SOME ELECTRIC PROPERTIES OF NUCLEI

by

Arnold Rudolph Bodmer.

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SUMMARY.

In Sections I and II, the volume dependent isotope shift and the elastic scattering of electrons by nuclei are treated by a method with which they can be simply related.

In Section I, the result already obtained by Feshbach that the s wave scattering due to the finite nuclear size depends only on the volume integral of the potential due to the nuclear charge distribution is derived in a simple manner with a clear indication of the limitations. It is shown that this is then the only information which can be obtained for energies at which only the s wave scattering is important.

In Section II, the isotope shift is calculated taking into account the distortion of the electronic wave function by the nuclear charge distribution using a non-perturbation method due to Broch and is reduced somewhat below that obtained with the simple perturbation method. It is shown by the same method as that used for the electron scattering that the isotope shift depends essentially only on the above volume integral and on its difference between two isotopes. Using the results obtained from the scattering of electrons by Ag and Au by Lyman, Hanson and Scott and assuming that the nuclear radius increases proportionally to $A^{\frac{1}{3}}$ the isotope shift is brought into considerably better agreement with the experimental data.

In Section III, the suggestion of Brix and Kopfermann of the effect of a difference in the deformations between two isotopes on the isotope shift is considered, the method of Section II being used to calculate this. It is shown that in those cases where it is possible to estimate these deformations from the spectroscopic quadrupole moments, the difference between this and the intrinsic quadrupole moment must be taken into account, and in general increases the effect greatly above the estimates of Brix and Kopfermann. A number of tentative

applications are made which suggest that this deformation dependent isotope shift may perhaps be responsible for the large variations observed in the data.

I.

THE ELASTIC SCATTERING OF ELECTRONS BY NUCLEI

The scattering of electrons by nuclei deviates from pure Coulomb scattering, i.e. if the nuclei could be represented as point charges, for energies at which the electron wavelength λ becomes of the same order of magnitude as the nuclear dimensions. If for the nuclear radius we take $R_0 = \frac{e^2}{2mc^2} A^{\frac{1}{3}}$, then for energies considerably in excess of the rest mass $R_0/\lambda = \frac{\varepsilon A^{1/3}}{137}$, where $\varepsilon = E/mc^2$ is the energy in relativistic units. For heavier nuclei, the scattering at large angles can then be expected to be influenced by the finite nuclear size at energies greater than about 5 MeV. For lighter nuclei, observable effects will be at somewhat higher energies. For such energies which must not be so great that the wavelength becomes of the same order as the internucleon distance, the scattering will be almost wholly elastic, and can be considered as due to the charge distribution of the nucleus as a whole. The scattering process will be adequately described by considering the electrons to move in the potential of the nuclear charge distribution. In this work we restrict ourselves to energies for which such a description of the scattering can be considered as valid. Radiative effects are quite appreciable and it is essential that they are allowed for in order that the scattering may be interpreted in terms of the nuclear charge distribution.

Calculations using the Born approximation have been made by Rose (1948), Elton (1950) and Parzen (1950), which however are only accurate for light elements. Detailed calculations of the scattering cross sections making a phase shift analysis have been made for a uniform and a surface charge distribution by Elton and Achesen (1951) for energies of about 20 MeV.

Feshbach has derived some general properties of the phase shifts for the extreme relativistic case when the rest mass of the electron can be neglected. In particular, he has shown that for not too high energies the s wave shift, which for energies in the region of 20 MeV is sufficient to describe the deviation from pure Coulomb scattering, depends essentially only on $\int_0^r V r'^2 dr'$ where V is the potential due to the nuclear charge distribution and r is sufficiently far outside this distribution for the potential to be Coulomb. It follows that scattering experiments can then determine only this volume integral of the nuclear potential giving one condition for the nuclear charge distribution. Feshbach assumes the condition $\varepsilon \frac{R_0}{\hbar/mc} \ll 1$ for his proof. However, for the extreme relativistic case this is just equal to R_0/λ, which as we have seen must be of the order of unity for finite nuclear size effects to be appreciable. In fact for heavier nuclei for energies somewhat greater than 20 MeV this expression will actually become greater than one. Hence it is of importance to see exactly what the limitations of the above dependence of the phase shift on the volume integral of the potential are. In this work a method is developed with which Feshbach's result may be derived in a simple manner as well as additional terms of what is essentially an expansion in powers of $\frac{\varepsilon R_0}{\hbar/mc}$. Above about 50 MeV this expansion ceases to converge and the whole basis of Feshbach's result breaks down. When, however, only the comparison of two charge distributions is of interest, it will be shown that it is sufficient to use only the volume integral of the potential up to about 50 MeV.

The method can also be applied to higher phase shifts which can be expected to become important at energies of about 30 MeV.

CALCULATIONS OF PHASE SHIFTS.

We consider an electron, energy E, moving in a central field with potential energy V. On separating out the angular dependence, the small and large radial functions $\frac{f_k}{\xi}$, $\frac{g_k}{\xi}$ may be obtained from the solutions of the equations

$$\frac{df_k}{d\xi} = \frac{k f_k}{\xi} - (\varepsilon - 1 - U) g_k \qquad (1)$$
$$\frac{dg_k}{d\xi} = (\varepsilon + 1 - U) f_k - \frac{k}{\xi} g_k$$

where $k = -(j + \frac{1}{2}) = -(\ell + 1)$, for $j = \ell + \frac{1}{2}$

$k = +(j + \frac{1}{2}) = \ell$ for $j = \ell - \frac{1}{2}$

We use relativistic units, $\hbar = m = c = 1$, and $\varepsilon = \frac{E}{mc^2}$, $U = \frac{V}{mc^2}$, $\xi = \frac{r}{\hbar/mc}$.

For the scattering of an electron by the field of a finite size nucleus, charge Ze, the asymptotic expressions for the regular solutions of fk , gk, in the form given by Feshbach, are

$$f_k \sim (\varepsilon - 1)^{1/2} \sin\left\{p\xi + \left(\frac{a\varepsilon}{p}\right) \log 2p\xi - (\ell+1)\frac{\pi}{2} + \eta_k\right\} \qquad (2)$$
$$g_k \sim (\varepsilon + 1)^{1/2} \cos\left\{p\xi + \left(\frac{a\varepsilon}{p}\right) \log 2p\xi - (\ell+1)\frac{\pi}{2} + \eta_k\right\}$$

where $a = Z\alpha$, $\alpha = \frac{e^2}{\hbar c}$ and p is the electron momentum in relativistic units.

In the more usual notation (Mott and Massey, 1949), the phase shift for the state $j = \ell + \frac{1}{2}$ is denoted by η_ℓ and that for $j = \ell - \frac{1}{2}$ by $\eta - (\ell + 1)$.

We may write

$$\eta_k = \eta^{(c)}_{k,R} + \delta_k \qquad (3)$$

$\eta^{(c)}_{k,R}$ is the phase shift for pure Coulomb scattering, the corresponding regular solutions being devoted by $f^{(c)}_{k,R}$, $g^{(c)}_{k,R}$, while δ_k is the extra phase shift due to the finite nuclear size and gives the deviation from pure Coulomb

scattering. The irregular Coulomb functions are denoted by $f^{(c)}_{k,I}$, $g^{(c)}_{k,I}$ and have the same asymptotic forms as the regular solutions except for a different phase shift $\eta^{(c)}_{k,I}$. Explicit expressions for $\eta^{(c)}_{k,R}$, $\eta^{(c)}_{k,I}$ and also for the differential cross-section in terms of the phase shifts are given by Parzen and Elton, but are not necessary for our purpose.

For the extreme relativistic case, $\varepsilon \gg 1$, where the rest mass may be neglected in the above equations, Acheson and Feshbach have pointed out the following relation between the phase shifts,

$$\eta_k = \eta_{-k} \tag{4}$$

Since this holds separately for the Coulomb phase shifts, also $\delta_k = \delta_{-k}$. Thus for extreme relativistic energies states with the same j but opposite spin orientations with respect to the orbital angular momentum have the same phase shifts. The numerical analysis of Elton and Parzen using the accurate equation (1) are in good agreement with this result. In addition for energies of the order of 20 MeV, the calculations of Elton and Acheson show that only the phase shifts δ_{-1}, δ_1 for the $s_{1/2}$, $p_{1/2}$ states differ appreciably from zero.

It is therefore only necessary to calculate δ_{-1} in this region of energies. However, for an energy of 100 MeV Parzen's results show that the phase shifts up to δ_{-5}, δ_5 must be included. As an approximate criterion we may take δ_k to become important for $k R_0 > \ell$. Thus δ_{-2}, δ_2 i.e. the $p_{3/2}$, $d_{3/2}$ states may be expected to become significant at energies of about 30 MeV. Convenient expressions for the cross-section in terms of the phase shifts for $\varepsilon \gg 1$ have been given by Acheson.

Feshbach has brought attention to the special class of potentials

$$V = \frac{F\left(\frac{r}{r_0}, \beta\right)}{r_0} \tag{5}$$

where r_0, β are parameters describing the charge distribution, r_0 being a length of the order of the nuclear dimensions β_1 β any additional parameters needed to specify the potential. From Poisson's equation it is seen that the corresponding charge density must have the form

$$\rho = \frac{\eta\left(\frac{r}{r_0}, \beta\right)}{r_0^3} \tag{6}$$

For this type of charge distribution Feshbach has proved that for $\varepsilon >> 1$ the phase shifts depend on ε, r_0 only through εr_0. This may be seen by rewriting equations (1) with the rest mass neglected. The importance of this result is in extending calculations already made to other radii or energies.

In order to calculate δ_k we consider the regular solutions of the radial functions at a point r_1, sufficiently far outside the nuclear charge distribution for the potential to be Coulomb,

$$\begin{aligned} f_k(r_1) &= C_1 f^{(c)}_{k,R}(r_1) + C_2 f^{(c)}_{k,I}(r_1) \\ g_k(r_1) &= C_1 g^{(c)}_{k,R}(r_1) + C_2 g^{(c)}_{k,I}(r_1) \end{aligned} \tag{7}$$

C_2/C_1 measures the admixture of the irregular to the regular solution due to the deviation from a Coulomb field in the nuclear interior. From (7) and using the asymptotic forms (2),

$$\tan \delta_k = \frac{\frac{C_2}{C_1} \sin\left(\eta^{(c)}_{k,I} - \eta^{(c)}_{k,R}\right)}{\frac{C_2}{C_1} \cos\left(\eta^{(c)}_{k,I} - \eta^{(c)}_{k,R}\right) - 1} \tag{8}$$

C_2/C_1 is obtained using both equations (7)

$$\frac{C_2}{C_1} = \frac{K_{k,\varepsilon}(r_1)\, g^{(c)}_{k,R}(r_1) - f^{(c)}_{k,R}(r_1)}{f^{(c)}_{k,I}(r_1) - K_{k,\varepsilon}(r_1)\, g^{(c)}_{k,I}(r_1)} \tag{9}$$

where $K_{k,\varepsilon}(r_1) = f_k(r_1)/g_k(r_1)$

Kk is written with the suffix ε where we specially wish to emphasise that it depends on the energy. For our purpose it is sufficient to note that for a given energy δ_k depends on the nuclear charge distribution only through $K_{k,\varepsilon}(r_1)$. As is evident from the above derivation C_2/C_1, and hence δ_k are independent of r_1, so long as this is outside the charge distribution. Thus two different charge distributions will give the same scattering if the $K_{k,\varepsilon}(r_1)$ for all the important δ_k are the same for both. In particular at energies for which the phase shifts $\delta_{-1} = \delta_1$ are sufficient, two distributions will give the same scattering at a given energy if $K(r_1)$ is the same, where for the s state the suffix -1 will for convenience be omitted.

METHOD FOR OBTAINING K_k

By differentiating $K_k = f_k/g_k$ with respect to ξ and using the equation (1), the following Riccati equation is obtained for K_k,

$$\frac{dK_k}{d\xi} = \frac{2k}{\xi} K_k - (\varepsilon + 1 - u) K_k^2 - (\varepsilon - 1 - u) \tag{10}$$

For extreme relativistic case, $\varepsilon >> 1$, this becomes

$$\frac{dK_k}{d\xi} = \frac{2k}{\xi} K_k - (\varepsilon - u)(1 + K_k^2) \tag{11}$$

For the potential (5), it is then seen by rewriting this equation that for $\varepsilon >> 1$,

$$K_{k,\varepsilon} = K_k(\varepsilon\xi, \varepsilon\xi_0, \beta) \tag{12}$$

which is in agreement with Feshbach's result that for potentials of this type, the phase shift depends on ε , ξ_0 only through $\varepsilon\,\xi_0$. We may then for future reference note that for the charge distributions (6) which in addition are such that the charge density is zero outside r_0, where r_0 can now be considered as the nuclear radius, the total differential of K_k with respect to ξ_0 , using (12) is given by

$$\frac{dK_{k,\varepsilon}(\xi_0)}{d\xi_0} = \left[\frac{\partial K_{k,\varepsilon}}{\partial \xi}\right]_{\xi=\xi_0} + \left[\frac{\partial K_{k,\varepsilon}}{\partial \xi_0}\right]_{\xi=\xi_0} = \frac{\varepsilon}{\xi_0}\left[\frac{\partial K_{k,\varepsilon}}{\partial \varepsilon}\right]_{\xi=\xi_0} \tag{13}$$

The required solution of K_k is determined by the behaviour near the origin. We require those solutions of (10) which correspond to the regular solutions for f_k, g_k. For potentials less rapidly divergent at the origin than a Coulomb potential it is seen from equations (1) that the indicial behaviour of K_k corresponding to these regular solutions is

$$\lim_{\xi\to 0} K_k = \frac{\varepsilon - 1 - u(0)}{2k-1}\,\xi \qquad \text{for } k < 0 \tag{14}$$

$$\lim_{\xi\to 0} K_k = \frac{(2k+1)}{(\varepsilon + 1 - u(0))}\,\frac{1}{\xi} \qquad \text{for } k > 0$$

We consider first the case $k < 0$. If we rewrite (10) in the form,

$$(1 - K_k^2) + (u - \varepsilon)(1 + K_k^2) = \frac{dK_k}{d\xi} - \frac{2k}{\xi}K_k$$

this may be transformed into an integral equation by multiplying by the integrating factor ξ^{-2k}

$$K_k(\xi) = \xi^{2k}\int_0^{\xi}\left\{(u - \varepsilon + 1) + (u - \varepsilon - 1)K_k^2\right\}\xi'^{-2k}\,d\xi' \tag{15}$$

If $K_k^2 < 1$ this may be solved by successive iteration starting with $K_k = 0$. The solution obtained in this way is then seen to have the appropriate

indicial behaviour corresponding to $k < 0$.

