2} \text{curl}(\text{curl } \underline{u}) + \underline{u} = \underline{0}. \quad (2.1.11)$$

11.2 A Reciprocal Theorem

This section is devoted to the statement and proof of the equivalent of Green's theorem in elastodynamics. This theorem, known as the reciprocal theorem, connects two vector fields \underline{u} and \underline{v} defined in a region D with boundary ∂D . First, we state the theorem

Theorem 2.2.1 (reciprocal theorem)

Let \underline{u} and \underline{v} be two vector fields, with $\underline{\tau}$ and $\underline{\sigma}$ the corresponding stress tensors, such that \underline{u} and \underline{v} and their first and second derivatives are continuous in D . Then

$$\int_{\partial D} \{ \underline{u} \cdot \underline{\sigma} - \underline{v} \cdot \underline{\tau} \} \cdot \underline{\hat{n}} \, ds = - \int_D \{ \underline{u} \cdot \text{div } \underline{\sigma} - \underline{v} \cdot \text{div } \underline{\tau} \} \, dv$$

where $\underline{\hat{n}}$ is the unit normal pointing into D . In particular the right hand side of this equation vanishes when both \underline{u} and \underline{v} satisfy Navier's equations (2.1.4).

$$\begin{aligned} \text{proof: } & \int_D \{ \underline{u} \cdot \text{div } \underline{\sigma} - \underline{v} \cdot \text{div } \underline{\tau} \} \, dv \\ &= \int_D \text{div} \{ \underline{u} \cdot \underline{\sigma} - \underline{v} \cdot \underline{\tau} \} \, dv + \int_D \{ \text{grad } \underline{v} : \underline{\tau} - \text{grad } \underline{u} : \underline{\sigma} \} \, dv \\ &= - \int_{\partial D} \{ \underline{u} \cdot \underline{\sigma} - \underline{v} \cdot \underline{\tau} \} \cdot \underline{\hat{n}} \, ds + \int_D \{ \text{grad } \underline{v} : \underline{\underline{C}} : \text{grad } \underline{u} - \text{grad } \underline{u} : \underline{\underline{C}} : \text{grad } \underline{v} \} \, dv \end{aligned}$$

where we have used the divergence theorem. The second integral on the right hand side vanishes because of the symmetry conditions (2.1.5) on the stiffness tensor. The volume integral is easily seen to vanish if both \underline{u} and \underline{v} satisfy (2.1.4). This completes the proof of the theorem.

In the next section, the reciprocal theorem will be extended to unbounded regions.

II.3 Radiation Conditions

If the domain D is unbounded, restrictions on the behaviour of the scattered components of the displacements at infinity are needed in order to ensure uniqueness. Physically, they are interpreted as representing outgoing waves at infinity. These restrictions, known as the radiation conditions, have been fully discussed by Kupradze (1965, chap. III). With these radiation conditions, the reciprocal theorem can be extended to unbounded regions. The proof has been given for a harmonic time-dependence by Kupradze (1965, Chap. III).

In order to give a precise formulation of the radiation conditions for an isotropic solid, write

$$\underline{u} = \underline{u}' + \underline{u}''$$

where \underline{u}' , \underline{u}'' are determined uniquely from (2.1.11) by

$$\underline{u}' = -\frac{1}{k^2} \text{grad}(\text{div } \underline{u}) \quad \underline{u}'' = \frac{1}{k^2} \text{curl}(\text{curl } \underline{u})$$

and hence satisfy $\text{curl } \underline{u}' = 0$, $\text{div } \underline{u}'' = 0$. \underline{u}' and \underline{u}'' are respectively called the potential and solenoidal components of the vector \underline{u} . They satisfy the following Helmholtz equations

$$(\nabla^2 + k^2)\underline{u}'(\underline{r}) = \underline{0}, \quad (\nabla^2 + k^2)\underline{u}''(\underline{r}) = \underline{0}.$$

It can be shown that each component must satisfy conditions similar to those given by Sommerfeld for the scalar Helmholtz equation (see Jones (1964), p.56) in order to preserve uniqueness. These are

$$\lim_{r \rightarrow \infty} \underline{u}'(\underline{r}) = \underline{0}, \quad (2.3.1)$$

$$\lim_{r \rightarrow \infty} r^{\frac{n-1}{2}} \left\{ \frac{\partial \underline{u}'(\underline{r})}{\partial r} - i k \underline{u}'(\underline{r}) \right\} = \underline{0}, \quad (2.3.2)$$

$$\lim_{r \rightarrow \infty} \underline{u}''(\underline{r}) = \underline{0} \quad (2.3.3)$$

and

$$\lim_{r \rightarrow \infty} r^{\frac{n-1}{2}} \left\{ \frac{\partial \underline{u}''(\underline{r})}{\partial r} - i k \underline{u}''(\underline{r}) \right\} = \underline{0} \quad (2.3.4)$$

where $r = |\underline{r}|$ and n ($= 2, 3$) denotes the dimension of space. It should be noted that conditions (2.3.1) and (2.3.3) can be derived respectively from (2.3.2) and (2.3.4) (see Colton and Kress (1983, p.71)) and so need not be forced.

11.4 Boundary-Value Problem

Consider a three-dimensional linearly elastic solid of unbounded extent, containing an infinite vertical cylindrical cavity of arbitrary smooth cross-section. The solid is also assumed to be homogeneous and isotropic. Suppose the cavity is irradiated by time-harmonic stress waves of frequency ω , propagating perpendicular to the vertical axis of the cylinder so that the material is in a state of plane strain. The three-dimensional problem, therefore, reduces to a two-dimensional one; i.e. one only needs to consider a plane perpendicular to the vertical axis of the cylindrical cavity (see Fig. 2.4.1).

We shall denote by D the domain formed by the projection of the region occupied by the elastic medium, onto a plane perpendicular to the vertical axis of the cylindrical cavity (see Fig. 2.4.1). D_- and ∂D denote, respectively, the interior of D and the boundary separating D_- and D . A system of Cartesian coordinates is taken with the origin O in D_- as shown in Fig. 2.4.1.

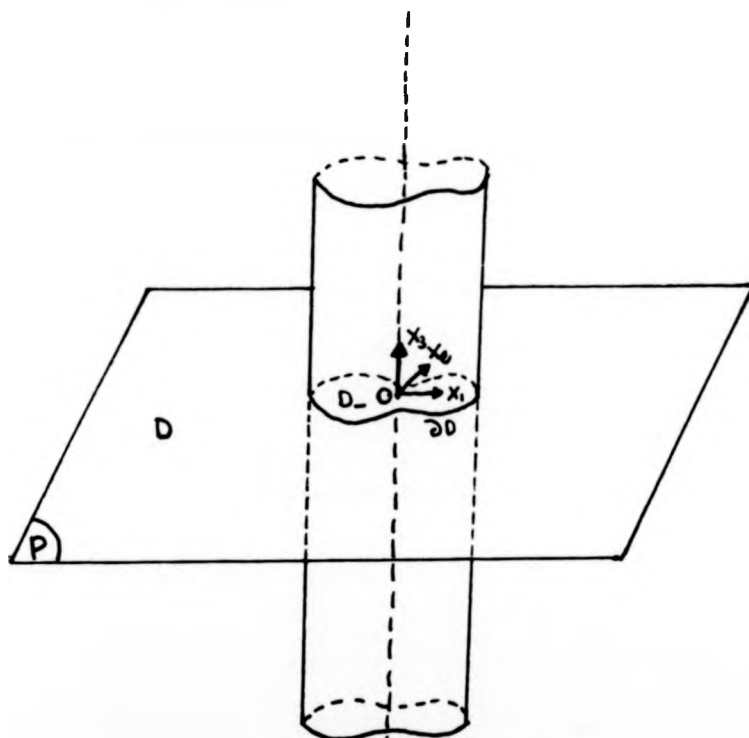


Fig. 2.4.1

We shall also use the following notation:

Capital letters P, Q denote points of D ; lower-case letters p, q denote points of ∂D ; P_-, Q_- denote points of D_- and $\hat{n}(p)$ the unit normal to the boundary at the point p pointing into D . \underline{u}^{inc} and $\underline{\tau}^{inc}$ denote the incident displacements and stresses, respectively.

The decomposition of the total displacement \underline{u} and thus of the total stress $\underline{\tau}$ in the form

$$\underline{u} = \underline{u}^{inc} + \underline{u}^{sc}$$

$$\underline{\tau} = \underline{\tau}^{inc} + \underline{\tau}^{sc}$$

defines the scattered displacements and stresses \underline{u}^{sc} and $\underline{\tau}^{sc}$, respectively. The scattering boundary-value problem can now be formulated.

Suppose the incident field is defined by the function \underline{u}^{inc} which satisfies the equations of motion (2.1.11) everywhere in $D \cup \partial D$, except possibly at a finite number of isolated points in D ; thus, e.g. \underline{u}^{inc} might correspond to a plane wave, or to a line source at a point in D . We wish to determine the scattered field \underline{u}^{sc} subject to the requirement that the total field \underline{u} leaves the boundary stress-free. In terms of equations, this can be stated as follows:

Boundary-value problem: $S(\underline{u}^{inc})$

Determine a function \underline{u}^{sc} for $P \in D$, satisfying

- i) Elastodynamic equations of motion in D

$$\frac{1}{k^2} \text{grad}(\text{div } \underline{u}^{sc}(P)) - \frac{1}{k^2} \text{curl}(\text{curl } \underline{u}^{sc}(P)) + \underline{u}^{sc}(P) = \underline{0} \quad P \in D, \quad (2.4.1)$$

- ii) Stress-free boundary condition on ∂D

$$\underline{T}^{sc}(p) = \underline{\tau}^{sc}(p) \cdot \hat{n}(p) = - \underline{\tau}^{inc}(p) \cdot \hat{n}(p) = - \underline{T}^{inc}(p) \quad P \in \partial D, \quad (2.4.2)$$

where T is defined as the traction operator which acts on $\underline{u}(p)$ with respect to the point p . To avoid ambiguity, we shall write T_p or T_q when

the operator acts on a function of the positions of more than one point. For an isotropic material, T takes the form

$$T \equiv T_p \equiv \lambda \hat{n}(p) \operatorname{div} + 2\mu \frac{\partial}{\partial n_p} + \mu \hat{n}(p) \wedge \operatorname{curl}. \quad (2.4.3)$$

iii) The radiation conditions at infinity, as defined in section 11.3.

It is known that $S(\underline{u}^{\text{inc}})$ has at most one solution:

Uniqueness theorem 1:

The solution, regular in D , of the equation

$$\frac{1}{k^2} \operatorname{grad}(\operatorname{div} \underline{u}) - \frac{1}{k^2} \operatorname{curl}(\operatorname{curl} \underline{u}) + \underline{u} = \underline{0}$$

is identically zero if it satisfies the radiation conditions and one of the following homogeneous conditions on the boundary ∂D :

- i) $\underline{u} = \underline{0}$
- ii) $T\underline{u} = \underline{0}$
- iii) $T\underline{u} + \sigma \underline{u} = \underline{0}$, where $\sigma = \sigma_1 + i\sigma_2$, $\sigma_2 \leq 0$.

proof: This theorem is proved by Kupradze (1979, pp. 132-136) in three dimensions, but the proof remains valid in two dimensions.

Kupradze (1979, pp. 431-437) has also proved an existence theorem for $S(\underline{u}^{\text{inc}})$. Therefore, we can conclude that $S(\underline{u}^{\text{inc}})$ has precisely one solution, for all frequencies.

A standard method for solving $S(\underline{u}^{\text{inc}})$ is the boundary integral equation method. An integral representation of the solution is chosen, from which a boundary integral equation is derived. In the next chapter,

we shall derive integral representations and the corresponding boundary integral equations for $S(\underline{u}^{inc})$ and discuss the solvability of these integral equations.

We conclude this chapter, by formulating an interior homogeneous boundary-value problem, the use of which will be apparent in the next chapter.

Boundary-value problem: I_h

Consider a region R bounded by a curve C . We wish to determine a regular solution, \underline{u} , of

$$\frac{1}{k^2} \text{grad}(\text{div } \underline{u}) - \frac{1}{k^2} \text{curl}(\text{curl } \underline{u}) + \underline{u} = \underline{0}$$

In R , such that $\underline{u} = \underline{0}$ on C . k^2 and k^2 are real and given by (2.1.10).

This is an eigenvalue problem: the solution of I_h is $\underline{u} \equiv \underline{0}$ unless ω^2 takes on one of a discrete set of values (see Kupradze (1979, pp. 405-408)).

CHAPTER III

INTEGRAL REPRESENTATIONS AND THE PROBLEM OF
IRREGULAR FREQUENCIES

III.1 Green's Displacement Tensor

To obtain integral representations for the solution of $S(u^{inc})$, a fundamental singular solution of (2.1.4) is needed. This solution, known as the Green's displacement tensor, is a two-point singular tensor, $\underline{G}^f(P;Q)$, such that $\underline{G}^f(P;Q)$ satisfies (2.1.4), except when $P=Q$, and the radiation conditions. It can be shown (see Tan (1975)) that for a three-dimensional unbounded homogeneous elastic medium, this tensor can, at least in principle, be obtained using Fourier transforms. The two-dimensional form is obtained from its three-dimensional counterpart by performing the integration with respect to the vertical coordinate x_3 (see Tan (1976)). For a homogeneous isotropic solid, this is given by

$$G_{ij}^f(P;Q) = \frac{i}{4\mu} \{ \psi \delta_{ij} + \frac{1}{k^2} \frac{\partial^2 (\psi - \phi)}{\partial x_i \partial x_j} \} \quad (3.1.1)$$

with the corresponding stress tensor given by

$$\Sigma_{ijk}^f(P;Q) = \frac{i}{4} \left\{ \frac{2}{k^2} \frac{\partial^3 (\psi - \phi)}{\partial x_i \partial x_j \partial x_k} + \frac{k^2 - 2k^2}{k^2} \delta_{ij} \frac{\partial \psi}{\partial x_k} + \delta_{jk} \frac{\partial \psi}{\partial x_i} + \delta_{ik} \frac{\partial \psi}{\partial x_j} \right\} \quad (3.1.2)$$

where

$$\psi = H_0^{(1)}(kR)$$

and

$$\phi = H_0^{(1)}(kR),$$

$H_0^{(1)}(.)$ being the zeroth-order Hankel function of the first kind and R the

distance between the points P and Q. Here, δ_{ij} and $\frac{\partial}{\partial x_i}$ denote, respectively, the Kronecker symbol and differentiation with respect to the i^{th} -coordinate of the point P.

III.2 Integral Representations and Boundary Integral Equations

Having defined the fundamental solution, the integral representations for the solution of $S(\underline{u}^{\text{inc}})$ can now be derived. There are two methods for obtaining these integral representations, the direct method and the indirect method. The direct method, based on the use of the reciprocal theorem, formulates the problem in terms of variables which have definite physical meanings, while the indirect method uses variables whose physical meaning are not always clear. Both methods will be used.

We begin with the indirect method, which was extensively used by Kupradze (1965 and 1979). The scattered displacement is represented as an elastic single layer,

$$\underline{u}^{\text{sc}}(P) = \int_{\partial D} \underline{G}^f(P; q) \cdot \underline{p}(q) ds_q, \quad P \in D, \quad (3.2.1)$$

where $\underline{p}(q)$ is a Hölder-continuous unknown density defined on ∂D . However, it should be noted, as we shall see later, that this representation is not always valid (see Kupradze (1965, p. 41)). This representation satisfies equation (2.4.1), the radiation conditions and is continuous in $D \cup \partial D_-$ (see Kupradze (1965, p.33)). If we implement the boundary condition (2.4.2) and use the jump property of the elastic single layer as P approaches ∂D (see Kupradze (1965, p. 35)), we obtain

$$\underline{T}\underline{u}^{\text{sc}}(p) = -\frac{1}{2}\underline{p}(p) + \int_{\partial D} \underline{T}\underline{G}^f(p; q) \cdot \underline{p}(q) ds_q = -\underline{T}\underline{u}^{\text{inc}}(p) \quad p, q \in \partial D. \quad (3.2.2)$$

This is an integral equation of the second kind for the values of $\underline{\rho}$. Once $\underline{\rho}(q)$ has been found, the displacement at any point in D is given by (3.2.1).

In the direct method, the reciprocal theorem is applied to the solution \underline{u}^{SC} of $S(\underline{u}^{inc})$ and \underline{G}^f in the region between ∂D and a large circle S_∞ of radius r_∞ and centre O (see Fig. 3.2.1).

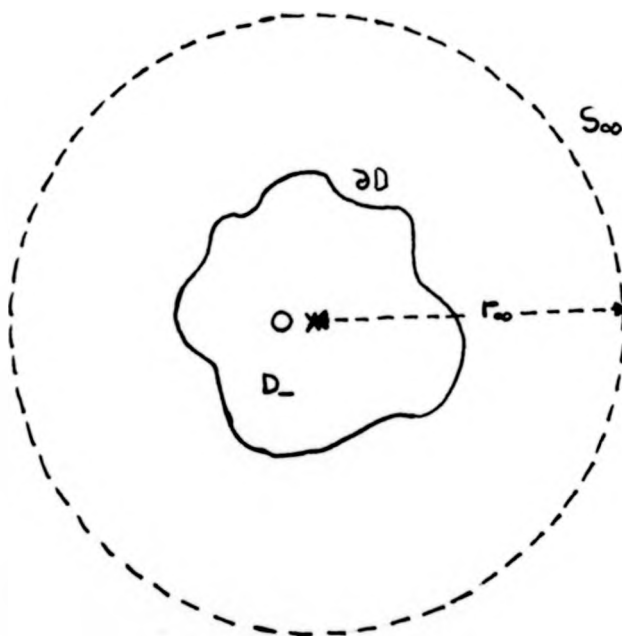


Fig. 3.2.1

Since \underline{G}^f and \underline{u}^{SC} both satisfy the radiation conditions at infinity, the contribution from integrating over S_∞ vanishes as $r_\infty \rightarrow \infty$ (see Kupradze (1965, p. 49)). Hence we obtain

$$\underline{u}^{SC}(P) = \int_{\partial D} \{ \underline{u}^{SC}(q) \cdot T_q \underline{G}^f(q; P) - T_q \underline{u}^{SC}(q) \cdot \underline{G}^f(q; P) \} ds_q, \quad (3.2.3)$$

$$\underline{u}^{SC}(P) = \int_{\partial D} \{ \underline{u}^{SC}(q) \cdot T_q \underline{G}^f(q; P) - T_q \underline{u}^{SC}(q) \cdot \underline{G}^f(q; P) \} ds_q, \quad (3.2.4)$$

$$\underline{0} = \int_{\partial D} \{ \underline{u}^{sc}(q) \cdot T_{q\equiv} \underline{G}^f(q; P_-) - T \underline{u}^{sc}(q) \cdot \underline{G}^f(q; P_-) \} ds_q. \quad (3.2.5)$$

Similarly, we apply the reciprocal theorem in D_- to \underline{G}^f and \underline{u}^{inc} . Since \underline{u}^{inc} is regular and satisfies (2.4.1) in D_- , we obtain

$$\underline{0} = \int_{\partial D} \{ \underline{u}^{inc}(q) \cdot T_{q\equiv} \underline{G}^f(q; P) - T \underline{u}^{inc}(q) \cdot \underline{G}^f(q; P) \} ds_q, \quad (3.2.6)$$

$$-\frac{1}{2} \underline{u}^{inc}(P) = \int_{\partial D} \{ \underline{u}^{inc}(q) \cdot T_{q\equiv} \underline{G}^f(q; P) - T \underline{u}^{inc}(q) \cdot \underline{G}^f(q; P) \} ds_q, \quad (3.2.7)$$

$$-\underline{u}^{inc}(P_-) = \int_{\partial D} \{ \underline{u}^{inc}(q) \cdot T_{q\equiv} \underline{G}^f(q; P_-) - T \underline{u}^{inc}(q) \cdot \underline{G}^f(q; P_-) \} ds_q. \quad (3.2.8)$$

Adding equations (3.23-3.25) to equations (3.2.6-3.2.8) respectively, and implementing the boundary condition (2.4.2) yields

$$\underline{u}^{sc}(P) = \int_{\partial D} \underline{u}(q) \cdot T_{q\equiv} \underline{G}^f(q; P) ds_q, \quad (3.2.9)$$

$$\frac{1}{2} \underline{u}(P) - \int_{\partial D} \underline{u}(q) \cdot T_{q\equiv} \underline{G}^f(q; P) ds_q = \underline{u}^{inc}(P), \quad (3.2.10)$$

$$\underline{u}^{inc}(P_-) + \int_{\partial D} \underline{u}(q) \cdot T_{q\equiv} \underline{G}^f(q; P_-) ds_q = \underline{0}. \quad (3.2.11)$$

Equation (3.2.9) is an integral representation for \underline{u}^{sc} as an elastic double layer, whilst equation (3.2.10) is a boundary integral equation of the second kind for the boundary values of the total displacement \underline{u} . Once $\underline{u}(q)$ has been found, the displacement at any point in D is given by (3.2.9). As for the last equation (3.2.11), it asserts that the field induced at any point in D_- by the elastic double layer is exactly cancelled by the field induced by the incident wave \underline{u}^{inc} .

In the next section, we shall discuss the solvability of the boundary integral equations (3.2.2) and (3.2.10).

III.3 Solvability of the Boundary Integral Equations

Consider the two boundary integral equations given by (3.2.2) and (3.2.10), which we rewrite as follows:

$$\frac{1}{2}\underline{\rho}(p) - \int_{\partial D} \underline{K}(p;q) \cdot \underline{\rho}(q) ds_q = T\underline{u}^{inc}(p) \quad (3.3.1)$$

and

$$\frac{1}{2}\underline{u}(p) - \int_{\partial D} \underline{u}(q) \cdot \underline{K}(q;p) ds_q = \underline{u}^{inc}(p), \quad (3.3.2)$$

where

$$\underline{K}(p;q) = T_p G^f(p;q).$$

The kernels of the boundary integral equations (3.3.1) and (3.3.2) are singular. We isolate the singularity by rewriting $\underline{K}(p;q)$, say, as

$$\underline{K}(p;q) = \underline{K}^o(p;q) + \underline{K}^*(p;q), \quad (3.3.3)$$

where $\underline{K}^o(p;q)$ is continuous for $(p;q) \in \partial D \times \partial D$ and $\underline{K}^*(p;q)$ is the singular part, given by (see Appendix A1)

$$K_{ij}^*(p;q) = \frac{\tau^2}{2\pi} \left(\hat{n}_i(p) \frac{\partial \ln R}{\partial x_j} - \hat{n}_j(p) \frac{\partial \ln R}{\partial x_i} \right), \quad (3.3.4)$$

with

$$\tau = k/K$$

and $\hat{n}_i(p)$ is the i^{th} -component of the unit normal at the point p . Let s_p denote the distance measured, along the boundary and in the anti-clockwise direction, from some fixed point on ∂D to the point p . From (3.3.4), it can be seen that the singularity of the kernels is of a Cauchy type. Therefore, the integrals in (3.3.1) and (3.3.2) do not converge in the ordinary sense. However, under the assumptions that $\underline{\rho}(q)$, $\underline{u}(q)$ and $(s_p - s_q)\underline{K}(p;q)$ are Hölder-continuous on the boundary, which itself is

assumed to be twice-differentiable, these integrals can be interpreted in the Cauchy principal sense (see Smirnov (1964 , pp. 103-119)).

As pointed out in the previous section, (3.3.1) and (3.3.2) are integral equations of the second kind. It is known that in order to discuss the solvability of this type of equation, the use of Fredholm theorems is required. However, these theorems can only be applied to a certain class of integral equations, namely the Fredholm-type equation. A Fredholm-type equation has a kernel $\underline{\underline{K}}^F(p;q)$, say, which satisfies the condition

$$\lim_{p \rightarrow q} R \underline{\underline{K}}^F(p;q) = \underline{\underline{0}}. \quad (3.3.5)$$

Since the singularity of our kernels is of a Cauchy type, clearly these will not satisfy condition (3.3.5). It follows, therefore, that (3.3.1) and (3.3.2) are not Fredholm-type equations. Thus, it appears that the Fredholm theorems are not applicable. However, it can be shown (see Muskhelishvili (1953, Chap. 19)) that the use of the Fredholm theorems can be extended to a certain class of Cauchy-type singular integral equations. To this end, we rewrite (3.3.1) as

$$\underline{\underline{A}}(p) \cdot \underline{\underline{p}}(p) + \frac{\underline{\underline{B}}(p)}{\pi i} \int_{\partial D} \frac{\underline{\underline{p}}(q)}{(s_q - s_p)} ds_q - \int_{\partial D} \underline{\underline{N}}(p;q) \cdot \underline{\underline{p}}(q) ds_q = T \underline{\underline{u}}^{inc}(p), \quad (3.3.6)$$

where

$$\underline{\underline{A}}(p) = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (3.3.7)$$

$$\underline{\underline{N}}(p;q) = (\underline{\underline{L}}(p;q) - \underline{\underline{L}}(p;p)) / (s_q - s_p),$$

$$\underline{\underline{L}}(p;q) = (s_q - s_p) \cdot \underline{\underline{K}}(p;q)$$

and

$$\begin{aligned}
\underline{\underline{B}}(p) &= -\pi i \underline{\underline{L}}(p;p) = -\pi i \lim_{q \rightarrow p} (s_q - s_p) \underline{\underline{K}}(p;q) \\
&= -\pi i \lim_{q \rightarrow p} (s_q - s_p) \underline{\underline{K}}^*(p;q) \\
&= \frac{i}{2} \tau^2 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\end{aligned} \tag{3.3.8}$$

Note $\underline{\underline{A}}(p)$ and $\underline{\underline{B}}(p)$ are actually independent of p . At this point we introduce an integer, known as the index of the integral equation (3.3.6), which plays an important role in the theory of singular integral equation. This is denoted and defined by (see Muskhelishvili (1953) p. 419))

$$\chi = \frac{1}{2\pi} \left[\arg \left\{ \frac{\det(\underline{\underline{A}} - \underline{\underline{B}})}{\det(\underline{\underline{A}} + \underline{\underline{B}})} \right\} \right]_{\partial D}, \tag{3.3.9}$$

where $\det(\cdot)$ denotes the determinant and $[\cdot]_{\partial D}$ the increment, suffered by the function in braces, on one circuit of ∂D in the anti-clockwise direction. It is known (see Muskhelishvili (1953, Chap. 6. §56)) that if the index of an integral equation is zero, then the Fredholm theorems hold. For Fredholm-type equation, the index is zero and it is for this reason that singular integral equations with zero index are known as Quasi-Fredholm integral equations. Using (3.3.7), (3.3.8) and (3.3.9) it can clearly be seen that the index of (3.3.6) is zero. Similarly, we can show that the index of (3.3.2) is also zero. Therefore, the Fredholm theorems can be applied to (3.3.1) and (3.3.2).

III.4 The Irregular Frequencies

The boundary integral equations (3.3.1) and (3.3.2) are adjoint integral equations. It follows, from the Fredholm theorems, that if one of them is uniquely solvable so is the other one. Therefore, only one of them needs to be discussed. Furthermore, the Fredholm alternative

stipulates that for the inhomogeneous equation to be uniquely solvable, the corresponding homogeneous equation must have trivial solutions only. Consider, therefore, the homogeneous equation corresponding to (3.3.1),

$$\frac{1}{2}\underline{v}(p) - \int_{\partial D} \underline{K}(p;q) \cdot \underline{v}(q) ds_q = \underline{0}. \quad (3.4.1)$$

If (3.4.1) has no non-trivial solutions, it follows that (3.3.1) is uniquely solvable for every suitable $T\underline{u}^{inc}(p)$ and consequently (3.3.2) is also uniquely solvable for every suitable $\underline{u}^{inc}(p)$.

Suppose now that (3.4.1) has a non-trivial solution $\underline{p}(p)$ so that

$$\frac{1}{2}\underline{p}(p) - \int_{\partial D} \underline{K}(p;q) \cdot \underline{p}(q) ds_q = \underline{0}. \quad (3.4.2)$$

Then the corresponding representation $\underline{u}(P)$ defined by (3.2.1) has a vanishing traction $T\underline{u}(p) = \underline{0}$; thus, by the uniqueness theorem 1 (see section II.4), the exterior displacement $\underline{u}(P)$ vanishes everywhere in D . In particular, $\underline{u}(p)$ vanishes on ∂D . Consider now the displacement $\underline{u}(P_-)$ defined by (3.2.1) but with P_- lying in D_- . Since across an elastic single layer the displacement is known to be continuous, it follows that $\underline{u}(p_-) = \underline{u}(p) \equiv \underline{0}$ on ∂D . If now ω^2 is not an eigenvalue of I_h (see section II.4), it follows from $\underline{u}(p_-) = \underline{0}$ that $\underline{u}(P_-) = \underline{0}$ for all $p_- \in D_-$, and in particular that, on ∂D ,

$$T\underline{u}(p_-) = \frac{1}{2}\underline{p}(p) + \int_{\partial D} \underline{K}(p;q) \cdot \underline{p}(q) ds_q = \underline{0}. \quad (3.4.3)$$

Adding (3.4.2) to (3.4.3) yields $\underline{p} \equiv \underline{0}$. This contradicts the assumption that $\underline{p} \neq \underline{0}$, and it follows that (3.4.1) is non-singular, unless ω^2 is an eigenvalue of I_h . Thus, (3.3.1) and (3.3.2) fail to be uniquely solvable whenever ω^2 coincides with an eigenvalue of I_h . These values of the frequency are known as the irregular frequencies.

Summing up what has been said so far, we have the following:

If ω^2 is not an irregular frequency, the elastic single layer (3.2.1) solves $S(\underline{u}^{inc})$, where \underline{u} solves (3.3.1). Similarly, the elastic double layer (3.2.9) also solves $S(\underline{u}^{inc})$, where \underline{u} solves (3.3.2); this is not obvious, as it is necessary to verify that the boundary condition (2.4.2) is satisfied, but this can be shown by adapting the argument given by Kleirman and Roach (1974).

If ω^2 is an irregular frequency, then, as mentioned in section III.2, the representation (3.2.1) fails to remain valid, whereas it can be shown that (3.2.9) is still valid (see section IV.2), even though we cannot now obtain $\underline{u}(q)$ by solving (3.3.2).

We conclude this chapter, with the following remark: Despite the failure of the boundary integral equations (3.3.1) and (3.3.2) to be uniquely solvable at the irregular frequencies, the physical problem is known to have a unique solution (see section II.4) at all values of the frequency. The irregular frequencies are, therefore, unphysical. They arise from the method of solution. In the next two chapters, we shall show how this difficulty can be overcome, in two ways, yielding integral equations which are uniquely solvable at all frequencies.

CHAPTER IV

MODIFIED GREEN'S TENSOR USING A DISSIPATIVE BOUNDARY

IV.1 Modified Theory

In this chapter, we describe the first method which renders the boundary integral equations non-singular. The irregular frequencies no longer appear. This method, first developed for exterior problems in acoustics by Ursell (1973), is extended here to exterior problems in elastodynamics. It is based on the modification of the fundamental solution and its replacement by a new Green's tensor which satisfies all the conditions of the non-modified tensor and an additional condition, similar to that of Ursell (1973), on a boundary inside D_- .

Let (r, θ) denote the polar coordinates of any point in $D \cup \partial D \cup D_-$. Consider a circle C of centre O and radius a such that C lies inside the inscribed circle to D (see Fig. 4.1.1).

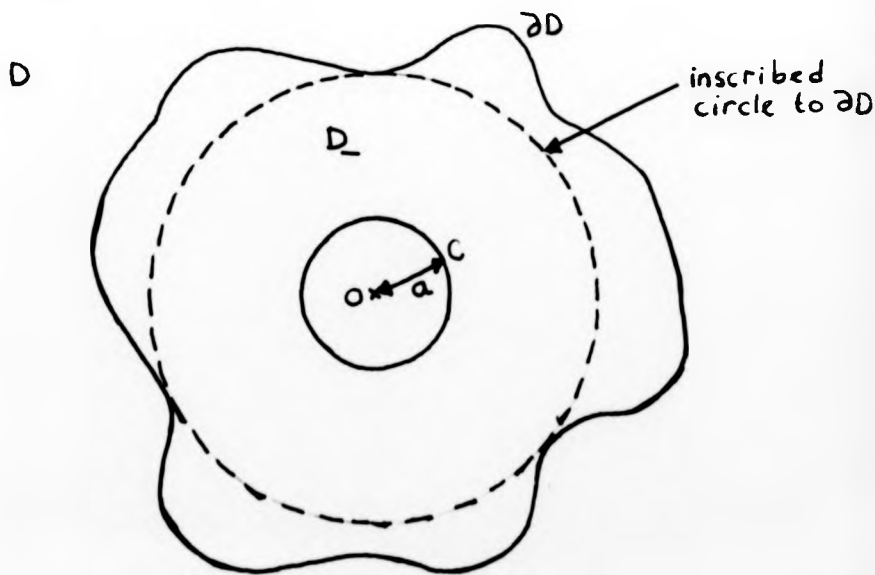


Fig. 4.1.1

We denote the region $r > a$ and extending to infinity by D_a . Then the modified Green's tensor is chosen to be

$$\underline{\underline{G}}^1(P;Q) = \underline{\underline{G}}^f(P;Q) + \underline{\underline{I}}(P;Q), \quad (4.1.1)$$

where $\underline{\underline{I}}(P;Q)$ is assumed to be a regular solution of (2.4.1) in $D_a \cup C$. Note that $\underline{\underline{I}}(P;Q)$ may have singularities inside C . $\underline{\underline{G}}^1(P;Q)$ is required to satisfy the radiation condition at infinity, and also the additional boundary condition

$$T_P \underline{\underline{G}}^1(P;Q) + \underline{\underline{F}} \cdot \underline{\underline{G}}^1(P;Q) = \underline{\underline{0}} \quad P \in C, \quad (4.1.2)$$

where $\underline{\underline{F}}$ is an arbitrary second-order tensor satisfying some suitable conditions which will be specified in due course. Note that $\underline{\underline{F}}$ may depend on P , although it will be assumed constant for the sake of simplicity.

We end this section by looking at the symmetry properties of the non-modified tensor $\underline{\underline{G}}^f$ and discuss the conditions under which the modified tensor $\underline{\underline{G}}^1$ can satisfy, if at all, some or all of these symmetry properties. Note, however, that some of these symmetry properties may not be necessary,

From (3.1.1) it can readily be seen that $\underline{\underline{G}}^f$ satisfies

$$G_{ij}^f(P;Q) = G_{ji}^f(Q;P) \quad (4.1.3)$$

and

$$G_{ij}^f(P;Q) = G_{ij}^f(Q;P). \quad (4.1.4)$$

We can ensure that $\underline{\underline{G}}^1$ also satisfies (4.1.3) if we require that $\underline{\underline{F}}$ be symmetric. To see this, apply the reciprocal theorem to $\underline{\underline{G}}^1(Q;P_1)$ and

$\underline{G}^1(Q;P_2)$ in D_a , where P_1 and P_2 are two distinct points in D_a . We get

$$\begin{aligned} G_{k1}^1(P_2;P_1) - G_{1k}^1(P_1;P_2) \\ = \int_C \{G_{i1}^1(Q;P_1)\Sigma_{ijk}^1(Q;P_2) - G_{ik}^1(Q;P_2)\Sigma_{ij1}^1(Q;P_1)\}\hat{n}_j(Q)ds_q. \end{aligned}$$

Using (4.1.2), leads to

$$G_{k1}^1(P_2;P_1) - G_{1k}^1(P_1;P_2) = (F_{ij} - F_{ji}) \int_C G_{ik}^1(Q;P_2)G_{j1}^1(Q;P_1)ds_q.$$

It can be seen that if we choose \underline{F} so that

$$F_{ij} = F_{ji}, \quad (4.1.5)$$

this will yield

$$G_{k1}^1(P_1;P_2) = G_{1k}^1(P_2;P_1), \quad (4.1.6)$$

and this is the same symmetry property as (4.1.3) satisfied by \underline{G}^f . Note that this symmetry implies that $\underline{G}^1(P;Q)$ will satisfy the governing equations of motion and the radiation conditions with respect to both P and Q . Since we shall require this property, condition (4.1.5) will therefore be imposed. Note that we are not able to say anything a priori about the second property, (4.1.4) indeed, we shall show that our particular \underline{G}^1 (corresponding to $F_{ij} = F_{ji}$), constructed in section IV.3, does not satisfy (4.1.4).

IV.2 Modified Boundary Integral Equations

Having defined the new Green's tensor, the integral representations and the corresponding boundary integral equations are modified as follows:

In the indirect method, equation (3.2.1) becomes

$$\underline{u}^{SC}(P) = \int_{\partial D} \underline{G}^1(P; q) \cdot \underline{e}(q) ds_q \quad P \in D, \quad (4.2.1)$$

where the fundamental solution \underline{G}^f has been replaced by the new tensor \underline{G}^1 . Since $\underline{r}(P; Q)$ is a regular solution of (2.4.1) in $D \cup C$, and \underline{G}^1 satisfies the radiation conditions, clearly $\underline{u}^{SC}(P)$ defined by (4.2.1) is also a regular solution of (2.4.1) in D and satisfies the radiation conditions. The implementation of the boundary condition (2.4.2) leads to

$$T\underline{u}^{SC}(p) = -\frac{1}{2}\underline{e}(p) + \int_{\partial D} T_{\underline{p}} \underline{G}^1(p; q) \cdot \underline{e}(q) ds_q = -T\underline{u}^{inc}(p) \quad p, q \in \partial D. \quad (4.2.2)$$

Moreover, it follows from (4.1.2) that the corresponding displacement $\underline{u}^{SC}(P_-)$ defined by (4.2.1) with P_- lying in $D \cap D_a$ satisfies

$$T\underline{u}^{SC}(P_-) + F \cdot \underline{u}^{SC}(P_-) = \underline{0} \quad P_- \in C. \quad (4.2.3)$$

In the direct method, the application of the reciprocal theorem to $\underline{u}^{SC}(P)$ and $\underline{G}^1(P; Q)$ in D and to $\underline{u}^{inc}(P)$ and $\underline{G}^1(P; Q)$ in $D \cap D_a$ leads, after implementation of the boundary conditions (2.4.2), to

$$\begin{aligned} \underline{u}^{SC}(P) = & \int_{\partial D} \underline{u}(q) \cdot T_q \underline{G}^1(q; P) ds_q + \int_C \{ \underline{u}^{inc}(q) \cdot T_q \underline{G}^1(q; P) - T\underline{u}^{inc}(q) \cdot \\ & \underline{G}^1(q; P) \} ds_q \quad P \in D \end{aligned} \quad (4.2.4)$$

and

$$\frac{1}{2}\underline{u}(p) - \int_{\partial D} \underline{u}(q) \cdot \underline{T}_q \underline{G}^1(q;p) ds_q = \underline{u}^{inc}(p) + \int_C \{\underline{u}^{inc}(q) \cdot \underline{T}_q \underline{G}^1(q;p) - \underline{T}_u^{inc}(q) \cdot \underline{G}^1(q;p)\} ds_q \quad p \in \partial D. \quad (4.2.5)$$

If $p \in D_a$, it can be seen that

$$\int_C \{\underline{u}^{inc}(q) \cdot \underline{T}_q \underline{G}^f(q;p) - \underline{T}_u^{inc}(q) \cdot \underline{G}^f(q;p)\} ds_q = 0 \quad p \in D_a. \quad (4.2.6)$$

Hence, (4.2.4) and (4.2.5) become

$$\underline{u}^{sc}(p) = \int_{\partial D} \underline{u}(q) \cdot \underline{T}_q \underline{G}^1(q;p) ds_q + \int_C \{\underline{u}^{inc}(q) \cdot \underline{T}_q \underline{G}^1(q;p) - \underline{T}_u^{inc}(q) \cdot \underline{G}^1(q;p)\} ds_q \quad p \in D, \quad (4.2.7)$$

$$\frac{1}{2}\underline{u}(p) - \int_{\partial D} \underline{u}(q) \cdot \underline{T}_q \underline{G}^1(q;p) ds_q = \underline{u}^{inc}(p) + \int_C \{\underline{u}^{inc}(q) \cdot \underline{T}_q \underline{\Gamma}(q;p) - \underline{T}_u^{inc}(q) \cdot \underline{\Gamma}(q;p)\} ds_q \quad p \in \partial D. \quad (4.2.8)$$

Consider now the modified boundary integral equations (4.2.2) and (4.2.8), which we rewrite as

$$\frac{1}{2}\underline{p}(p) - \int_{\partial D} \underline{K}^1(p;q) \cdot \underline{p}(q) ds_q = \underline{T}_u^{inc}(p) \quad p \in \partial D, \quad (4.2.9)$$

$$\frac{1}{2}\underline{u}(p) - \int_{\partial D} \underline{u}(q) \cdot \underline{K}^1(q;p) ds_q = \underline{u}^{inc}(p) \quad p \in \partial D, \quad (4.2.10)$$

where

$$\underline{K}^1(p;q) = \underline{T}_p \underline{G}^1(p;q)$$

and

$$\underline{u}^{inc}(p) = \underline{u}^{inc}(p) + \int_C \{\underline{u}^{inc}(q) \cdot \underline{T}_q \underline{\Gamma}(q;p) - \underline{T}_u^{inc}(q) \cdot \underline{\Gamma}(q;p)\} ds_q \quad p \in \partial D. \quad (4.2.11)$$

The modified kernels are singular. In fact, the singular term is the same as in the non-modified kernels. It follows, as before, that under the same assumptions as in section III.3, except that $\underline{K}(p,q)$ should be replaced by $\underline{K}^1(p;q)$, the integrals in (4.2.9) and (4.2.10) can be interpreted as Cauchy principal values. Furthermore, it can be shown, in a similar fashion as in section III.3, that the Fredholm theorems still apply to these modified integral equations. Since (4.2.9) and (4.2.10) are adjoint integral equations, we need to discuss the solvability of (4.2.9) only.

Before we re-examine the arguments of section III.4 for the modified integral equation (4.2.9) and with the interior D_- replaced by $D \cap D_a$, we state and prove the following uniqueness theorem:

Uniqueness theorem 2:

Suppose that a region R is bounded externally by a curve C_1 and internally by a curve C_2 . Suppose that in R the function \underline{u} is a regular solution of

$$\frac{1}{k^2} \text{grad}(\text{div } \underline{u}) - \frac{1}{k^2} \text{curl}(\text{curl } \underline{u}) + \underline{u} = \underline{0},$$

where k^2 and k^2 are real and given by (2.1.10), satisfying the boundary condition

$$\underline{u} = \underline{0} \quad \text{or} \quad T\underline{u} = \underline{0} \quad \text{on } C_1 \quad (4.2.12)$$

and the boundary condition

$$T\underline{u} + \underline{F} \cdot \underline{u} = \underline{0} \quad \text{on } C_2 \quad (4.2.13)$$

where T is the traction operator and \underline{F} is an arbitrary second-order tensor whose elements F_{ij} are constants satisfying the following conditions:

- i) $\text{Im}(F_{11}) \cdot \text{Im}(F_{22}) > 0$, Im denotes the imaginary part.
- ii) $F_{12} = F_{21}^*$, the $*$ denotes the conjugate.

Then \underline{u} vanishes identically in T for all values of the frequency.

Proof: By the reciprocal theorem applied to \underline{u} and its conjugate \underline{u}^* , we have

$$\begin{aligned}
 0 &= \int_{C_1 \cup C_2} \{ \underline{u} \cdot T \underline{u}^* - \underline{u}^* \cdot T \underline{u} \} ds \\
 &= \int_{C_1} \{ \underline{u} \cdot T \underline{u}^* - \underline{u}^* \cdot T \underline{u} \} ds + \int_{C_2} \{ \underline{u} \cdot T \underline{u}^* - \underline{u}^* \cdot T \underline{u} \} ds \\
 &= \int_{C_2} \{ \underline{u} T \underline{u}^* - \underline{u}^* T \underline{u} \} ds \\
 &= \int_{C_2} \{ \underline{u}^* \cdot \underline{F} \cdot \underline{u} - \underline{u} \cdot \underline{F}^* \cdot \underline{u}^* \} ds \\
 &= \int_{C_2} \{ u_i^* F_{ij} u_j - u_i F_{ij}^* u_j^* \} ds \\
 &= (F_{ij} - F_{ji}^*) \int_{C_2} u_i^* u_j ds \\
 &= (F_{11} - F_{11}^*) \int_{C_2} |u_1|^2 ds + (F_{22} - F_{22}^*) \int_{C_2} |u_2|^2 ds \\
 &\quad + (F_{12} - F_{21}^*) \int_{C_2} u_1^* u_2 ds + (F_{21} - F_{12}^*) \int_{C_2} u_2^* u_1 ds \\
 &= 2i \text{Im}(F_{11}) \int_{C_2} |u_1|^2 ds + 2i \text{Im}(F_{22}) \int_{C_2} |u_2|^2 ds.
 \end{aligned}$$

Thus, we must have $u_1 = u_2 = 0$ on C_2 and therefore $\underline{u} = \underline{0}$ on C_2 , and from (4.2.13) we have $T\underline{u} = \underline{0}$ on C_2 . It follows, by analytical continuation, that $\underline{u} \equiv \underline{0}$ in R . Note that this theorem still holds for F_{ij} non-constant.

Consider now the modified homogeneous integral equation corresponding to (4.2.9).

$$\frac{1}{2}v(p) - \int_D K^1(p;q) \cdot v(q) ds_q = 0. \quad (4.2.14)$$

If (4.2.14) has no non-trivial solution, the conclusion is the same as in section III.4.

Suppose now that (4.2.14) has a non-trivial solution $\underline{p}(p)$, then the corresponding exterior displacement $\underline{u}(P)$ defined by (4.2.1) vanishes in D as in section III.4, and therefore $\underline{u}(p)$ vanishes on ∂D . Therefore, $\underline{u}(p_-)$ vanishes as before, and from uniqueness theorem 2 it now follows that $\underline{u}(P_-)$ vanishes identically in $D \cap D_a$, whatever the value of ω^2 . Then it follows, as in III.4, that $\underline{p} \equiv 0$, and we have a contradiction. Hence, (4.2.14) has no non-trivial solution whatever the value of ω^2 . It follows, by the Fredholm theorems, that (4.2.9) is uniquely solvable and consequently (4.2.10) is also uniquely solvable for all ω^2 . We have then proved the following:

The representation (4.2.1) solves $S(\underline{u}^{inc})$, where \underline{p} is the unique solution of (4.2.2). Note that this representation is always valid, unlike (3.2.1). Similarly, (4.2.7) also solves $S(\underline{u}^{inc})$, where \underline{u} is the unique solution of (4.2.8). To show this, we first note that $\underline{u}^{sc}(P)$, defined by (4.2.7), does satisfy the equations of motion and the radiation conditions; it is here that we need the symmetry property (4.1.6). It is also necessary to verify that the boundary condition (2.4.2) is satisfied. This leads to

$$\begin{aligned} T \underline{u}^{inc}(p) + T_p \int_{\partial D} \underline{u}(q) \cdot T_{q\bar{q}}^1(q;p) ds_q + T_p \int_C \{ \underline{u}^{inc}(q) \cdot T_{q\bar{q}}^1(q;p) \\ - T \underline{u}^{inc}(q) \cdot \underline{\Gamma}(q;p) \} ds_q = 0 \quad p \in \partial D. \end{aligned} \quad (4.2.15)$$

This is a compatibility condition which $\underline{u}(p)$ has to satisfy. In order to show that this is so, let us define a function in $D \cap D_a$ by

$$\begin{aligned} \underline{U}(P_-) &= \underline{u}^{inc}(P_-) + \int_{\partial D} \underline{u}(q) \cdot T_{\underline{q}} G^1(q; P_-) ds_q + \int_C \{ \underline{u}^{inc}(q) \cdot T_{\underline{q}} G^1(q; P_-) \\ &\quad - T_{\underline{u}^{inc}(q)} G^1(q; P_-) \} ds_q, \quad P_- \in D \cap D_a. \end{aligned} \quad (4.2.16)$$

$\underline{U}(P_-)$ satisfies (2.4.1) in $D \cap D_a$ and, if we let P_- approach ∂D , we find that

$$\begin{aligned} \underline{U}(p) &= \underline{u}^{inc}(p) - \frac{1}{2} \underline{u}(p) + \int_{\partial D} \underline{u}(q) \cdot T_{\underline{q}} G^1(q; p) ds_q + \int_C \{ \underline{u}^{inc}(q) \cdot T_{\underline{q}} G^1(q; p) \\ &\quad - T_{\underline{u}^{inc}(q)} G^1(q; p) \} ds_q = 0, \text{ by (4.2.5).} \end{aligned}$$

Moreover $\underline{U}(P_-)$ satisfies (4.1.2) on C . It follows, from uniqueness theorem 2, that $\underline{U}(P_-) \equiv 0$ in $D \cap D_a$ and, in particular, $T\underline{U}(p) = 0$ on ∂D . Hence,

$$\begin{aligned} T\underline{U}(p) &= 0 = T\underline{u}^{inc}(p) + T_p \int_{\partial D} \underline{u}(q) \cdot T_{\underline{q}} G^1(q; p) ds_q + T_p \int_C \{ \underline{u}^{inc}(q) \cdot \\ &\quad T_{\underline{q}} G^1(q; p) - T_{\underline{u}^{inc}(q)} G^1(q; p) \} ds_q. \end{aligned}$$

Using (4.2.6), it can be seen that (4.2.15) is satisfied. Furthermore, it can be shown that (4.2.7) can be simplified to yield

$$\underline{u}^{sc}(p) = \int_{\partial D} \underline{u}(q) \cdot T_{\underline{q}} G^f(q; p) ds_q, \quad (4.2.17)$$

where $\underline{u}(q)$ is the unique solution of (4.2.8). To this aim, we first need to show that

$$\underline{u}^{sc}(p) = \underline{u}(p) - \underline{u}^{inc}(p), \quad (4.2.18)$$

where $\underline{u}^{sc}(p)$ is defined by (4.2.7). Letting P approach ∂D in this formula, we have

$$\begin{aligned} \underline{u}^{SC}(p) &= \frac{1}{2} \underline{u}(p) + \int_{\partial D} \underline{u}(q) \cdot \underline{T}_{q\underline{G}}^1(q;p) ds_q + \int_C \{ \underline{u}^{inc}(q) \cdot \underline{T}_{q\underline{\Gamma}}(q;p) \\ &\quad - \underline{T}_{\underline{u}^{inc}}(q) \cdot \underline{\Gamma}(q;p) \} ds_q = \underline{u}(p) - \underline{u}^{inc}(p), \text{ by (4.2.8),} \end{aligned}$$

as required. We now apply the reciprocal theorem to $\underline{u}^{SC}(P)$ and $\underline{\Gamma}(P;Q)$ in D and to $\underline{u}^{inc}(P)$ and $\underline{\Gamma}(P;Q)$ in $D_+ \cap D_-$; this leads to

$$\int_{\partial D} \{ \underline{u}^{SC}(q) \cdot \underline{T}_{q\underline{\Gamma}}(q;P) - \underline{T}_{\underline{u}^{SC}}(q) \cdot \underline{\Gamma}(q;P) \} ds_q = 0 \quad (4.2.19)$$

and

$$\int_{C \cup \partial D} \{ \underline{u}^{inc}(q) \cdot \underline{T}_{q\underline{\Gamma}}(q;P) - \underline{T}_{\underline{u}^{inc}}(q) \cdot \underline{\Gamma}(q;P) \} ds_q = 0 \quad (4.2.20)$$

Adding (4.2.19) to (4.2.20) and implementing (2.4.2) yields

$$\int_{\partial D} \underline{u}(q) \cdot \underline{T}_{q\underline{\Gamma}}(q;P) ds_q + \int_C \{ \underline{u}^{inc}(q) \cdot \underline{T}_{q\underline{\Gamma}}(q;P) - \underline{T}_{\underline{u}^{inc}}(q) \cdot \underline{\Gamma}(q;P) \} ds_q = 0. \quad (4.2.21)$$

Substituting (4.2.21) in (4.2.7) leads to (4.2.17). Note that the representation (4.2.17) does not involve \underline{G}^1 explicitly. This confirms what we said earlier in section III.4.

IV.3 Construction of the Modified Tensor

In this section, we construct the modified Green's tensor when \underline{F} is chosen to be $\underline{F} = F \underline{I}$, where \underline{I} is the identity tensor and $F = |F|e^{i\delta}$ is an arbitrary constant such that $0 < \delta < \pi$. Clearly this choice of \underline{F} satisfies the condition (4.1.5) and the conditions required by the uniqueness theorem 2. The condition $0 < \delta < \pi$, which appears to be more restrictive than actually required, will be needed in our construction.

The modified Green's tensor $\underline{\underline{G}}^1(P;Q)$ must satisfy the Navier equation

$$\frac{1}{k^2} \text{grad}_P(\text{div}_P \underline{\underline{G}}^1(P;Q)) - \frac{1}{k^2} \text{curl}_P(\text{curl}_P \underline{\underline{G}}^1(P;Q)) + \underline{\underline{G}}^1(P;Q) = \underline{\underline{0}} \quad \text{when } P, Q \in D_a, \quad (4.3.1)$$

except at $P = Q$ when $\underline{\underline{G}}^1(P;Q)$ has a singularity such that

$$\underline{\underline{G}}^1(P;Q) \text{ remains bounded as } P \rightarrow Q. \quad (4.3.2)$$

At infinity $\underline{\underline{G}}^1(P;Q)$ must satisfy the radiation conditions as defined in section II.3. On the circle C there is the dissipative boundary condition

$$T_P \underline{\underline{G}}^1(P;Q) + F \underline{\underline{G}}^1(P;Q) = \underline{\underline{0}}, \quad (4.3.3)$$

where P is on C and Q is on the boundary ∂D . Since $r_P < r_Q$, we have the following expansion for $\underline{\underline{G}}^f(P;Q)$ (see, e.g., V. Varatharajulu and Y. H. Pao (1976)):

$$\underline{\underline{G}}^f(P;Q) = \frac{i}{4\mu k^2} \sum_{\sigma=1}^2 \sum_{m=0}^{\infty} \left[\underline{\underline{\nabla}} \hat{\phi}_m^{\sigma}(P) \otimes \underline{\underline{\nabla}} \hat{\phi}_m^{\sigma}(Q) + \underline{\underline{\nabla}} \cdot (\hat{\psi}_m^{\sigma}(P) \underline{\underline{e}}_3) \otimes \underline{\underline{\nabla}} \cdot (\hat{\psi}_m^{\sigma}(Q) \underline{\underline{e}}_3) \right], \quad (4.3.4)$$

where

$$\begin{aligned} \hat{\phi}_m^{\sigma}(P) &= J_m(kr_P) E_m^{\sigma}(\theta_P), \\ \hat{\psi}_m^{\sigma}(P) &= J_m(kr_P) E_m^{\sigma}(\theta_P), \\ \hat{\phi}_m^{\sigma}(P) &= H_m^{(1)}(kr_P) E_m^{\sigma}(\theta_P), \\ \hat{\psi}_m^{\sigma}(P) &= H_m^{(1)}(kr_P) E_m^{\sigma}(\theta_P), \\ E_m^{\sigma}(\theta) &= \sqrt{\epsilon_m} \begin{cases} \cos m \theta & \sigma=1 \\ \sin m \theta & \sigma=2, \end{cases} \\ \epsilon_m &= \begin{cases} 1 & m=0 \\ 2 & m>0, \end{cases} \end{aligned}$$

\underline{e}_3 : unit vector in the x_3 -direction

\otimes denotes the tensor product,

$H_m^{(1)}(.)$ denotes the m^{th} -order Hankel function of the first kind

and

$J_m(.)$ denotes the m^{th} -order Bessel function of the first kind.

We choose $\underline{G}^1(P;Q)$ to be

$$\underline{G}^1(P;Q) = \underline{G}^f(P;Q) + \frac{1}{4\mu K^2} \sum_{\sigma=1}^2 \sum_{m=0}^{\infty} [\underline{\nabla} \Phi_m^{\sigma}(P) \otimes \underline{A}_m^{\sigma} + \underline{\nabla} \wedge (\Psi_m^{\sigma}(P) \underline{e}_3) \otimes \underline{B}_m^{\sigma}], \quad (4.3.5)$$

where \underline{A}_m^{σ} and \underline{B}_m^{σ} are unknown vectors independent of the position of P .

This expansion satisfies identically the equations (4.3.1), (4.3.2) and also the radiation conditions. Implementing the boundary condition (4.3.3) yields, after a long calculation,

$$\underline{A}_m^{\sigma} = \alpha_m^1 \underline{\nabla} \Phi_m^{\sigma}(Q) - \beta_m^1 (-1)^{\sigma} \underline{\nabla} \wedge (\Psi_m^{3-\sigma}(Q) \underline{e}_3) \quad (4.3.6)$$

and

$$\underline{B}_m^{\sigma} = \alpha_m^2 \underline{\nabla} \wedge (\Psi_m^{\sigma}(Q) \underline{e}_3) + \beta_m^2 (-1)^{\sigma} \underline{\nabla} \Phi_m^{3-\sigma}(Q), \quad (4.3.7)$$

where

$$\begin{aligned} \alpha_m^1 &= (C_m \hat{B}_m - D_m \hat{A}_m) / \Delta_m, \\ \alpha_m^2 &= (B_m \hat{C}_m - A_m \hat{D}_m) / \Delta_m, \\ \beta_m^1 &= (C_m \hat{D}_m - D_m \hat{C}_m) / \Delta_m, \\ \beta_m^2 &= (B_m \hat{A}_m - A_m \hat{B}_m) / \Delta_m, \end{aligned}$$

and

$$\Delta_m = A_m D_m - B_m C_m,$$

with

$$A_m = [2\tau^2 H_m^{(1)}(ka) + (2\tau^2 - 1) H_m^{(1)}(ka)] + \frac{F}{\mu k^2} k H_m^{(1)}(ka),$$

$$B_m = \frac{2m}{(ka)^2} [ka H_m^{(1)}(ka) - H_m^{(1)}(ka)] + \frac{F}{\mu k^2} \frac{m}{a} H_m^{(1)}(ka),$$

$$C_m = \frac{2m}{(ka)^2} [ka H_m^{(1)}(ka) - H_m^{(1)}(ka)] + \frac{F}{\mu k^2} \frac{m}{a} H_m^{(1)}(ka),$$

$$D_m = [2 H_m^{(1)}(ka) + H_m^{(1)}(ka)] + \frac{F}{\mu k^2} k H_m^{(1)}(ka),$$

here the ' denotes the derivative. The \hat{A}_m , \hat{B}_m , \hat{C}_m and \hat{D}_m are defined as above, but with the Hankel function $H_m^{(1)}(.)$ replaced by $J_m(.)$. The modified tensor can thus be written as follows:

$$\begin{aligned} \underline{\underline{G}}^1(P;Q) &= \underline{\underline{G}}^f(P;Q) + \frac{i}{4\mu k^2} \sum_{\sigma=1}^2 \sum_{m=0}^{\infty} \{ \alpha_m^1 \nabla \phi_m^{\sigma}(P) \otimes \nabla \phi_m^{\sigma}(Q) \\ &\quad + \alpha_m^2 \nabla \wedge (\psi_m^{\sigma}(P) \underline{e}_3) \otimes \nabla \wedge (\psi_m^{\sigma}(Q) \underline{e}_3) \\ &\quad - \beta_m^1 (-1)^{\sigma} \nabla \phi_m^{\sigma}(P) \otimes \nabla \wedge (\psi_m^{3-\sigma}(Q) \underline{e}_3) \\ &\quad + \beta_m^2 (-1)^{\sigma} \nabla \wedge (\psi_m^{\sigma}(P) \underline{e}_3) \otimes \nabla \phi_m^{3-\sigma}(Q) \}. \end{aligned} \quad (4.3.8)$$

Two questions remain to be answered:

- i) When does this expansion converge?
- ii) Does Δ_m , the common denominator of α_m^1 , α_m^2 , β_m^1 and β_m^2 , ever vanish?

The questions are answered as follows: the expansion (4.3.8) converges whenever $r_P \cdot r_Q > a^2$ (see Appendix A2). Δ_m can be shown (see Appendix A3) to never vanish, it is here that the condition $0 < \delta < \pi$ is required.

The expressions for the numerator of β_m^1 and β_m^2 can be made much simpler. Using some transformations involving the Wronskian of Bessel functions, we get

$$\beta_m^1 = \beta_m^2 = -\frac{i}{\pi} \frac{2m}{(Ka)^4} \left[\left\{ \frac{a}{\mu} F-2 \right\}^2 + 2 \{ (Ka)^2 - 2m^2 \} \right] / \Delta_m. \quad (4.3.9)$$

As expected, it can be seen, using (4.3.8) and (4.3.9) that

$$G_{ij}^1(P;Q) = G_{ji}^1(Q;P).$$

This concludes the construction of the modified tensor.

In the next chapter, we shall discuss another method for avoiding the irregular frequencies.

CHAPTER V

MODIFIED GREEN'S TENSOR USING MULTIPOLES

V.1 Introduction

In the previous chapter, we have shown how the fundamental solution can be modified yielding integral equations which are uniquely solvable at all frequencies. This chapter deals with a second method which removes the irregular frequencies. It involves the addition of a finite series of multipoles to the fundamental solution \underline{G}^f . With some mild conditions on the coefficients of these multipoles, this frees an interval of irregular frequencies. This method has an advantage over the previous one in that it only requires the addition of a finite number of terms. It has the disadvantage that uniqueness is guaranteed only for a restricted range of the frequency instead of for all frequencies. However, in numerical work and practical applications, we are usually concerned with a finite range of the frequency so that this is not a serious drawback.