For $k > 0$, if we put $K_k = \frac{1}{\chi_k}$ we obtain a Riccati equation for χ_k similar to that for K_k. As above this may be transformed into the integral equation

$$\chi_k(\xi) = \frac{1}{\xi^{2k}} \int_0^{\xi} \left\{ (\varepsilon + 1 - u) + (\varepsilon - 1 - u)\chi_k^2 \right\} \xi'^{2k} d\xi' \qquad (16)$$

The solution which corresponds to solving this by successive iteration starting with $\chi_k = 0$ gives then the correct indicial behaviour for K_k when $k > 0$. From the Riccati equations for χ_k, K_k it is readily seen that for $\varepsilon >> 1$, the solutions as obtained above satisfy $K_k K_{-k} = -1$, which is of course the counterpart of the result $\delta_{-k} = \delta_k$. In what follows below we shall then restrict ourselves to $k < 0$.

The first iteration of (15) is

$$K_k^{(1)}(\xi) = \xi^{2k} \int_0^{\xi} \xi'^{-2k} u \, d\xi' + \frac{(\varepsilon - 1)\xi}{2k - 1} \qquad (17)$$

Thus for the s wave within the validity of the first iteration, the phase shift depends only on the volume integral of the potential which is just Feshbach's result. By integrating by parts twice and using Poisson's equation, the integral in (17) may be expressed in terms of an integral over the charge density. Thus

$$\int_0^r V r'^{-2k} dr' = \frac{2\pi}{k(2k-1)} \int_0^{\infty} e\rho \, r'^{-2k+2} dr' + \frac{Ze^2 r^{-2k}}{2k} \qquad (18)$$

The upper limit in the integral on the right is actually r but may, since r is effectively outside the charge distribution, be replaced by ∞. Thus if the first iteration gives a sufficiently good approximation for K_k, δ_k will be the same for two charge distributions if $\int_0^{\infty} \rho \, r^{-2k-2} dr$ is the same for

both. For the s wave this integral just becomes $\int_0^\infty \rho\, r^4\, dr$.

CALCULATION OF K_ε FOR $\varepsilon >> 1$

We now consider in more detail the case of the s wave, k = -1, for $\varepsilon >> 1$. (15) then becomes

$$K_\varepsilon(\xi) = \frac{1}{\xi^2}\int_0^\xi \xi'^2 (u-\varepsilon)(1+K_\varepsilon^2)\, d\xi' \tag{19}$$

For the first iteration we have

$$K_\varepsilon^{(1)}(\xi) = \frac{1}{\xi^2}\int_0^\xi u\, \xi'^2 d\xi' - \frac{\varepsilon\xi}{3} \tag{20}$$

and for the second,

$$K_\varepsilon^{(2)}(\xi) = K_\varepsilon^{(1)}(\xi) + \frac{1}{\xi^2}\int_0^\xi (u-\varepsilon)\left[K_\varepsilon^{(1)}(\xi')\right]^2 \xi'^2 d\xi' \tag{21}$$

This process converges rapidly if $K_\varepsilon^2 << 1$, and we may then write

$$K_\varepsilon(\xi) = K_\varepsilon^{(1)}(\xi) + \sum_{n=2}^{\infty} \chi_\varepsilon^{(n)}(\xi) \tag{22}$$

where $\chi^{(n)}$ is the extra term given by the n'th iteration. If the potential is of the type (5) and proportional to a, then the above iteration process will also give an expansion in homogeneous polynomials in a and $\varepsilon\xi_0$ of increasing degree. Thus the first iteration will contain terms in a, $\varepsilon\xi_0$, the second terms in a^3, $a^2\varepsilon\xi_0$, $a(\varepsilon\xi_0)^2$, $(\varepsilon\xi_0)^3$. Such successive homogeneous polynomials do not however correspond to successive iterations after the second since K_ε occurs as the square in the iteration.

We may expect that above a certain energy the iteration process will no longer converge and the basis for Feshbach's result with break down. This will occur when $K^{(1)}(\xi_0)$, which since U is proportional to a, is of the order of $(a + \varepsilon\xi_0)$ becomes of the order of unity.

In order to obtain a more precise idea of the convergence and of the importance of the additional term given by the second iteration, we consider the charge distribution used by Rosenthal and Breit (1932),

$$\rho = \frac{3}{4\pi}\left(\frac{Ze}{r_0^3}\right)\left[\left(\frac{n+1}{3}\right)\left(\frac{r}{r_0}\right)^{n-2}\right] \quad ; \; r < r_0$$
$$\rho = 0 \quad ; \; r > r_0 \tag{23}$$

The corresponding potential energy of the electron is

$$V = -\left(\frac{n+1}{n}\right)\left[1 - \frac{1}{(n+1)}\left(\frac{r}{r_0}\right)^{n}\right]\frac{Ze^2}{r_0} \quad ; \; r < r_0$$
$$V = -\frac{Ze^2}{r} \quad ; \; r > r_0 \tag{24}$$

n can vary from -1 for a point charge through $n = 2$ for a uniform charge distribution to $n = \infty$ when all the charge is on the surface. For the first iteration we have,

$$K_\varepsilon^{(1)}(\xi) = \left(\frac{\xi_0}{\xi}\right)^2 K_{\varepsilon=0}^{(1)}(\xi_0) - \frac{a}{2}\left[1 - \left(\frac{\xi_0}{\xi}\right)^2\right] - \frac{\varepsilon\xi}{3} \quad ; \; \xi > \xi_0 \tag{25}$$

$$K_{\varepsilon=0}^{(1)}(\xi_0) = -\frac{a}{3}\left(\frac{n+4}{n+3}\right) \tag{26}$$

For the second iteration

$$K_\varepsilon^{(2)}(\xi) = K_\varepsilon^{(1)}(\xi) + \left(\frac{\xi_0}{\xi}\right)^2 \lambda_\varepsilon^{(2)}(\xi_0) + \frac{1}{\xi^2}\int_{\xi_0}^{\xi}(n-\varepsilon)\left[K_\varepsilon^{(1)}(\xi')\right]^2 \xi'^2 d\xi' ; \quad \xi > \xi_0 \tag{27}$$

$$\lambda_\varepsilon^{(2)}(\xi_0) = -\frac{a^3}{9}\left(\frac{n+1}{3}\right)^3\left\{\frac{1}{5} - \frac{(n+9)}{(n+1)(n+3)(n+5)} + \frac{3(2n+9)}{(n+1)^2(n+3)^2(2n+5)} - \frac{9}{(n+1)^3(n+3)^2(3n+5)}\right\}$$
$$\frac{\varepsilon\xi_0 a^2}{9}\left(\frac{n+1}{n}\right)^2\left\{\frac{3}{5} - \frac{2(n+9)}{(n+1)(n+3)(n+5)} + \frac{3(2n+9)}{(n+1)^2(n+3)^2(2n+5)}\right\}$$
$$\frac{(\varepsilon\xi_0)^2 a}{9}\left\{\frac{3n+2}{5n} - \frac{6}{n(n+3)(n+5)} - \frac{1}{(n+1)(n+5)}\right\} - \frac{(\varepsilon\xi_0)^3}{45} \tag{28}$$

$$\frac{1}{\xi^2}\int_{\xi_0}^{\xi}(u-\varepsilon)\left[K_\varepsilon^{(1)}(\xi')\right]^2\xi'^2 d\xi' = -\frac{a^3}{4}\left\{\frac{1}{2}\left[1-\left(\frac{\xi_0}{\xi}\right)^2\right] + \right.$$

$$\left. + \frac{2}{3}\left(\frac{n+1}{n+3}\right)\left(\frac{\xi_0}{\xi}\right)^2 \log\left(\frac{\xi_0}{\xi}\right) + \frac{1}{18}\left(\frac{n+1}{n+3}\right)^2\left(\frac{\xi_0}{\xi}\right)^2\left[1-\left(\frac{\xi_0}{\xi}\right)^2\right]\right\}$$

$$\frac{a^2\varepsilon\xi}{18}\left\{\frac{1}{2}\left(\frac{n+1}{n+3}\right)^2\left(\frac{\xi_0}{\xi}\right)^3\left[1-\left(\frac{\xi_0}{\xi}\right)\right] - 5\left(\frac{n+1}{n+3}\right)\left(\frac{\xi_0}{\xi}\right)^2\left[1-\left(\frac{\xi_0}{\xi}\right)\right]\right.$$

$$\left. + \frac{7}{2}\left[1-\left(\frac{\xi_0}{\xi}\right)^3\right]\right\}$$

$$\frac{a(\varepsilon\xi)^2}{9}\left\{\left[1-\left(\frac{\xi_0}{\xi}\right)^4\right] - \frac{1}{2}\left(\frac{n+1}{n+3}\right)\left(\frac{\xi_0}{\xi}\right)\left[1-\left(\frac{\xi_0}{\xi}\right)^2\right]\right\}$$

$$- \frac{(\varepsilon\xi)^3}{45}\left[1-\left(\frac{\xi_0}{\xi}\right)^5\right] \qquad (29)$$

For constant density, $n = 2$, we have from the above formulae,

$$K_\varepsilon^{(2)}(\xi_0) = -\left[\frac{2}{5}a + \frac{\varepsilon\xi_0}{3}\right] - \left[0{\cdot}042a^3 + 0{\cdot}102a^2\varepsilon\xi_0 + 0{\cdot}074a(\varepsilon\xi_0)^2 + 0{\cdot}022(\varepsilon\xi_0)^3\right] \qquad (30)$$

For a surface charge distribution; $n = \infty$, the solution for $K(\xi_0)$ may be obtained explicitly,

$$K(\xi_0) = -\frac{1}{(a+\varepsilon\xi_0)}\left[1 - (a+\varepsilon\xi_0)\cot(a+\varepsilon\xi_0)\right]$$

Expanding $\cot(a+\varepsilon\xi_0)$ this becomes

$$K(\xi_0) = -\frac{1}{3}(a+\varepsilon\xi_0)\left[1 + \frac{1}{15}(a+\varepsilon\xi_0)^2 + \frac{2}{315}(a+\varepsilon\xi_0)^4 + \cdots\right] \qquad (31)$$

The first two terms of this expression are identical with those derived from the first two iterations, while the third term can easily be obtained from the third iteration. We see that above about 50 MeV the second iteration increases very rapidly and the method ceases to be reliable. Thus even at 15 MeV the second iteration contributes almost 10% to $K(\xi_0)$ for a surface charge distribution.

COMPARISON OF CHARGE DISTRIBUTIONS.