This method was first used by Jones (1974) in acoustics, following Ursell's (1973) work with the first method. However, the argument involved is difficult to follow. Ursell (1978) shortened and simplified the proof of a key theorem in Jones' paper. Since then, this method has been investigated by several authors; a detailed discussion of these investigations is given in chapter I. However, in the context of elastodynamics, we know of only two contributions. Kobayashi & Nishimura (1982) followed Jones' argument closely, for two-dimensional problems. They considered a modified Green's function of a similar form to (4.3.8) but took only a finite number of terms and did not include any cross-terms; i.e. $\beta_m^1 = \beta_m^2 = 0$. They also assumed that all the coefficients α_m^1 and α_m^2

of the multipoles were equal and independent of m . Here again, the argument is difficult to follow. Very recently, Jones (1984) has considered three-dimensional problems, using Ursell's (1978) simplification. As Kobayashi & Nishimura (1982), he did not include any cross-terms, however, he did not truncate the series and the coefficients were not assumed to be equal or independent of m . Independently, we have adopted a similar approach for two-dimensional problems, but with a few differences. As Kobayashi & Nishimura, we take a finite series. However, we do include cross-terms, making our modified Green's function more general. We shall be comparing our results with those of Kobayashi & Nishimura and Jones (1984).

V.2 Modified Green's Tensor

The modified Green's tensor $\underline{\underline{G}}^2(P;Q)$ will consist of the unmodified Green's tensor $\underline{\underline{G}}^f(P;Q)$ and an additional term $\underline{\underline{L}}(P;Q)$:

$$\underline{\underline{G}}^2(P;Q) = \underline{\underline{G}}^f(P;Q) + \underline{\underline{L}}(P;Q). \quad (5.2.1)$$

We shall require, for the same reasons as in the previous chapter, that $\underline{\underline{G}}^2(P;Q)$ satisfies the governing equations of motion and the radiation conditions at infinity with respect to both P and Q . To achieve this, a sufficient condition is that $\underline{\underline{L}}(P;Q)$ satisfies the governing equations of motion and the radiation conditions with respect to P , and the following symmetry condition (see section IV.1):

$$L_{ij}(P;Q) = L_{ji}(Q;P). \quad (5.2.2)$$

Taking into account (4.3.8), we choose the following form for $\underline{\underline{L}}(P;Q)$:

$$\begin{aligned} \underline{L}(P;Q) = & \frac{i}{4\mu K^2} \sum_{\sigma=1}^2 \sum_{m=0}^M [a_m^\sigma \nabla \Phi_m^\sigma(P) \otimes \nabla \Phi_m^\sigma(Q) + b_m^\sigma \nabla \cdot (\Psi_m^\sigma(P) \underline{e}_3) \otimes \nabla \cdot (\Psi_m^\sigma(Q) \underline{e}_3) \\ & + c_m (-1)^\sigma \{ \nabla \cdot (\Psi_m^\sigma(P) \underline{e}_3) \otimes \nabla \Phi_m^{3-\sigma}(Q) - \nabla \Phi_m^\sigma(P) \otimes \nabla \cdot (\Psi_m^{3-\sigma}(Q) \underline{e}_3) \}], \end{aligned} \quad (5.2.3)$$

where M is a positive integer and a_m^σ , b_m^σ and c_m are arbitrary coefficients satisfying some suitable conditions to be specified later. Note that c_m , the coefficient of the cross-terms, is independent of σ . This is a necessary condition for (5.2.2) to hold. For the sake of simplicity, we shall suppress the dependence on σ of a_m^σ and b_m^σ and simply write a_m and b_m . This choice of $\underline{L}(P;Q)$ does, indeed, satisfy the equations of motion in $D \setminus \partial D \cup \{0\}$ and the radiation conditions at infinity with respect to P , and the symmetry condition (5.2.2). Note that if M tends to infinity, some suitable restrictions, on the coefficients a_m , b_m and c_m , are required so as to secure the convergence of (5.2.3).

V.3 Modified Boundary Integral Equations

Because of the singularities of $\underline{L}(P;Q)$ at the origin, we need to isolate it by taking a circle C_ϵ of centre 0 and radius ϵ . This circle lies entirely inside D_- . Denote the region outside this circle and extending to infinity by D_ϵ . The integral representations and their corresponding boundary integral equations can be obtained in the same way as in section IV.2.

In the indirect method, the single layer formulation enables us to write

$$\underline{u}^{sc}(P) = \int_{\partial D} \underline{G}^2(P,q) \cdot \underline{p}(q) ds_q. \quad (5.3.1)$$

This representation satisfies equation (2.4.1), the radiation conditions

at infinity and is continuous in $D \cup \partial D \cup D_-$, except at the origin. The implementation of the boundary condition (2.4.2) leads to

$$T u^{sc}(p) = -\frac{1}{2} \rho(p) + \int_{\partial D} T_{p \equiv} G^2(p; q) \cdot \rho(q) ds_q = -T u^{inc}(p) \quad p, q \in \partial D. \quad (5.3.2)$$

The direct method yields the following equations:

$$\begin{aligned} u^{sc}(p) &= \int_{\partial D} u(q) \cdot T_{q \equiv} G^2(q; p) ds_q + \int_{\epsilon} \{ u^{inc}(q) \cdot T_{q \equiv} L(q; p) - \\ &\quad T u^{inc}(q) \cdot L(q; p) \} ds_q \quad p \in D \end{aligned} \quad (5.3.3)$$

and

$$\frac{1}{2} u(p) - \int_{\partial D} u(q) \cdot T_{q \equiv} G^2(q; p) ds_q = \tilde{u}^{inc}(p) \quad p \in \partial D, \quad (5.3.4)$$

where

$$\tilde{u}^{inc}(p) = u^{inc}(p) + \int_{\epsilon} \{ u^{inc}(q) \cdot T_{q \equiv} L(q; p) - T u^{inc}(q) \cdot L(q; p) \} ds_q. \quad (5.3.5)$$

Consider now the modified boundary integral equations (5.3.2) and (5.3.4), which we rewrite as follows:

$$\frac{1}{2} \rho(p) - \int_{\partial D} K^2(p; q) \cdot \rho(q) ds_q = T u^{inc}(p) \quad p \in \partial D \quad (5.3.6)$$

and

$$\frac{1}{2} u(p) - \int_{\partial D} u(q) \cdot K^2(q; p) ds_q = \tilde{u}^{inc}(p) \quad p \in \partial D, \quad (5.3.7)$$

where

$$K^2(p; q) = T_{p \equiv} G^2(p; q).$$

Since $\underline{L}(p;q)$ is not singular on ∂D , it follows that the singular term in the kernels of (5.3.6) and (5.3.7) is the same as in section III.3. Therefore, under the same conditions as before, the integrals in (5.3.6) and (5.3.7) can be interpreted in the Cauchy principal sense. It can also be shown, as before, that the Fredholm theorems can be used. It follows, since (5.3.6) and (5.3.7) are adjoint integral equations, that only one of these equations needs to be discussed.

Let us now re-examine the arguments for the solvability of the integral equation (5.3.6) in the light of the modified Green's tensor. To do this, consider the homogeneous equation corresponding to (5.3.6)

$$\frac{1}{2}\underline{v}(p) - \int_{\partial D} \underline{K}^2(p;q) \cdot \underline{v}(q) ds_q = \underline{0}. \quad (5.3.8)$$

If (5.3.8) has trivial solutions only, it follows, from the Fredholm theorems, that (5.3.6) and (5.3.7) are uniquely solvable.

Suppose now that (5.3.8) has a non-trivial solution $\underline{v}(p)$. Then the corresponding exterior displacement $\underline{u}(P)$ defined by (5.3.1) vanishes, as before, in D . It follows, in the same manner as before, that $\underline{u}(p) = \underline{u}(p_-) = \underline{0}$. Consider now the corresponding displacement $\underline{u}(P_-)$ defined by (5.3.1), but with P_- lying in D_- . $\underline{u}(P_-)$ satisfies the equation of motion in D_- , except at the origin, and the boundary condition $\underline{u}(p_-) = \underline{0}$ on ∂D . It does not however follow that $\underline{u}(P_-)$ is an eigenfunction of the interior problem I_h (see section II.4) since $\underline{G}^2(P_-;q)$ has singularities at the origin. Consider the expansion of $\underline{u}(P_-)$ inside any circle with centre 0 and lying inside D_- . Using (4.3.4) and (5.2.3), we get

$$\underline{u}(P_-) = \sum_{\sigma=1}^2 \sum_{m=0}^{\infty} [A_m^{\sigma} \hat{\Phi}_m^{\sigma}(P_-) + B_m^{\sigma} \hat{\Psi}_m^{\sigma}(P_-) \mathbf{e}_3]$$

$$\begin{aligned}
& + \sum_{\sigma=1}^2 \sum_{m=0}^M \{ \{ a_m A_m^{\sigma} - c_m (-1)^{\sigma} B_m^{3-\sigma} \} \nabla_m^{\sigma}(\underline{P}_{-}) \\
& + \{ b_m B_m^{\sigma} + c_m (-1)^{\sigma} A_m^{3-\sigma} \} \nabla_m^{\sigma}(\underline{P}_{-}) \underline{e}_3 \} \},
\end{aligned} \tag{5.3.9}$$

where

$$A_m^{\sigma} = \frac{i}{4\mu K^2} \int_{\partial D} \nabla_m^{\sigma}(q) \cdot \underline{\rho}(q) ds_q \tag{5.3.10}$$

and

$$B_m^{\sigma} = \frac{i}{4\mu K^2} \int_{\partial D} \nabla_m^{\sigma}(q) \underline{e}_3 \cdot \underline{\rho}(q) ds_q. \tag{5.3.11}$$

Apply now the reciprocal theorem to $\underline{u}(\underline{P}_{-})$ and its conjugate $\underline{u}^*(\underline{P}_{-})$ in $D \cap D_{\epsilon}$; we get

$$\begin{aligned}
0 &= \int_{\partial D \cup C_{\epsilon}} \{ \underline{u}(\underline{P}_{-}) \cdot \underline{T} \underline{u}^*(\underline{P}_{-}) - \underline{u}^*(\underline{P}_{-}) \cdot \underline{T} \underline{u}(\underline{P}_{-}) \} ds \\
&= \int_{\partial D} \{ \underline{u}(\underline{P}_{-}) \cdot \underline{T} \underline{u}^*(\underline{P}_{-}) - \underline{u}^*(\underline{P}_{-}) \cdot \underline{T} \underline{u}(\underline{P}_{-}) \} ds + \int_{C_{\epsilon}} \{ \underline{u}(\underline{P}_{-}) \cdot \underline{T} \underline{u}^*(\underline{P}_{-}) \\
&\quad - \underline{u}^*(\underline{P}_{-}) \cdot \underline{T} \underline{u}(\underline{P}_{-}) \} ds \\
&= \int_{C_{\epsilon}} \{ \underline{u}(\underline{P}_{-}) \cdot \underline{T} \underline{u}^*(\underline{P}_{-}) - \underline{u}^*(\underline{P}_{-}) \cdot \underline{T} \underline{u}(\underline{P}_{-}) \} ds \\
&= -18\mu K^2 \sum_{\sigma=1}^2 \sum_{m=0}^M [|A_m^{\sigma}|^2 \cdot \{ |a_m + \frac{1}{2}|^2 + |c_m|^2 - \frac{1}{4} \} \\
&\quad + |B_m^{\sigma}|^2 \cdot \{ |b_m + \frac{1}{2}|^2 + |c_m|^2 - \frac{1}{4} \} \\
&\quad + 2(-1)^{\sigma} \text{Re} \{ [c_m^*(a_m + \frac{1}{2}) + c_m(b_m^* + \frac{1}{2})] A_m^{3-\sigma} B_m^{\sigma} \}],
\end{aligned} \tag{5.3.12}$$

where we have used the boundary condition $\underline{u}(\underline{p}_{-}) = 0$ on ∂D , the expansion (5.3.9) on C_{ϵ} and the following orthogonality relations:

$$\oint_{\epsilon} \{ \hat{F}_{-m}^{\sigma\ell}(P) \cdot \hat{T}F_{-n}^{\nu k}(P) - \hat{T}F_{-m}^{\sigma\ell}(P) \cdot \hat{F}_{-n}^{\nu k}(P) \} ds_P = 0, \quad (5.3.13)$$

$$\oint_{\epsilon} \{ \hat{F}_{-m}^{\sigma\ell}(P) \cdot \hat{T}F_{-n}^{\nu k}(P) - \hat{T}F_{-m}^{\sigma\ell}(P) \cdot \hat{F}_{-n}^{\nu k}(P) \} ds_P = i4\mu K^2 \delta_{mn} \delta_{\sigma\nu} \delta_{\ell k} \quad (5.3.14)$$

and

$$\oint_{\epsilon} \{ \hat{F}_{-m}^{\sigma\ell}(P) \cdot \hat{T}F_{-n}^{\nu k}(P) - \hat{T}F_{-m}^{\sigma\ell}(P) \cdot \hat{F}_{-n}^{\nu k}(P) \} ds_P = i8\mu K^2 \delta_{mn} \delta_{\sigma\nu} \delta_{\ell k}, \quad (5.3.15)$$

with

$$\underline{F}_{-m}^{\sigma 1}(P) = \underline{\nabla} \hat{\phi}_{-m}^{\sigma}(P), \quad \underline{F}_{-m}^{\sigma 2}(P) = \underline{\nabla} \wedge (\hat{\psi}_{-m}^{\sigma}(P) \underline{e}_3),$$

$$\hat{\underline{F}}_{-m}^{\sigma 1}(P) = \underline{\nabla} \hat{\phi}_{-m}^{\sigma}(P) \text{ and } \hat{\underline{F}}_{-m}^{\sigma 2}(P) = \underline{\nabla} \wedge (\hat{\psi}_{-m}^{\sigma}(P) \underline{e}_3).$$

These orthogonality relations can easily be established using the reciprocal theorem (see, e.g. Pao (1978)).

By an appropriate choice of the coefficients a_m , b_m and c_m , we wish, using (5.3.12), to force the following conditions:

$$A_m^{\sigma} = B_m^{\sigma} = 0 \text{ for } \sigma = 1, 2 \text{ and } m = 0, 1, \dots, M. \quad (5.3.16)$$

This will then make $\underline{u}(P_-)$ a regular solution of (2.4.1) in all D_- . The choice of a_m , b_m and c_m which leads to (5.3.16) will be discussed later. Assuming that (5.3.16) holds, (5.3.9) becomes

$$\underline{u}(P_-) = \sum_{\sigma=1}^2 \sum_{m=M+1}^{\infty} [A_m^{\sigma} \hat{\underline{\nabla}} \hat{\phi}_{-m}^{\sigma}(P_-) + B_m^{\sigma} \hat{\underline{\nabla}} \wedge (\hat{\psi}_{-m}^{\sigma}(P_-) \underline{e}_3)]. \quad (5.3.17)$$

$\underline{u}(P_-)$ now satisfies the equation of motion in all D_- and the boundary condition $\underline{u}(p_-) = \underline{0}$. Furthermore, a close examination of (5.3.17) shows that $\underline{u}(P_-)$ has a zero, in r at the origin, of at least order M . Denote by $\omega_{M,1}$ the lowest value of the frequency for which $\underline{u}(P_-)$ can, if at all,

be an eigenfunction. If we now choose the frequency ω in the integral equation (5.3.8) such that

$$\omega < \omega_{M,1} \quad (5.3.18)$$

this cannot be an eigenvalue for $\underline{u}(P_-)$, and therefore $\underline{u}(P_-)$ must vanish identically in all D_- . Hence, the integral equation (5.3.8) can only have trivial solutions. It follows that (5.3.6) and (5.3.7) are uniquely solvable for all the frequencies satisfying (5.3.18). Note that if we increase M , the order of the zero in r at the origin of $\underline{u}(P_-)$ increases. It follows, from a general principle (see Courant & Hilbert (1953, p.407)) which states that strengthening the constraints of $\underline{u}(P_-)$ increases or at any rate does not decrease the value of $\omega_{M,1}$, that more irregular frequencies are eliminated. However, it should be pointed out that we do not, at present, have a criterion for estimating $\omega_{M,1}$ or indeed for deciding how large M should be before a particular irregular frequency is eliminated. We have thus proved the following:

Subject to (5.3.16), the integral equations (5.3.6) and (5.3.7) are uniquely solvable for all frequencies satisfying (5.3.18). It follows, that the representation (5.3.1) solves $S(\underline{u}^{inc})$, where \underline{u} is the unique solution of (5.3.2). Note that this representation is only valid for that particular range of frequencies. Similarly, (5.3.3) also solves $S(\underline{u}^{inc})$, where \underline{u} is the unique solution of (5.3.4). This can be shown in the same manner as in section IV.2. Moreover, it can also be shown, as in section IV.2, that (5.3.3) simplifies to give

$$\underline{u}^{sc}(P) = \int_{\partial D} \underline{u}(q) \cdot T_q G^f(q;P) ds_q.$$

Let us now discuss some possible choices of a_m , b_m and c_m which ensures that (5.3.16) holds. So, consider equation (5.3.12), which we rewrite here as

$$\sum_{\sigma=1}^2 \sum_{m=0}^M [|A_m^\sigma|^2 \cdot (|a_m + \frac{1}{2}|^2 + |c_m|^2 - \frac{1}{4}) + |B_m^\sigma|^2 \cdot (|b_m + \frac{1}{2}|^2 + |c_m|^2 - \frac{1}{4}) + 2(-1)^\sigma \text{Re}\{ [c_m^* (a_m + \frac{1}{2}) + c_m (b_m^* + \frac{1}{2})] A_m^{3-\sigma} B_m^\sigma \}] = 0. \quad (5.3.19)$$

There is more than one way, as we shall see later, of choosing a_m , b_m and c_m . One obvious choice is

$$c_m^* (a_m + \frac{1}{2}) + c_m (b_m^* + \frac{1}{2}) = 0 \text{ for } m=0, \dots, M, \quad (5.3.20)$$

$$|a_m + \frac{1}{2}|^2 + |c_m|^2 - \frac{1}{4} > 0 \text{ for } m=0, \dots, M \quad (5.3.21)$$

and

$$|b_m + \frac{1}{2}|^2 + |c_m|^2 - \frac{1}{4} > 0 \text{ for } m=0, \dots, M, \quad (5.3.22)$$

which clearly leads to (5.3.16). Note that the inequalities in (5.3.21) and (5.3.22) can be reversed provided it is done for all m . (This will also apply to the results below). Equations (5.3.20), (5.3.21) and (5.3.22) can be satisfied in several ways; let us consider some of them.

i) $c_m = 0$, i.e. there are no cross-terms. (5.3.20) is identically satisfied and (5.3.21) and (5.3.22) take the form

$$|a_m + \frac{1}{2}| - \frac{1}{2} > 0 \quad m=0, \dots, M \quad (5.3.23)$$

and

$$|b_m + \frac{1}{2}| - \frac{1}{2} > 0 \quad m=0, \dots, M. \quad (5.3.24)$$

These are the same conditions as those derived in three-dimensions by Jones (1984). Moreover, if we assume that $a_m = b_m = a$, say, (5.3.23) and (5.3.24) become

$$\operatorname{Re}\{a\}/|a|^2 \neq -1. \quad (5.3.25)$$

This is equivalent to the condition obtained by Kobayashi & Nishimura (1982).

ii) $c_m \neq 0$, but pure imaginary. (5.3.20) leads to

$$a_m = b_m^*,$$

whilst (5.3.21) and (5.3.22) become

$$|a_m + \frac{1}{2}|^2 + |c_m|^2 - \frac{1}{4} > 0.$$

iii) $a_m = b_m = -\frac{1}{2}$. (5.3.20) is satisfied and (5.3.21) and (5.3.22) yield

$$|c_m| > \frac{1}{2}.$$

iv) $a_m = b_m = 0$, i.e. there are only cross-terms. This implies, using (5.3.20), that c_m must be pure imaginary and (5.3.21) and (5.3.22) give

$$|c_m| \neq 0.$$

Note that this is a special case of (ii).

As we mentioned earlier, (5.3.20), (5.3.21) and (5.3.22) are, by no means, the only conditions which lead to (5.3.16). To see this, we rewrite (5.3.19) as follows:

$$\begin{aligned} & \sum_{\sigma=1}^2 \sum_{m=0}^M [\operatorname{Re}\{a_m\} \cdot |A_m^\sigma|^2 + \operatorname{Re}\{b_m\} \cdot |B_m^\sigma|^2 \\ & + |a_m(-1)^\sigma A_m^{3-\sigma} + c_m B_m^\sigma|^2 + |c_m(-1)^\sigma A_m^{3-\sigma} + b_m B_m^\sigma|^2 \\ & + 2\operatorname{Re}\{c_m\} \cdot (-1)^\sigma \cdot \operatorname{Re}\{A_m^{3-\sigma} \cdot B_m^{\sigma*}\}] = 0. \end{aligned} \quad (5.3.26)$$

It follows that, if we choose a_m , b_m and c_m such that

$$\operatorname{Re}\{a_m\} > 0, \quad m=0, \dots, M \quad (5.3.27)$$

$$\operatorname{Re}\{b_m\} > 0, \quad m=0, \dots, M \quad (5.3.28)$$

and

$$\operatorname{Re}\{c_m\} = 0, \quad m=0, \dots, M \quad (5.3.29)$$

this will lead to (5.3.16). (Note that the inequalities in (5.3.27) and (5.3.28) cannot be reversed).

We have, thus, shown that the coefficients of the multipoles can be chosen, in more than one way, so that (5.3.16) holds, which consequently leads, as we have shown, to the unique solvability of the integral equations (5.3.2) and (5.3.4).

CHAPTER VI

NUMERICAL SOLUTION OF THE MODIFIED AND NON-MODIFIED BOUNDARY INTEGRAL EQUATIONS

VI.1 Introduction

It is known that integral equations cannot, generally, be solved in closed forms and therefore resort must be made to numerical methods of solution. In this chapter, we are concerned with the numerical solution of the boundary integral equations derived in the previous three chapters. We shall consider the integral equations derived from the direct method, as in this case the unknown has a definite physical meaning. (Similar methods can be used to treat the integral equations derived from the indirect method). This enables us to compare the numerical solution, in the case of a circular cavity, with the exact solution obtained from the method of separation of variables. We shall consider, in particular, the non-modified integral equation (3.2.10), where the existence of the irregular frequencies will be confirmed, and the modified integral equation (5.3.4) which will remove them. We shall not, however, include any cross-terms in the modified Green's tensor, \underline{G}^2 , i.e. we shall take $c_m = 0$ in (5.2.3).

VI.2 Discretization of the Boundary Integral Equations

Consider the non-modified boundary integral equation (3.2.10), which we rewrite here as follows:

$$\frac{1}{2}\underline{u}(p) - \int_{\partial D} \underline{u}(q) \cdot \underline{K}(q;p) ds_q = \underline{u}^{inc}(p) \quad p \in \partial D. \quad (6.2.1)$$

There are several methods for solving integral equations of this type. One popular method is the boundary element method; see e.g. Cruse and Rizzo (1975), Jaswon and Symm (1977) and Tanaka (1983). We shall adopt this method here.

The first step in the boundary element method is the subdivision of the boundary ∂D into N suitably small smooth intervals or 'elements' $\Delta_1, \Delta_2, \dots, \Delta_N$, so that $\partial D = \bigcup_{j=1}^N \Delta_j$. (6.2.1), then, becomes

$$\frac{1}{2} \underline{u}(p) - \sum_{j=1}^N \int_{\Delta_j} \underline{u}(q) \cdot \underline{K}(q;p) ds_q = \underline{u}^{inc}(p) \quad p \in \partial D. \quad (6.2.2)$$

Denote the end points of the interval Δ_j by q_{2j-1} and q_{2j+1} . Inside each interval Δ_j , we take a point q_{2j} such that q_{2j} does not coincide with q_{2j-1} or q_{2j+1} (see Fig. 6.2.1).

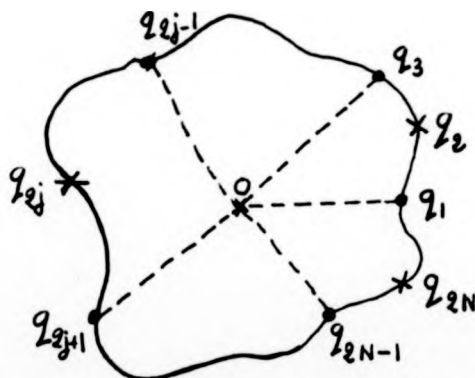


Fig. 6.2.1 Subdivision of the boundary

The second step, is the approximation of the unknown $\underline{u}(q)$ over each interval Δ_j . The simplest of these approximations is the piece-wise constant, i.e.

$$\underline{u}(q) = \underline{u}^j \quad \text{over} \quad \Delta_j. \quad (6.2.3)$$

where \underline{u}^j is a constant vector over Δ_j . A better approximation is the linear one, where $\underline{u}(q)$ is represented, over Δ_j , as a linear combination of the values of $\underline{u}(q)$ at the end points of Δ_j , namely

$$\underline{u}(q) = A_j(q)\underline{u}(q_{2j-1}) + B_j(q)\underline{u}(q_{2j+1}), \quad (6.2.4)$$

where

$$A_j(q) = (s_q - s_{q_{2j+1}})/(s_{q_{2j-1}} - s_{q_{2j+1}})$$

and

$$B_j(q) = -(s_q - s_{q_{2j-1}})/(s_{q_{2j-1}} - s_{q_{2j+1}}).$$

(s_q is defined in section III.3). However, we have found that the quadratic approximation, where $\underline{u}(q)$ is expressed as a linear combination of the values of $\underline{u}(q)$ at the end points and the point inside Δ_j , i.e.

$$\underline{u}(q) = A_{2j-1}(q)\underline{u}(q_{2j-1}) + B_{2j}(q)\underline{u}(q_{2j}) + C_{2j+1}(q)\underline{u}(q_{2j+1}), \quad (6.2.5)$$

where

$$A_{2j-1}(q) = \frac{(s_q - s_{q_{2j}})(s_q - s_{q_{2j+1}})}{(s_{q_{2j-1}} - s_{q_{2j}})(s_{q_{2j-1}} - s_{q_{2j+1}})},$$

$$B_{2j}(q) = \frac{(s_q - s_{q_{2j-1}})(s_q - s_{q_{2j+1}})}{(s_{q_{2j}} - s_{q_{2j-1}})(s_{q_{2j}} - s_{q_{2j+1}})},$$

and

$$C_{2j+1}(q) = \frac{(s_q - s_{q_{2j-1}})(s_q - s_{q_{2j}})}{(s_{q_{2j+1}} - s_{q_{2j-1}})(s_{q_{2j+1}} - s_{q_{2j}})},$$

gives better results with less elements than the simpler approximations, even for fairly high values of the frequency. We shall, therefore, only

use the quadratic approximation. Note that higher-order approximations can be considered, although, we have found that this was not necessary. The final step, is to evaluate equation (6.2.2) at the points q_i ('collocation'), i.e. we take $p = q_i$ $i = 1, 2, \dots, 2N$, leading to

$$\frac{1}{2} \underline{u}(q_i) - \sum_{j=1}^N \int_{\Delta_j} \underline{u}(q) \cdot \underline{K}(q; q_i) ds_q = \underline{u}^{inc}(q_i) \quad q_i \in \partial D$$

$$i = 1, 2, \dots, 2N. \quad (6.2.6)$$

Substituting (6.2.5) into (6.2.6), we get the following linear system of algebraic equations:

$$\underline{M} \cdot \underline{U} = \underline{U}^{inc}, \quad (6.2.7)$$

where

$$\underline{U} = \begin{pmatrix} u_1(q_1) \\ \vdots \\ u_1(q_{2N}) \\ u_2(q_1) \\ \vdots \\ u_2(q_{2N}) \end{pmatrix} \quad \text{and} \quad \underline{U}^{inc} = \begin{pmatrix} u_1^{inc}(q_1) \\ \vdots \\ u_1^{inc}(q_{2N}) \\ u_2^{inc}(q_1) \\ \vdots \\ u_2^{inc}(q_{2N}) \end{pmatrix}; \quad (6.2.8)$$

here, $u_1(q)$ and $u_2(q)$ denote the components of the displacement in the x_1 and x_2 directions, respectively. \underline{M} is given by

$$\underline{M} = \begin{pmatrix} \frac{1}{2}I - M_{11} & -M_{12} \\ -M_{21} & \frac{1}{2}I - M_{22} \end{pmatrix}, \quad (6.2.9)$$

I is a $2N \times 2N$ identity matrix and $M_{\alpha k}$ is a $2N \times 2N$ matrix whose elements are defined as follows:

$$M_{lk}^{i1} = \int_{\Delta_N} C_{2N+1}(q) K_{lk}(q; q_i) ds_q + \int_{\Delta_1} A_1(q) K_{lk}(q; q_i) ds_q \quad (6.2.10)$$

$$M_{lk}^{i(2j-1)} = \int_{\Delta_{j-1}} C_{2j-1}(q) K_{lk}(q; q_i) ds_q + \int_{\Delta_j} A_{2j-1}(q) K_{lk}(q; q_i) ds_q \quad j=2,3,\dots,N \quad (6.2.11)$$

and

$$M_{lk}^{i,2j} = \int_{\Delta_j} B_{2j}(q) K_{lk}(q; q_i) ds_q \quad j=1,\dots,N. \quad (6.2.12)$$

Having approximated our integral equation with the linear system of algebraic equations (6.2.7), this is then solved numerically. But before this is done, we need to compute the elements of the matrix M which involve the evaluation of the integrals given by (6.2.10-12). To be consistent, all integrals should be evaluated with an error which is not worse than the error incurred by using the quadratic approximation to $\underline{u}(q)$. Therefore, it seems reasonable to approximate Δ_j by some simple curve; however, we evaluate integrals over the original elements, and do not approximate Δ_j . When these integrals are not singular, i.e. $i \neq j$ or $l \neq k$, they are evaluated using an 8-point Gaussian quadrature. If $i=j$ and $l \neq k$, we know that the singularity of the integrand is of a Cauchy-type, and so we use a special algorithm (D01AQF from the NAG Library). Most of the computing time is taken up by the evaluation of the elements of the matrix M . In fact, the time taken to actually solve (6.2.7), once M is available, is practically negligible in comparison.

VI.3 Numerical Solution of the Non-modified Integral Equation for Scattering by a Circular Cavity

A computer program for solving (6.2.7), given the geometry of the cross-section and the incident field, was written. As an illustration of the foregoing theory and in order to test our program, we investigated the scattering of a plane P-wave and a plane S-wave by a circular cavity. The plane P-wave and S-wave are defined by

$$\underline{u}^{inc} = U^P \underline{\nabla}(e^{ik\hat{\alpha} \cdot \underline{r}}) \text{ and } \underline{u}^{inc} = U^S \underline{\nabla} \wedge (e^{ik\hat{\alpha} \cdot \underline{r}} \underline{e}_3), \quad (6.3.1)$$

respectively, where U^P and U^S are constants, $\hat{\alpha}$ and \underline{r} are given by

$$\hat{\alpha} = \cos \alpha \underline{i} + \sin \alpha \underline{j}$$

and

$$\underline{r} = x\underline{i} + y\underline{j},$$

where α is the angle of incidence, which is taken to be zero because of the symmetry in our configuration.

Once the solution of (6.2.7) is obtained, it is compared to the exact one which is obtained from the method of separation of variables. The error in the numerical solution is then computed using the following formula:

$$E(u_i) = \frac{\int_{\partial D} |u_i^{app} - u_i^{ex}|^2 ds}{\int_{\partial D} |u_i^{ex}|^2 ds}, \quad (6.3.2)$$

where

$E(u_i)$ denotes the error in the i^{th} -component of the solution,

u_i^{app} is the i^{th} -component of the solution obtained from (6.2.7).

and

u_i^{ex} is the i^{th} -component of the exact solution.