Although it would appear from the above that the first iteration is not a very good approximation for K even at rather low energies and breaks down completely at higher energies, nevertheless when the equivalence of two charge distributions is being considered it is sufficient to use only the first iteration. We consider as an example the equivalent radii, r_s and r_c, of a surface and a uniform charge distribution. The simplest way to obtain the relation between r_s and r_c is to equate the expressions (27) for $n = 2$ and $n = \infty$ with $\xi = \xi_c$,

$$\left(\frac{r_s}{r_c}\right)^2 = \frac{3}{5}\left[1 - \left(0{\cdot}012\,a^2 + 0{\cdot}023\,a\,(\varepsilon\xi_c)^2 + 0{\cdot}012\,(\varepsilon\xi_c)^2\right]\right. \quad (32)$$

The first term is due to the first iteration and can immediately be obtained by equating $\int_0^\infty \rho r^4 dr = \frac{Ze}{4\pi}\left(\frac{n+1}{n+3}\right) r_0^2$ for $n = 2$ and $n = \infty$. The remaining terms from the second iteration make only a very small contribution. Thus for energies for which the iteration procedure does not break down entirely i.e. up to energies of about 50 MeV, the use of the first iteration is a very good approximation when the equivalence between the two charge distributions and not the actual value of K is being considered. Since the difference between a uniform and a surface charge distribution can be considered as an extreme the magnitude of the terms in (32) due to the second iteration may be regarded as

in the nature of an upper limit to these terms in the expression for the radius of the uniform charge distribution equivalent to some actual distribution. The reason for the near cancellation of the terms from the second iteration is seen from (21). Thus if $K^{(1)}(\xi_1)$ is equal for two charge distributions, $K^{(1)}(\xi)$ and $U(\xi)$ will not be very different for $0 < \xi \leq \xi_1$, and hence the integrand occurring in $\chi_\varepsilon^{(2)}(\xi_1)$ and hence $\chi_\varepsilon^{(2)}(\xi_1)$ itself will not differ very much for the two distributions. We see then that for any given charge distribution ρ, the radius of the equivalent uniform charge distribution will be given with good accuracy by

$$\frac{Ze}{4\pi}\,\frac{3}{5}\,r_c^2 = \int_0^\infty \rho\, r^4\, dr \tag{33}$$

SIGNIFICANCE FOR THE INTERPRETATION OF THE EXPERIMENTAL RESULTS

SIGNIFICANCE FOR THE INTERPRETATION OF THE EXPERIMENTAL RESULTS

The significance of the result just obtained is in the interpretation of the experiments. The calculations of Acheson and Elton extended by using Feshbach's result that the scattering depends only on εr_0, may be used to determine the radius $r_0(\varepsilon)$ of the equivalent uniform charge distribution from the scattering at an energy ε. Because of the near cancellation of terms in the second iteration, this equivalent radius for the actual but unknown distribution will only be very slightly energy dependent. Thus from (32) for Pb the ratio between the equivalent radii for a surface and a uniform charge distribution, $\left(\frac{r_s}{r_c}\right)$, differs by less than .25% from that given by the first iteration for $\varepsilon = 0$, and by less than 4% for an energy of 50 MeV. Since at energies at which r_0 can be expected to become appreciably energy dependent, higher phase shifts become important, the equivalent radius may be taken as energy independent for the energies of interest since variations of the above amount

would hardly be experimentally significant. This implies that for energies for which only the s wave shift is important, the only information which can be deduced from scattering experiments at different energies is the equivalent radius at zero energy. From (33) knowledge of r_c then implies knowledge of $\int_0^\infty \rho r^4 dr$.

More detailed information about the form of the nuclear charge distribution could be obtained from the scattering at somewhat higher energies when the $p_{3/2}$ (k = -2), $d_{3/2}$ (k = 2) states become important. In a similar way as above for not too high energies, the phase shifts for these states would depend on the charge distribution only through $\int_0^\infty \rho r^6 dr$. This could again be expressed in terms of the radius of an equivalent uniform charge distribution which would not of course in general be the same as that for the $S_{\frac{1}{2}}$, $p_{1/2}$ states.

Of course, for purposes of comparison, any of the distributions (23) and more especially a surface distribution could have been used; however, it seems most natural to use a uniform one.

The only relevant experiments so far available are those of Lyman, Hanson and Scott (1951) with 15.7 MeV electrons. Their results interpreted using the calculations of Acheson indicate that for Ag and Au with a radius of 1.45×10^{-13} $A^{\frac{1}{3}}$ cm. the charge is more concentrated towards the centre than for a uniform charge distribution. They find that the effective radius, assuming constant charge density, needed to fit the experimental results is about 20% smaller than the above value, i.e. about 1.15×10^{-13} $A^{\frac{1}{3}}$ cm. These results must be considered as somewhat uncertain as the radiative correction was allowed for using the result of Schwinger (1949). This is of doubtful accuracy for heavy elements since the radiative scattering is calculated for a Coulomb field using

the Born approximation and is given relative to the Born approximation expression for the elastic scattering.

NOTE. It recently came to the writer's attention that Rose and Newton (1951) have considered the Riccati equation (10) in connection with the nodal properties of the radial functions f_{12}, g_{12}. They do not however use this to obtain an explicit solution.

Rose (1951) has also considered an integral form of the radial equations together with an iteration procedure for solving these. With this the use of the first iteration for both functions then gives just the result (17) for $K^{(0)}$, Since however both functions enter explicitly in Rose's treatment, it is more convenient especially for the second iteration to consider K, which is the quantity of immediate interest in our consideration, directly.

II.

THEORY OF THE ISOTOPE SHIFT.

In the usual derivation of the volume dependent isotope shift (I.S.) the charge in binding energy of an electron due to the difference in its electrostatic interaction with a point charge and the same charge spread over the nuclear volume is calculated using a perturbation method. In this the difference between two isotopes of the potential energy of the electron in the nuclear region is averaged over the relativistic charge density when the electron is moving in the field of a point nucleus. If it is assumed that the charge is uniformly distributed throughout the nucleus and that the increase in volume between two isotopes is proportional to the increase in mass, then with the nuclear radius given by $r_0 = 1.9 \times 10^{-13} A^{\frac{1}{3}}$ cm., it is found that the I.S. calculated in this way is greater than the experimental I.S. by a factor of 2-3 except in the region of $Z = 60$, where the isotope shifts are anomalously large (Brix and Kopfermann, 1949, 1952).

It might be expected however that the spreading of the charge would strongly distort the wave function of the electron from its Coulomb form just inside the nuclear region where the perturbation takes place and thus to considerably affect the I.S. Hence it is of some interest to investigate whether a more accurate calculation taking into account this distortion of the wave function might not remove some of the discrepancy. An approximate estimate of the effect of the distortion when all the charge is on the surface was made by Rosenthal and Breit (1932). Crawford and Schalow (1949) made an estimate based on the work of Broch (1945) for the two cases of constant charge density, and all the charge on the surface which should be exact, but with which we disagree. They do not examine other charge distributions or the dependence on Z, and their method

cannot be extended to an arbitrary distribution.

Using the method of Broch (1945), which avoids perturbation theory altogether, it is shown that the I.S. of an s electron depends on the charge distribution only through $K_{\varepsilon=1}$, $\delta K_{\varepsilon=1}$, where the latter is the charge in $K_{\varepsilon=1}$ between the two isotopes. The method developed in I. may then be applied to calculate the I.S. for any given charge distribution. It is then also possible to establish a simple correction with the elastic scattering of electrons by nuclei.

PERTURBATION THEORY OF THE I.S.

Using the perturbation method, Rosenthal and Breit (1932) and Racah (1932) obtain for the I.S. due to a charge of potential energy δV of the electron in the field of the nucleus,

$$\delta(\Delta E)_{pert.} = \int_0^\infty \delta V \left(f_{k,R}^{(0)\,2} + g_{k,R}^{(0)\,2} \right) dr \tag{34}$$

where ΔE is the term shift due to the effect of the finite nuclear size. Since δV is only different from zero in a region of the dimensions of the nuclear radius, we may for the wave functions in this region neglect the binding energy as compared with the rest mass of the electron. It is then convenient for the present purpose to write the radial equations in the form

$$\begin{aligned} \frac{df_k}{dx} &= \frac{k}{x} f_k + \frac{U}{2a} g_k \\ \frac{dg_k}{dx} &= \frac{2-U}{2a} f_k - \frac{k}{x} g_k \end{aligned} \tag{35}$$

where $x = \frac{2Zr}{a_H} = 2a\xi$

and $a_H = \hbar^2/me^2$ is the Bohr radius.

With $U = -\frac{2a^2}{x}$, the functions $f_{k,R}^{(0)}$, $g_{k,R}^{(0)}$ are then given by (43)

with $C_2 = 0$. Since the value of x corresponding to the nuclear radius is $\sim 0.6 \times 10^{-4} A^{\frac{1}{3}} Z$, it is sufficiently accurate to retain only the lowest power of x. We have then

$$f_{k,R}^{(0)2} + g_{k,R}^{(0)2} = \frac{2C^2}{\Gamma^2(1+2\sigma)} k(k-\sigma) x^{2\sigma} \tag{36}$$

where $\sigma = \sqrt{k^2 - a^2}$

Thus

$$\delta(\Delta E)_{pert.} = N. \frac{k(k-\sigma)}{\Gamma^2(1+2\sigma)} \int_0^\infty \frac{\delta u}{2a^2} x^{2\sigma} dx \tag{37}$$

with $$N = 8\pi Z e^2 C^2. \tag{38}$$

The normalisation is that used by Rosenthal and Breit

$$4\pi \int_0^\infty (f_k^2 + g_k^2)\, dr = 1 \tag{39}$$

C^2 is determined from the normalisation by putting g_k/r asymptotically equal to the radial Schrödinger wave function for larger r. In this way, Rosenthal and Breit obtain for an s electron

$$C^2 = \frac{R}{2L^2 e^2} a_H^3 \psi^2(0) \tag{40}$$

where R is the Rydberg constant and $\psi(0)$ the value of the Schrödinger wave function at the origin.

With the charge density (6), Rosenthal and Breit obtain for the I.S. of an s electron

$$\delta(\Delta E)_{pert.} = N. \frac{1+\sigma}{\Gamma^2(1+2\sigma)} \frac{u+1}{(2\sigma+u+1)(2\sigma+1)} x_0^{2\sigma} \frac{\delta x_0}{x_0} \tag{41}$$

Brix and Kopfermann have taken as a standard with which to compare the I.S. as determined from experiment the case of constant charge density, $u = 2$, as given

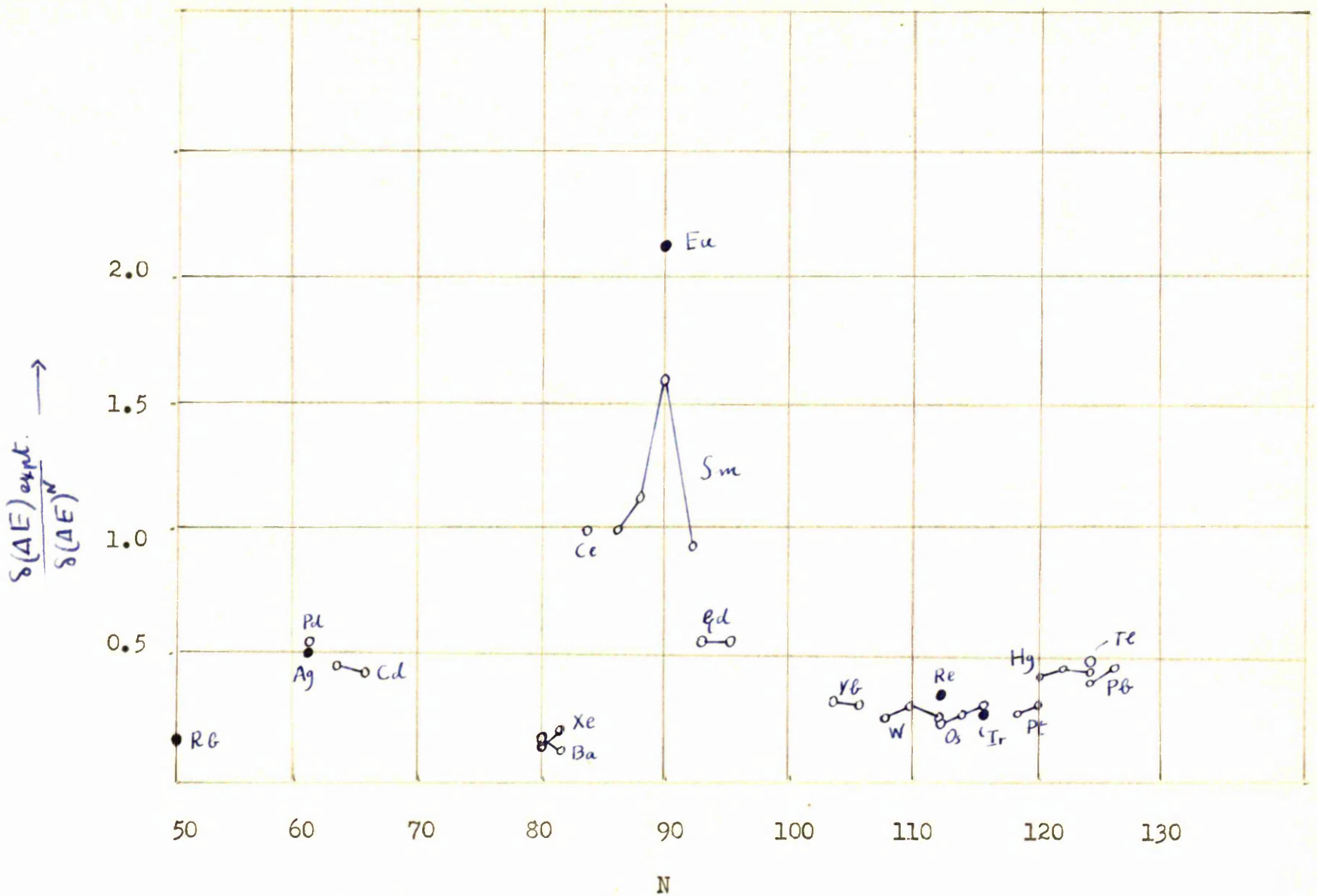

Fig. 1. $\frac{\delta(\Delta E)_{expt.}}{\delta(\Delta E)^{N}}$ as a function of the neutron number N for isotope pairs with N - 2 and N neutrons. $\delta(\Delta E)_{expt.}$ is the experimentally determined I.S. apart from rather uncertain shielding corrections, $\delta(\Delta E)^{N}$ the I.S. of the standard model of Brix and Kopfermann with the correction due to distortion included. Only the data for isotopes with even N are shown. Circles: Z even, Points: Z odd.

by (41), together with the assumption that the nuclear radius is proportional to $A^{\frac{1}{3}}$, and that therefore $\frac{\delta x_0}{x_0} = \frac{\delta A}{3A}$. For the actual value of the nuclear radius they use $1.4 \times 10^{-13} A^{\frac{1}{3}}$ cm.

Fig. 1 shows the results for the I.S. data, apart from rather uncertain corrections due to the shielding of the inner electrons by the valence electron (Crawford & Schalow (1949) and Humbach (1952)), according to the most recent presentation of Brix and Kopfermann (1951). In this a correction for the distortion of the wave function has already been made based on the work of Humbach (1952), which has only just come to the writer's notice. $\delta(\Delta E)^N$ is then the I.S. of the standard model of Brix and Kopfermann with the correction due to distortion already included, and not as calculated by the simple perturbation treatment.

Our results for the effect of the distortion of the wave function are in agreement with those of Humbach who has also used Broch's method. However, he does not give any simple procedure for calculating the I.S. for any given charge distribution such as is derived in the following, nor of course the relation with the electron scattering.

NON-PERTURBATION METHOD FOR THE I.S.

By a method avoiding the use of perturbation theory, Broch obtains for the difference in energy between an electron in the field of a finite size nucleus and a point nucleus,

$$\Delta E = -N\left(\frac{C_2}{C_1}\right)\frac{\sigma}{\Gamma(1+2\sigma)\Gamma(1-2\sigma)} \tag{42}$$

where N is given by (38) with C_1 instead of C. C_1, C_2 are the coefficients of the regular and irregular solutions respectively in the general solution for the two radial functions in the region exterior to the charge distribution where the electron is moving in a Coulomb field. For this region and where r is still sufficiently small that the binding energy may be neglected,

$$\begin{aligned} f_k(x) &= a\left[C_1 J_{2\sigma}(2x^{1/2}) + C_2 J_{-2\sigma}(2x^{1/2})\right] \\ g_k(x) &= C_1 A_{2\sigma}(2x^{1/2}) + C_2 A_{-2\sigma}(2x^{1/2}) \end{aligned} \tag{43}$$

where
$$A_{2\sigma} = (k-\sigma)J_{2\sigma} + x^{1/2} J_{2\sigma+1}$$
$$A_{-2\sigma} = (k+\sigma)J_{-2\sigma} + x^{1/2} J_{-2\sigma+1}$$

For a point nucleus $C_2 = 0$. Following Broch, the condition for the continuity of f_k/g_k at a point x_1, where the potential is Coulomb, then gives the admixture of the irregular relative to the regular solution due to the deviation from a Coulomb field in the region $x < x_1$,

$$\frac{C_2}{C_1} = \frac{a\, J_{2\sigma}(2x_1^{1/2}) - K_k(x_1)\, A_{2\sigma}(2x_1^{1/2})}{K_k(x_1)\, A_{-2\sigma}(2x_1^{1/2}) - a\, J_{2\sigma}(2x_1^{1/2})} \tag{44}$$

where $K_k(x_1)$ is now assumed evaluated neglecting the binding energy and is determined from the solution interior to x_1. It must be emphasised that the only condition on x_1 is that the potential is Coulomb at x_1. Thus if the charge density does not fall abruptly to zero, x_1 must be sufficiently far out

to satisfy this condition. As is evident from the derivation, C_2/C_1 is then independent of x_1. Expanding the Bessel functions and retaining only the lowest power of x_1,

$$\frac{C_2}{C_1} = -\frac{\Gamma(1-2\sigma)}{\Gamma(1+2\sigma)}\left\{\frac{a - K_k(x_1)(k-\sigma)}{a - K_k(x_1)(x+\sigma)}\right\} x_1^{2\sigma} \tag{45}$$

In this approximation it can easily be verified directly that C_2/C_1 is independent of x_1 by showing that $\frac{d}{dx}\left(\frac{C_2}{C_1}\right) = 0$ using equation (49) together with $U = -\frac{2a^2}{x_1}$. All the results derived below from (45) are then independent of x_1. Substituting (45) in (42),

$$\Delta E = N.\ \frac{\sigma}{\Gamma^2(1+2\sigma)}\left\{\frac{a - K_k(x_1)(k-\sigma)}{a - K_k(x_1)(k+\sigma)}\right\} x_1^{2\sigma} \tag{46}$$

The normalisation is given by (39). For an s electron C_1^2 is again given by (40) since the part of the normalisation integral extending to x_1 may be neglected because of the smallness of x_1, and outside x_1 that part due to C_2 can also be neglected again as a result of the smallness of x_1.

The I.S. is

$$\delta(\Delta E) = N.\ \frac{2\sigma^2}{\Gamma^2(1+2\sigma)}\ \frac{a.\ \delta K_k(x_1)}{\left[a - K_k(x_1)(k+\sigma)\right]^2}\ x_1^{2\sigma} \tag{47}$$

Thus for two charge distributions to give the same I.S. $K_k(x_1)$, $\delta K_k(x_1)$ must be the same for both.

In order to obtain K_k, the same method may be used as in I. If the binding energy is neglected, i.e. with $\varepsilon = 1$, the Riccati equation for K_k becomes

$$\frac{dK_k}{dx} = \frac{2k}{x}K_k + \frac{(U-2)}{2a}K_k^2 + \frac{U}{2a} \tag{48}$$

The indicial behaviour of the required solutions is obtained from (14) by

putting $\varepsilon = 1$. If in the nuclear region it is also permissible to write $\frac{u}{2a}$ instead of $\frac{(u-2)}{2a}$ for the factor multiplying K_k^2, (48) becomes

$$\frac{dK_k}{dx} = \frac{2k}{x} K_k + \frac{u}{2a}(1 + K_k^2) \tag{49}$$

This is formally equivalent to the equation for the extreme relativistic case if in this we put $\varepsilon = 0$. The approximation just made is equivalent to assuming $\frac{x_0}{a^2} \ll 1$ for the factor multiplying K_k^2. This is somewhat more stringent, especially for lighter elements, than the condition $x_0 \ll 1$ assumed previously. However, for $k < 0$ we see from the iteration procedure for obtaining K_k that the term neglected only affects K_k in the second iteration, the error involved in the additional term due to this being less than 10%. The only important case for the I.S. is in fact that of an s electron, $k = -1$, and since as we shall see the additional term due to the second iteration makes only a relatively small contribution for this, the error in K due to the above approximation is very small. For $k > 0$ we see from (16) that the approximation made above would cause an error in K_k already in the first iteration. Thus for $p_{1/2}$ states which are the only other states for which the I.S. is at all appreciable, it would be necessary to use equation (48) if we wish to obtain a more accurate value of K_1. However, even in this case if we are only interested in comparing charge distributions which give the same K_1 it is again only necessary to compare the volume integrals of the potentials as for $k = -1$, since the term neglected would only give a contribution independent of the potential in the first iteration.

In what follows we shall consider only s states and assume that it is sufficiently accurate to use (49). If we assume the potential has the form

(5), then by rewriting (49) it is seen that

$$K = K\left(\frac{x}{x_0}, \beta\right) \tag{50}$$

and then

$$\frac{\partial K}{\partial x_0} = -\frac{x}{x_0^2} K' \quad \text{and} \quad \frac{\partial K}{\partial x} = \frac{1}{x_0} K'$$

where the $'$ denoted differentiation with respect to $\left(\frac{x}{x_0}\right)$. Hence

$$\frac{\partial K}{\partial x_0} = -\left(\frac{x}{x_0}\right) \frac{\partial K}{\partial x}$$

Using (49),

$$\frac{\partial K}{\partial x_0} = -\left(\frac{x}{x_0}\right)\left\{-\frac{2K}{x} + \frac{U}{2a}(1+K^2)\right\} \tag{51}$$

With x_1 outside the charge distribution where $U = -\frac{2a^2}{x_1}$

$$\frac{\partial K}{\partial x_0} = \frac{1}{x_0}\left[a(1+K^2) + 2K\right] = \frac{1}{a x_0}\left[a + K(1+\sigma)\right]\left[a + K(1-\sigma)\right] \tag{52}$$

If we now consider a charge distribution of the form (6) but depending only on x_0, then using (52) and substituting for $\delta K = \frac{\partial K}{\partial x_0}\delta x_0$ in (47)

$$\delta(\Delta E) = N \cdot \frac{2\sigma^2}{r^2(1+2\sigma)} \frac{\left[a + K(x_1)(1+\sigma)\right]}{\left[a + K(x_1)(1-\sigma)\right]} x_1^{2\sigma} \frac{\delta x_0}{x_0} \tag{53}$$

Thus if $V = \frac{F(r/r_0)}{r_0}$ for both isotopes except for different values of r_0, then the I.S. depends on the change in charge distribution between the two isotopes only through $\frac{\delta r_0}{r_0}$.

It is of particular interest to consider the special class of charge distributions (6) but in addition such that the charge density is zero outside x_0, where x_0 can now be considered as the nuclear radius. The distribution (23) is of this type. An alternative way of obtaining the I.S. in this case

is by evaluating C_2/C_1 at $x_1 = x_0$ for the lighter isotope and at $x_1 = x_0 + \delta x_0$ for the heavier, i.e. we do not keep x_1 fixed. We then obtain $\delta(\Delta E)$ by taking the total differential with respect to x_0 of (46) with $x_1 = x_0$. From (13) we have with $\varepsilon = 0$

$$\frac{d\,K(x_0)}{dx_0} = 0$$

and we immediately obtain (53) with $x_1 = x_0$. For this type of charge distribution $K(x_0)$ is then independent of x_0 and the I.S. depends on the radius x_0 only through $x_0^{2\sigma}$ in addition to the relative charge of radius $\frac{\delta x_0}{x_0}$.

EVALUATION OF K.

Since for the I.S. of an s electron K can be considered as satisfying the same equation as for the extreme relativistic case with $\varepsilon = 0$, all the results obtained in I for $k = -1$, $\varepsilon \gg 1$ can immediately be used for the I.S. if we put $\varepsilon = 0$.

Since with relativistic units the potential U is proportional to a, the iteration procedure will also give K as an expansion in powers of a,

$$K = K^{(1)}\left[1 + \sum_{n=2}^{\infty} \gamma_n\, a^{2(n-1)}\right] \tag{54}$$

As already pointed out in I successive γ_n do not correspond to successive iterations except for $\gamma_2 = K^{(2)}/K^{(1)}$.

For the potentials (24), $K^{(1)}$, γ_2 are given by I (25, 26, 27, 28, 29) with $\varepsilon = 0$. $\gamma_3(r_0)$ is obtained from the third iteration and can be calculated without too much difficulty; the general expression for any n is however rather long. We give some special cases,

$$n = 0;\quad K^{(2)}(r_0) = -\frac{4a}{9}\left(1 + 0.15a^2\right)$$

$$n = 1;\quad K^{(3)}(r_0) = -\frac{5a}{12}\left(1 + 0.12a^2 + 0.021a^4\right)$$

$$n = 2;\quad K^{(3)}(r_0) = -\frac{2a}{5}\left(1 + 0.106a^2 + 0.0105a^4\right) \tag{55}$$