At an irregular frequency (6.2.7) becomes ill-conditioned. This is characterised by the modulus of the determinant of the matrix M becoming very small. Therefore, we may expect the error (6.3.2) to be large in a neighbourhood of that irregular frequency. In the special case of a circular cavity, the irregular frequencies are known; they are the roots of the following equation (see, e.g., Vekua (1968), p.235):

$$J_{n-1}(ka)J_{n+1}(Ka) + J_{n-1}(Ka)J_{n+1}(ka) = 0 \quad \text{for } n = 0, 1, 2, \dots \quad (6.3.3)$$

where a denotes the radius of the cavity and $k/K = \tau$ depends only on Poisson's ratio, ν :

$$\nu = \frac{1}{2}(1 - 2\tau^2)/(1 - \tau^2).$$

If we denote by $Ka_{n,m}$ the m^{th} -root of (6.3.3) corresponding to a fixed n , and set $ka_{n,m} = \tau Ka_{n,m}$, the first few irregular frequencies, for $\nu = 0.25$, are

①	$ka_{1,1} = 1.942691$;	$Ka_{1,1} = 3.364839$
②	$ka_{0,1} = 2.212236$;	$Ka_{0,1} = 3.831706$
③	$ka_{2,1} = 3.015134$;	$Ka_{2,1} = 5.222365$
④	$ka_{1,2} = 3.105644$;	$Ka_{1,2} = 5.379133$
⑤	$ka_{0,2} = 3.831706$;	$Ka_{0,2} = 6.636709$
⑥	$ka_{3,1} = 3.905663$;	$Ka_{3,1} = 6.764807$
⑦	$ka_{2,2} = 4.004479$;	$Ka_{2,2} = 6.935962$
⑧	$ka_{0,3} = 4.050451$;	$Ka_{0,3} = 7.015587$

A table of these irregular frequencies for $Ka \leq 20$ and the same Poisson's ratios is given in Niwa, Kobayashi and Kitahara (1982).

The computations have been performed for different values of Ka ranging between 0.1 and 7.5, and for $\nu = 0.25$. In particular, we have included the values of Ka which correspond to irregular frequencies. For both types of incident waves and for all values of Ka , the number of elements taken is $N = 30$. However, the numerical results seem to suggest that the number of elements required depends on the wavelength of the incident field. Indeed, we have found that if we choose the length of an element to be about $1/10$ of the wave length of the incident field, the error in the solution is reasonably small (provided we are not near an irregular frequency).

We have plotted the errors in both the radial and tangential components of the solution, $E(u_r)$ and $E(u_\theta)$, for both types of incident waves, as functions of Ka . The results are shown in Fig. 6.3.(a-d). In Fig. 6.3, a) and b) show $E(u_r)$ and $E(u_\theta)$, respectively, when the incident field is a P-wave, while c) and d) are the corresponding results for an incident S-wave. Note that, as expected, the errors become large when Ka is in the neighbourhood of an irregular frequency. Furthermore, in Fig. 6.3, a comparison of a) with b) and c) with d) shows that the radial component of the displacement, unlike the tangential one, is not affected by the presence of the irregular frequencies ② and ⑧, while at the irregular frequency ⑤ the roles are reversed. This is explained as follows. At an irregular frequency the solution of the non-modified integral equation is not unique. It consists of a solution of the inhomogeneous equation plus a linear combination of the solutions of the corresponding homogeneous equation. At the irregular frequencies ② and ⑧ the radial component of the non-trivial solution of the related interior

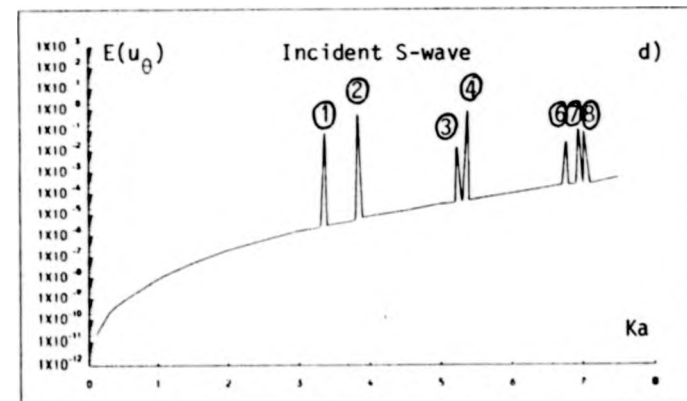
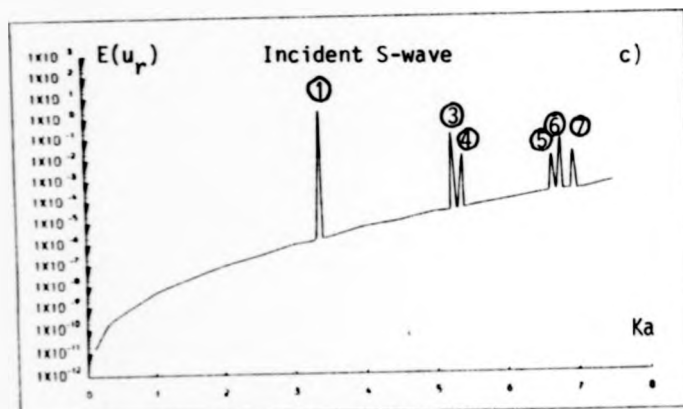
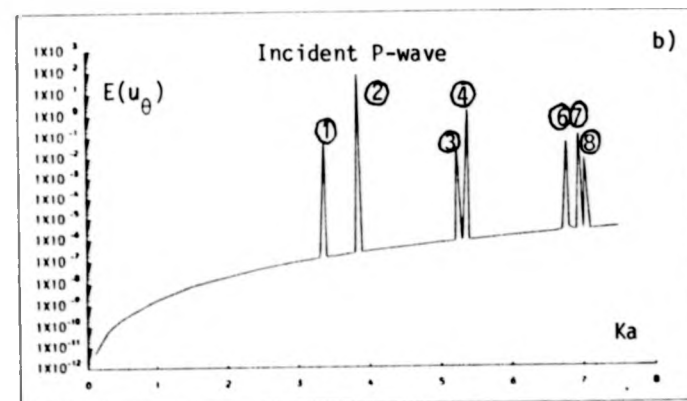
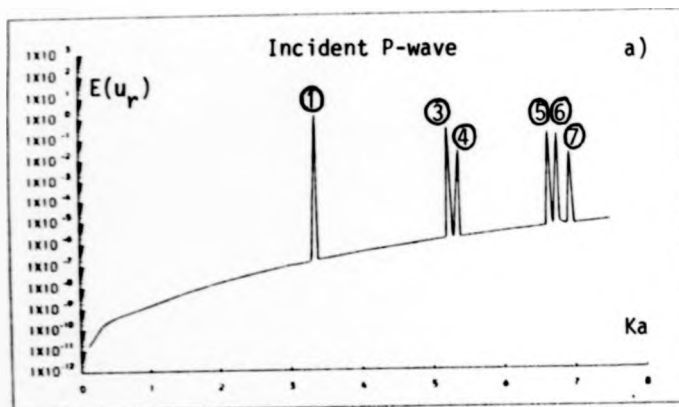


Fig. 6.3 Errors in the numerical solution of the non-modified integral equation (6.2.1) as a function of Ka for a circular cavity. Number of elements, $N = 30$.

problem I_h is identically zero (see Appendix A4) while at the irregular frequency ⑤ it is the tangential component that is identically zero (see Appendix A4). It follows (see Appendix A4) that at the irregular frequencies ② and ⑧ the radial component of the solutions of the homogeneous equation is zero, while at the irregular frequency ⑤ it is the tangential component that is zero. It can thus be concluded that at ② and ⑧ the radial component of the solution of the inhomogeneous equation is not affected, while at ⑤ the roles are reversed.

VI.4 Numerical Solution of the Modified Integral Equation for Scattering by a Circular Cavity

In this section we present numerical results obtained with the modified integral equation (5.3.4) and examine how the results of the previous section are affected. No cross-terms are included and, for the sake of simplicity, all the coefficients of the multipoles will be assumed equal and independent of m , i.e.

$$a_m = b_m = \gamma. \quad (6.4.1)$$

The location of the origin O , where the multipoles are singular, is characterised by the distance d over which it has been shifted in the x -direction, away from the centre of the circular cavity (see Fig. 6.4.1).

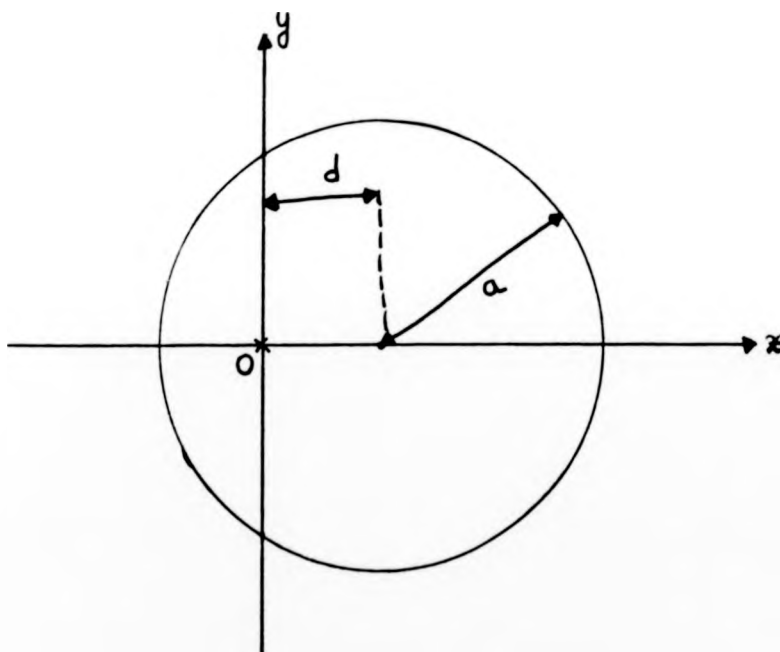


Fig. 6.4.1 Configuration of the scatterer.

The incident waves are the same as in the previous section, i.e. a P-wave and an S-wave, defined by (6.3.1). These can be expanded as follows (see, e.g., Abramowitz & Stegun (1964), p.361).

$$\underline{u}^{inc}(p) = \begin{cases} U^P \sum_{\sigma=1}^2 \sum_{m=0}^{\infty} A_m^{\sigma} \underline{\nabla} \hat{\phi}_m^{\sigma}(p) & \text{for a P-wave} \\ U^S \sum_{\sigma=1}^2 \sum_{m=0}^{\infty} A_m^{\sigma} \underline{\nabla} \cdot (\hat{\psi}_m^{\sigma}(p) \underline{e}_3) & \text{for an S-wave,} \end{cases} \quad (6.4.2)$$

where

$$A_m^{\sigma} = i^m E_m^{\sigma}(\alpha). \quad (6.4.3)$$

Substituting (6.4.2) in (5.3.5), yields for the right-hand side of (5.3.4)

$$\underline{\hat{u}}^{inc}(p) = \underline{u}^{inc}(p) + \gamma \begin{cases} U^P \sum_{\sigma=1}^2 \sum_{m=0}^M A_m^{\sigma} \underline{\nabla} \phi_m^{\sigma}(p) & \text{for a P-wave} \\ U^S \sum_{\sigma=1}^2 \sum_{m=0}^M A_m^{\sigma} \underline{\nabla} \cdot (\psi_m^{\sigma}(p) \underline{e}_3) & \text{for an S-wave.} \end{cases}$$

Because of the symmetry in our configuration, the angle of incidence α is taken to be zero.

Equation (5.3.4) is now discretized in the manner described in section VI.2, leading to a linear system of algebraic equations similar to (6.2.7). This is then solved numerically in the same way and under the same conditions as outlined in the previous two sections. The solution is compared to the exact one by computing the error defined in (6.3.2). The number of elements taken is $N = 30$ and Poisson's ratio is still 0.25. The results obtained are discussed below.

In Figs. 6.4.2-5 the errors in both components of the solution and for both types of incident waves are plotted against Ka . The range for Ka

is the same as before, i.e. (0.1, 7.5). The parameters γ and d/a have been assigned the following values: $\gamma = 0.5$ and $d/a = 0$. As required in the theory (see previous chapter), the choice of γ does satisfy condition (5.3.25), i.e.

$$\operatorname{Re}\{\gamma\}/|\gamma|^2 \neq -1. \quad (6.4.4)$$

The effect of adding one, two, three and four terms to the non-modified fundamental solution \underline{G}^f is presented in Figs. 6.4.2-5. Adding the first term, i.e. $M = 0$, removes peaks ②, ⑤ and ⑧ as shown in Figs. 6.4(2-5).a. When both the first and second terms are added, i.e. $M = 1$, peaks ① and ④ are also eliminated as depicted in Figs. 6.4.(2-5).b. If the third term is added, i.e. $M = 2$, peaks ③ and ⑦ are also removed as shown in Figs. 6.4.(2-5).c. Finally, adding the fourth term, i.e. $M = 3$ frees the entire interval of all the irregular frequencies as seen in Figs. 6.4.(2-5).d.

In table 6.4.6 (6.4.7) the errors, $E(u_r)$ and $E(u_\theta)$, in the numerical solution of the modified integral equation (5.3.4), when the incident field is a P-wave (an S-wave), are given as evaluated at the values of Ka which corresponds to the irregular frequencies in the range (0.1, 7.5). This is done for several values of M and d/a , while γ remains unchanged, i.e. $\gamma = \frac{1}{2}$. In the entries 1, 2, 3 and 4 we have $d/a = 0$ and $M = 0, 1, 2$ and 3 respectively. Note, as we have seen earlier, that four terms are required to free the interval considered of all the irregular frequencies. However, in the entries 5 and 6, where $d/a = 0.4$ and 0.8 respectively, it can be seen that the addition of only the first term, i.e. $M = 0$, suffices to eliminate all the irregular frequencies in the range. We may therefore conclude that the effect of shifting the origin away from the centre of the cavity eliminates more irregular frequencies with fewer additional terms.

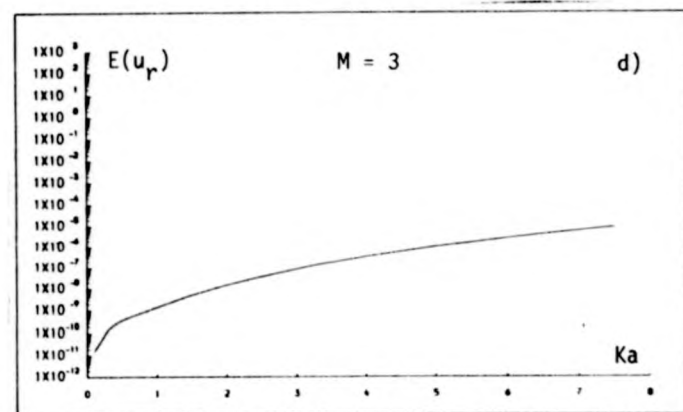
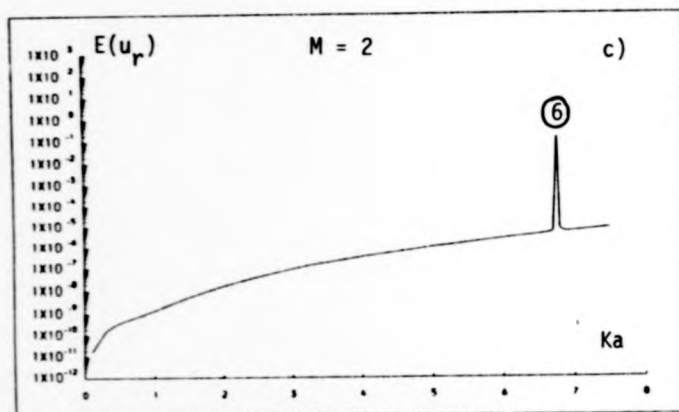
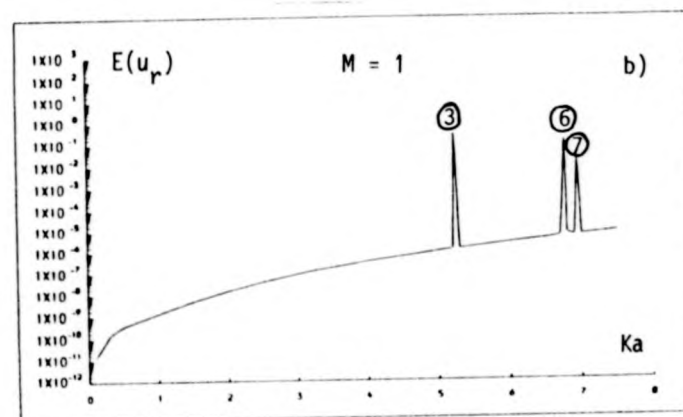
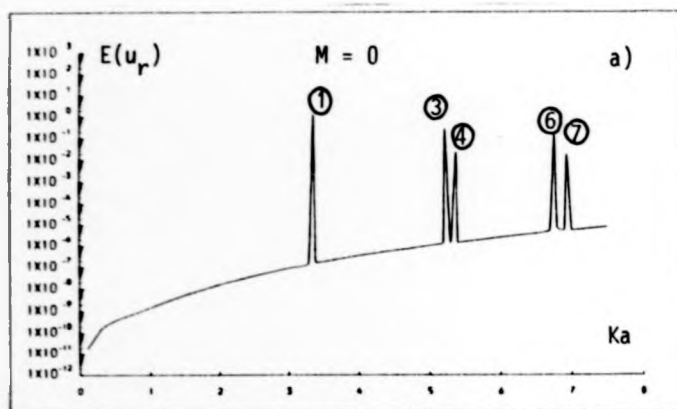


Fig. 6.4.2 Effect of adding $M+1$ terms to \underline{G}^f on the error $E(u_r)$, as a function of Ka , for a circular cavity. Number of elements, $N=30$. Incident P-wave. $d/a = 0$. $\gamma = \frac{1}{2}$.

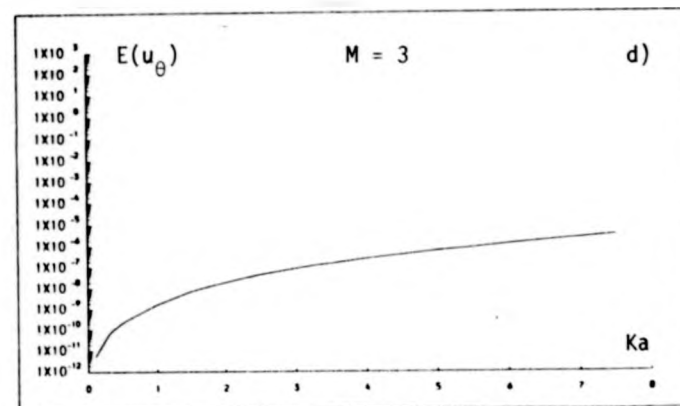
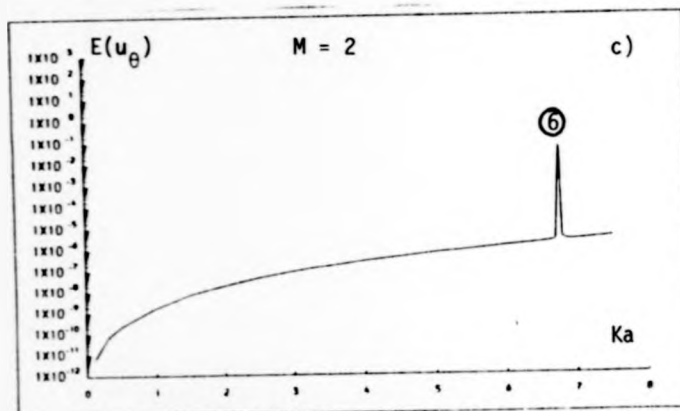
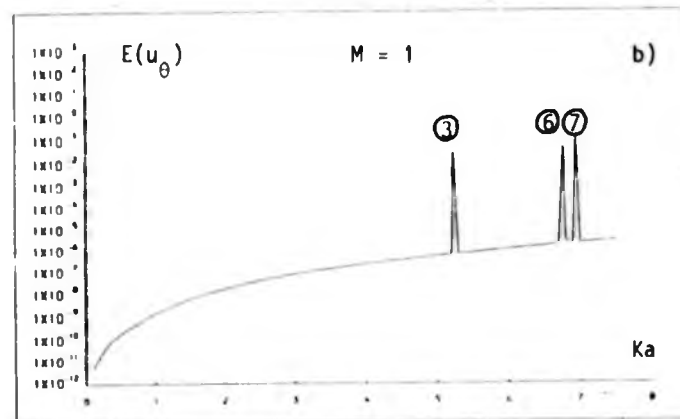
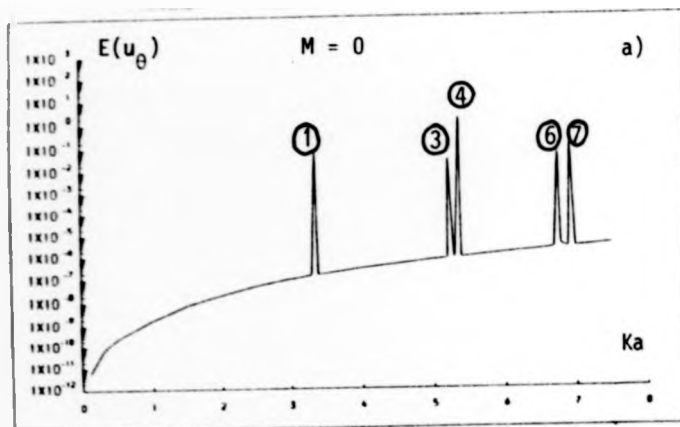


Fig. 6.4.3 Effect of adding $M+1$ terms to G^f on the error $E(u_\theta)$, as a function of Ka , for a circular cavity. Number of elements, $N=30$. Incident P-wave. $d/a = 0$. $\gamma = \frac{1}{2}$.

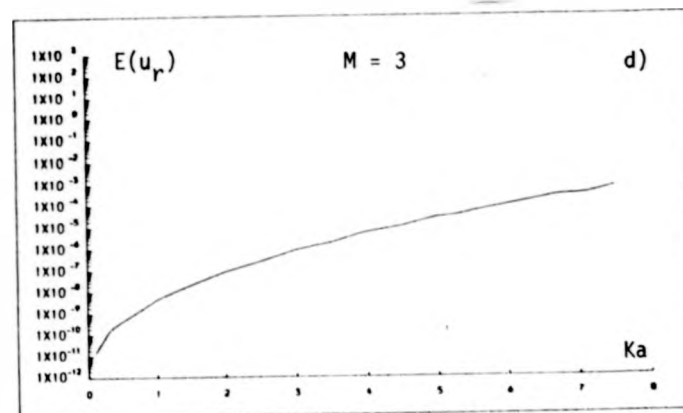
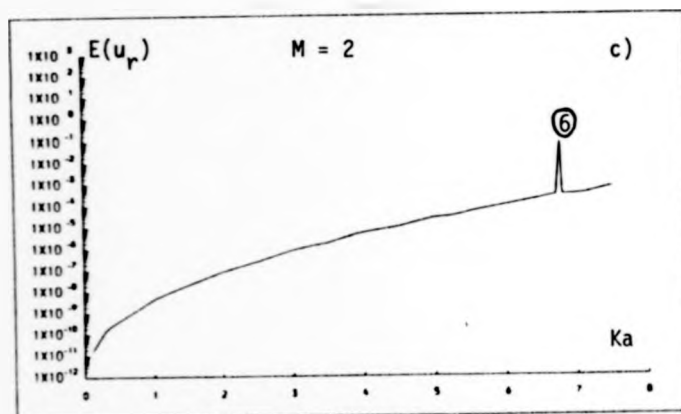
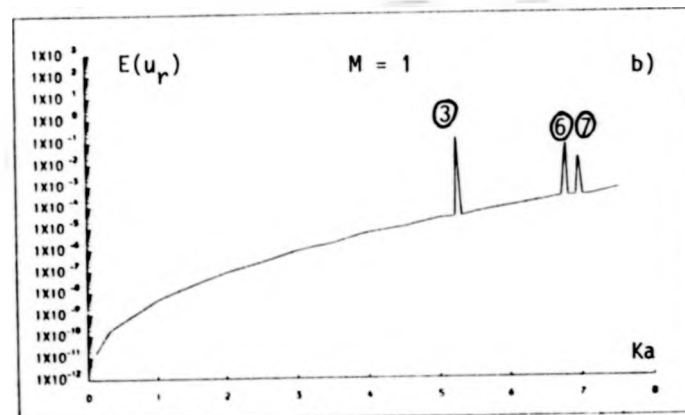
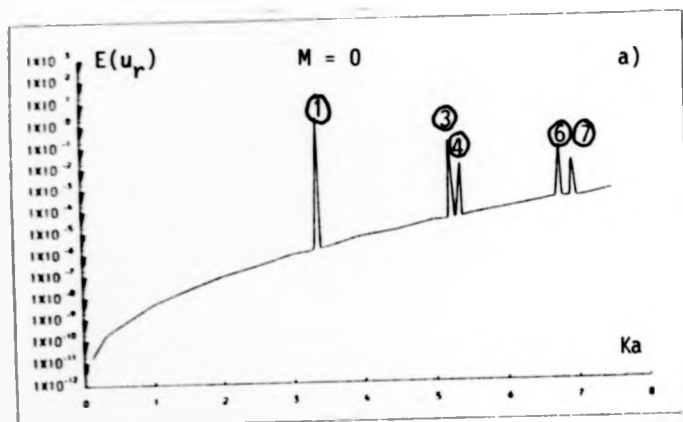


Fig. 6.4.4 Effect of adding $M+1$ terms to \underline{G}^f on the error $E(u_r)$, as a function of Ka , for a circular cavity. Number of elements, $N = 30$. Incident S-wave. $d/a = 0$. $\gamma = \frac{1}{2}$.

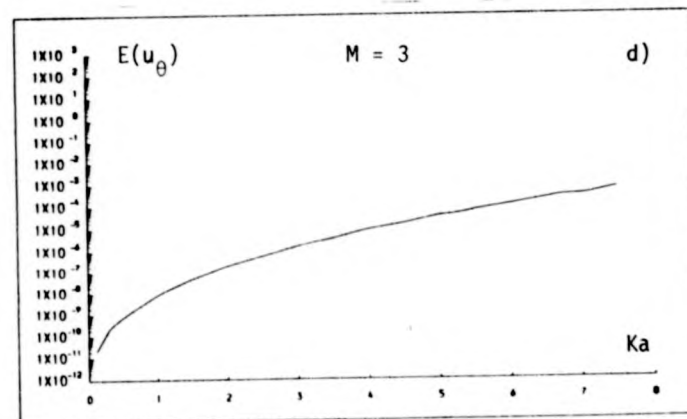
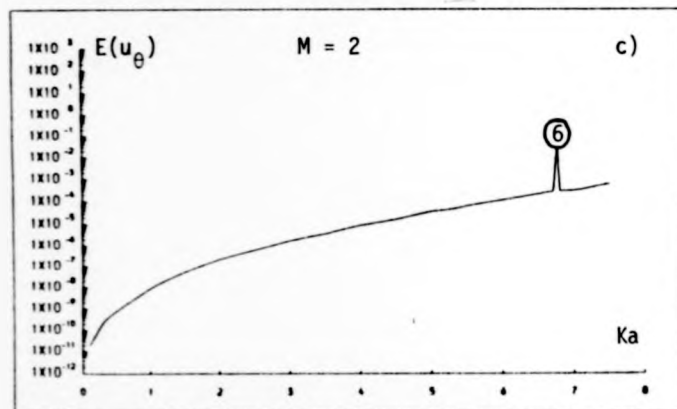
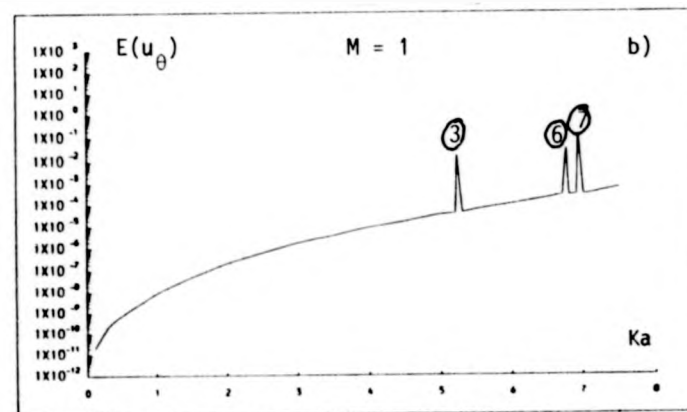
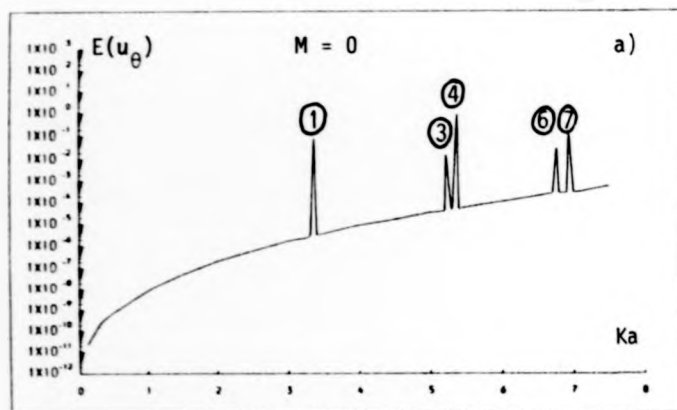


Fig. 6.4.5 Effect of adding $M+1$ terms to G^f on the error $E(u_\theta)$, as a function of Ka , for a circular cavity. Number of elements, $N=30$. Incident S-wave. $d/a = 0$. $\gamma = \frac{1}{2}$.

Entries	M	d/a	①	②	③	④	⑤	⑥	⑦	⑧
1	0	0	1.4	2.7 (-7)	2.8 (-1)	2.4 (-2)	4.6 (-6)	1.4 (-1)	1.8 (-2)	5.9 (-6)
			5.8	2.2 (-7)	2.9 (-2)	1.5	2.0 (-6)	5.1 (-2)	1.2 (-1)	2.4 (-6)
2	1	0	1.5 (-7)	2.7 (-7)	2.8 (-1)	1.4 (-6)	4.6 (-6)	1.4 (-1)	1.8 (-2)	5.9 (-6)
			1.4 (-7)	2.2 (-7)	2.9 (-2)	8.1 (-7)	2.0 (-6)	5.1 (-2)	1.2 (-1)	2.4 (-6)
3	2	0	1.5 (-7)	2.7 (-7)	1.2 (-6)	1.4 (-6)	4.6 (-6)	1.4 (-1)	5.6 (-6)	5.9 (-6)
			1.4 (-7)	2.2 (-7)	7.3 (-7)	8.1 (-7)	2.0 (-6)	5.1 (-2)	2.4 (-6)	2.4 (-6)
4	3	0	1.5 (-7)	2.7 (-7)	1.2 (-6)	1.4 (-6)	4.4 (-6)	4.8 (-6)	5.5 (-6)	5.8 (-6)
			1.4 (-7)	2.2 (-7)	7.3 (-7)	8.1 (-7)	1.9 (-6)	2.1 (-6)	2.3 (-6)	2.4 (-6)
5	0	0.4	1.5 (-7)	2.7 (-7)	1.4 (-6)	1.4 (-6)	4.6 (-6)	3.8 (-4)	5.6 (-6)	5.9 (-6)
			1.4 (-7)	2.2 (-7)	7.4 (-7)	8.3 (-7)	2.0 (-7)	1.4 (-4)	2.4 (-6)	2.4 (-6)
6	0	0.8	1.5 (-7)	2.7 (-7)	1.2 (-6)	1.4 (-6)	5.0 (-6)	5.2 (-6)	5.6 (-6)	5.9 (-6)
			1.4 (-7)	2.2 (-7)	7.3 (-7)	8.8 (-7)	2.0 (-6)	2.2 (-6)	2.6 (-6)	3.9 (-6)

Table 6.4.6 Effect of shifting the origin on the errors, $E(u_r)$ and $E(u_\theta)$, at the irregular frequencies. In each case, the upper (lower) entry is $E(u_r)$ ($E(u_\theta)$). Number of elements, $N = 30$. Incident P-wave. $\gamma = \frac{1}{2}$.