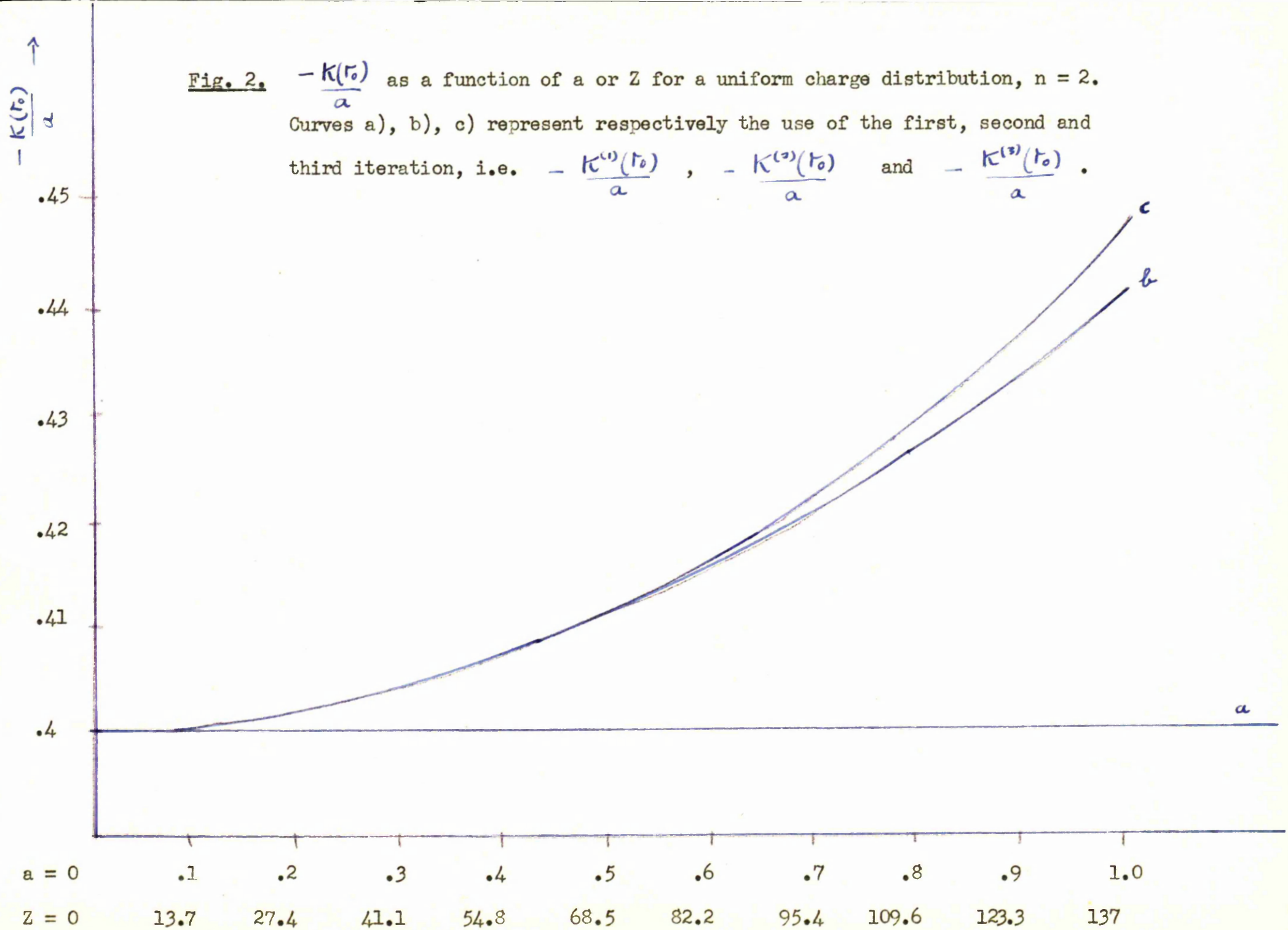

Fig. 2. $-\frac{K(r_0)}{a}$ as a function of a or Z for a uniform charge distribution, n = 2. Curves a), b), c) represent respectively the use of the first, second and third iteration, i.e. $-\frac{K^{(1)}(r_0)}{a}$, $-\frac{K^{(2)}(r_0)}{a}$ and $-\frac{K^{(3)}(r_0)}{a}$.

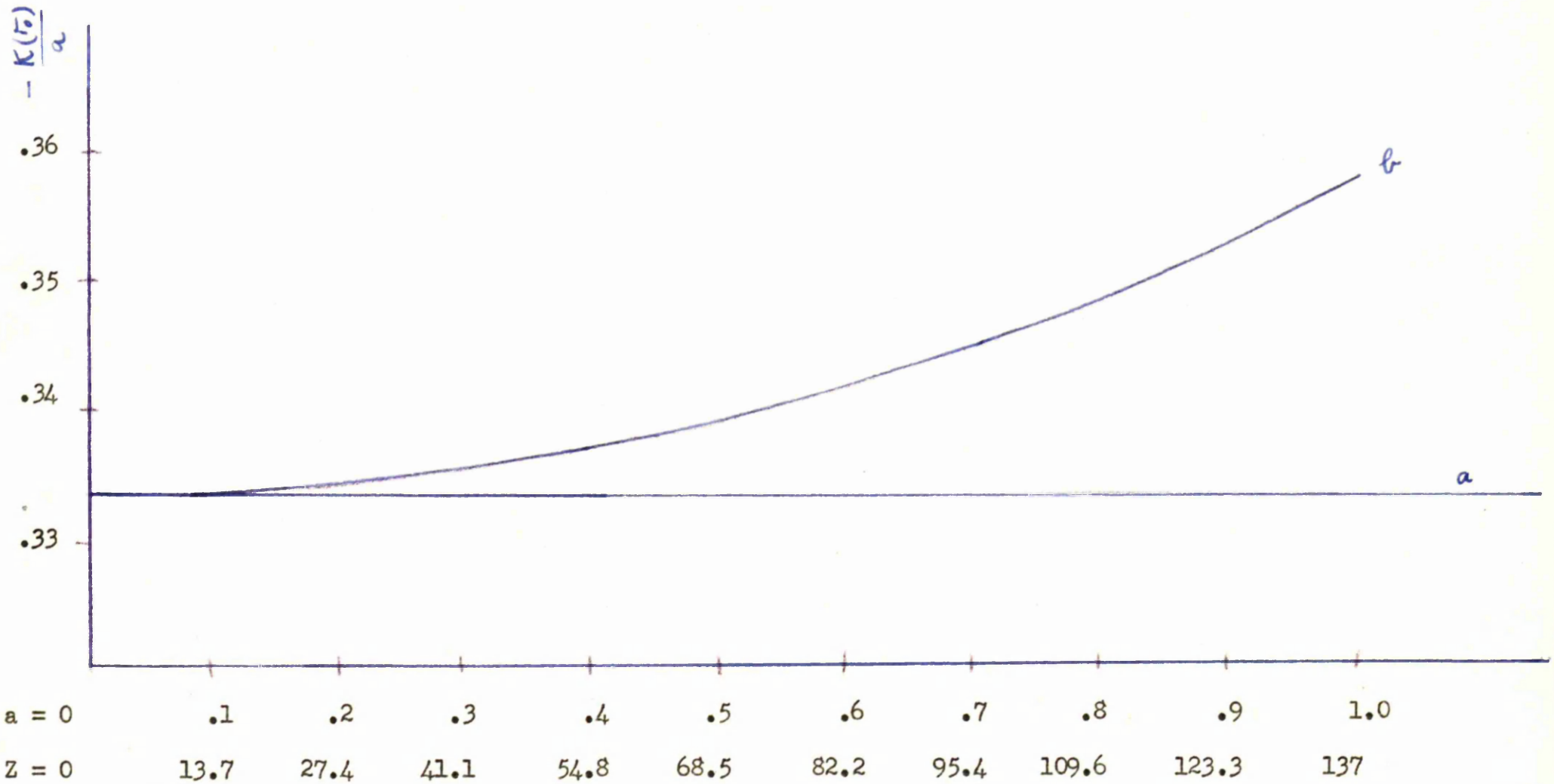

Fig. 3. $-\frac{K(r_0)}{a}$ as a function of a or Z for a surface charge distribution, $n = \infty$.

Curves a) and b) represent $-\frac{K^{(1)}(r_0)}{a}$ and $-\frac{K^{(3)}(r_0)}{a}$ respectively.

$$n = 3;\quad K^{(3)}(r_0) = -\frac{7a}{15}\left(1 + 0.087a^2 + 0.014a^4\right)$$
$$n = 4;\quad K^{(3)}(r_0) = -\frac{8a}{21}\left(1 + 0.082a^2 + 0.0135a^4\right)$$
$$n = \infty;\quad K^{(3)}(r_0) = -\frac{a}{3}\left(1 + \frac{a^2}{15} + \frac{2a^4}{315}\right)$$

$n = 0$ is a limiting case; the potential becomes logarithmic and the calculations must be carried out separately. In figures (2) and (3), $-K(r_0)/a$ is shown plotted against a for $n = 2$ and $n = \infty$. The straight line in each case represents the second iteration and the upper curve when $\mathcal{J}_3(r_0)$ has also been included. It is seen that this last term makes only a very small contribution to $K(r_0)$ for all elements, i.e. $Z \leq 92$. We may thus consider the second iteration to be a very good approximation for all elements. Since the additional term due to the second iteration is only a relatively minor part of K, the use of the equation (49) for K is seen to be justified.

It is of interest to compare the above results for K with those obtained by solving the radial equations directly. These calculations were made before the above iteration method was devised. The equations (35) are transformed into second order linear equations each involving only one of the functions f_k, g_k. Thus for f_k,

$$\frac{d^2 f_k}{dx^2} = \left[\frac{k(k-1)}{x^2} - \frac{U(U-2)}{4a^2} - \frac{k}{x}\frac{1}{U}\frac{dU}{dx}\right] f_k + \frac{1}{U}\frac{dU}{dx}\frac{df_k}{dx} \qquad (56)$$

If this is solved g_k may be obtained from the first of the equations (35) and K_k may be found. For the potentials (24) with n an integer greater than zero, the singularity at the origin is regular and a series solution in x is possible. Due to the smallness of x_0 it is found that the terms in x_0 in $K(x_0)$ may be neglected and that hence $\frac{dK(x_0)}{dx_0} = 0$ as already proved more generally above. For $k = -1$, the term in $K(x_0)$ in a is just (26). Successive terms in ascending powers of a^2 may be obtained from the recurrence relations. The coefficient of each power in a is now a not very rapidly converging series. The terms up to a^5 were calculated for $n = 1, 2, 3, 4, \infty$ and are in agreement with the ones given

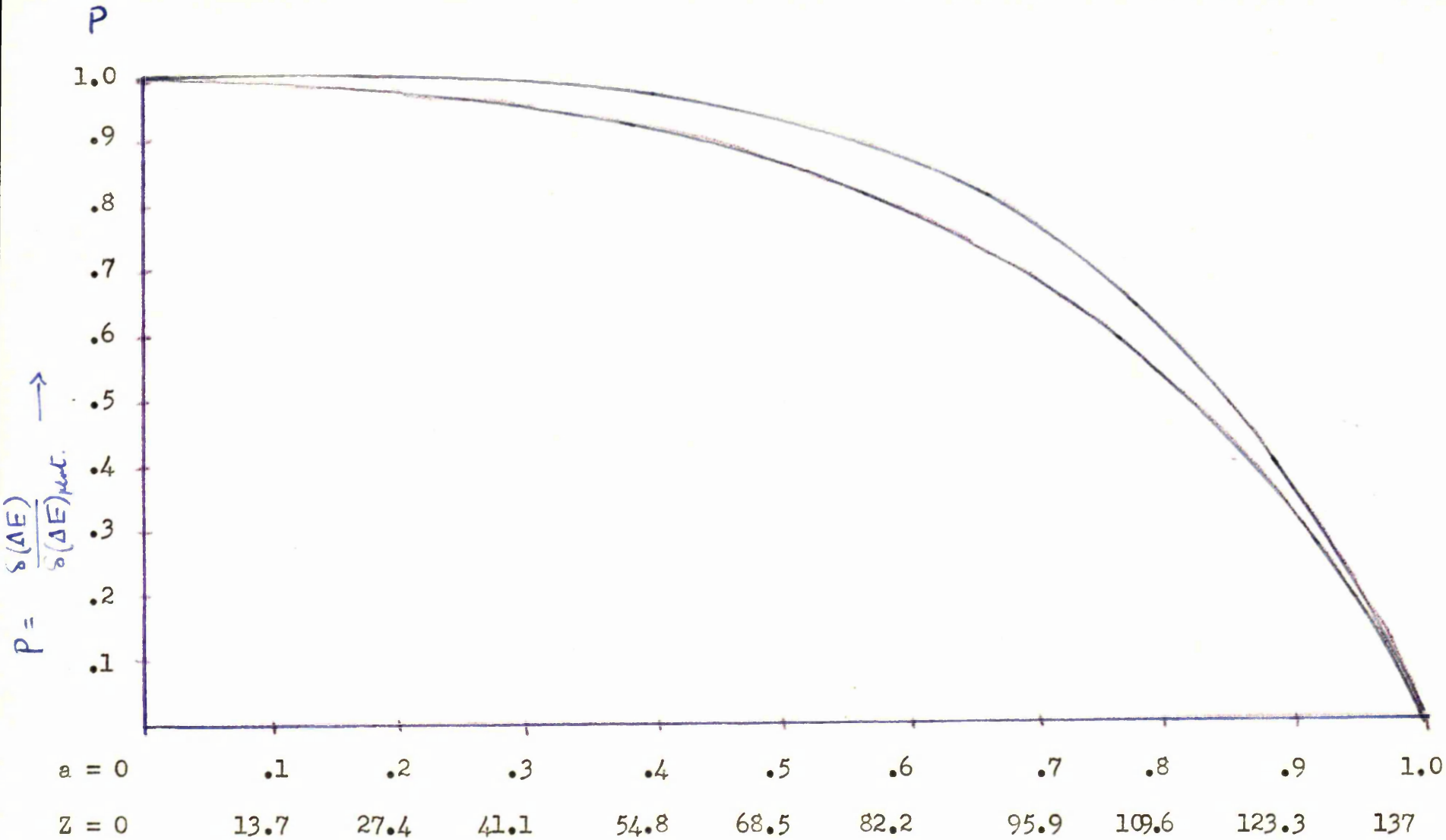

Fig. 4. $P = \frac{\delta(\Delta E)}{\delta(\Delta E)_{pert.}}$ as a function of a or Z for a uniform charge distribution, $n = 2$. The upper curve corresponds to the use of the first iteration $K^{(1)}(r_0)$ for $K(r_0)$. and the lower to the use of the third iteration, $K^{(3)}(r_0)$. The curve obtained using $K^{(2)}(r_0)$ is almost identical with the latter.