Entries	M	d/a	①	②	③	④	⑤	⑥	⑦	⑧
1	0	0	1.9	3.9 (-6)	1.3 (-1)	1.2 (-2)	2.4 (-4)	7.0 (-2)	1.6 (-2)	2.9 (-4)
			8.1 (-2)	6.6 (-6)	1.5 (-2)	8.6 (-1)	2.4 (-4)	2.9 (-2)	1.2 (-1)	3.0 (-4)
2	1	0	1.5 (-6)	3.9 (-6)	1.3 (-1)	3.7 (-5)	2.4 (-4)	7.0 (-2)	1.6 (-2)	2.9 (-4)
			3.0 (-6)	6.6 (-6)	1.5 (-2)	4.9 (-5)	2.4 (-4)	2.9 (-2)	1.2 (-1)	3.0 (-4)
3	2	0	1.5 (-6)	3.9 (-6)	3.0 (-5)	3.7 (-5)	2.4 (-4)	7.0 (-2)	2.8 (-4)	2.9 (-4)
			3.0 (-6)	6.6 (-6)	4.0 (-5)	4.9 (-5)	2.4 (-4)	2.9 (-2)	2.9 (-4)	3.0 (-4)
4	3	0	1.5 (-6)	3.9 (-6)	3.0 (-5)	3.7 (-5)	2.4 (-4)	2.8 (-4)	2.8 (-4)	2.9 (-4)
			3.0 (-6)	6.6 (-6)	4.0 (-5)	4.9 (-5)	2.4 (-4)	2.8 (-4)	2.9 (-4)	3.0 (-4)
5	0	0.4	1.5 (-6)	3.9 (-6)	3.0 (-5)	3.7 (-5)	2.4 (-4)	2.7 (-4)	2.8 (-4)	2.9 (-4)
			3.0 (-6)	6.6 (-6)	4.0 (-5)	4.9 (-5)	2.4 (-4)	2.7 (-4)	2.9 (-4)	3.0 (-4)
6	0	0.8	1.5 (-6)	3.9 (-6)	3.0 (-5)	3.7 (-5)	2.4 (-4)	2.7 (-4)	2.8 (-4)	2.9 (-4)
			3.0 (-6)	6.6 (-6)	4.0 (-5)	4.9 (-5)	2.4 (-4)	2.7 (-4)	2.9 (-4)	3.1 (-4)

Table 6.4.7 Effect of shifting the origin on the errors, $E(u_r)$ and $E(u_\theta)$, at the irregular frequencies. In each case, the upper (lower) entry is $E(u_r)(E(u_\theta))$.

Number of elements, $N = 30$. Incident S-wave. $\gamma = \frac{1}{2}$.

We now consider the case when the choice of γ does not satisfy the required condition (6.4.4). In table 6.4.8 the errors, $E(u_r)$ and $E(u_\theta)$, in the numerical solution of the modified integral equation (5.3.4), for both an incident P-wave and an incident S-wave, are computed at the irregular frequency ① i.e. $Ka = 3.364839$. Entries 1-3 and 4-6 correspond to an incident P-wave and an incident S-wave respectively. In all entries we have $M = 1$ and $d/a = 0$, while γ takes the indicated values. Although the choices of γ do not satisfy condition (6.4.4), the irregular frequency ① is still eliminated. It may therefore be concluded that condition (6.4.4) is not under all circumstances a necessary one to eliminate the difficulties observed at the irregular frequencies. This can also be seen in the proof given in section V.3.

Entries	γ	① $ka = 1.942691, Ka = 3.364839$	
		$E(u_r)$	$E(u_\theta)$
1	-1.0	1.5 (-7)	1.4 (-7)
2	$\frac{1}{2}(-1+i)$	1.5 (-7)	1.4 (-7)
3	$\frac{1}{4}(\sqrt{3}-2+i)$	1.5 (-7)	1.4 (-7)
4	-1.0	1.5 (-6)	3.0 (-6)
5	$\frac{1}{2}(-1+i)$	1.5 (-6)	3.0 (-6)
6	$\frac{1}{4}(\sqrt{3}-2+i)$	1.5 (-6)	3.0 (-6)

Table 6.4.8 Effect on the errors, $E(u_r)$ and $E(u_\theta)$,
at the irregular frequency ① when
 γ does not satisfy (6.4.4).

Entries 1-3: Incident P-wave.

Entries 4-6: Incident S-wave.

Number of elements, $N = 30$.

$M = 1$. $\frac{d}{a} = 0$.

VI.5 Some Numerical Results for the Scattering by an Elliptical Cavity

We conclude this chapter by presenting some numerical results for the scattering by an elliptical cavity. The incident fields considered are the same as before, i.e. a P-wave and an S-wave as defined by (6.3.1). The elliptical cavity is characterised by the ratio, H , of its semi-major axis, a , to its semi-minor axis, b , i.e. $H = a/b$ (see Fig. 6.5.1).

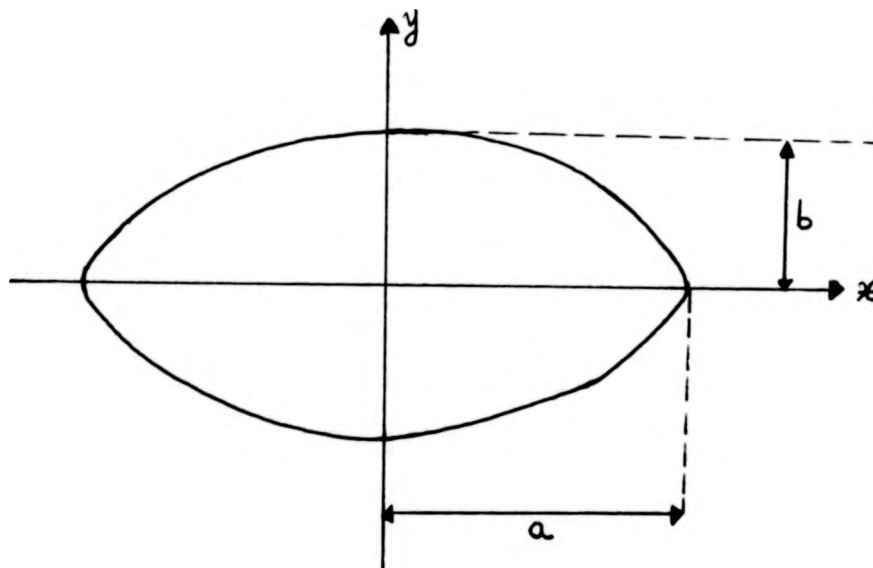


Fig. 6.5.1 Configuration of the scatterer.

The results are obtained from the numerical solution of the non-modified integral equation (6.2.1). The solution, i.e. the complex displacement, is made dimensionless by dividing by the complex amplitude of the incident field, i.e. ikU^P for the P-wave and $-ikU^S$ for the S-wave. As the explicit expression of the exact solution cannot be obtained in this case (see, e.g., Pao & Mow (1973), p.451), the modulus of the determinant of the matrix M (see section VI.2) is used to ensure that we are not near an irregular frequency.

The computations have been performed for two values of Ka , 2.0 and 4.0, and two angles of incidence, $\alpha = 0^\circ$ and 45° , while Poisson's ratio ν and the parameter H are fixed at 0.25 and 2.0 respectively. The number of elements N is chosen so that the length of an element of the boundary is approximately $1/10$ of the shear-wave length. The results are presented in tables 6.5.2-5 and given to three (3) decimal places. In the first column, we have the angle η (in degrees) which determines the position of the station on the boundary. This angle, however, is not the polar angle but is a plane elliptical coordinate, defined by

$$x = a \cos \eta$$

and

$$y = b \sin \eta,$$

where x and y are the cartesian coordinates of the station. The second and third columns represent the dimensionless components of the displacement in the x and y -directions, respectively, for the incident P-wave. The fourth and fifth columns correspond to the incident S-wave. In tables 6.5.2 and 3, where the angle of incidence is $\alpha = 0^\circ$, the results are given for the upper half of the boundary only, since there is a symmetry about the x -axis. However, in tables 6.5.4 and 5, where the angle of incidence is $\alpha = 45^\circ$, the results are given for both the upper and lower half of the boundary, as in this case there is no symmetry.

The results given in tables 6.5.2-5 for the incident P-wave were obtained using more elements than actually required by the criterion mentioned in section VI.3. This criterion suggests taking $N \approx 12$ for $Ka = 2.0$ and $N \approx 24$ for $Ka = 4.0$. Some results with these values of N are given in table 6.5.6, where the angle of incidence is $\alpha = 0^\circ$, and these should be compared with the corresponding results in tables 6.5.2 and 3. It can be seen that they agree to at least two (2) decimal places.

η (°)	Incident P-wave		Incident S-wave	
	u_x/iku^P	u_y/iku^P	$-u_x/iku^S$	$-u_y/iku^S$
0	-0.453	0.000	0.000	-0.897
	0.742	0.000	0.000	-0.206
9	-0.423	-0.073	-0.225	-0.875
	0.741	0.000	-0.241	-0.162
18	-0.332	-0.142	-0.448	-0.804
	0.736	0.003	-0.452	-0.039
27	-0.187	-0.204	-0.663	-0.672
	0.720	0.008	-0.600	0.155
36	0.008	-0.255	-0.856	-0.466
	0.685	0.019	-0.664	0.394
45	0.238	-0.296	-1.009	-0.177
	0.620	0.033	-0.622	0.641
54	0.488	-0.323	-1.097	0.178
	0.515	0.050	-0.480	0.844
63	0.735	-0.340	-1.096	0.570
	0.366	0.068	-0.248	0.945
72	0.957	-0.346	-1.002	0.939
	0.171	0.085	0.033	0.906
81	1.131	-0.343	-0.813	1.217
	-0.062	0.101	0.318	0.704
90	1.238	-0.331	-0.563	1.353
	-0.320	0.113	0.562	0.368
99	1.272	-0.310	-0.287	1.313
	-0.585	0.123	0.721	-0.052
108	1.231	-0.281	-0.028	1.111
	-0.838	0.127	0.791	-0.482
117	1.130	-0.244	0.173	0.789
	-1.064	0.128	0.768	-0.853
126	0.985	-0.204	0.306	0.410
	-1.250	0.122	0.681	-1.125
135	0.818	-0.160	0.360	0.035
	-1.392	0.111	0.558	-1.284
144	0.651	-0.121	0.352	-0.292
	-1.493	0.096	0.422	-1.348
153	0.502	-0.083	0.296	-0.546
	-1.558	0.075	0.296	-1.345
162	0.387	-0.052	0.210	-0.721
	-1.596	0.052	0.183	-1.312
171	0.314	-0.024	0.108	-0.821
	-1.616	0.026	0.088	-1.280
180	0.289	0.000	0.000	-0.854
	-1.621	0.000	0.000	-1.267

Table 6.5.2 Values of the complex displacements at different stations on the boundary. In each case, the upper (lower) entry is the real (imag.) part.
 Number of elements, $N = 20$. $Ka = 2.0$. $\nu = 0.25$.
 Angle of incidence, $\alpha = 0^\circ$. $H = 2.0$.
 $|\text{Determinant}(M)| = 0.284$.

η (°)	Incident P-wave		Incident S-wave	
	u_x/ikU^P	u_y/ikU^P	$-u_x/ikU^S$	$-u_y/ikU^S$
0	-0.422	0.000	0.000	1.028
	-0.410	0.000	0.000	0.251
9	-0.417	0.004	0.522	0.946
	-0.358	-0.178	0.169	0.161
18	-0.396	-0.012	0.950	0.688
	-0.207	-0.335	0.226	-0.080
27	-0.342	-0.059	1.180	0.241
	0.027	-0.449	0.094	-0.386
36	-0.233	-0.138	1.116	-0.348
	0.313	-0.500	-0.227	-0.609
45	-0.051	-0.235	0.711	-0.922
	0.602	-0.478	-0.634	-0.580
54	0.212	-0.320	0.031	-1.223
	0.826	-0.387	-0.933	-0.211
63	0.537	-0.365	-0.712	-1.012
	0.915	-0.252	-0.924	0.391
72	0.875	-0.355	-1.221	-0.274
	0.811	-0.109	-0.529	0.916
81	1.154	-0.301	-1.262	0.683
	0.498	0.008	0.129	1.013
90	1.297	-0.230	-0.815	1.354
	0.017	0.087	0.766	0.549
99	1.248	-0.172	-0.107	1.360
	-0.544	0.134	1.106	-0.259
108	0.996	-0.138	0.534	0.707
	-1.069	0.163	1.041	-0.983
117	0.584	-0.120	0.871	-0.245
	-1.463	0.183	0.674	-1.274
126	0.091	-0.103	0.864	-1.058
	-1.673	0.193	0.215	-1.057
135	-0.398	-0.079	0.636	-1.472
	-1.705	0.188	-0.154	-0.505
144	-0.819	-0.051	0.350	-1.483
	-1.608	0.167	-0.351	0.128
153	-1.138	-0.025	0.126	-1.248
	-1.446	0.131	-0.384	0.656
162	-1.350	-0.008	0.006	-0.952
	-1.284	0.089	-0.305	1.009
171	-1.469	0.000	-0.022	-0.729
	-1.168	0.044	-0.164	1.198
180	-1.507	0.000	0.000	-0.648
	-1.127	0.000	0.000	1.255

Tables 6.5.3 Values of the complex displacements at different stations on the boundary. In each case, the upper (lower) entry is the real (imag.) part.
 Number of elements, $N = 40$. $Ka = 4.0$. $\nu = 0.25$.
 Angle of incidence, $\alpha = 0^\circ$. $H = 2.0$.
 $|\text{Determinant}(M)| = 0.046$.

η (°)	Incident P-wave		Incident S-wave	
	u_x/ikU^P	u_y/ikU^P	$-u_x/ikU^S$	$-u_y/ikU^S$
0	0.113	0.224	0.584	0.391
	0.764	0.214	-0.013	0.863
18	-0.173	-0.142	0.654	-0.064
	0.710	0.094	0.119	0.447
36	-0.317	-0.417	0.532	-0.412
	0.551	-0.038	0.003	-0.040
54	-0.288	-0.526	0.223	-0.626
	0.337	-0.076	-0.328	-0.368
72	-0.088	-0.412	-0.239	-0.603
	0.123	0.046	-0.664	-0.328
90	0.225	-0.104	-0.719	-0.264
	-0.052	0.251	-0.719	0.038
108	0.546	0.261	-0.994	0.263
	-0.196	0.349	-0.332	0.380
126	0.763	0.524	-0.919	0.667
	-0.347	0.213	0.307	0.374
144	0.819	0.613	-0.559	0.722
	-0.534	-0.112	0.882	0.047
162	0.741	0.568	-0.130	0.472
	-0.743	-0.485	1.199	-0.349
180	0.609	0.488	0.180	0.115
	-0.950	-0.806	1.276	-0.648
198	0.513	0.477	0.293	-0.173
	-1.137	-1.036	1.224	-0.868
216	0.517	0.605	0.205	-0.290
	-1.285	-1.158	1.116	-1.088
234	0.645	0.876	-0.053	-0.154
	-1.348	-1.137	0.939	-1.305
252	0.865	1.208	-0.404	0.283
	-1.261	-0.949	0.645	-1.380
270	1.087	1.452	-0.699	0.929
	-0.977	-0.634	0.230	-1.127
288	1.197	1.495	-0.762	1.504
	-0.529	-0.289	-0.205	-0.508
306	1.115	1.325	-0.525	1.703
	-0.028	-0.004	-0.485	0.255
324	0.850	1.009	-0.099	1.453
	0.399	0.178	-0.496	0.834
342	0.483	0.623	0.319	0.937
	0.669	0.250	-0.281	1.034

Table 6.5.4 Values of the complex displacements at different stations on the boundary. In each case, the upper (lower) entry is the real (imag.) part.
 Number of elements, $N = 30$. $Ka = 2.0$. $\nu = 0.25$.
 Angle of incidence, $\alpha = 45^\circ$. $H = 2.0$.
 $|\text{Determinant}(M)| = 0.284$.

η ($^{\circ}$)	Incident P-wave		Incident S-wave	
	u_x/ikU^P	u_y/ikU^P	$-u_x/ikU^S$	$-u_y/ikU^S$
0	-0.314	-0.007	0.093	-1.146
	0.208	0.536	0.315	0.031
18	-0.527	-0.094	-0.470	-0.814
	-0.241	0.199	-0.032	-0.231
36	-0.478	0.128	-0.517	0.0318
	-0.392	-0.074	-0.162	0.120
54	-0.200	0.283	0.083	0.785
	-0.065	-0.202	0.124	0.362
72	0.134	-0.012	0.650	0.386
	0.467	-0.217	0.147	-0.146
90	0.353	-0.364	0.133	-0.680
	0.644	-0.050	-0.440	-0.416
108	0.488	-0.092	-0.962	-0.451
	0.283	0.228	-0.485	0.258
126	0.593	0.447	-0.936	0.612
	-0.307	0.181	0.492	0.451
144	0.492	0.508	0.231	0.624
	-0.809	-0.284	1.174	-0.195
162	0.122	0.109	1.169	-0.201
	-1.098	-0.767	0.880	-0.513
180	-0.319	-0.340	1.353	-0.807
	-1.216	-1.045	0.237	-0.234
198	-0.610	-0.567	1.180	-1.039
	-1.291	-1.211	-0.143	0.151
216	-0.600	-0.456	1.021	-1.194
	-1.410	-1.372	-0.100	0.248
234	-0.218	-0.040	0.884	-1.311
	-1.525	-1.460	0.277	-0.224
252	0.422	0.453	0.539	-0.995
	-1.475	-1.326	0.643	-1.177
270	0.984	0.859	-0.060	0.121
	-1.122	-0.935	0.530	-1.611
288	1.204	1.209	-0.389	1.391
	-0.490	-0.368	-0.051	-0.628
306	1.061	1.330	0.017	1.498
	0.202	0.232	-0.421	0.917
324	0.645	1.006	0.636	0.372
	0.629	0.653	-0.142	1.460
342	0.119	0.433	0.656	-0.763
	0.600	0.744	0.308	0.844

Table 6.5.5 Values of the complex displacements at different stations on the boundary. In each case, the upper (lower) entry is the real (imag.) part.
 Number of elements, $N = 40$. $Ka = 4.0$. $\nu = 0.25$.
 Angle of incidence, $\alpha = 45^{\circ}$. $H = 2.0$.
 $|\text{Determinant}(M)| = 0.046$.

η ($^{\circ}$)	$N = 12, Ka = 2.0$		$N = 24, Ka = 4.0$	
	u_x/ikU^p	u_y/ikU^p	u_x/ikU^p	u_y/ikU^p
0	-0.452	0.000	-0.422	0.000
	0.743	0.000	-0.410	0.000
45	0.238	-0.297	-0.050	-0.235
	0.620	0.031	0.602	-0.477
90	1.238	-0.331	1.297	-0.230
	-0.320	0.112	0.017	0.086
135	0.818	-0.159	-0.398	-0.080
	-1.392	0.111	-1.705	0.189
180	0.290	0.000	-1.506	0.000
	-1.621	0.000	-1.127	0.000

Table 6.5.6 Values of the complex displacements at a number of selected stations on the boundary. In each case, the upper (lower) entry is the real (imag.) part.

Incident P-wave. $\nu = 0.25$. $H = 2.0$.

Angle of incidence, $\alpha = 0^{\circ}$.

$$|\text{Determinant } (M)| = \begin{cases} 1.035 & \text{for } N = 12. \\ 0.607 & \text{for } N = 24. \end{cases}$$

CHAPTER VII

LOW-FREQUENCY MATCHED ASYMPTOTIC EXPANSIONS

VII.1 Introduction

In this chapter, we set out to solve the boundary value problem $S(\underline{u}^{inc})$ for low frequencies, using the method of matched asymptotic expansions (M.A.E.). This method, extensively used in fluid dynamics, has only recently been adopted to study the scattering of elastic waves by small cylindrical inhomogeneities as reviewed by Datta (1978). Buchwald (1978) developed a method for studying the diffraction of elastic waves by a small circular cylindrical cavity in an otherwise unbounded domain. His method is based on the establishment of a relationship between the equations of plane elastodynamics and elastostatics. Later Buchwald & Tran Cong (1984) extended this method, using Muskhelishvili's (1963) conformal mapping method, to the diffraction of elastic waves by a small elliptic cylindrical cavity. Here, we shed some light on some aspects of Buchwald & Tran Cong's (1984) work and extended the use of this method to the diffraction of elastic waves by a small cylindrical cavity whose smooth arbitrary cross-section can be mapped onto a circle by one of a certain class of mappings (see (7.4.8) below). Some numerical results for the elliptic case are presented and compared with those obtained from the boundary integral equation (B.I.E.) method. We shall adopt the notation of Buchwald & Tran Cong (1984) in this chapter.

VII.2 Basic Formulation

It is known that the solution of equation (2.4.1) can be expressed in terms of two potentials which satisfy the Helmholtz equation. Following Buchwald (1978), this expression has the form

$$\mu u_x^{SC} = \mu \frac{\partial \phi^{SC}}{\partial x} - \frac{\partial \psi^{SC}}{\partial y}, \quad (7.2.1a)$$

$$\mu u_y^{SC} = \mu \frac{\partial \phi^{SC}}{\partial y} + \frac{\partial \psi^{SC}}{\partial x}, \quad (7.2.1b)$$

where u_x^{SC} and u_y^{SC} are the scattered components of the displacements in the x and y-directions, respectively, and ϕ^{SC} , ψ^{SC} are the corresponding potentials. λ' , μ' are dimensionless constants given in terms of the Lamé constants λ, μ by

$$\lambda' = \lambda / (\lambda + 2\mu) = (1 - 2\tau^2), \quad (7.2.2a)$$

$$\mu' = \mu / (\lambda + 2\mu) = \tau^2, \quad (7.2.2b)$$

so that $\lambda' + 2\mu' = 1$.

Substituting (7.2.1) into (2.4.1) yields, after some manipulation,

$$\nabla^2 \left(\frac{\partial \phi^{SC}}{\partial x} - \frac{\partial \psi^{SC}}{\partial y} \right) + k^2 \left(\mu \frac{\partial \phi^{SC}}{\partial x} - \frac{\partial \psi^{SC}}{\partial y} \right) = 0, \quad (7.2.3a)$$

$$\nabla^2 \left(\frac{\partial \phi^{SC}}{\partial y} + \frac{\partial \psi^{SC}}{\partial x} \right) + k^2 \left(\mu \frac{\partial \phi^{SC}}{\partial y} + \frac{\partial \psi^{SC}}{\partial x} \right) = 0, \quad (7.2.3b)$$

where ∇^2 is the two-dimensional Laplacian operator.

If \hat{n} is the unit normal at a point on the boundary and \hat{s} is the unit tangent at the same point, such that the three vectors (\hat{n}, \hat{s}, e_3) form a right-handed local orthonormal coordinate system, the boundary condition (2.4.2) can then be expressed as

$$\tau_{nn}^{sc} = -\tau_{nn}^I, \quad \tau_{sn}^{sc} = -\tau_{sn}^I, \quad (7.2.4)$$

where $\tau_{nn}^{sc}, \tau_{sn}^{sc}$ are the scattered components of the stress tensor with respect to that local coordinate system and τ_{nn}^I, τ_{sn}^I correspond to the incident field.

VII.3 The Inner Problem and Expansion

We introduce the inner variables (x', y') and the corresponding polar coordinates (r', θ') such that

$$x' = x/L, \quad y' = y/L, \quad r' = r/L \quad \text{and} \quad \theta' = \theta, \quad (7.3.1)$$

where L is a characteristic constant length of the scatterer. In terms of these dimensionless variables, equation (7.2.3) becomes

$$\nabla'^{-2} \left(\frac{\partial \phi^{sc}}{\partial x'} - \frac{\partial \psi^{sc}}{\partial y'} \right) + (KL)^2 \left(\mu \frac{\partial \phi^{sc}}{\partial x'} - \frac{\partial \psi^{sc}}{\partial y'} \right) = 0, \quad (7.3.2a)$$

$$\nabla'^{-2} \left(\frac{\partial \phi^{sc}}{\partial y'} + \frac{\partial \psi^{sc}}{\partial x'} \right) + (KL)^2 \left(\mu \frac{\partial \phi^{sc}}{\partial y'} + \frac{\partial \psi^{sc}}{\partial x'} \right) = 0. \quad (7.3.2b)$$

Here, ∇'^{-2} denotes the two-dimensional dimensionless Laplacian operator.

As $KL \ll 1$, we assume the following asymptotic expansions for ϕ^{sc} and ψ^{sc} :

$$\phi^{SC} = \phi_0^{SC} + iKL \phi_1^{SC} + (iKL)^2 \phi_2^{SC} + \dots, \quad (7.3.3a)$$

$$\psi^{SC} = \psi_0^{SC} + iKL \psi_1^{SC} + (iKL)^2 \psi_2^{SC} + \dots, \quad (7.3.3b)$$

Substituting (7.33) into (7.3.2), and comparing coefficients of powers of iKL , we get

$$\nabla^2 \left(\frac{\partial \phi_l^{SC}}{\partial x} - \frac{\partial \psi_l^{SC}}{\partial y} \right) = \left(\mu \frac{\partial \phi_{l-2}^{SC}}{\partial x} - \frac{\partial \psi_{l-2}^{SC}}{\partial y} \right), \quad (7.3.4a)$$

$$\nabla^2 \left(\frac{\partial \phi_l^{SC}}{\partial y} + \frac{\partial \psi_l^{SC}}{\partial x} \right) = \left(\mu \frac{\partial \phi_{l-2}^{SC}}{\partial y} + \frac{\partial \psi_{l-2}^{SC}}{\partial x} \right), \quad (7.3.4b)$$

for $l = 0, 1, 2, \dots$ and the right-hand sides are zero when $l = 0, 1$.

It follows that

$$\nabla^4 \phi_l^{SC} = \mu \nabla^2 \phi_{l-2}^{SC} \quad \text{for } l \geq 2, \quad (7.3.5a)$$

$$\nabla^4 \psi_l^{SC} = \nabla^2 \psi_{l-2}^{SC} \quad \text{for } l \geq 2, \quad (7.3.5b)$$

and, for $l = 0, 1$, the functions ϕ_l^{SC}, ψ_l^{SC} satisfy the biharmonic equation

$$\nabla^4 \phi_l^{SC} = 0, \quad \nabla^4 \psi_l^{SC} = 0 \quad \text{for } l = 0, 1. \quad (7.3.6)$$

In order to avoid solving the inhomogeneous equations (7.3.5), we shall only consider equations (7.3.6), and so we are restricting ourselves to the first two terms in the expansions (7.3.3). The solution of (7.3.6) is obtained using Muskhelishvili's (1963) technique. Let

$$W_l^{SC} = \phi_l^{SC}(x^*, y^*) + i \psi_l^{SC}(x^*, y^*) \quad \text{for } l = 0, 1. \quad (7.3.7)$$

The solution can then be expressed as follows (see Buchwald (1978)):

$$w_{\ell}^{SC} = \bar{z} \Omega_{\ell}(z) + \int \omega_{\ell}(z) dz \quad \ell = 0, 1, \quad (7.3.8)$$

where $\Omega_{\ell}(z)$, $\omega_{\ell}(z)$ are functions of the complex variable $z = x' + iy'$ which are analytic in the appropriate domain, and $\bar{z} = x' - iy'$.

We now introduce the following expressions which are obtained after a substitution of (7.3.8) into the expressions of the Cartesian components of the stress tensor and the displacements:

$$\begin{aligned} \Theta_{\ell}^{SC} &= (\tau_{xx}^{SC} + \tau_{yy}^{SC})_{\ell} = \frac{2}{L^2} (1 - \mu') \nabla^2 \phi_{\ell}^{SC} = \frac{8}{L^2} (1 - \mu') \frac{\partial^2 \phi_{\ell}^{SC}}{\partial z \partial \bar{z}} \\ &= \frac{4}{L^2} (1 - \mu') \left\{ \frac{\partial^2 w_{\ell}^{SC}}{\partial z \partial \bar{z}} + \frac{\partial^2 \bar{w}_{\ell}^{SC}}{\partial z \partial \bar{z}} \right\} = \frac{4}{L^2} (1 - \mu') \{ \Omega_{\ell}'(z) + \bar{\Omega}_{\ell}'(\bar{z}) \}, \quad (7.3.9) \end{aligned}$$

$$\begin{aligned} \phi_{\ell}^{SC} &= (\tau_{xx}^{SC} - \tau_{yy}^{SC} + 2i\tau_{xy}^{SC})_{\ell} = \frac{4}{L^2} \left\{ (1 + \mu') \frac{\partial^2 w_{\ell}^{SC}}{\partial z^2} - (1 - \mu') \frac{\partial^2 \bar{w}_{\ell}^{SC}}{\partial \bar{z}^2} \right\} \\ &= -\frac{4}{L^2} (1 - \mu') \{ z \bar{\Omega}_{\ell}''(\bar{z}) + \bar{\omega}_{\ell}'(z) \}, \quad (7.3.10) \end{aligned}$$

$$\begin{aligned} D_{\ell} &= \mu(u_x^{SC} + iu_y^{SC})_{\ell} / (1 - \mu') = \frac{1}{L} \left\{ \kappa \frac{\partial w_{\ell}^{SC}}{\partial \bar{z}} - \frac{\partial \bar{w}_{\ell}^{SC}}{\partial z} \right\} \\ &= \frac{1}{L} \{ \kappa \Omega_{\ell}(z) - z \bar{\Omega}_{\ell}'(\bar{z}) - \bar{\omega}_{\ell}(z) \}, \quad (7.3.11) \end{aligned}$$

where $\Omega_{\ell}'(z)$, $\bar{\Omega}_{\ell}'(\bar{z})$ are the first and second derivatives of $\Omega_{\ell}(z)$ with respect to z and

$$\kappa = (1 + \mu') / (1 - \mu') = (1 + \tau^2) / (1 - \tau^2).$$

At this point, it should be noted that if C and E are constants

$$\Omega_R(z) = Eiz, \quad \omega_R(z) = C, \quad W_R = Eiz\bar{z} + Cz = \phi^R + i\psi^R, \quad (7.3.12)$$

corresponds to a rigid body translation with zero stresses. Furthermore, given any analytic function $\chi(z)$, a substitution in (7.3.9-11) shows that

$$W^* = \phi^* + i\psi^* = \kappa\chi(z) + \overline{\chi(z)} \quad (7.3.13)$$

corresponds to zero displacements and stresses.

We shall now give the expression of the boundary condition in terms of the complex potentials $\Omega_\ell(z)$, $\omega_\ell(z)$. If β is the angle the unit normal, at a point on the boundary, makes with the x-axis, it can be shown that

$$(\tau_{nn}^{SC} + i\tau_{sn}^{SC})_\ell = \frac{1}{2}\Theta_\ell^{SC} + \frac{1}{2}\Phi_\ell^{SC} e^{-2i\beta}. \quad (7.3.14)$$

The boundary condition (7.2.4) is then given by

$$(\tau_{nn}^{SC} + i\tau_{sn}^{SC})_\ell = \frac{1}{2}\Theta_\ell^{SC} + \frac{1}{2}\Phi_\ell^{SC} e^{-2i\beta} = -\tau_\ell^I, \quad (7.3.15)$$

where

$$\tau_\ell^I = (\tau_{nn}^I + i\tau_{sn}^I)_\ell. \quad (7.3.16)$$

Substituting for Θ_ℓ^{SC} and Φ_ℓ^{SC} , leads to

$$\{\Omega_\ell^*(z) + \overline{\Omega_\ell^*(z)}\} - \{z\overline{\Omega_\ell^*(z)} + \overline{\omega_\ell^*(z)}\} e^{-2i\beta} = -L^2\tau_\ell^I/2(1-\mu^*). \quad (7.3.17)$$

It can be shown (see, e.g., England (1971, pp. 40-41)) that (7.3.17) can be rewritten as

$$\frac{d}{dz} \{ \Omega_\ell(z) + z \overline{\Omega_\ell(z)} + \overline{\omega_\ell(z)} \} = -L^2 \cdot \tau_\ell^I / 2(1-\mu^2). \quad (7.3.18)$$

Denote by z_0 and z the complex numbers which corresponds to two points on the boundary such that z_0 is fixed and z is variable. Integrating (7.3.18) over the boundary anti-clockwise from z_0 to z yields

$$\Omega_\ell(z) + z \overline{\Omega_\ell(z)} + \overline{\omega_\ell(z)} = R_\ell(z) + \text{constant}, \quad (7.3.19)$$

where the constant is the value of the left-hand side expression for $z = z_0$ and $R_\ell(z)$ is given by

$$R_\ell(z) = -\frac{L^2}{2(1-\mu^2)} \cdot \int_{z_0}^z \tau_\ell^I dz. \quad (7.3.20)$$

VII.4 Determination of the Complex Potentials $\Omega_\ell(z)$, $\omega_\ell(z)$

In this section we describe a method for determining the complex potentials $\Omega_\ell(z)$, $\omega_\ell(z)$ through the implementation of the boundary condition (7.3.19). This method is discussed in detail by England (1971, Chapter V). But first, we consider the conformal transformation $z = m(\xi)$ which maps the exterior of the unit ^{circle} C_1 in the ξ -plane, on to the exterior of the boundary ∂D in the z -plane (see Fig. 7.4.1).

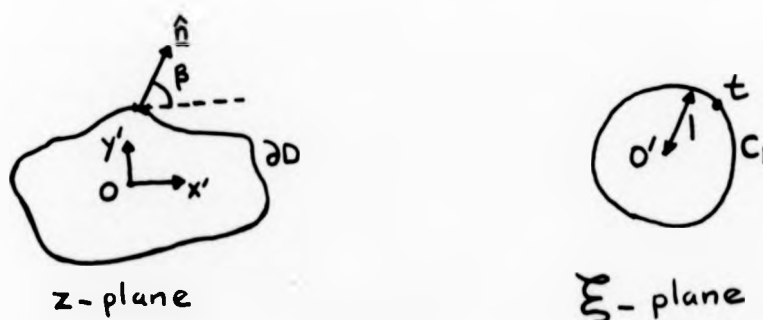


Fig. 7.4.1

$m(\xi)$ is assumed to be single valued and analytic in the domain exterior to C_1 . Moreover, we also assume that the derivative of $m(\xi)$ with respect to ξ is non-zero for $|\xi| \geq 1$. When ξ is on C_1 , it will be denoted by t . In the ξ -plane, the boundary condition (7.3.19) is then expressed as

$$\Omega_\ell(t) + \frac{m(t)}{m^*(t)} \cdot \overline{\Omega_\ell^*(t)} + \overline{\omega_\ell(t)} = R_\ell(t) + \text{constant}. \quad (7.4.1)$$

It can be shown (see, e.g. England (1971, p.138)) that the complex potentials are of the form

$$\Omega_\ell(\xi) = -\frac{i}{2\pi} \cdot \frac{(X_\ell + iY_\ell)}{(1+\kappa)} \text{Ln}\xi + \Omega_\ell^*(\xi), \quad (7.4.2)$$

$$\omega_\ell(\xi) = -\frac{i}{2\pi} \cdot \kappa \cdot \frac{(X_\ell - iY_\ell)}{(1+\kappa)} \text{Ln}\xi + \omega_\ell^*(\xi), \quad (7.4.3)$$

where $\Omega_\ell^*(\xi)$, $\omega_\ell^*(\xi)$ are analytic and single valued for $|\xi| \geq 1$ and are bounded as $|\xi| \rightarrow \infty$. $X_\ell + iY_\ell$ is the resultant force over the hole and is given by

$$X_\ell + iY_\ell = -\frac{L^2}{2(1-\mu^2)} \int_{\partial D} \tau_\ell^I dz. \quad (7.4.4)$$

Note that the complex potentials given by (7.4.2) and (7.4.3) are multiple-valued and analytic functions for $|\xi| \geq 1$. However, the complex displacement and stresses they generate are single valued for $|\xi| \geq 1$.

A substitution of (7.4.2) and (7.4.3) into (7.4.1) gives

$$\Omega_\ell^*(t) + \frac{m(t)}{m^*(t)} \overline{\Omega_\ell^*(t)} + \overline{\omega_\ell^*(t)} = F_\ell(t), \quad (7.4.5)$$

where

$$F_{\ell}(t) = R_{\ell}(t) + \frac{i}{2\pi}(X_{\ell} + iY_{\ell})\ln t - \frac{i}{2\pi} \cdot \frac{(X_{\ell} - iY_{\ell})}{(1+\kappa)} \cdot \frac{m(t)}{t \overline{m}(t)} \\ + \text{constant}$$

and is single valued on C_1 . On integrating through (7.4.5) with

$$\frac{1}{2\pi i} \int_{C_1} \frac{dt}{t - \xi} \quad \text{and} \quad |\xi| \geq 1$$

we find

$$\frac{1}{2\pi i} \int_{C_1} \left\{ \Omega_{\ell}^*(t) + \frac{m(t)}{\overline{m}(t)} \overline{\Omega_{\ell}^*(t)} + \overline{\omega_{\ell}^*(t)} \right\} \frac{dt}{t - \xi} = \frac{1}{2\pi i} \int_{C_1} \frac{F_{\ell}(t) dt}{t - \xi}.$$

As $\Omega_{\ell}^*(\xi)$ and $\omega_{\ell}^*(\xi)$ are single-valued analytic functions for $|\xi| \geq 1$ including the point at infinity, we have

$$\frac{1}{2\pi i} \int_{C_1} \frac{\overline{\omega_{\ell}^*(t)}}{t - \xi} dt = 0 \quad \text{for} \quad |\xi| \geq 1,$$

$$\frac{1}{2\pi i} \int_{C_1} \frac{\Omega_{\ell}^*(t)}{t - \xi} dt = -\Omega_{\ell}^*(\xi) + \Omega_{\ell}^*(\infty) \quad \text{for} \quad |\xi| \geq 1.$$

It follows that

$$\Omega_{\ell}^*(\xi) = \Omega_{\ell}^*(\infty) + \frac{1}{2\pi i} \int_{C_1} \frac{m(t) \overline{\Omega_{\ell}^*(t)}}{\overline{m}(t) (t - \xi)} dt - \frac{1}{2\pi i} \int_{C_1} \frac{F_{\ell}(t)}{t - \xi} dt \\ \text{for} \quad |\xi| \geq 1. \quad (7.4.6)$$

Similarly, we have

$$\omega_{\ell}^*(\xi) = \omega_{\ell}^*(\infty) + \frac{1}{2\pi i} \int_{C_1} \frac{\overline{m(t)} \Omega_{\ell}^*(t)}{m^*(t)(t-\xi)} dt - \frac{1}{2\pi i} \int_{C_1} \frac{\overline{F_{\ell}(t)}}{t-\xi} dt$$

for $|\xi| \geq 1$. (7.4.7)

In order to evaluate the integrals involving $\Omega_{\ell}^*(t)$ in (7.4.6) and (7.4.7), we need to specify the function $m(\xi)$. This is chosen to be

$$z = m(\xi) = \xi + \frac{m}{\xi^n} \quad (0 \leq m < \frac{1}{n}), \quad (7.4.8)$$

where m is real and n an integer. The condition $0 \leq m < \frac{1}{n}$ ensures that the boundary ∂D does not have loops or cusps. This conformal transformation maps the exterior of the unit circle C_1 onto the exterior of a hypotrochoid. When $n = 1, 2$ or 3 the unit circle C_1 in the ξ -plane is mapped onto an ellipse, a curvilinear triangle or a curvilinear square, respectively (see Fig. 7.4.2).

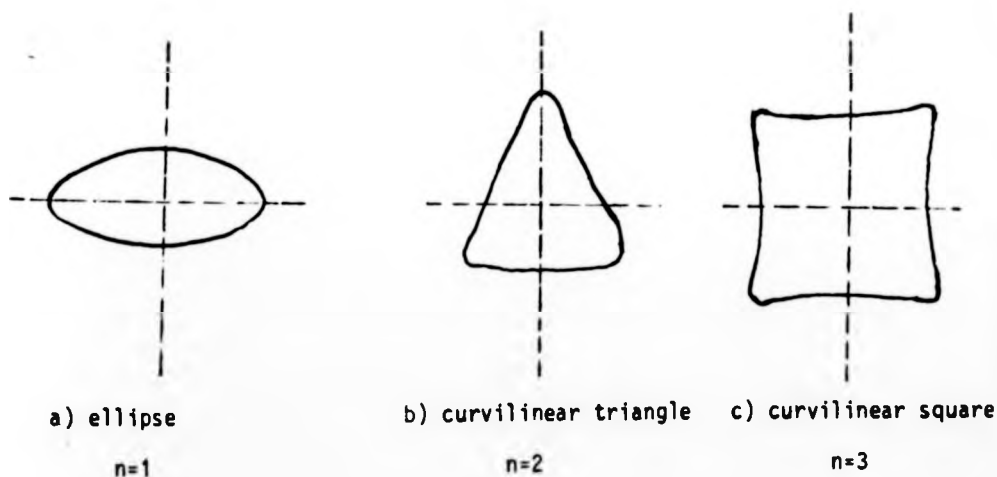


Fig. 7.4.2

From (7.4.8) it can be verified that $m(\xi)$ is a single-valued analytic function and its derivative $m'(\xi) \neq 0$ for $|\xi| \geq 1$.

Having specified $m(\xi)$, it can be shown (see, e.g., England (1971), pp.143-144)) that the integrals involving $\Omega_k^*(t)$ can be simplified to give

$$\frac{1}{2\pi i} \int_{C_1} \frac{\overline{m(t)\Omega_k^*(t)}}{\overline{m^*(t)}(t-\xi)} dt = m \sum_{i=1}^{n-2} i \frac{\alpha_k^i}{\xi} \xi^{n-(i+1)} \quad |\xi| \geq 1, \quad (7.4.9)$$

$$\frac{1}{2\pi i} \int_{C_1} \frac{\overline{m(t)\Omega_k^*(t)}}{\overline{m^*(t)}(t-\xi)} dt = -\frac{\overline{m(1/\xi)}}{\overline{m^*(\xi)}} \cdot \Omega_k^*(\xi) - m \sum_{i=1}^{n-1} i \alpha_k^i \xi^{n-(i+1)} \quad (7.4.10)$$

$$|\xi| \geq 1,$$

where α_k^i are the unknown coefficients in the expansion of $\Omega_k^*(\xi)$ for large $|\xi|$, i.e.

$$\Omega_k^*(\xi) = \sum_{i=0}^{\infty} \alpha_k^i \xi^{-i} \quad \text{for } |\xi| \rightarrow \infty. \quad (7.4.11)$$

The summation in (7.4.9) and (7.4.10) are zero when the upper limit is < 1 . Using (7.4.9) and (7.4.10), (7.4.6) and (7.4.7) become

$$\Omega_k^*(\xi) = -\frac{1}{2\pi i} \int_{C_1} \frac{F_k(t)}{t-\xi} dt + m \sum_{i=1}^{n-2} i \frac{\alpha_k^i}{\xi} \xi^{n-(i+1)} \quad (7.4.12)$$

$$\text{for } |\xi| \geq 1$$

$$\omega_k^*(\xi) = -\frac{1}{2\pi i} \int_{C_1} \frac{\overline{F_k(t)}}{\overline{t-\xi}} dt - \frac{\overline{m(1/\xi)}}{\overline{m^*(\xi)}} \cdot \Omega_k^*(\xi) - m \sum_{i=1}^{n-1} i \alpha_k^i \xi^{n-(i+1)} \quad (7.4.13)$$

$$\text{for } |\xi| \geq 1,$$

where we have assumed, without loss of generality, that $\Omega_k^*(\infty) = \omega_k^*(\infty) = 0$. It can be seen from (7.4.12) and (7.4.13) that the determination of the functions $\Omega_k^*(\xi)$, $\omega_k^*(\xi)$ is not completed yet. They are expressed in terms

of the unknown coefficients α_ℓ^i . However, these coefficients can be determined by letting $|\xi| \rightarrow \infty$ in (7.4.12) and replacing the function $\Omega_\ell^*(\xi)$ in the left-hand side by its asymptotic expansion given by (7.4.11). A comparison of the coefficients of equal powers of $1/\xi$ leads to a system of linear equations for α_ℓ^i whose solution is

$$\alpha_\ell^i = \begin{cases} -\frac{\beta_\ell^{i-1} + m[n-(i+1)]a_{n-2}\bar{\beta}^{n-(i+2)}}{i[n-(i+1)]m^2 - 1} & \text{for } 1 \leq i \leq n-2 \\ \beta_\ell^{i-1} & \text{for } i > n-2, \end{cases} \quad (7.4.14)$$

where

$$\beta_\ell^i = \frac{1}{2\pi i} \int_{C_1} F_\ell(t) t^i dt \quad 0 \leq i < \infty \quad (7.4.15)$$

and

$$a_n = \begin{cases} 0 & n \leq 0 \\ 1 & n > 0. \end{cases}$$

This completes the determination of the complex potentials $\Omega_\ell(\xi)$, $\omega_\ell(\xi)$.

We shall now consider the case of an incident P-wave propagating along the x-axis, i.e.

$$\phi^I = e^{ikx} \quad \text{and} \quad \psi^I = 0. \quad (7.4.16)$$

τ_ℓ^I is therefore given by

$$\tau_\ell^I = -\frac{(1-\mu^*)}{L^2} \cdot (kL)^2 \{1 + \gamma_1 e^{-2i\beta}\} \cdot \begin{cases} 1 & \text{for } \ell=0 \\ \sqrt{\mu^*} x & \text{for } \ell=1, \end{cases} \quad (7.4.17)$$

where

$$\gamma_1 = \mu^*/(1 - \mu^*).$$

In terms of $m(\xi)$, $e^{-2i\beta}$ is given by (see, e.g., England (1971, p.131))

$$e^{-2i\beta} = \frac{\overline{\xi m^*(\xi)}}{\xi m(\xi)}. \quad (7.4.18)$$

From (7.4.4), (7.4.17) and (7.4.18), we get for $X_\ell + iY_\ell$

$$X_\ell + iY_\ell = \begin{cases} 0 & \ell=0 \\ 2\pi i(1 + \kappa)\delta_1 & \ell=1, \end{cases} \quad (7.4.19)$$

with

$$\delta_1 = \sqrt{\mu^*}(1 - nm)^2(kL)^2/8. \quad (7.4.20)$$

The complex potentials $\Omega_\ell(\xi)$, $\omega_\ell(\xi)$, in this case, are given by

$$\Omega_0(\xi) = \frac{(kL)^2}{2} \cdot \left\{ \frac{\gamma_1}{(n-2)m^2 a_{n-2} - 1} \cdot \frac{1}{\xi} + \frac{m\gamma_1 a_{n-2}}{(n-2)m^2 a_{n-2} - 1} \cdot \frac{1}{\xi^{n-2}} + \frac{m}{\xi^n} \right\}, \quad (7.4.21)$$

$$\omega_0(\xi) = \frac{(kL)^2}{2} \left[(a_{n-2} - 1)m\gamma_1 \xi^{n-2} + (1 + nm^2)/\xi - m\gamma_1/\xi^n + \frac{(1 + nm^2)}{(\xi^{n+1} - mn)} \cdot \left\{ \frac{\gamma_1}{(n-2)m^2 a_{n-2} - 1} \cdot \xi^{n-2} + \frac{a_{n-2} \cdot m(n-2)\gamma_1}{(n-2)m^2 a_{n-2} - 1} \cdot \xi + mn/\xi \right\} \right], \quad (7.4.22)$$

$$\begin{aligned}
\Omega_1(\xi) = & \delta_1 \ln \xi + \sqrt{\mu} \frac{(kL)^2}{8} \left[\frac{\gamma_1}{2(n-3)m^2 a_{n-3} - 1} \cdot \frac{1}{\xi^2} \right. \\
& + \frac{2m\gamma_1 a_{n-3}}{2(n-3)m^2 a_{n-3} - 1} \cdot \frac{1}{\xi^{n-3}} + m(1 + nm^2) a_{n-1} / \xi^{n-1} \\
& \left. + \frac{2m}{n+1} (n-\gamma_1) / \xi^{n+1} + m^2 / \xi^{2n} \right], \quad (7.4.23)
\end{aligned}$$

$$\begin{aligned}
\omega_1(\xi) = & -\kappa \delta_1 \ln \xi + \sqrt{\mu} \frac{(kL)^2}{8} \left[2(a_{n-2} - 1)m\gamma_1 \xi^{n-3} \right. \\
& + \{1 + 2m^2(n-\gamma_1)\} / \xi^2 + \frac{2m}{n+1} (1 + n(n+1)m^2 - n\gamma_1) / \xi^{n+1} \\
& - \gamma_1 \cdot \{2ma_{n-1} / \xi^{n-1} + m^2 / \xi^{2n}\} \\
& + \frac{(1 + nm^2)}{(\xi^{n+1} - mn)} \cdot \left\{ \frac{2\gamma_1}{2(n-3)m^2 a_{n-3} - 1} \cdot \xi^{n-3} - (1 - nm^2) \xi^{n-1} \right. \\
& + \frac{2m(n-3)\gamma_1 a_{n-3}}{2(n-3)m^2 a_{n-3} - 1} \cdot \xi^2 + (n-1)(1 + nm^2)m \\
& \left. \left. + 2m(n-\gamma_1) / \xi^2 + 2nm^2 / \xi^{n+1} \right\} \right]. \quad (7.4.24)
\end{aligned}$$

VII.5 The Outer Problem and Expansion

We introduce the outer variables (X', Y') and the corresponding polar coordinates (R, Θ) such that

$$X' = K.X, Y' = K.Y, R = Kr \text{ and } \Theta = \theta. \quad (7.5.1)$$

Following Sabina & Willis (1975), the general solution of the outer problem, for small KL , can be expressed, in terms of the outer coordinates,

as follows:

$$\phi^{SC} = \sum_{j=0}^{\infty} \epsilon^j [A_j \cos j\theta + A_j^* \sin j\theta] H_j^{(1)}(\sqrt{\mu} R), \quad (7.5.2)$$

$$\psi^{SC} = \sum_{j=0}^{\infty} \epsilon^j [B_j \cos j\theta + B_j^* \sin j\theta] H_j^{(1)}(R), \quad (7.5.3)$$

where the notation $\epsilon = KL$ has been used and will be kept hereafter.

$H_j^{(1)}(.)$ denotes the Hankel function of the first kind and order j and A_j , A_j^* , B_j and B_j^* are unknown constants which may depend on ϵ if the matching requires it. They are assumed to be of $O(1)$ or smaller for $j > 0$ and $O(\epsilon)$ or smaller for $j = 0$. As Sabina & Willis (1975), we write the unknown constants as follows:

$$A_j = a_j^{(0)} + \epsilon a_j^{(1)} + \epsilon^2 a_j^{(2)} + \dots, \quad (7.5.4a)$$

$$A_j^* = a_j^{*(0)} + \epsilon a_j^{*(1)} + \epsilon^2 a_j^{*(2)} + \dots, \quad (7.5.4b)$$

$$B_j = b_j^{(0)} + \epsilon b_j^{(1)} + \epsilon^2 b_j^{(2)} + \dots, \quad (7.5.4c)$$

$$B_j^* = b_j^{*(0)} + \epsilon b_j^{*(1)} + \epsilon^2 b_j^{*(2)} + \dots, \quad (7.5.4d)$$

where we take $a_0^{(0)} = b_0^{(0)} = 0$.

It can be verified that (7.5.2-3) do satisfy the equations of motion and the radiation conditions at infinity. The boundary condition on ∂D is redundant.

VII.6 The Matching of the Inner and Outer Expansions

We shall now proceed to relate the inner and outer expansions in order to determine the unknown constants of the outer expansions. This is done using the asymptotic matching principle described by Crighton & Leppington (1973) and which is briefly outlined below.

We write $\phi_{\text{inner}}, \psi_{\text{inner}}$ for the potentials of the inner solution and $\phi_{\text{outer}}, \psi_{\text{outer}}$ for those corresponding to the outer solution. Note that each potential is expressed in terms of its corresponding coordinates, i.e. inner coordinates for $\phi_{\text{inner}}, \psi_{\text{inner}}$ and outer coordinates for $\phi_{\text{outer}}, \psi_{\text{outer}}$. We introduce the notation $\phi_{\text{inner}}^{(p)}, \psi_{\text{inner}}^{(p)}$ for the asymptotic expansion of $\phi_{\text{inner}}, \psi_{\text{inner}}$ up to and including all terms $O(\epsilon^p)$ for fixed inner coordinates; and we write $\phi_{\text{inner}}^{(p,q)}, \psi_{\text{inner}}^{(p,q)}$ for the result of rewriting $\phi_{\text{inner}}^{(p)}, \psi_{\text{inner}}^{(p)}$ in terms of outer coordinates and expanding up to and including all terms $O(\epsilon^q)$ for fixed outer coordinates. Similarly $\phi_{\text{outer}}^{(q)}, \psi_{\text{outer}}^{(q)}$ will denote the asymptotic expansion of $\phi_{\text{outer}}, \psi_{\text{outer}}$ through $O(\epsilon^q)$ as $\epsilon \rightarrow 0$ for fixed outer coordinates; and $\phi_{\text{outer}}^{(q,p)}, \psi_{\text{outer}}^{(q,p)}$ will denote the result of expressing $\phi_{\text{outer}}^{(q)}, \psi_{\text{outer}}^{(q)}$ in inner coordinates, which are then held fixed as $\epsilon \rightarrow 0$, and expanding through $O(\epsilon^p)$. Note that terms which are $O(\epsilon \ln \epsilon)$ and $O(\epsilon)$ are here both regarded as $O(\epsilon)$. With this notation, the matching principle is expressed as follows:

$$\phi_{\text{inner}}^{(p,q)} = \phi_{\text{outer}}^{(q,p)}, \quad (7.6.1a)$$

$$\psi_{\text{inner}}^{(p,q)} = \psi_{\text{outer}}^{(q,p)}. \quad (7.6.1b)$$

Note that a transformation of $\phi_{\text{inner}}^{(p,q)}, \psi_{\text{inner}}^{(p,q)}$ back into inner coordinates or $\phi_{\text{outer}}^{(q,p)}, \psi_{\text{outer}}^{(q,p)}$ back into outer coordinates must be made before the identification is performed.

The potentials ϕ_{inner} , ψ_{inner} , ϕ_{outer} , ψ_{outer} in our problem are given by

$$\phi_{\text{inner}} = \phi_0 + iKL \phi_1 + \dots + \phi^* + \phi^R, \quad (7.6.2a)$$

$$\psi_{\text{inner}} = \psi_0 + iKL \psi_1 + \dots + \psi^* + \psi^R, \quad (7.6.2b)$$

$$\phi_{\text{outer}} = \sum_{j=0}^{\infty} \epsilon^j \{A_j \cos j\theta + A_j^* \sin j\theta\} H_j^{(1)}(\sqrt{\mu}R), \quad (7.6.3a)$$

$$\psi_{\text{outer}} = \sum_{j=0}^{\infty} \epsilon^j \{B_j \cos j\theta + B_j^* \sin j\theta\} H_j^{(1)}(R), \quad (7.6.3b)$$

where the potentials ϕ^* , ψ^* , ϕ^R , ψ^R , introduced in section VII.3, are included to enable us to eliminate or add terms if the matching requires it. However, these potentials must satisfy the conditions mentioned in section VII.3, i.e. ϕ^* , ψ^* generate no displacements and no stresses and ϕ^R , ψ^R represent a rigid body translation. The potentials ϕ^* , ψ^* will be, hereafter, referred to as the null-potentials.

As only ϕ_0 , ψ_0 , ϕ_1 , ψ_1 are known, which incidentally are all of order ϵ^2 (see section VII.4), the inner potentials ϕ_{inner} , ψ_{inner} can only be expanded up to and including $O(\epsilon^3)$, i.e.

$$\phi_{\text{inner}}^{(3)} = \phi_0 + iKL \phi_1 + \phi^* + \phi^R, \quad (7.6.4a)$$

$$\psi_{\text{inner}}^{(3)} = \psi_0 + iKL \psi_1 + \psi^* + \psi^R. \quad (7.6.4b)$$

This, as it will be seen later, puts a restriction on the approximation to which the coefficients of the outer expansions can be determined.

As an illustration, we shall now apply the matching principle for the special case $n = 1$. The expressions for the complex potentials

$\Omega_0(\xi)$, $\omega_0(\xi)$, $\Omega_1(\xi)$ are (see (7.4.21-24))

$$\Omega_0(\xi) = \beta_1/\xi, \quad (7.6.5a)$$

$$\omega_0(\xi) = (1 + m^2)\beta_1/\xi(\xi^2 - m) + \beta_2/\xi, \quad (7.6.5b)$$

where

$$\beta_1 = \frac{1}{2}(m - \gamma_1)(kL)^2, \quad \beta_2 = \frac{1}{2}(1 + m^2 - 2m\gamma_1)(kL)^2,$$

and

$$\Omega_1(\xi) = \delta_1 \ln \xi + \delta_2/\xi^2, \quad (7.6.6a)$$

$$\omega_1(\xi) = \delta_3 \ln \xi + \delta_5/\xi^2 + \delta_6/(\xi^2 - m) + \delta_7/\xi^2(\xi^2 - m), \quad (7.6.6b)$$

where

$$\delta_1 = \sqrt{\mu^*}(1 - m^2)(kL)^2/8; \quad \delta_2 = \sqrt{\mu^*}(1 + m)\beta_1/4;$$

$$\delta_2 = -(1 + 2\gamma_1)\delta_1; \quad \delta_5 = \sqrt{\mu^*}(1 + m)(1 + 2m^2 - 3m\gamma_1)(kL)^2/8;$$

$$\delta_6 = -(1 + m^2)\delta_1; \quad \delta_7 = 2(1 + m^2)\delta_2.$$

Note that Buchwald & Tran Cong (1984) have an additional term $+\delta_4$ in their expression for $\omega_1(\xi)$; we discuss this at the end of the present section.