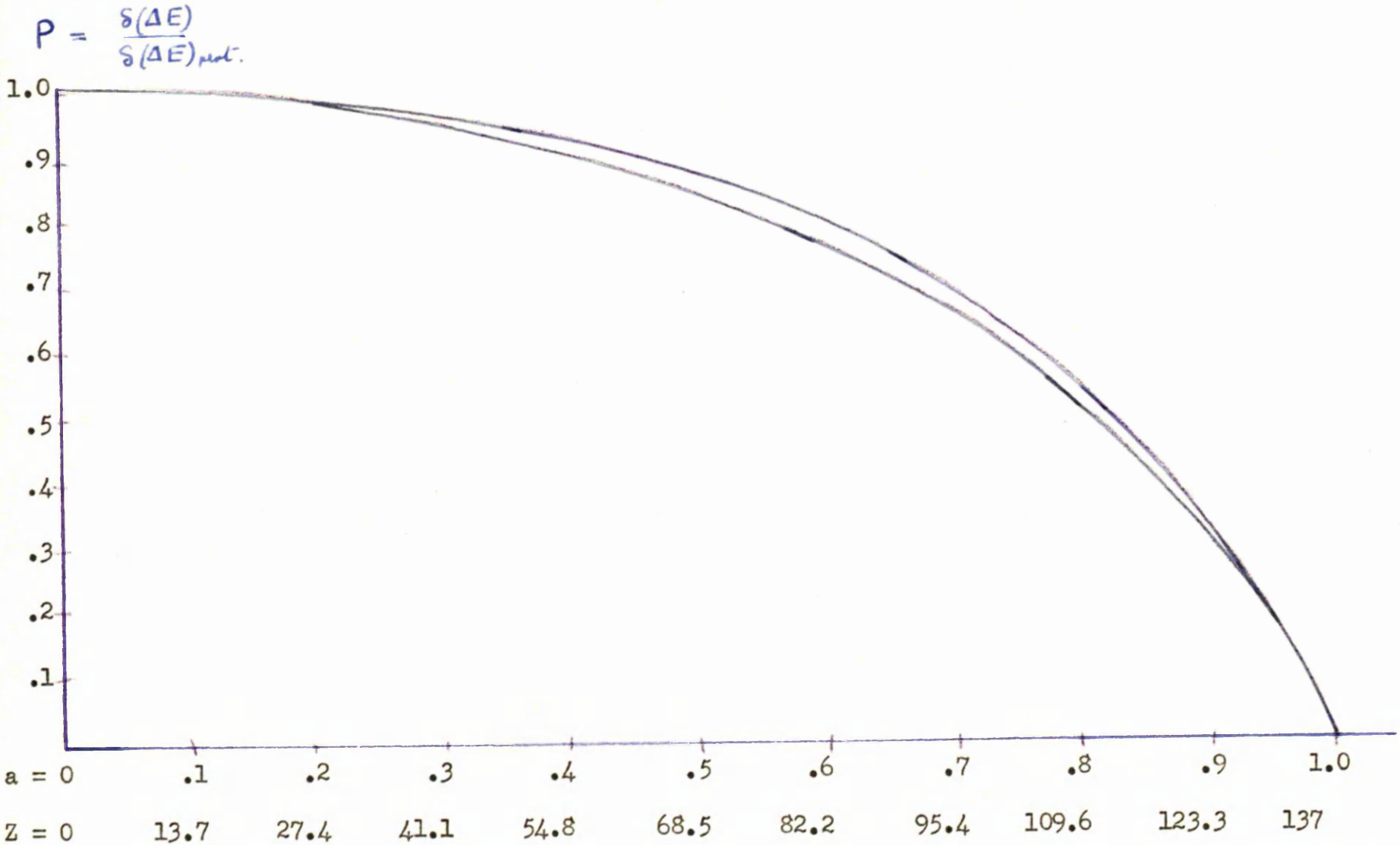

Fig. 5. $P = \frac{\delta(\Delta E)}{\delta(\Delta E)_{pert.}}$ as a function of a or Z for a surface charge distribution, $n = 2\infty$. The upper curve corresponds to the use of the first iteration $K^{(1)}(r_0)$ for $K(r_0)$ and the lower to the use of the third iteration $K^{(3)}(r_0)$. The curve obtained using $K^{(2)}(r_0)$ is almost identical with the latter.

above. This is essentially the method used by Crawford and Schalow to investigate the cases of uniform and surface distributions.

It is seen that the iteration method is much less tedious and can be applied to any charge distribution, whereas only in exceptional cases can a series solution in x be found for equation (56).

EVALUATION OF I.S.

The I.S. for the potentials (24) is obtained if the value of $K(r_0)$ is substituted in (53) with $x_1 = x_0$. In the approximation where only the first iteration (26) for $K(r_0)$ is used, the I.S. for a uniform charge distribution becomes

$$\delta(AE) = N \cdot \frac{2\sigma^2}{r^2(1+2\sigma)} \left(\frac{3 \quad 2\sigma}{3+2\sigma} \right) x_0^{2\sigma} \cdot \frac{3x_0}{x_0} \tag{57}$$

The ratio of the I.S. as given by (53) to that calculated using the perturbation method (41), is

$$P = \frac{\delta(AI)}{\delta(AE)_{pert.}} = \frac{2\sigma^2(1+2\sigma)}{1+\sigma} \left\{ \frac{u+K(r_0)(1+\sigma)}{u+K(r_0)(1-\sigma)} \right\} \left\{ \frac{2\sigma+u+1}{u+1} \right\} \tag{58}$$

P was calculated using (55) for u = 1,2,3,4, ∞ and in Figures (4) and (5) is shown plotted against Z for $n = 2$ and $n = \infty$. For the lower curve in each case, the terms up to a^5 have been included in $K(r_0)$, while the upper curve corresponds to the use of the first iteration only. The effect of $\gamma_3(r_0)$ is very small and on the scale used the curve for P using only the second iteration is almost identical with that obtained if $\gamma_3(r_0)$ is also included.

It is seen that the effect of the distortion of the wave function is to decrease the I.S. below its value as calculated by the perturbation method and that this difference increases with Z as is to be expected. P varies only slightly with n, i.e. with the concentration of charge. Thus for Z = 82, P = 0.78 for a uniform charge distribution and 0.76 when all the charge is on the surface. The use of the first iteration is already a considerable improvement on the perturbation

treatment and becomes a progressively better approximation as the charge moves outward.

It is seen then that the effect of distortion although appreciable and in the right direction could explain only a part of the discrepancy between the uniform model of Brix and Kopfermann and the experimental data.

With the help of the results just obtained, the I.S. may be calculated for any given charge distribution. Thus the uniform charge distribution which gives the same I.S. must have the same values of $K(r_1)$, $\delta K(r_1)$ as the given distribution and the problem is then just to calculate this equivalent uniform charge distribution. The equivalence of K considered in I was seen to be conveniently specified by the radius r_c of this equivalent distribution. It was shown that if r_c is obtained from (33) it has almost the same value as if the second iteration is used for its determination. Similarly, using only the first iteration $\frac{\delta r_c}{r_c}$ may be determined from

$$\frac{Ze}{2\pi}\,\frac{3}{5}\,r_c^2\,\frac{\delta r_c}{r_c} = \int_0^\infty \delta\rho\, r^4\, dr \tag{59}$$

and will again be accurate also within the use of the second iteration.

It may be noted that for the charge distributions (6) which in addition do not depend on further parameters β, we have from (46) and (53) that $\delta(\Delta E) = \Delta E \cdot 2\sigma \cdot \frac{\delta x_0}{x_0}$ and that therefore $\frac{\delta r_c}{r_c} = \frac{\delta r_0}{r_0}$ for these distributions. This also follows from (33) and (59).

The I.S. for the given distribution may now be obtained from (53) with and $x_1 = x_c$ and using for $K(r_0)$ the values already obtained for a uniform charge distribution. Alternatively we may use the results obtained for P, which depends only on Z, together with the I.S. as calculated by the perturbation method with the values r_c and $\frac{\delta r_c}{r_c}$.

The important point is that using only the first iteration for a given

charge distribution, the I.S. may be calculated accurate to within the use of the second iteration if the results for a uniform charge distribution already obtained with this, are used. The I.S. will then depend besides on Z only on r_c as $\frac{\delta r_c}{r_c}$.

RELATION WITH ELECTRON SCATTERING

It was shown in I that the radius r_c of the uniform charge distribution needed to describe the s wave scattering was almost independent of energy and in particular equal to r_c for $\varepsilon = 0$. Thus r_c as obtained from experiments on electron scattering can be used for the I.S. in a way which follows immediately from the discussion at the end of the previous section. The data on the I.S. will then give effective values of $\frac{\delta r_c}{r_c}$ which are related to the change in the charge distribution between the two isotopes by (59). If the charge distribution is of the type (6) and also does not depend on β, then $\frac{\delta r_c}{r_c}$ would have the direct interpretation as the relative charge of the parameter occurring in the potential. Any such information would be a valuable test for theories of nuclear structure. Thus on the basis of an extended individual particle model, the change in charge distribution would have to be ascribed to a change in the range and strength and also of the shape of the collective potential in which the nucleons move. The addition of any neutrons must change in some way the extent and depth of this potential and hence also the proton distribution. On the other hand, there would also according to Rainwater (1950) be a change in the shape of the individual nucleon potential and consequently a change in deformation of the nucleus. Any such difference in deformation between two isotopes would then according to Brix and Kopfermann (1949) also give rise to an I.S.

Since Ag and Au are near the lighter and heavier end respectively of the elements for which the volume dependent I.S. is known, we take as a tentative estimate of the equivalent radius r_c of these elements the value $1.15 \times 10^{-13} A^{\frac{1}{3}}$ cm. as determined from the scattering of electrons by Ag and Au by Lyman, Hanson and

Scott. The I.S. will then be reduced by a factor $\left(\frac{1.15}{1.4}\right)^{2.6}$ from the value obtained if a radius of $1.4 \times 10^{-13} A^{\frac{1}{3}}$ cm. is used. This is equal to .7 for Z = 55 and to .74 for Z = 82. Including the effect of the distortion of the wave function, the value of the I.S. as obtained with the standard model of Brix and Kopfermann using perturbation theory is then reduced by .64 for Z = 55 and by .58 for Z = 82. This would account for a considerable part of the discrepancy between the data and the model of Brix and Kopfermann.

To take the radius as increasing proportionally to $A^{\frac{1}{3}}$ is in general insufficient to specify the change in the charge distribution between two isotopes, especially if deformation effects are likely to be important as for instance is strongly suggested for the exceptionally large I.S. of Sm and Eu (Brix and Kopfermann, 1949).

III.

I.S. AND NUCLEAR DEFORMATIONS.

Brix and Kopfermann (1949, 1951) have suggested that a change in charge distribution due to a difference in shape between two isotopes will lead to an I.S. in addition to that due to an increase in volume usually considered. The effective potential energy of an $s_{1/2}$ or $p_{1/2}$ electron in the nuclear interior will then be obtained by averaging the deformed nucleus over all directions. If the isotopes have a spin $I \geqslant 1$, they will possess a spectroscopic quadrupole moment defined with respect to fixed axes in space for the substate $M = I$. Brix and Kopfermann have assumed that if these quadrupole moments are too large to be due to the motion of a single proton it is justified to describe them by an ellipsoidal deformation from spherical shape of the nucleus as a whole, it then being possible to make a quantitative estimate of the deformations and hence also of the extra potential arising from these.

It is important when calculating the deformation from the spectroscopic quadrupole moment to take into account that this may be considerably smaller than the intrinsic quadrupole moment referred to the axis of the nucleus. This difference will have an important influence especially for smaller I in view of the fact that the deformation occurs as the square in the extra potential to which it gives rise. Brix and Kopfermann have especially considered the connection of the I.S. between Sm^{150} and Sm^{152} with the quadrupole moments of Eu^{151} and Eu^{153} which have one extra proton relative to the corresponding Sm isotopes. Both the I.S. between Eu^{151} and Eu^{153} and that between Sm^{150} and Sm^{152} are exceptionally large. It is especially noteworthy that the latter is nearly twice the already large I.S. between any other two adjacent even Sm isotopes, thus the relative spacing of the Sm I.S. are,

Sm^{144}	Sm^{147}	Sm^{148}	Sm^{149}	Sm^{150}	Sm^{152}	Sm^{154}
0	1.41	2	2.26	3.14	4.81	5.72

Assuming that the deformations of Sm^{150} and Sm^{152} are approximately the same as those of the corresponding Eu isotopes and calculating the deformations of the latter from their known quadrupole moments, Brix and Kopfermann obtain an estimate for the contribution of the deformation dependent part of the I.S. between Eu^{151} and Eu^{153} and hence also of that between Sm^{150} and Sm^{152}. As a result of their calculation which ignores the difference between the spectroscopic and intrinsic quadrupole moments, they find that the deformation dependent contribution of the I.S. is rather smaller than the difference of the I.S. between Sm^{150} and Sm^{152} and that between other adjacent even Sm isotopes; their interpretation then being that in view of the uncertainties involved just this I.S. jump is explained as due to a difference in deformation. When however the difference between the spectroscopic and intrinsic quadrupole moments are taken into account, it will be shown that the deformation dependent part of the I.S. is of the order of magnitude of the whole of the anomalously large I.S. of Eu and hence with above assumptions of Brix and Kopfermann regarding the deformations of Sm^{150} and Sm^{152} also of the order of magnitude of the whole of the I.S. between these.