Noting that the inverse of (7.4.8), for $n=1$, is

$$\xi = \frac{1}{2}[z + (z^2 - 4m)^{\frac{1}{2}}] \quad (7.6.7)$$

it is found when expressing in terms of the outer coordinates and expanding in powers of ϵ that

$$\xi^{-1} = \frac{\epsilon}{Z} + m \frac{\epsilon^3}{Z^3} + 2m^2 \frac{\epsilon^5}{Z^5} + 5m^3 \frac{\epsilon^7}{Z^7} + O(\epsilon^9),$$

$$(\xi^2 - m)^{-1} = \frac{\epsilon^2}{Z^2} + 3m \frac{\epsilon^4}{Z^4} + 10m^2 \frac{\epsilon^6}{Z^6} + 35m^3 \frac{\epsilon^8}{Z^8} + O(\epsilon^{10}),$$

$$\text{Ln} \xi = \text{Ln}(Z\epsilon) - m \frac{\epsilon^2}{Z^2} - \frac{3}{2} m^2 \frac{\epsilon^4}{Z^4} - \frac{10}{3} m^3 \frac{\epsilon^6}{Z^6} + O(\epsilon^8),$$

where $Z = R e^{i\theta}$. Substituting the above expressions into the complex potentials, the following asymptotic expansions, in terms of the outer coordinates, for the potentials $\phi_0, \psi_0, \phi_1, \psi_1$ are eventually obtained:

$$\begin{aligned} \phi_0 = & C_1 \text{Ln} R + (C_2 + C_3 \frac{\epsilon^2}{R^2}) \cos 2\theta + m(C_2 + C_4 \frac{\epsilon^2}{R^2}) \frac{\epsilon^2}{R^2} \cos 4\theta \\ & + m^2(2C_2 + C_5 \frac{\epsilon^2}{R^2}) \frac{\epsilon^4}{R^4} \cos 6\theta + C_6 \frac{\epsilon^6}{R^6} \cos 8\theta + O(\epsilon^{10}), \end{aligned} \quad (7.6.8a)$$

$$\begin{aligned} \psi_0 = & C_1 \theta - (C_2 + C_3 \frac{\epsilon^2}{R^2}) \sin 2\theta - m(C_2 + C_4 \frac{\epsilon^2}{R^2}) \frac{\epsilon^2}{R^2} \sin 4\theta \\ & - m^2(2C_2 + C_5 \frac{\epsilon^2}{R^2}) \frac{\epsilon^4}{R^4} \sin 6\theta - C_6 \frac{\epsilon^6}{R^6} \sin 8\theta + O(\epsilon^{10}), \end{aligned} \quad (7.6.8b)$$

where

$$C_1 = \beta_2 ; C_2 = \beta_1 ; C_3 = -\frac{1}{2}(m\beta_2 + (1+m^2)\beta_1) ;$$

$$C_4 = -\frac{m}{2}\beta_2 + (1+m^2)\beta_1 ; C_5 = -\frac{5}{2}\frac{m}{3}\beta_2 + (1+m^2)\beta_1 ;$$

$$C_6 = 5m^3\beta_1 ,$$

and

$$\begin{aligned} \phi_1 = & D_1 \epsilon^{-1} R \theta \sin \theta + \{m D_2 R \epsilon^{-1} \text{Ln}(R \epsilon^{-1}) + D_3 R \epsilon^{-1} + D_4 \frac{\epsilon}{R}\} \cos \theta \\ & + \{D_5 + D_6 \frac{\epsilon^2}{R^2}\} \frac{\epsilon}{R} \cos 3\theta + \{D_7 + D_8 \frac{\epsilon^2}{R^2}\} \frac{\epsilon^3}{R^3} \cos 5\theta + D_9 \frac{\epsilon^5}{R^5} \cos 7\theta + O(\epsilon^9), \end{aligned} \quad (7.6.9a)$$

$$\begin{aligned}
\psi_1 = & -\mu D_1 \epsilon^{-1} R^0 \cos \theta + \{D_2 R \epsilon^{-1} \ln(R \epsilon^{-1}) + D_3 R \epsilon^{-1} - D_4 \frac{\epsilon}{R}\} \sin \theta \\
& - \{D_5 + D_6 \frac{\epsilon^2}{R^2} \frac{\epsilon}{R}\} \sin 3\theta - \{D_7 + D_8 \frac{\epsilon^2}{R^2} \frac{\epsilon}{R^3}\} \sin 5\theta \\
& - D_9 \frac{\epsilon^5}{R^5} \sin 7\theta + O(\epsilon^9),
\end{aligned} \tag{7.6.9b}$$

where

$$D_1 = 2(1 + \gamma_1) \delta_1; \quad D_2 = -D_1; \quad D_3 = \kappa \delta_1; \quad D_4 = (m \delta_3 - \delta_5 - \delta_6);$$

$$D_5 = \delta_2 - m \delta_1; \quad D_6 = \frac{1}{6} \{3m^2 \delta_3 - 4m \delta_5 - 6m \delta_6 - 2\delta_7\};$$

$$D_7 = \frac{m}{2} \{4\delta_2 - 3m \delta_1\}; \quad D_8 = \frac{m}{3} \{2m^2 \delta_3 - 3m \delta_5 - 6m \delta_6 - \delta_7\};$$

$$D_9 = 5m^2 \{\delta_2 - \frac{2}{3} m \delta_1\}.$$

A comparison of (7.6.8-9) with (7.6.3) shows that, to this approximation, we may express the outer solution as

$$\phi_{\text{outer}} = \sum_{j=0}^8 \epsilon^{j A_j H_j^{(1)}} (\sqrt{\mu} R) \cos j\theta, \tag{7.6.10a}$$

$$\psi_{\text{outer}} = \sum_{j=1}^8 \epsilon^{j B_j H_j^{(1)}} (R) \sin j\theta. \tag{7.6.10b}$$

Let us now apply (7.6.1) with $p = 3$ and $q = 4$. Using (7.6.4), (7.6.8) and (7.6.9), we get for $\phi_{\text{inner}}^{(3,4)}$ and $\psi_{\text{inner}}^{(3,4)}$

$$\begin{aligned}
\phi_{\text{inner}}^{(3,4)} = & C_1 \ln R + i D_1 R \sin \theta + i [\mu D_2 R \ln(R \epsilon^{-1}) + D_3 R + D_4 \epsilon^2 / R] \cos \theta \\
& + [C_2 + C_3 \frac{\epsilon^2}{R^2}] \cos 2\theta + i D_5 \frac{\epsilon^2}{R} \cos 3\theta + m C_2 \frac{\epsilon^2}{R^2} \cos 4\theta \\
& + \phi^* + \phi^R,
\end{aligned} \tag{7.6.11a}$$

$$\begin{aligned}
 \psi_{\text{inner}}^{(3,4)} = & C_1 \theta - i\mu D_1 R \theta \cos \theta + i[D_2 R \ln(R\epsilon^{-1}) + D_3 R \\
 & - D_4 \epsilon^2/R] \sin \theta - [C_2 + C_3 \epsilon^2/R^2] \sin 2\theta - iD_5 \frac{\epsilon^2}{R} \sin 3\theta \\
 & - mC_2 \frac{\epsilon^2}{R^2} \sin 4\theta + \psi^* + \psi^R.
 \end{aligned} \quad (7.6.11b)$$

Rewriting (7.6.11) back into inner coordinates gives

$$\begin{aligned}
 \phi_{\text{inner}}^{(3,4)} = & C_1 \ln \epsilon + iD_1 \epsilon r \theta \sin \theta + C_1 \ln r' + i\epsilon[\mu D_2 r \ln r' \\
 & + D_3 r' + D_4/r'] \cos \theta + [C_2 + C_3/r'^2] \cos 2\theta \\
 & + iD_5 \epsilon \frac{\cos 3\theta}{r'} + mC_2 \frac{\cos 4\theta}{r'^2} + \phi^* + \phi^R,
 \end{aligned} \quad (7.6.12a)$$

$$\begin{aligned}
 \psi_{\text{inner}}^{(3,4)} = & C_1 \theta - i\mu D_1 \epsilon r \theta \cos \theta + i\epsilon[D_2 r \ln r' + D_3 r' - D_4/r'] \sin \theta \\
 & - [C_2 + C_3/r'^2] \sin 2\theta - iD_5 \epsilon \frac{\sin 3\theta}{r'} - mC_2 \frac{\sin 4\theta}{r'^2} \\
 & + \psi^* + \psi^R.
 \end{aligned} \quad (7.6.12b)$$

For the outer expansions $\phi_{\text{outer}}^{(4)}$, $\psi_{\text{outer}}^{(4)}$ we have

$$\begin{aligned}
 \phi_{\text{outer}}^{(4)} = & \{ \epsilon a_0^{(1)} + \dots + \epsilon^4 a_0^{(4)} \} H_0^{(1)}(\sqrt{\mu} R) \\
 & + \epsilon \{ a_1^{(0)} + \dots + \epsilon^3 a_1^{(3)} \} H_1^{(1)}(\sqrt{\mu} R) \cos \theta \\
 & + \epsilon^2 \{ a_2^{(0)} + \dots + \epsilon^2 a_2^{(2)} \} H_2^{(1)}(\sqrt{\mu} R) \cos 2\theta \\
 & + \epsilon^3 \{ a_3^{(0)} + \epsilon a_3^{(1)} \} H_3^{(1)}(\sqrt{\mu} R) \cos 3\theta \\
 & + \epsilon^4 a_4^{(0)} H_4^{(1)}(\sqrt{\mu} R) \cos 4\theta,
 \end{aligned} \quad (7.6.13a)$$

$$\begin{aligned}
\psi_{\text{outer}}^{(4)} = & \epsilon \{b_1^{(0)} + \dots + \epsilon^3 b_1^{(3)}\} H_1^{(1)}(R) \sin \theta \\
& + \epsilon^2 \{b_2^{(0)} + \dots + \epsilon^2 b_2^{(2)}\} H_2^{(1)}(R) \sin 2\theta \\
& + \epsilon^3 \{b_3^{(0)} + \epsilon b_3^{(1)}\} H_3^{(1)}(R) \sin 3\theta \\
& + \epsilon^4 b_4^{(0)} H_4^{(1)}(R) \sin 4\theta.
\end{aligned} \tag{7.6.13b}$$

Rewriting (7.6.13) into inner coordinates and expanding up to and including all terms $O(\epsilon^3)$ leads to

$$\begin{aligned}
\psi_{\text{outer}}^{(4,3)} = & \{ \epsilon a_0^{(1)} + \epsilon^2 a_0^{(2)} + \epsilon^3 a_0^{(3)} \} \left[1 + \frac{2i}{\pi} (\gamma + \text{Ln}(\frac{\sqrt{\mu} r'})}{2}) \right] \\
& - \epsilon a_0^{(1)} \left[1 + \frac{2i}{\pi} (\gamma - 1 + \text{Ln}(\frac{\sqrt{\mu} r'})}{2}) \right] \mu^{-\frac{2}{3}} \frac{\epsilon^2 r^2}{4} \\
& + \left[-\frac{2i}{\pi} \{a_1^{(0)} + \dots + \epsilon^3 a_1^{(3)}\} \frac{1}{\sqrt{\mu} r'} + \frac{i}{2\pi} \{a_1^{(0)} + \epsilon a_1^{(1)}\} \right. \\
& \left. [2\gamma - \pi i - 1 + 2\text{Ln}(\sqrt{\mu} \epsilon r'/2)] \sqrt{\mu} \epsilon^2 r^2 \cos \theta - \frac{i}{\pi} \right. \\
& \left. [\{a_2^{(0)} + \dots + \epsilon^2 a_2^{(2)}\} \frac{4}{\mu^{\frac{2}{3}} r^2} + \epsilon^2 \{a_2^{(0)} + \epsilon a_2^{(1)}\} \right. \\
& \left. \cos 2\theta - \frac{i}{\pi} \{a_3^{(0)} + \epsilon a_3^{(1)}\} \frac{16}{\mu^{\frac{2}{3}} \sqrt{\mu} r^3} + \epsilon^2 \{a_3^{(0)} + \epsilon a_3^{(1)}\} \right. \\
& \left. \frac{2}{\sqrt{\mu} r'} \cos 3\theta - \frac{2i}{\pi} \{a_4^{(0)} \frac{48}{\mu^{\frac{2}{3}} r^4} + \epsilon^2 a_4^{(0)} \frac{4}{\mu^{\frac{2}{3}} r^2} \} \cos 4\theta, \tag{7.6.14a}
\end{aligned}$$

$$\begin{aligned}
\psi_{\text{outer}}^{(4,3)} = & \left[-\frac{2i}{\pi} \{b_1^{(0)} + \dots + \epsilon^3 b_1^{(3)}\} \cdot \frac{1}{r^2} + \frac{i}{2\pi} \{b_1^{(0)} + \epsilon b_1^{(1)}\} [2\gamma - 1 \right. \\
& - \pi i + 2\text{Ln}(\epsilon r/2)] \epsilon^2 r \sin \theta - \frac{i}{\pi} \{b_2^{(0)} + \dots + \epsilon^2 b_2^{(2)}\} \\
& \cdot \frac{4}{r^2} + \epsilon^2 \{b_2^{(0)} + \epsilon b_2^{(1)}\} \sin 2\theta - \frac{i}{\pi} \{b_3^{(0)} + \epsilon b_3^{(1)}\} \frac{16}{r^3} \\
& + \epsilon^2 \{b_3^{(0)} + \epsilon b_3^{(1)}\} \frac{2}{r} \sin 3\theta - \frac{2i}{\pi} [b_4^{(0)} \frac{48}{r^4} + \epsilon^2 b_4^{(0)} \frac{4}{r^2}] \\
& \left. \sin 4\theta, \right] \quad (7.6.14b)
\end{aligned}$$

where we have used the following asymptotic expansions of the Hankel functions for small argument x :

$$\begin{aligned}
H_0^{(1)}(x) = & (1 - x^2/4) + \frac{2i}{\pi} [(1 - x^2/4)(\gamma + \text{Ln}(x/2)) + x^2/4] \\
& + O(x^4 \text{Ln} x),
\end{aligned}$$

$$H_1^{(1)}(x) = -\frac{2i}{\pi} \cdot \frac{1}{x} + [2\gamma - \pi i - 1 + 2\text{Ln}(x/2)] \frac{i}{2\pi} x + O(x^3 \text{Ln} x),$$

$$H_j^{(1)}(x) = -\frac{i}{\pi} (j-2)! [(j-1) \left(\frac{2}{x}\right)^j + \left(\frac{2}{x}\right)^{j-2}] + O(x^{4-j}) \text{ for } j \geq 2,$$

γ , here, denotes Euler's constant.

An examination of (7.6.12), (7.6.14) seems to suggest the following form for the null-potential ϕ^* , ψ^* :

$$\begin{aligned}
\phi^* = & x_0 + x_1 \text{Ln} r + x_2 (r' \theta \sin \theta - r \text{Ln} r \cos \theta) + (x_3 + x_4/r) \cos \theta \\
& + x_5 \cos 2\theta / r^2 + x_6 \cos 3\theta / r^3 + x_7 \cos 4\theta / r^4, \quad (7.6.15a)
\end{aligned}$$

$$\begin{aligned}
\psi^* = & \mu^* [x_1 \theta - x_2 (r' \theta \cos \theta + r \text{Ln} r \sin \theta) + (x_3 - x_4/r) \sin \theta \\
& - x_5 \sin 2\theta / r^2 - x_6 \sin 3\theta / r^3 - x_7 \sin 4\theta / r^4], \quad (7.6.15b)
\end{aligned}$$

while for ϕ^R and ψ^R we have

$$\phi^R = C_R r \cos \theta \quad \text{and} \quad \psi^R = C_R r \sin \theta. \quad (7.6.16)$$

X_0, X_1, \dots, X_7 and C_R are constants to be determined by the matching process. Note that the expressions given by (7.6.15) satisfy the required conditions for ϕ^*, ψ^* , i.e. they generate no displacements or stresses, while the expressions given by (7.6.16) give rise to a rigid body translation but no stresses. Using (7.6.1), (7.6.12), (7.6.14), (7.6.15) and (7.6.16), and identifying coefficients of the same type leads to a set of equations, which when solved yields the following:

$$A_0 = i\pi/2 + O(\epsilon^4); \quad \epsilon A_1 = 2\pi\delta_1/\sqrt{\mu^*} - \pi\sqrt{\mu^*}D_4\epsilon^2/2 + O(\epsilon^5);$$

$$\epsilon^2 A_2 = i\pi C_2 + i\pi\mu^* C_3 \epsilon^2/4 + O(\epsilon^5);$$

$$\epsilon^3 A_3 = -\pi\sqrt{\mu^*}D_5\epsilon^2 + O(\epsilon^5); \quad \epsilon^4 A_4 = i\pi\mu^* m C_2 \epsilon^2/8 + O(\epsilon^5);$$

$$\epsilon B_1 = -2\pi\delta_1 + \pi D_4 \epsilon^2/2 + O(\epsilon^5); \quad \epsilon^2 B_2 = -i\pi C_2 - i\pi C_3 \epsilon^2/4 + O(\epsilon^5);$$

$$\epsilon^3 B_3 = \pi D_5 \epsilon^2/2 + O(\epsilon^5); \quad \epsilon^4 B_4 = -i\pi m C_2 \epsilon^2/8 + O(\epsilon^5);$$

with

$$X_0 = \frac{i\pi}{2} \frac{C_1}{\gamma_1} \left[1 + \frac{2i}{\pi} \{ \gamma + \text{Ln}(\sqrt{\mu^*}/2) + (1 + \gamma_1) \text{Ln} \epsilon \} \right];$$

$$X_1 = -\frac{1}{\mu^*} C_1; \quad X_2 = -iD_1 \epsilon; \quad X_3 = \frac{2i\delta_1 \epsilon}{(1-\mu^*)} [\text{Ln}(\sqrt{\mu^*} \epsilon^2) - b^*];$$

$$X_4 = -4i\delta_1/\mu^* \epsilon; \quad X_5 = 4C_2/\mu^* \epsilon^2; \quad X_6 = 8iD_5/\mu^* \epsilon;$$

$$X_7 = 12\pi C_2/\mu^* \epsilon^2;$$

$$C_R = i\delta_1 \epsilon \{ (1 + \gamma_1) \{ b^* (1 + \mu^*) - 2\text{Ln} \epsilon - 2\mu^* \text{Ln}(\sqrt{\mu^*} \epsilon) \} - \kappa \},$$

where $b^* = \pi i + 1 + 2\text{Ln} 2 - 2\gamma$.

From (7.6.13a) and (7.6.14a), it can be seen that no matter how far we expand $\phi_{\text{outer}}^{(q)}$, the coefficient A_0 of the outer expansion can only be determined up to the order ϵ^3 . This is due to the fact that in the expansion of $\phi_{\text{outer}}^{(q,p)}$, the value of p cannot go beyond 3 for reasons which were mentioned earlier in this section. It follows, therefore, that for consistency all terms of order ϵ^4 and higher in the remaining coefficients should be dropped. As a result of this, the expression (7.6.10) for the scattered potentials become

$$\phi_{\text{outer}} = \sum_{j=0}^2 \epsilon^j A_j H_j^{(1)}(\sqrt{\mu} R) \cos j\theta, \quad (7.6.17a)$$

$$\psi_{\text{outer}} = \sum_{j=1}^2 \epsilon^j B_j H_j^{(1)}(R) \sin j\theta, \quad (7.6.17b)$$

with the coefficients $\epsilon^j A_j$, $\epsilon^j B_j$ given by

$$A_0 = \frac{i\pi}{2} \frac{C_1}{\gamma_1} + O(\epsilon^4);$$

$$\epsilon A_1 = 2\pi\delta_1/\sqrt{\mu} + O(\epsilon^4); \quad \epsilon B_1 = -2\pi\delta_1 + O(\epsilon^4)$$

$$\epsilon^2 A_2 = i\pi C_2 + O(\epsilon^4); \quad \epsilon^2 B_2 = -i\pi C_2 + O(\epsilon^4).$$

The above results are the same as those derived by Buchwald & Tran Cong (1984). However, in the expression of the complex potentials $\omega_1(\xi)$, given by (7.6.6b), we do not have a constant term which Buchwald & Tran Cong (1984) have included in their expression. The value assigned to that constant is $-m\delta_1$. We have not been able to explain how that constant was obtained. However, it can be seen from our systematic calculation that its absence does not affect in any way the results obtained.

VII.7 Results of the matching for $n > 1$

In this section we present the results of the matching for $n > 1$. The details of these calculations are not included as they are similar to those for the case treated in the previous section ($n=1$). In this case ($n > 1$), however, the explicit expression of the inverse transformation of (7.4.8) cannot be obtained. Nevertheless, it can be seen from the previous section that only the asymptotic expansion, in terms of the outer coordinates, is required. This can be shown to be (see Appendix A5) of the following form:

$$\begin{aligned} \xi = Z\epsilon^{-1} &+ \frac{a_n}{Z^n} \epsilon^n + \frac{a_{2n+1}}{Z^{2n+1}} \epsilon^{2n+1} + \frac{a_{3n+2}}{Z^{3n+2}} \epsilon^{3n+2} \\ &+ \frac{a_{4n+3}}{Z^{4n+3}} \epsilon^{4n+3} + O(\epsilon^{5n+4}), \end{aligned} \quad (7.7.1)$$

where $Z = Re^{i\theta}$ and

$$a_n = -m ; \quad a_{2n+1} = -nm^2 ; \quad a_{3n+2} = -\frac{n(3n+1)}{2} m^3 ;$$

$$a_{4n+3} = -\frac{n(2n+1)(4n+1)}{3} m^4.$$

Note that (7.7.1) is also valid for $n = 1$.

To our order of approximation, the scattered potentials are given by

$$\phi_{\text{outer}} = \sum_{j=0}^3 \epsilon^j A_j H_j^{(1)}(\sqrt{\mu} R) \cos j\theta, \quad (7.7.2a)$$

$$\psi_{\text{outer}} = \sum_{j=1}^3 \epsilon^j B_j H_j^{(1)}(R) \sin j\theta, \quad (7.7.2b)$$

where for $n = 2$ we have

$$A_0 = i\pi(C_1 + iD_4\epsilon)/2\gamma_1;$$

$$\epsilon A_1 = 2\pi\delta_0/\sqrt{\mu} + i\pi\sqrt{\mu}C_2\epsilon/2;$$

$$\epsilon B_1 = -(2\pi\delta_0 + i\pi C_2\epsilon/2);$$

$$\epsilon^2 A_2 = i\pi(C_3 + iD_6\epsilon); \quad \epsilon^2 B_2 = -i\pi(C_3 + iD_6\epsilon);$$

$$\epsilon^3 A_3 = i\pi\sqrt{\mu}C_5\epsilon/2; \quad \epsilon^3 B_3 = -i\pi C_5\epsilon/2,$$

all with errors of $O(\epsilon^4)$, with

$$C_1 = (1 + 2m^2)\beta_3; \quad C_2 = -m\beta_1; \quad C_3 = \beta_1; \quad C_5 = \beta_2;$$

$$D_4 = \delta_6; \quad D_6 = \delta_1$$

and

$$\beta_1 = -\gamma_1(kL)^2/2; \quad \beta_2 = m(kL)^2/2; \quad \beta_3 = (kL)^2/2;$$

$$\delta_0 = \sqrt{\mu}(1 - 2m^2)(kL)^2/8; \quad \delta_1 = \sqrt{\mu}m(1 + 2m^2)(kL)^2/8;$$

$$\delta_6 = -\sqrt{\mu}m\gamma_1(kL)^2/2.$$

For $n > 2$, the coefficients $\epsilon^j A_j$, $\epsilon^j B_j$ are given in the table below, again with errors of $O(\epsilon^4)$:

n	3	4	≥ 5
A_0	$i\pi C_1/2\gamma_1$	$i\pi C_1/2\gamma_1$	$i\pi C_1/2\gamma_1$
ϵA_1	$2\pi\delta_0/\sqrt{\mu'}$	$2\pi\delta_0/\sqrt{\mu'}$	$2\pi\delta_0/\sqrt{\mu'}$
ϵB_1	$-2\pi\delta_0$	$-2\pi\delta_0$	$-2\pi\delta_0$
$\epsilon^2 A_2$	$i\pi(C_2 + C_4)$	$i\pi C_2 - \pi D_7 \epsilon$	$i\pi C_2$
$\epsilon^2 B_2$	$-i\pi(C_2 + C_4)$	$-i\pi C_2 + \pi D_7 \epsilon$	$-i\pi C_2$
$\epsilon^3 A_3$	0	$i\pi\sqrt{\mu'} C_4 \epsilon/2$	0
$\epsilon^3 B_3$	0	$-i\pi C_4 \epsilon/2$	0

where

$$C_1 = (1 + nm^2)\beta_3; \quad C_2 = \beta_1; \quad C_4 = m\beta_1; \quad D_7 = \delta_2;$$

with

$$\beta_1 = \frac{\gamma_1}{(n-2)m^2 - 1} \cdot (kL)^2/2; \quad \beta_3 = (kL)^2/2;$$

$$\delta_0 = \sqrt{\mu'}(1 - nm^2)(kL)^2/8; \quad \delta_2 = \sqrt{\mu'} \cdot \frac{2m\gamma_1 a_{n-3}}{2(n-3)m^2 a_{n-3} - 1} \cdot (kL)^2/8$$

and

$$a_n = \begin{cases} 0 & n \leq 0 \\ 1 & n > 0. \end{cases}$$

The constant C_R in the rigid body translation potentials is given by

$$C_R = i\delta_0 \epsilon [(1 + \gamma_1) \{b^*(1 + \mu') - 2\text{Ln}\epsilon - 2\mu' \text{Ln}(\sqrt{\mu'}\epsilon)\} - \kappa],$$

where

$$\delta_0 = \sqrt{\mu'}(1 - nm^2)(kL)^2/8$$

and $b^* = i\pi + 1 + 2\text{Ln}2 - 2\gamma$. We do not record the lengthy expressions for

the null-potentials here; they do not contribute to the displacement field itself.

VII.8 Comparison of the Numerical Results of the M.A.E. Method with those of the B.I.E. Method for the Special Case of the Ellipse (n=1)

We conclude this chapter by presenting some numerical results obtained from the M.A.E. method for the special case of an elliptic cavity (n=1). The ellipse is characterized by the ratio $H = a/b$ of its semi-major axis, a , to its semi-minor axis, b . The characteristic length L is chosen to be a . The incident field is as defined in section VII.4, i.e. a P-wave at zero incidence.

The quantities computed are the total dimensionless complex components in the x and y directions of the surface displacements, which are given by

$$u_x^* = \frac{L\mu u_x}{i\mu^* kL} = e^{ikL(1+m)\cos\eta} + \text{Real}(D_0)/i\gamma_1 kL + \text{Real}(D_1)/\gamma_1 \sqrt{\mu^*} - C_R/i\gamma_1 kL. \quad (7.8.1a)$$

$$u_y^* = \frac{L\mu u_y}{i\mu^* kL} = \text{Im}(D_0)/i\gamma_1 kL + \text{Im}(D_1)/\gamma_1 \sqrt{\mu^*}, \quad (7.8.1b)$$

where Real and Im denote the real and imaginary parts,

$$D_0 = \kappa\beta_1\bar{\xi} + (m\beta_1 - \beta_2)\xi,$$

$$D_1 = -m\delta_1 + \kappa\delta_2\bar{\xi}^2 - (\delta_1 + \delta_5)\xi^2 - [(m\delta_1 - 2\delta_2 + \delta_7)\xi^2 + m(m\delta_1 - 2\delta_2) + \delta_6]/(\bar{\xi}^2 - m),$$

$$C_R = i\delta_1\epsilon[(1 + \gamma_1)\{b^*(1 + \mu^*) - 2\text{Ln}\epsilon - 2\mu^*\text{Ln}(\sqrt{\mu^*}\epsilon)\} - \kappa].$$

The constants $\beta_1, \beta_2, \delta_1, \delta_2, \dots, \delta_7$ are given in section VII.6, $\xi = e^{i\eta}$ with $0 \leq \eta \leq 2\pi$ and $\bar{\xi}$ denotes the conjugate of ξ . In terms of H , m is expressed as

$$m = \frac{1}{2}(H - 1)/H.$$

Numerical results from the B.I.E. method are also presented and a comparison of these results with those obtained from the M.A.E. method is made. The number of elements N used in computing the solution from the B.I.E. method is 10. Results are presented for $Ka = 0.1$ and $H = 1.5$ and 3.0, while Poisson's ratio ν is fixed at 0.25. As D_0 and D_1 are of order ϵ^2 and C_R of order ϵ^3 , it can be seen from (7.8.1) that we expect the results from the M.A.E. method to agree with those of the B.I.E. method to the order of ϵ^2 .

In table 7.6.1a(b) we have the results for $\epsilon = 0.1$ and $H = 1.5$ (3.0). In the first column we have the angle η which determines the position of the station on the boundary. In the second and third columns we have the real and imaginary parts of the complex component of the displacements in the x -direction and in the fourth and fifth columns we have those corresponding to the y -direction. At every station, the upper (lower) entry corresponds to the results from the B.I.E. (M.A.E.) method. Note that since there is a symmetry about the x -axis, the results are only given for the upper half of the boundary. It can clearly be seen that the results agree to the order expected, i.e. two decimal places.

$$C_R = i\delta_1\epsilon[(1 + \gamma_1)\{b^*(1 + \mu^*) - 2\text{Ln}\epsilon - 2\mu^*\text{Ln}(\sqrt{\mu^*}\epsilon)\} - \kappa].$$

The constants $\beta_1, \beta_2, \delta_1, \delta_2, \dots, \delta_7$ are given in section VII.6, $\xi = e^{i\eta}$ with $0 \leq \eta \leq 2\pi$ and $\bar{\xi}$ denotes the conjugate of ξ . In terms of H , m is expressed as

$$m = \frac{1}{2}(H - 1)/H.$$

Numerical results from the B.I.E. method are also presented and a comparison of these results with those obtained from the M.A.E. method is made. The number of elements N used in computing the solution from the B.I.E. method is 10. Results are presented for $Ka = 0.1$ and $H = 1.5$ and 3.0, while Poisson's ratio ν is fixed at 0.25. As D_0 and D_1 are of order ϵ^2 and C_R of order ϵ^3 , it can be seen from (7.8.1) that we expect the results from the M.A.E. method to agree with those of the B.I.E. method to the order of ϵ^2 .

In table 7.6.1a(b) we have the results for $\epsilon = 0.1$ and $H = 1.5$ (3.0). In the first column we have the angle η which determines the position of the station on the boundary. In the second and third columns we have the real and imaginary parts of the complex component of the displacements in the x -direction and in the fourth and fifth columns we have those corresponding to the y -direction. At every station, the upper (lower) entry corresponds to the results from the B.I.E. (M.A.E.) method. Note that since there is a symmetry about the x -axis, the results are only given for the upper half of the boundary. It can clearly be seen that the results agree to the order expected, i.e. two decimal places.

While the B.I.E. method provides the solution for any type of boundary, no significant simplifications can be made when ϵ becomes small. Furthermore, there is a limit on how small ϵ can be before the solution from the B.I.E. method becomes numerically unstable ($\epsilon = 10^{-3}$ in our case). Although this difficulty may be overcome, it may lead to an increase in the time and hence cost in computing the solution. The M.A.E. method, as we have seen, gives results with a reasonable error (of order ϵ^3) and ϵ can be taken as small as we wish. However, as ϵ gets bigger the error incurred is no longer negligible. In this case, one can always resort to the B.I.E. method.

$\eta (^{\circ})$	Real (u_x^*)	Im (u_x^*)	Real (u_y^*)	Im (u_y^*)
0	0.9868 0.9878	0.1553 0.1536	0.0000 0.	0.0000 0.
18	0.9873 0.9883	0.1475 0.1459	0.0000 -0.0001	0.0044 0.0045
36	0.9888 0.9896	0.1247 0.1233	0.0000 -0.0002	0.0082 0.0085
54	0.9907 0.9913	0.0893 0.0882	0.0000 -0.0002	0.0115 0.0117
72	0.9923 0.9926	0.0446 0.0440	0.0000 -0.0001	0.0133 0.0137
90	0.9932 0.9931	-0.0050 -0.0051	0.0003 0.0000	0.0142 0.0144
108	0.9930 0.9926	-0.0545 -0.0541	0.0004 0.0001	0.0133 0.0137
126	0.9920 0.9913	-0.0993 -0.0984	0.0005 0.0002	0.0115 0.0116
144	0.9906 0.9896	-0.1345 -0.1335	0.0004 0.0002	0.0082 0.0085
162	0.9894 0.9883	-0.1575 -0.1560	0.0002 0.0001	0.0044 0.0045
180	0.9890 0.9878	-0.1653 -0.1638	0.0000 0.0000	0.0000 0.0000

Table 7.6.1a Numerical values of the displacements for $K_a = 0.1$ and $H = 1.5$. The upper (lower) entry corresponds to the B.I.E. (M.A.E.) method.

$\eta(^{\circ})$	Real (u_x^*)	Im (u_x^*)	Real (u_y^*)	Im (u_y^*)
0	0.9874	0.1404	0.0000	0.0000
	0.9879	0.1396	0.	0.
18	0.9879	0.1334	-0.0004	0.0090
	0.9884	0.1325	-0.0003	0.0089
36	0.9894	0.1128	-0.0006	0.0171
	0.9898	0.1121	-0.0005	0.0170
54	0.9913	0.0808	-0.0006	0.0237
	0.9915	0.0802	-0.0005	0.0233
72	0.9928	0.0403	-0.0004	0.0277
	0.9929	0.0399	-0.0003	0.0274
90	0.9935	-0.0046	-0.0001	0.0293
	0.9935	-0.0046	0.0000	0.0289
108	0.9932	-0.0494	0.0002	0.0277
	0.9929	-0.0492	0.0003	0.0274
126	0.9920	-0.0899	0.0005	0.0237
	0.9915	-0.0895	0.0005	0.0233
144	0.9904	-0.1219	0.0005	0.0171
	0.9898	-0.1214	0.0005	0.0170
162	0.9891	-0.1425	0.0003	0.0090
	0.9884	-0.1419	0.0003	0.0089
180	0.9886	-0.1496	0.0000	0.0000
	0.9879	-0.1489	0.0000	0.0000

Table 7.6.1b Numerical values of the displacements for $K_a = 0.1$ and $H = 3.0$. The upper (lower) entry corresponds to the B.I.E. (M.A.E.) method.

CHAPTER VIII

CONCLUSION

The problem of scattering of time-harmonic stress waves by an infinite cylindrical cavity of arbitrary smooth cross-section, in an otherwise unbounded, homogeneous, isotropic linearly elastic solid, was considered. The direction of propagation was chosen perpendicular to the axis of the cylinder so that the elastic solid is in a state of plane strain. The three-dimensional problem was, thus, reduced to a two-dimensional one.

The method chosen to solve this problem is the boundary integral equation method (B.I.E.M.), using the fundamental solution \underline{G}^f . This reduces the problem to solving a Fredholm integral equation of the second kind on the boundary. However, this integral equation was found not to have a unique solution at the so called irregular frequencies (I.F.). As these are known to arise from the method of solution rather than the nature of the problem, since this latter is known to be uniquely solvable at all frequencies, we developed two methods for eliminating them. Both methods are based on modifying the fundamental solution \underline{G}^f .

The first method, which we presented in Chapter IV, consists of adding an infinite series of multipoles to \underline{G}^f . The modified fundamental solution \underline{G}^1 was then made to satisfy a dissipative type of condition on a circular boundary inside the scatterer. This, we showed, led to a modified integral equation which was uniquely solvable at all frequencies. The explicit construction of \underline{G}^1 involved terms, which we called cross-terms, that were not expected, since they do not appear in the corresponding expansion of \underline{G}^f . The complicated expressions for the coefficients of the multipoles make this method unsuitable for numerical work.

The second method, presented in Chapter V, consists of adding a finite series of multipoles. The additional finite series was chosen to be of the same form as in the previous method. However, the choice of the coefficients of the multipoles was practically at our disposal. With some mild restrictions on these coefficients, we showed that a restricted range of frequency is freed from I.F. We also deduced that this result remains true even if the cross-terms are not included and thus we recovered the results of Jones (1984) when adapted to two dimensions. When we further assumed the coefficients of the multipoles to be equal and independent of the summation index, we recovered the results of Kobayashi & Nishimura (1982). Furthermore, we showed that uniqueness still holds when only cross-terms are used.

Chapter VI was mainly devoted to the numerical solutions of the modified and non-modified boundary integral equations for the special case of a circular cavity. The numerical solutions were compared with the exact solution obtained by separation of variables. The results from the numerical solution of the non-modified boundary integral equation confirmed the existence of I.F., while those of the modified boundary integral equation, using our second method, showed that they were eliminated. These results also led to the conclusion that the conditions on the coefficients of the multipoles were not necessary to eliminate the I.F. Furthermore, it was also found that the position of the multipoles inside the scatterer affects the number of additional terms of multipoles required to eliminate a particular I.F. The chapter was concluded with some numerical results for the scattering by an elliptic cavity.

Chapter VII was devoted to low frequency asymptotic solutions using the method of matched asymptotic expansions. The work presented here was based on two papers by Buchwald (1978) and Buchwald & Tran Cong

(1984) who studied the diffraction of elastic waves by a small circular cavity and a small elliptic cavity, respectively, in an otherwise unbounded domain. In this chapter, we clarified and systematized some aspects of their work and extended it to the diffraction of elastic waves by a small cylindrical cavity with a hypotrochoidal boundary. Results were presented for the special case of an incident P-wave. Finally, these were compared, for the special case of an elliptic cavity, with the results from the numerical solution of the boundary integral equation and good agreement was found.

Further work would involve establishing whether the modified solution \underline{G}^1 , from the first method, falls into the class of modified fundamental solutions from the second method. In other words, do the coefficients of the multipoles in \underline{G}^1 satisfy the conditions derived for the coefficients of the multipoles in the second method? This is known to be the case in acoustics (see, e.g., Martin(1982)).

It would also be interesting to devise some criteria, similar to those in acoustics (see, e.g., Kleinman & Roach(1982)), for choosing the coefficients of the multipoles in the second method. However, this may lead, as in acoustics, to complicated expressions for these coefficients.

APPENDIX A1

Consider the expression for the kernel of the boundary integral equation (3.3.1). This is given by

$$\underline{K}(p; q) = T_p G^f(p; q). \quad (A1.1)$$

Using (3.1.1), this kernel can be written, in Cartesian coordinates, as follows:

$$K_{ln}(p; q) = \sum_{lmn}^f (p; q) \hat{n}_m(p), \quad (A1.2)$$

where $\sum_{lmn}^f (p; q)$ is given by (3.1.2) and $\hat{n}_m(p)$ is the m^{th} -component of the unit normal at the point p on ∂D . Performing the differentiation in $\sum_{lmn}^f (p; q)$ and rearranging leads to

$$\begin{aligned} K_{ln}(p; q) = & A_{ln} [\{ H_0^{(1)}(KR) - 2H_1^{(1)}(KR)/KR \} - \tau^2 \{ H_0^{(1)}(KR) \\ & - 2H_1^{(1)}(KR)/KR \}] + B_{ln} KR H_1^{(1)}(KR) \\ & + C_{ln} KR H_1^{(1)}(KR), \end{aligned} \quad (A1.3)$$

where

$$A_{ln} = \frac{1}{2} [\{ 4R_l R_n / R^2 - \delta_{ln} \} R_m \hat{n}_m(p) / R^2 - \{ \hat{n}_l(p) R_n + \hat{n}_n(p) R_l \} / R^2],$$

$$B_{ln} = \frac{1}{4} [\{ 2R_l R_n / R^2 - \delta_{ln} \} R_m \hat{n}_m(p) / R^2 - \hat{n}_n(p) R_l / R^2],$$

$$C_{ln} = \frac{1}{4} [(2\tau^2 - 1) \hat{n}_l(p) R_n / R^2 - 2\tau^2 \frac{R_l R_n}{R^2} - \frac{R_m \hat{n}_m(p)}{R^2}],$$

with

$H_n^{(1)}(.)$: n^{th} -order Hankel function of the first kind

R : distance between the points p and q

$R_\ell = x_\ell(p) - x_\ell(q)$

$x_\ell(p)$: ℓ^{th} -coordinate of the point p

$\delta_{\ell n}$: Kronecker symbol.

Using the series expansions of the Hankel functions, $K_{\ell n}(p; q)$ is rewritten as

$$K_{\ell n}(p; q) = K_{\ell n}^*(p; q) + K_{\ell n}^0(p; q), \quad (\text{A1.4})$$

where

$$K_{\ell n}^*(p; q) = \frac{1}{2\pi} \{ \hat{n}_\ell(p) R_n - \hat{n}_n(p) R_\ell \} / R^2 \quad (\text{A1.5})$$

and $K_{\ell n}^0(p; q) = K_{\ell n}(p; q) - K_{\ell n}^*(p; q)$ is continuous over $\partial D \times \partial D$.

Using the fact that

$$\partial \ln R / \partial x_\ell = R_\ell / R^2,$$

(A1.5) becomes

$$K_{\ell n}^*(p; q) = \frac{1}{2\pi} \{ \hat{n}_\ell(p) \partial \ln R / \partial x_n - \hat{n}_n(p) \partial \ln R / \partial x_\ell \}. \quad (\text{A1.6})$$

Note that $K_{\ell n}^*(p; q)$ is only singular when $\ell \neq n$ and vanishes when $\ell = n$.

APPENDIX A2

Consider the additional infinite series in (4.3.8), which we rewrite here as follows:

$$\underline{\underline{L}}(P;Q) = \underline{\underline{L}}^1(P;Q) + \underline{\underline{L}}^2(P;Q) + \underline{\underline{L}}^3(P;Q) + \underline{\underline{L}}^4(P;Q), \quad (\text{A2.1})$$

where

$$\underline{\underline{L}}^1(P;Q) = \frac{i}{4\mu K^2} \sum_{\sigma=1}^2 \sum_{m=0}^{\infty} \alpha_m^1 \nabla \phi_m^{\sigma}(P) \otimes \nabla \phi_m^{\sigma}(Q), \quad (\text{A2.2})$$

$$\underline{\underline{L}}^2(P;Q) = \frac{i}{4\mu K^2} \sum_{\sigma=1}^2 \sum_{m=0}^{\infty} \alpha_m^2 \nabla \wedge (\psi_m^{\sigma}(P) \underline{e}_3) \otimes \nabla \wedge (\psi_m^{\sigma}(Q) \underline{e}_3), \quad (\text{A2.3})$$

$$\underline{\underline{L}}^3(P;Q) = -\frac{i}{4\mu K^2} \sum_{\sigma=1}^2 \sum_{m=0}^{\infty} (-1)^{\sigma} \beta_m^1 \nabla \phi_m^{\sigma}(P) \otimes \nabla \wedge (\psi_m^{3-\sigma}(Q) \underline{e}_3), \quad (\text{A2.4})$$

$$\underline{\underline{L}}^4(P;Q) = \frac{i}{4\mu K^2} \sum_{\sigma=1}^2 \sum_{m=0}^{\infty} (-1)^{\sigma} \beta_m^2 \nabla \wedge (\psi_m^{\sigma}(P) \underline{e}_3) \otimes \nabla \phi_m^{3-\sigma}(Q). \quad (\text{A2.5})$$

α_m^1 , α_m^2 , β_m^1 and β_m^2 are as defined in section IV.3. We wish to determine the condition under which $\underline{\underline{L}}(P;Q)$ converges. If we show that $\underline{\underline{L}}^j(P;Q)$ ($j = 1, 2, 3$ and 4) converges, then clearly $\underline{\underline{L}}(P;Q)$ will also converge. Consider the series given by $\underline{\underline{L}}^1(P;Q)$. This is a tensor of order 2. To show that $\underline{\underline{L}}^1(P;Q)$ converges, we need to show that every element of this tensor is a convergent series. To fix ideas, consider the element $\underline{\underline{L}}_{r_P r_Q}^1(P;Q)$ given by

$$\underline{\underline{L}}_{r_P r_Q}^1(P;Q) = \sum_{m=0}^{\infty} u_m, \quad (\text{A2.6})$$

where

$$u_m = v_m \cos(\theta_P - \theta_Q), \quad (\text{A2.7})$$

$$v_m = \frac{i}{4\mu k^2} a_m \frac{\partial H_m^{(1)}(kr_P)}{\partial r_P} \cdot \frac{\partial H_m^{(1)}(kr_Q)}{\partial r_Q} \quad (A2.8)$$

and $(r_P, \theta_P), (r_Q, \theta_Q)$ are the polar coordinates of P and Q, respectively.

Since $|u_m| < |v_m|$, it follows, by the comparison test, that if

$\sum_{m=0}^{\infty} |v_m|$ is convergent, then $\sum_{m=0}^{\infty} u_m$ is absolutely convergent. So, let us

determine the condition under which $\sum_{m=0}^{\infty} |v_m|$ converges. To do so, we

shall make use of the ratio test, i.e. if

$$\lim_{m \rightarrow \infty} \left| \frac{v_{m+1}}{v_m} \right| \quad (A2.9)$$

exists and is less than one, then the series converges. When m is large, the asymptotic expression for v_m is

$$v_m \sim -\frac{m^3}{2\pi\mu(Ka)^2(ka)^2} \cdot \left(\frac{a^2}{r_P r_Q}\right)^{m+1}, \quad (A2.10)$$

where we have used the asymptotic expressions of the Bessel functions of order m and the Hankel functions of the first kind and order m. Hence

$$\lim_{m \rightarrow \infty} \left| \frac{v_{m+1}}{v_m} \right| = \frac{a^2}{r_P r_Q} \quad (A2.11)$$

and so, if $r_P r_Q > a^2$, then $\sum_{m=0}^{\infty} u_m$ converges absolutely. In a similar fashion, the remaining elements of $\underline{\underline{L}}^1(P;Q)$ can be shown to converge absolutely under the same condition. It follows therefore that the series given by $\underline{\underline{L}}^1(P;Q)$ converges absolutely for $r_P r_Q > a^2$.

Similarly, it can also be shown that $\underline{\underline{L}}^2(P;Q)$, $\underline{\underline{L}}^3(P;Q)$ and $\underline{\underline{L}}^4(P;Q)$ converge absolutely under the same condition. Hence, $\underline{\underline{L}}(P;Q)$ converges absolutely whenever $r_P r_Q > a^2$. Note that this is the same condition as that found by Ursell (1973).

APPENDIX A3

Consider equation (4.3.3), which we rewrite here

$$T_P \underline{G}^1(P; Q) + \underline{F} \cdot \underline{G}^1(P; Q) = \underline{0} \quad P \in C, \quad (A3.1)$$

where $\underline{G}^1(P; Q)$ is given by (4.3.8), $\underline{F} = F \underline{I}$ with $F = |F| e^{i\delta}$ and $0 < \delta < \pi$, Q is on boundary ∂D and P is on the circle C with radius a and is inside the scatterer. Substituting for $\underline{G}^1(P; Q)$ in (A3.1) leads to

$$\sum_{\sigma=1}^2 \sum_{m=0}^{\infty} [\underline{X}_m^{\sigma}(P) \otimes \underline{\nabla} \underline{\phi}_m^{\sigma}(Q) + \underline{Y}_m^{\sigma}(P) \otimes \underline{\nabla} \wedge (\underline{\psi}_m^{\sigma}(Q) \underline{e}_3)] = \underline{0}, \quad (A3.2)$$

where

$$\begin{aligned} \underline{X}_m^{\sigma}(P) = & \{T \underline{\nabla} \underline{\phi}_m^{\sigma}(P) + F \underline{\nabla} \underline{\phi}_m^{\sigma}(P)\} \alpha_m^1 - \{T \underline{\nabla} \wedge (\underline{\psi}_m^{3-\sigma}(P) \underline{e}_3) + F \underline{\nabla} \wedge (\underline{\psi}_m^{3-\sigma}(P) \underline{e}_3)\} \\ & (-1)^{\sigma} \beta_m^2 + \{T \underline{\nabla} \hat{\underline{\phi}}_m^{\sigma}(P) + F \underline{\nabla} \hat{\underline{\phi}}_m^{\sigma}(P)\} \end{aligned}$$

and

$$\begin{aligned} \underline{Y}_m^{\sigma}(P) = & \{T \underline{\nabla} \wedge (\underline{\psi}_m^{\sigma}(P) \underline{e}_3) + F \underline{\nabla} \wedge (\underline{\psi}_m^{\sigma}(P) \underline{e}_3)\} \alpha_m^2 + \{T \underline{\nabla} \underline{\phi}_m^{3-\sigma}(P) + F \underline{\nabla} \underline{\phi}_m^{3-\sigma}(P)\} \\ & (-1)^{\sigma} \beta_m^1 + \{T \underline{\nabla} \wedge (\hat{\underline{\psi}}_m^{\sigma}(P) \underline{e}_3) + F \underline{\nabla} \wedge (\hat{\underline{\psi}}_m^{\sigma}(P) \underline{e}_3)\}, \end{aligned}$$

with α_m^1 , α_m^2 , β_m^1 and β_m^2 defined in section IV.3.

Dotting (A3.2) on the right with $T \underline{\nabla} \hat{\underline{\phi}}_n^{\vee}(Q)$ gives

$$\sum_{\sigma=1}^2 \sum_{m=0}^{\infty} [\underline{X}_m^{\sigma}(P) (\underline{\nabla} \underline{\phi}_m^{\sigma}(Q) \cdot T \underline{\nabla} \hat{\underline{\phi}}_n^{\vee}(Q)) + \underline{Y}_m^{\sigma}(P) (\underline{\nabla} \wedge (\underline{\psi}_m^{\sigma}(Q) \underline{e}_3) \cdot T \underline{\nabla} \hat{\underline{\phi}}_n^{\vee}(Q))] = \underline{0}. \quad (A3.3)$$

Applying T_Q to (A3.2) and then dotting on the right with $\hat{\nabla}_n^v(Q)$ yields

$$\sum_{\sigma=1}^2 \sum_{m=0}^{\infty} [X_m^{\sigma}(P)(T\nabla\hat{\phi}_m^{\sigma}(Q) \cdot \hat{\nabla}_n^v(Q)) + Y_m^{\sigma}(P)(T\nabla\wedge(\psi_m^{\sigma}(Q)\underline{e}_3) \cdot \hat{\nabla}_n^v(Q))] = \underline{0}. \quad (A3.4)$$

Subtracting (A3.4) from (A3.3) and integrating over ∂D leads to

$$\sum_{\sigma=1}^2 \sum_{m=0}^{\infty} [X_m^{\sigma}(P) \int_{\partial D} \{\nabla\hat{\phi}_m^{\sigma}(Q) \cdot T\nabla\hat{\phi}_n^v(Q) - T\nabla\hat{\phi}_m^{\sigma}(Q) \cdot \hat{\nabla}_n^v(Q)\} ds_Q + Y_m^{\sigma}(P) \int_{\partial D} \{\nabla\wedge(\psi_m^{\sigma}(Q)\underline{e}_3) \cdot T\nabla\hat{\phi}_n^v(Q) - T\nabla\wedge(\psi_m^{\sigma}(Q)\underline{e}_3) \cdot \hat{\nabla}_n^v(Q)\} ds_Q] = \underline{0}. \quad (A3.5)$$

Using the identities given by (5.3.13-15), (A3.5) reduces to

$$X_m^{\sigma}(P) = \underline{0}, \text{ for } \sigma = 1, 2 \text{ and } m = 0, 1, \dots, \text{ and } P \in C. \quad (A3.6)$$

Similarly, it can be shown that

$$Y_m^{\sigma}(P) = \underline{0}, \text{ for } \sigma = 1, 2 \text{ and } m = 0, 1, \dots, \text{ and } P \in C. \quad (A3.7)$$

It can thus be concluded that (α_m^1, β_m^2) and (α_m^2, β_m^1) are the solutions of (A3.6) and (A3.7), respectively.

In order to show that the determinant Δ_m in the expression of $\alpha_m^1, \alpha_m^2, \beta_m^1$ and β_m^2 never vanishes, we need to show that equations (A3.6) and (A3.7) are uniquely solvable. To do so, all we have to show is that the corresponding homogeneous equations, i.e.

$$\{T\nabla\hat{\phi}_m^{\sigma}(P) + F\nabla\hat{\phi}_m^{\sigma}(P)\}x_m^1 - \{T\nabla\wedge(\psi_m^{3-\sigma}(P)\underline{e}_3) + F\nabla\wedge(\psi_m^{3-\sigma}(P)\underline{e}_3)\}(-1)^{\sigma}y_m^2 = \underline{0} \quad (A3.8)$$

and

$$\{T\underline{\nabla}^{\sigma}(\psi_m^{\sigma}(P)\underline{e}_3) + F\underline{\nabla}^{\sigma}(\psi_m^{\sigma}(P)\underline{e}_3)\}x_m^2 + \{T\underline{\nabla}^{3-\sigma}(P) + F\underline{\nabla}^{3-\sigma}(P)\}(-1)^{\sigma}y_m^1 = 0, \quad (A3.9)$$

where x_m^1 , x_m^2 , y_m^1 and y_m^2 are the unknowns, have only the trivial solution.

Consider equation (A3.8), which we rewrite as

$$T\underline{U}_m^{\sigma}(P) + F\underline{U}_m^{\sigma}(P) = 0 \quad P \in C, \quad (A3.10)$$

where

$$\underline{U}_m^{\sigma}(P) = \underline{\nabla}_m^{\sigma}(P)x_m^1 - (-1)^{\sigma}\underline{\nabla}_m^{3-\sigma}(P)\underline{e}_3 y_m^2. \quad (A3.11)$$

Denoting by $\underline{U}_m^{\sigma}(P)$ the conjuguate of $\underline{U}_m^{\sigma}(P)$ and using (A3.10), we have

$$T\underline{U}_m^{\sigma}(P) \cdot \underline{U}_m^{\sigma}(P) - T\underline{U}_m^{\sigma}(P) \cdot \underline{U}_m^{\sigma}(P) = -2i|F|\sin\delta|\underline{U}_m^{\sigma}(P)|^2. \quad (A3.12)$$

Substituting for $\underline{U}_m^{\sigma}(P)$ in (A3.12) and integrating over C yields, after simplification,

$$|x_m^1|^2 + |y_m^2|^2 + \frac{|F|\sin\delta}{4\mu K^2} \int_C |\underline{U}_m^{\sigma}(P)|^2 ds_P = 0, \quad (A3.13)$$

where we have used the identities given by (5.3.13-15). Since $0 < \delta < \pi$ and $\mu > 0$, it can be seen that (A3.13) is satisfied if and only if

$$x_m^1 = y_m^2 = 0.$$

Similarly, we can show that

$$x_m^2 = y_m^1 = 0.$$

Therefore the homogeneous equations (A3.8) and (A3.9) can only have the trivial solution. It follows that the solutions of (A3.6) and (A3.7) are unique and thus the determinant Δ_m never vanishes.

APPENDIX A4

The solution of the interior homogeneous boundary value problem I_h formulated in section II.4 can be sought in the form of a double layer, i.e.

$$\underline{u}(P_-) = \int_{\partial D} \underline{\mu}(q) \cdot \underline{T}_q \underline{G}^f(q; P_-) ds_q, \quad (A4.1)$$

where $\underline{\mu}(q)$ is an unknown density. Implementing the boundary condition $\underline{u}(p) = \underline{0}$ on ∂D , yields the integral equation

$$\underline{\mu}(p) - \int_{\partial D} \underline{\mu}(q) \cdot \underline{T}_q \underline{G}^f(q; p) ds_q = \underline{0}. \quad (A4.2)$$

This integral equation is the same as the homogeneous form of (6.2.1). Therefore, we can deduce properties of the latter by examining $\underline{\mu}$.

For P_- inside the inscribed circle to ∂D , we can use (4.3.4) in (A4.1) to give

$$\underline{u}(P_-) = \sum_{\sigma=1}^2 \sum_{m=0}^{\infty} [A_m^{\sigma} \hat{\nabla} \hat{\phi}_m^{\sigma}(P_-) + B_m^{\sigma} \hat{\nabla} \wedge (\hat{\psi}_m^{\sigma}(P_-) \underline{e}_3)], \quad (A4.3)$$

where

$$A_m^{\sigma} = \frac{i}{4\mu k^2} \int_{\partial D} \underline{\mu}(q) \cdot \underline{T} \hat{\nabla} \hat{\phi}_m^{\sigma}(q) ds_q \quad (A4.4a)$$

and

$$B_m^{\sigma} = \frac{i}{4\mu k^2} \int_{\partial D} \underline{\mu}(q) \cdot \underline{T} \hat{\nabla} \wedge (\hat{\psi}_m^{\sigma}(q) \underline{e}_3) ds_q. \quad (A4.4b)$$

In what follows, we shall assume ∂D to be a circle of radius a . We can then determine A_m^{σ} , B_m^{σ} by applying the boundary condition $\underline{u}(p) = \underline{0}$ on ∂D , to (A4.3). Once these are known, we can determine $\underline{\mu}$ from (A4.4).

A4.1 Determination of A_m^σ, B_m^σ

In polar coordinates, the radial and tangential components of \underline{u} are, respectively, given by

$$u_r(P_-) = \sum_{\sigma=1}^2 \sum_{m=0}^{\infty} [A_m^\sigma k J_m'(kr) E_m^\sigma(\theta) + B_m^\sigma (-1)^\sigma \frac{m}{r} J_m(kr) E_m^{3-\sigma}(\theta)], \quad (A4.1.1a)$$

$$u_\theta(P_-) = \sum_{\sigma=1}^2 \sum_{m=0}^{\infty} [A_m^\sigma (-1)^\sigma \frac{m}{r} J_m(kr) E_m^{3-\sigma}(\theta) - B_m^\sigma k J_m'(kr) E_m^\sigma(\theta)], \quad (A4.1.1b)$$

where (r, θ) are the polar coordinates of the point P_- and $J_m'(\cdot)$ denotes the derivative of the Bessel function $J_m(\cdot)$ with respect to the argument.

Here, the centre of the circular boundary is taken as the origin.

Applying the boundary condition $\underline{u}(p) = \underline{0}$ on ∂D to (A4.1.1), leads to the following linear system of algebraic equations:

$$\begin{aligned} ka J_m'(ka) A_m^\sigma - m J_m(ka) (-1)^\sigma B_m^{3-\sigma} &= 0, & \sigma=1,2 \\ m J_m(ka) A_m^\sigma - ka J_m'(ka) (-1)^\sigma B_m^{3-\sigma} &= 0. & \text{and} \\ & & m=0,1,\dots \end{aligned} \quad (A4.1.2)$$

The determinant of this system is given by

$$J_{m+1}(ka) J_{m-1}(ka) + J_{m+1}(ka) J_{m-1}(ka) = 0 \quad m=0,1,\dots \quad (A4.1.3)$$

This is the equation which gives the irregular frequencies (I.F.) for a circular boundary. Some of the solutions of this equation for $\nu = 1/4$ are given in section VI.3. For $m=0$, equations (A4.1.2) become

$$\begin{aligned} ka J_1(ka) A_0^\sigma &= 0, \\ ka J_1(ka) B_0^{3-\sigma} &= 0, \end{aligned} \quad \sigma=1,2 \quad (A4.1.4)$$

where we have used $J_0'(\cdot) = -J_1(\cdot)$, while the determinant given by (A4.1.3) becomes

$$J_1(ka)J_1(Ka) = 0. \quad (A4.1.5)$$

Any root of $J_1(ka) = 0$ or $J_1(Ka) = 0$ is also a root of (A4.1.5). The I.F. denoted by ② in section VI.3 is a root of $J_1(Ka) = 0$ but not $J_1(ka) = 0$. It follows, from (A4.1.4), that $A_0^\sigma = 0$ and B_0^σ remains arbitrary. Assuming that ② is not a root of (A4.1.3) for $m > 0$, it follows that

$$A_m^\sigma = B_m^\sigma = 0 \quad \text{for } m > 0 \text{ and } \sigma = 1, 2.$$

Thus, the radial and tangential components of \underline{u} defined by (A4.1.1) become

$$u_r(P_-) = 0 \quad \text{and} \quad u_\theta(P_-) = B_0^1 K J_1(Kr). \quad (A4.1.6)$$

Similarly, it can be shown that at the I.F. denoted by ③, the radial component of \underline{u} is zero, while at the I.F. denoted by ⑤ it is the tangential component that is zero.

A4.2 Determination of $\underline{\mu}$

Let μ_r, μ_θ denote the radial and tangential components of $\underline{\mu}$, respectively. μ_r and μ_θ can be expanded on ∂D into Fourier series as follows:

$$\mu_r = \sum_{v=1}^2 \sum_{n=0}^{\infty} \mu_n^v E_n^v(\theta), \quad (A4.2.1a)$$

$$\mu_\theta = \sum_{v=1}^2 \sum_{n=0}^{\infty} \tilde{\mu}_n^v E_n^v(\theta), \quad (A4.2.1b)$$

where $\mu_n^v, \tilde{\mu}_n^v$ are unknown coefficients. Substituting (A4.2.1) into (A4.4) leads to the following linear system of algebraic equations:

$$A_m \mu_m^\sigma + B_m (-1)^\sigma \mu_m^{3-\sigma} = -\frac{2i}{\pi a} A_m^\sigma, \quad (\text{A4.2.2a})$$

$$C_m \mu_m^\sigma + D_m (-1)^\sigma \mu_m^{3-\sigma} = \frac{2i}{\pi a} (-1)^\sigma B_m^{3-\sigma}, \quad (\text{A4.2.2b})$$

where A_m , B_m , C_m and D_m are as defined in section IV.3, with $F = 0$. The solution of this system is

$$\mu_m^\sigma = -\frac{2i}{\pi a} (A_m^\sigma D_m + (-1)^\sigma B_m^{3-\sigma} B_m) / \Delta_m^0, \quad (\text{A4.2.3a})$$

$$\mu_m^{\sigma} = -\frac{2i}{\pi a} ((-1)^\sigma A_m^{3-\sigma} C_m - B_m^\sigma A_m) / \Delta_m^0, \quad (\text{A4.2.3b})$$

where Δ_m^0 corresponds to the determinant Δ_m defined in section IV.3 for $F = 0$. It can be seen, from Appendix A3, that Δ_m^0 never vanishes. It follows that the solution given by (A4.2.3) is therefore unique.

At the I.F. ②, we have shown in the previous section that

$$A_m^\sigma = B_m^\sigma = 0 \quad \text{for } m > 0 \text{ and } \sigma = 1, 2$$

and

$$A_0^\sigma = 0 \quad \text{for } \sigma = 1, 2.$$

It then follows that

$$\mu_m^\sigma = \mu_m^{\sigma} = 0 \quad \text{for } m > 0 \text{ and } \sigma = 1, 2,$$

$$\mu_0^\sigma = \frac{2i}{\pi a} B_0^\sigma A_0$$

and $\mu_0^\sigma = 0$, since $B_0 = 0$. A substitution of the above results in (A4.2.1) shows that the radial component μ_r vanishes at the I.F. ②.

A similar result can be obtained at the I.F. ⑧, while at the I.F. ⑤ it can be shown that it is μ_θ that vanishes. These results account for the numerical observations in section VI.3.

APPENDIX A5

Consider the conformal transformation defined by (7.4.8), which we rewrite here

$$z = \xi + m/\xi^n, \quad (\text{A5.1})$$

where n is a positive integer and m is real with $0 \leq m < 1/n$. We wish to find the asymptotic expansion of the inverse transformation in terms of the outer coordinates. The equation which determines the inverse of (A5.1) has n solutions. Special care must, therefore, be taken in choosing the right one. This can be done by looking at the behaviour of z for large $|\xi|$. Taking this into account, we assume the following form:

$$\xi = z + \sum_{j=1}^{\infty} a'_j/z^j, \quad (\text{A5.2})$$

where a'_j are coefficients to be determined. Re-expressing (A5.2) in terms of outer coordinates gives

$$\xi = Z\epsilon^{-1} + \sum_{j=1}^{\infty} a'_j \epsilon^j/Z^j, \quad (\text{A5.3})$$

where ϵ and Z are as defined in Chapter VII. Substituting for ξ , given by (A5.3), into (A5.1) leads to

$$0 = \sum_{j=1}^{\infty} a'_j \epsilon^j/Z^j + m \left(\sum_{j=1}^{\infty} a'_j \epsilon^j/Z^j \right)^{-n}. \quad (\text{A5.4})$$

In order to determine the coefficients a'_j , we expand the expression multiplied by m in equation (A5.4) for small ϵ and then equate to zero all the coefficients of the powers of $1/Z$. This leads to an infinite system of equations which gives for the first few coefficients the following expressions:

$$a'_1 = a'_2 = \dots = a'_{n-1} = 0,$$

$$a'_n = -m,$$

$$a'_{n+1} = a'_{n+2} = \dots = a'_{2n} = 0,$$

$$a'_{2n+1} = -nm^2,$$

$$a'_{2n+2} = a'_{2n+3} = \dots = a'_{3n+1} = 0,$$

$$a'_{3n+2} = -n(3n+1)m^3/2,$$

$$a'_{3n+3} = a'_{3n+4} = \dots = a'_{4n+2} = 0,$$

$$a'_{4n+3} = -n(2n+1)(4n+1)m^4/3.$$

Substituting the above expressions into (A5.3) gives

$$\begin{aligned} \xi = & Z\epsilon^{-1} + a'_n \epsilon^n / Z^n + a'_{2n+1} \epsilon^{2n+1} / Z^{2n+1} \\ & + a'_{3n+2} \epsilon^{3n+2} / Z^{3n+2} + a'_{4n+3} \epsilon^{4n+3} / Z^{4n+3} \\ & + O(\epsilon^{5n+4}). \end{aligned} \quad (\text{A5.5})$$

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