In the following the deformation dependent I.S. is calculated with the method developed in II, using the same model as Brix and Kopfermann of a uniformly charged nucleus deformed into a rotational ellipsoid of the same volume as the undeformed spherical nucleus, the difference between the spectroscopic and intrinsic quadrupole moments being taken into account. Various tentative applications are considered, especially in connection with the large observed variations in the I.S.

CALCULATION OF POTENTIALS.

We consider the nuclear charge distribution to be represented by a uniformly charged body of revolution whose surface r_0' is given by

$$r_0' = r_0 \left[1 + \varepsilon u(\theta) + \varepsilon^2 w(\theta) + \cdots \right] \tag{60}$$

where θ is the angle between the axis of revolution and the radius vector r_0' from the origin to a point on the surface, the origin being taken as the centre of the sphere, radius r_0, which is obtained in the limit $\varepsilon = 0$. ε is a parameter which gives a measure of the deformation of the body relative to this sphere. In the following, it is assumed that ε is sufficiently small that only powers of ε up to ε^2 need be retained. Expanding in terms of spherical harmonics we write

$$u(\theta) = \sum_{n=0}^{\infty} \alpha_n P_n(\cos\theta)$$

$$w(\theta) = \sum_{n=0}^{\infty} \beta_n P_n(\cos\theta)$$

$$u^2(\theta) = \sum_{n=0}^{\infty} \gamma_n P_n(\cos\theta)$$

With constant charge density,

$$\rho = \frac{3}{4\pi} \frac{Ze}{r_0^3}$$

the total charge is Ze if the volume is the same as that of the sphere of radius r_0. The potential energy of an electron at a point r in the interior of the nucleus is then

$$V(r) = V^{(0)}(r) - \frac{Ze^2}{r_0} 3 \sum_{n=0}^{\infty} \frac{P_n(\cos\theta)}{2n+1} \left(\frac{r}{r_0}\right)^n \left\{ \varepsilon\alpha_n + \varepsilon^2 \left[\beta_n + \frac{\gamma_n}{2}(n-1) \right] \right\}$$

where

$$V^{(0)}(r) = -\frac{Ze^2}{r_0} \frac{3}{2} \left[1 - \frac{1}{3} \left(\frac{r}{r_0}\right)^2 \right]$$

is the potential energy due to the uniformly charged sphere, radius r_0.

For a $s_{1/2}$ or $p_{1/2}$ electron, the effective potential energy $V(r)$ is obtained by averaging $V(\underline{r})$ over all directions.

$$V(r) = V^{(0)}(r) + V_\varepsilon(r) \tag{61}$$

where

$$V_\varepsilon(r) = -\frac{2e^2}{r_0} 3 \left\{ \varepsilon \alpha_0 + \varepsilon^2 \left[\beta_0 + \frac{\delta_0}{2} \right] \right\} \tag{62}$$

is the potential energy due to the deformation. We now suppose that the nucleus may be described more specifically by an ellipsoid of revolution with semi-axes a, b along and perpendicular to the axis of revolution respectively. If the volume of the ellipsoid is the same as that of the sphere radius r_0, the deformation may be described in terms of the parameter $\varepsilon = \frac{a-b}{r_0}$ which is positive for a prolate and negative for an oblate ellipsoid of revolution. Using the equation for the surface of an ellipsoid,

$$\alpha_0 = 0 \quad , \quad \beta_0 = -\frac{4}{45} \quad , \quad \delta_0 = \frac{4}{945}$$

for both a prolate and oblate ellipsoid and from (62)

$$V_\varepsilon(r) = \frac{2e^2}{r_0} \frac{2}{15} \varepsilon^2 \tag{63}$$

The difference of potential energy in the nuclear interior between two isotopes is,

$$\delta V = \delta V_v + \delta V_\varepsilon + \delta V_{\varepsilon, v}$$

where

$$\delta V_v = \frac{2e^2}{r_0} \frac{\delta r_0}{r_0} \frac{3}{2} \left[1 - \left(\frac{r}{r_0} \right)^2 \right]$$

is the usual change in potential energy due to an increase of volume specified by δr_0 of a spherical nucleus radius r_0.

$$\delta V_\varepsilon = \frac{2e^2}{r_0} \frac{2}{15} \left(\varepsilon_2^2 - \varepsilon_1^2 \right) \tag{64}$$

is two isotopes. δV_ε ential energy due to differing deformations ε_2 , ε_1 of the two isotopes. δV_ε may either increase or decrease the I.S. due to δV_v depending on whether $|\varepsilon_2| > |\varepsilon_1|$ or $|\varepsilon_1| > |\varepsilon_2|$ respectively.

$$\delta V_{\varepsilon, v} = - \frac{2e^2}{r_0} \frac{\delta r_0}{r_0} \frac{2}{15} \varepsilon_2^2$$

is a small correction to the volume dependent charge of potential δV_v arising from the fact that for an ellipsoidal nucleus the charge is effectively concentrated more towards the centre. Even for $\varepsilon = \frac{1}{4}$ the I.S. due to $\delta V_{\varepsilon, v}$ is less than 2% of that due to δV_v and may therefore safely be neglected.

If the nucleus has a spin I $\geqslant$ 1, it will have a spectroscopic quadrupole moment Q which may then be used to calculate ε . If Q is so large that it must be interpreted as due to a large number of protons, the description of the nuclear charge distribution by a uniformly charged ellipsoid can be considered as reasonable especially if the cause of these large quadrupole moments is taken to be a deformation by the nucleons in unclosed shells of the core of the remaining nucleons as suggested by Rainwater (1950). The intrinsic quadrupole moment Q_0 defined with respect to the symmetry axis of the nucleus is then given in terms of the deformation ε by

$$Q_0 = \frac{4}{5} \varepsilon Z r_0^2$$

Q_0 will be larger than the spectroscopic quadrupole moment Q , the relation between the two being (see e.g. A. Bohr, 1951)

$$Q_0 = \gamma_I Q \quad ; \quad \gamma_I = \frac{(I+1)(2I+3)}{I(2I-1)} \tag{65}$$

When I becomes large γ_I tends to one, but for small I, γ_I may be quite large. Q is zero whatever the value of Q_0 if I = 0, $\frac{1}{2}$.

We then have

$$\varepsilon = \gamma_I \frac{5}{4Z} \cdot \frac{Q}{r_0^2} \tag{66}$$

and

$$\delta V_\varepsilon = \frac{Ze^2}{r_0} \frac{5}{24} \frac{(\gamma_{I_2}^2 Q_2^2 - \gamma_{I_1}^2 Q_1^2)}{Z^2 r_0^4} \tag{67}$$

Thus, for small I, γ_I will have a very large effect on δV_ε since it occurs as the square in the above expression, the omission of this factor by Brix and Kopfermann being in fact quite crucial. Even if the nucleus does not have a spectroscopic quadrupole moment it may still have a deformation, i.e. an intrinsic quadrupole moment with respect to the internal symmetry axis and hence also for even and spin $\frac{1}{2}$ isotopes δV_ε need not be zero. In this case, however, ε can only be obtained on the basis of some model of nuclear structure such as Rainwater's which allows for deformations. The importance of nuclei which have observable spectroscopic quadrupole moments and for which the I.S. is known is that ε and hence the I.S. due to δV_ε may be rather directly estimated from the spectroscopic quadrupole moments and that therefore a comparison with the experimentally observed I.S. may be made involving only a minimum of hypothesis.

CALCULATION OF THE I.S. DUE TO NUCLEAR DEFORMATION

To calculate the I.S. corresponding to δV_ε, we write this in the form,

$$\delta V_\varepsilon = \frac{Ze^2}{r_0} F$$

where

$$F = \frac{2}{15} (\varepsilon_2^2 - \varepsilon_1^2) \tag{68}$$

and does not depend on r.

For nuclei with spectroscopic quadrupole moments,

$$F = \frac{5}{24} \frac{\left(\gamma_{I_2}^2 Q_2^2 - \gamma_{I_1}^2 Q_1^2 \right)}{\lambda^2 r_0^4} \tag{69}$$

We denote the I.S. corresponding to $\delta V_v, \delta V_\varepsilon$ by $\delta(\Delta E_v), \delta(\Delta E_\varepsilon)$ respectively. Using the simple perturbation theory to calculate the I.S., we have from (37) for an s electron,

$$\delta(\Delta E_\varepsilon)_{pert} = N \cdot \frac{1+\sigma}{r^2(1+2\sigma)} \frac{F}{(1+2\sigma)} x_0^{2\sigma}$$

and with (41) for $\mu = 2$,

$$\frac{\delta(\Delta E_\varepsilon)_{pert}}{\delta(\Delta E_v)_{pert}} = \frac{2\sigma+3}{5} \cdot F \cdot \frac{r_0}{\delta r_0} \tag{70}$$

which is essentially the result given by Brix and Kopfermann (1949).

We now calculate the I.S. using the method developed in II. As was shown, it is sufficient to limit ourselves to the first iteration. For a uniform spherical charge distribution, radius r_0, we have from (25) and (26) with $\mu = 2$ for a change in radius δr_ε,

$$\delta K^{(1)}(r) = \frac{\delta r_\varepsilon}{r_0} \frac{a}{5} \left(\frac{r_0}{r} \right)^2 \quad ; \quad r > r_0 \tag{71}$$

The change in $K^{(1)}(r)$ corresponding to δV_ε is on the other hand from (20),

$$\delta K_\varepsilon^{(1)}(r) = \frac{aF}{3} \left(\frac{r_0}{r} \right)^2 \quad ; \quad r > r_0 \tag{72}$$

equating (71) and (72) we obtain the change in radius δr_ε of the uniform spherical charge distribution of radius r_0 which will give the same I.S. as ,

$$\frac{\delta r_\varepsilon}{r_0} = \frac{5}{3} F \tag{73}$$

And therefore from (53),

$$\delta(\Delta E_\varepsilon) = N \cdot \frac{2\sigma^2}{\Gamma^2(1+2\sigma)} \frac{[a + K(1+\sigma)]}{[a + K(1-\sigma)]} x_0^{2\sigma} \frac{5}{3} F$$

If the change of radius corresponding to $\delta(\Delta E_v)$ is δr_0 ,

$$\frac{\delta(\Delta E_\varepsilon)}{\delta(\Delta E_v)} = \frac{\delta r_\varepsilon}{r_0} \Big/ \frac{\delta r_0}{r_0} = \frac{5}{3} F \frac{r_0}{\delta r_0} \tag{74}$$

It is seen that the perturbation expansion (70) reduces to this in the limit $a \to 0$, i.e. $\sigma \to 1$.

We have

$$\frac{\delta(\Delta E_\varepsilon)}{\delta(\Delta E_v)} \Big/ \frac{\delta(\Delta E_\varepsilon)_{pert.}}{\delta(\Delta E_v)_{pert.}} = \frac{5}{5+2\sigma}$$

For Pb, $a = 0.6$, $\sigma = 0.8$, this is 1.1 and thus the perturbation expression for $\delta(\Delta E_\varepsilon)/\delta(\Delta E_v)$ is a rather good approximation.

For the ratio $\delta(\Delta E_\varepsilon)/\delta(\Delta E_\varepsilon)_{pert.}$ we have

$$\frac{\delta(\Delta E_\varepsilon)}{\delta(\Delta E_\varepsilon)_{pert.}} = \frac{5}{3+2\sigma}\left[\frac{\delta(\Delta E_v)}{\delta(\Delta E_v)_{pert.}}\right] = \frac{5}{3+2\sigma} P$$

For Pb, this is $\simeq$ 0.85. Assuming $r_0 = R_0 A^{1/3}$ and $\frac{\delta r_0}{r_0} = \frac{\delta A}{3A}$ we have from (69) and (74) for isotopes with a spectroscopic quadrupole moment,

$$\frac{\delta(\Delta E_\varepsilon)}{\delta(\Delta E_v)} = \frac{25}{24} \cdot \frac{1}{\delta A} \frac{(\gamma_{I_2}^2 Q_2^2 - \gamma_{I_1}^2 Q_1^2)}{A^{1/3} Z^2 R_0^4}$$

With $\delta A = 2$, $R_0 = 1.4 \times 10^{-13}$ cm. and Q expressed in barns,

$$\frac{\delta(\Delta E_\varepsilon)}{\delta(\Delta E_v)^N} = 1.35 \times 10^3 \frac{(\gamma_{I_2}^2 Q_2^2 - \gamma_{I_1}^2 Q_1^2)}{Z^2 A^{1/3}} \tag{75}$$

where the index N denotes that the calculations have been made assuming a uniformly charged nucleus of radius $r_0 = 1.4 \times 10^{-13} A^{\frac{1}{3}}$ cm., the change in radius being taken as $\frac{2}{3A}$. This is the standard model which has been used by Brix and Kopfermann (1949, 1951).

RESULTS AND DISCUSSION.

We make use of the ratio $s = Q_2/Q_1$, of the spectroscopic quadrupole moments of the two isotopes as this can be directly determined from the hyperfine structure, involving no detailed knowledge of the atomic wave functions, and is in general known with considerably greater accuracy than the absolute values of the quadrupole moments themselves. For two isotopes with the same spin, $\gamma_{I_1} = \gamma_{I_2} = \gamma_I$ and,

$$\frac{\delta(\Delta E_r)}{\delta(\Delta E_v)^N} = \frac{1.35 \times 10^3}{Z^2 A^{1/3}} \gamma_I^2 (s^2 - 1) Q_1^2 \tag{76}$$

In this expression apart from the assumptions regarding the constancy of the charge density and the magnitude of the nuclear radius, the principal uncertainty will be due to the error in Q_1.

The I.S. data, apart from rather uncertain corrections due to the shielding of the inner electrons by the valence electron have been given by Brix and Kopfermann in the form $\delta(\Delta E)/\delta(\Delta E_v)^N$ where $\delta(\Delta E)$ is the experimentally determined I.S. and $\delta(\Delta E_v)^N$ is the I.S. of their standard model. In their most recent presentation (1951) shown in Fig. 1, they have included the correction P due to distortion of the wave function as calculated by Humbach (1952), this being in agreement with the results obtained in II.

If we assume that effects of nuclear polarisation by the electron, discussed by Breit, Arfken and Clendenin (1950) are small, we may write

$$\delta(\Delta E) = \delta(\Delta E_v) + \delta(\Delta E_r)$$

where the quantities are supposed to be actual or calculated values as will be clear from the context. Then

$$\delta = \frac{\delta(\Delta E_r)}{\delta(\Delta E_v)^N} = \frac{\delta(\Delta E)}{\delta(\Delta E_v)^N} - \frac{\delta(\Delta E_r)}{\delta(\Delta E_v)^N}$$

Thus in order to test the above theory, we must see whether we obtain a reasonable

value for δ , the actual value of which must of course always be positive, with $\frac{\delta(\Delta E_\epsilon)}{\delta(\Delta E_\nu)^N}$ as determined by Brix and Kopfermann.

We consider the I.S. between the two Eu isotopes, Eu^{151} and Eu^{153} (Z = 63). The spin of both is $5/2$ and therefore from (65), $\gamma_{5/2} = \frac{14}{5}$, $\gamma'_{5/2} = 8$. The hyperfine structure and I.S. has recently been very thoroughly reinvestigated by Brix (1952), who gives s = 2.0 and $\delta(\Delta E)/\delta(\Delta E_\nu)^N = 2.3$, where this value includes an estimate of the effect of the shielding of the inner electrons by the valence electron. Schüler and Korsching (1936) give $Q_{Eu^{151}} = 1.2$ barns, this value being considered as somewhat uncertain as the terms which can be used for calculation are perturbed, the error being of the order of 20%. Hence from (76),

$$\frac{\delta(\Delta E_\epsilon)}{\delta(\Delta E_\nu)^N} = 0.5(s^2 - 1)\, Q^2_{Eu^{151}} \simeq 2.2 \pm 0.9$$

and $\delta = 2.3 - 2.2 = 0.1 \pm 0.9$.

In view of the rather large errors involved in $Q_{Eu^{151}}$ and $\delta(\Delta E)/\delta(\Delta E_\nu)^N$ and the uncertainty of the assumptions regarding the nuclear charge distribution, the actual numerical value of δ obtained cannot be considered as very significant. However, the above results show very clearly that the anomalously large I.S. between the Eu isotopes is readily explained as a consequence of a difference in deformations, if the experimentally determined spectroscopic quadrupole moments are used to make an estimate of these.

The whole and not merely the jump of the exceptionally large I.S. between Sm^{150} and Sm^{152} (Z = 62) is then readily explained if the difference between the spectroscopic and intrinsic quadrupole moments is taken into account and if the deformations are assumed to be somewhat less than those of the corresponding Eu isotopes. The still larger I.S. between Eu^{151} and Eu^{153} would then indicate that the extra proton increases the already large deformations of the Sm isotopes. In connection with these, it is interesting to consider the application of an

empirical rule given by A. Bohr (1951) to the isotopes Sm^{147}, Sm^{148}. This rule is that for two isotopes which differ by two neutrons and have the same value of I, the nucleus with the numerically smallest quadrupole moment has a magnetic moment closest to the appropriate Schmidt value. The theoretical justification for this according to Bohr is that the greater the deformation of the core the greater is the influence of the coupling of the angular momentum of the nucleons in unclosed shells with the motion of the core and the greater consequently is likely to be the deviation of the magnetic moment from the Schmidt value.

The spins of the odd neutron nuclei Sm^{147}, Sm^{149} are both 7/2, while the magnetic moments are -0.3 and -0.25 nuclear magnetons respectively, the quadrupole moments not being known. The I.S. is approximately the same as that between the other Sm isotope pairs (excepting Sm^{150} and Sm^{152}) and therefore rather large. The spins and magnetic moments agree with the configurations $(f_{7/2})^3$ and $(f_{7/2})^5$ predicted by the shell model, the magnetic moment of Sm^{147} lying rather closer to the appropriate Schmidt value than that of Sm^{149}. Bohr's rule thus predicts that the numerical value of the quadrupole moment of Sm^{149} is greater than that of Sm^{147} and that therefore the deformation dependent I.S. will tend to increase the volume dependent part. This would support the explanation of these rather large I.S. as due to a deformation dependent effect which then becomes exceptionally large for Sm^{150}, Sm^{152} and Eu^{151}, Eu^{153}.

The conjunction between the neutron numbers 88 and 90 of the I.S. jump observed both for Sm and Nd (Z = 60), together with the exceptionally large difference in quadrupole and magnetic moments of the Eu isotopes, suggests that there is some change in the neutron configuration on the addition of the 45th pair of neutrons.

The only other isotopes besides Eu^{151} and Eu^{153} for which both the spectroscopic quadrupole moments and the I.S. are known are Re^{185} and Re^{187} (Z = 75).

The spin of both isotopes is $5/_2$. Schüler and Korsching (1937) give the two values $Q_{Re^{185}}/Q_{Re^{187}}$ = 1.08 and 1.02 as obtained from two different lines, the former being claimed as the more accurate; $Q_{Re^{187}}$ = 2.6 barns, there being good agreement between the values obtained from the two different terms. The deformation dependent I.S. will thus act in the opposite direction to the volume dependent part. If we write $Q_{Re^{185}}/Q_{Re^{187}} = 1 + \beta$, $\beta \sim 1$, then

$$\frac{\delta(\Delta E_\varepsilon)}{\delta(\Delta E_\vartheta)^N} = -4.6\,\beta = \begin{cases} (-0.37; & \beta = 0.08 \\ (-0.09; & \beta = 0.02 \end{cases}$$

On the other hand, Brix and Kopfermann using the data of Schüler and Korsching give,

$$\delta(\Delta E)/\delta(\Delta E_\vartheta)^N = 0.35.$$ Thus,

$$\delta = .72 \quad \text{if } \beta = 0.08$$
$$\delta = .46 \quad \text{if } \beta = 0.02.$$

There is thus in general the possibility, also present for isotopes without spectroscopic quadrupole moments, that the deformation dependent I.S. is sufficiently negative to reduce the I.S. to a very small or perhaps even negative value. This may then perhaps be the reason for the rather small I.S. of the elements from W (Z = 74) to Pt (Z = 78) inclusive.

Kopfermann and Meyer (1947, 1951) similarly to their considerations of Brix and Kopfermann about the I.S. jump in Sm and the quadrupole moments of Eu, have suggested that the difference in the Re quadrupole moments may be connected with the decrease in the I.S. of the corresponding W (Z = 74) isotopes, W^{184} and W^{186}. The relative spacing of the I.S. of the W isotopes is,

W^{180}	W^{182}	W^{184}	W^{186}
-0.91	0	1	1.89

With the larger and more accurate value 1.08 for $Q_{Re^{185}}/Q_{Re^{187}}$ it would seem however that with deformations for W^{184}, W^{186} of approximately the same

value as those of the corresponding Re isotopes and including the difference between the spectroscopic and intrinsic quadrupole moments, $\delta(\Delta E_\varepsilon)$, $\delta(\Delta E_\nu)^N$ is too large to account for just the rather small jump in the I.S. It seems more likely to suppose, in a similar way as for the Sm isotopes, that the somewhat small I.S. of all the W isotopes as well as that of the Re isotopes is due to a negative deformation dependent I.S., and that the small I.S. jump between W^{184} and W^{186} is due to a change in this.

A case of some interest in connection with the possibility of very small I.S. is that of the Iodine isotopes I^{127}, I^{129} (Z = 53) for which the quadrupole moments but not the I.S. are known. The spin of I^{127} is $5/2$ and $Q_{I^{127}} = -0.75$ barns, the corresponding quantities for I^{129} being $7/2$, $\gamma_{7/2} = \frac{15}{7}$, $\gamma_{7/2}^2 = 4.6$ and $Q_{I^{129}} = -0.43$ barns. From (75),

$$\frac{\delta(\Delta E_\varepsilon)}{\delta(\Delta E_\nu)^N} = -0.35$$

It must be remarked that the model of an ellipsoidal nucleus is not likely to be a very good description in this case as the quadrupole moments are rather near the single particle values expected on the basis of the simple shell model. The order of magnitude of the deformation dependent I.S. obtained is nevertheless likely to be correct and the I.S. between I^{127} and I^{129} can be expected to be rather small. The reason for the very small I.S. of the elements Xe (Z = 54) and Ba (Z = 56) which follow Iodine may then perhaps be due to large negative deformation dependent I.S. In this connection it is interesting to consider the application of Bohr's rule to the odd neutron isotopes Ba^{135} and Ba^{137}. These each have a spin of $3/2$ and the magnetic moments 0.832 and 0.936 nuclear magnetons respectively, the quadrupole moments not being known. The values are not far from the $d_{3/2}$ Schmidt value suggesting the neutron configuration $(d_{3/2})^1$ and $(d_{3/2})^3$ which are in good agreement with the predictions of the shell

model. The magnetic moment of Ba^{137} is nearest the Schmidt value, this isotope having just one neutron less than the magic number 82, and Bohr's rule then predicts that $Q_{Ba^{137}} - Q_{Ba^{138}}$ and that therefore the deformation dependent I.S. will act in the opposite direction to the volume dependent part. This then would seem to support the explanation that the deformation dependent I.S. may at any rate be partly responsible for the small I.S. of Ba.

Certain indications may be obtained from the I.S. and magnetic moments of isotopes with a spin $\frac{1}{2}$, which of course have no spectroscopic quadrupole moments, if a plausible extension of Bohr's rule is made. This would then state that for two isotopes which differ by two neutrons and have the same value of I the nucleus with the magnetic moment closest to the appropriate Schmidt value would have the smaller deformation. The only isotopes with a spin $\frac{1}{2}$ which come into consideration at the present time are those of the odd proton elements Ag (Z = 47) and Tℓ (Z = 81). The magnetic moments of Ag^{187}, Ag^{109} are -0.159 and -0.086 nuclear magnetons respectively, both being near the $p_{1/2}$ Schmidt value in good agreement with the shell model. The magnetic moment of Ag^{109} is somewhat closer to the Schmidt value and therefore the deformation dependent I.S. should be negative. It is not however likely that this is very appreciable in view of the normal value of the I.S. and also of the fact that since the magnetic moments of both isotopes are rather close to the Schmidt value their deformations will probably be small.

The magnetic moments of $Tℓ^{203}$ and $Tℓ^{205}$ are 1.596 and 1.612 nuclear magnetons respectively, these being somewhat closer to the $s_{1/2}$ then the $p_{1/2}$ Schmidt value, this being reconcilable with the shell model. The magnetic moment of the heavier isotope is closest to the Schmidt value and the deformation dependent I.S. should be negative. There is no indication from the experimental I.S. that this is significant. Perhaps in view of the proximity of these isotopes to both the

neutron magic number 126 and the proton magic number 82 it is not unlikely that their deformations are small.

Finally we mention two cases which seem to be of some interest:

1) The spin and spectroscopic quadrupole moments of Sb^{121} and Sb^{123} (Z = 51) are $5/_2$, -0.3 barns and $7/_2$, -1.2 barns respectively, the I.S. not being known. From (75), $\delta(\Delta E_\varepsilon)/\delta(\Delta E_\nu)^N = 0.62$ and thus the I.S. can be expected to be rather large. This might indicate that the unknown I.S. in the region between Cd and Xe may have values quite different from those expected if the points for Cd and Xe in Fig. 1 are joined by a straight line.

2) The two isotopes Yb^{171}, Yb^{173} (Z = 70) have spins of $\frac{1}{2}$ and $5/_2$ respectively. $Q_{Yb^{173}} = +3.9$ barns and is exceptionally large. $\delta(\Delta E)/\delta(\Delta E_\nu)^N \simeq 0.3$ and although somewhat on the small side is not abnormally so, thus $\delta(\Delta E_\varepsilon)/\delta(\Delta E_\nu)^N$ cannot be expected to be numerically very large and will almost certainly be less than 0.5 in magnitude.

One obtains

$$\frac{\varepsilon_{Yb^{171}}}{\varepsilon_{Yb^{173}}} \simeq 1 - \frac{1}{12}\frac{\delta(\Delta E_\varepsilon)}{\delta(\Delta E_\nu)^N}$$

This shows that even though the deformation of Yb^{173} is so large, $\varepsilon_{Yb^{173}} \simeq \frac{1}{3}$, that of Yb^{171} can only differ very little. In fact since the relative spacing of the I.S. of the Yb isotopes is given by

Yb^{171}	Yb^{172}	Yb^{173}	Yb^{174}	Yb^{176}
0	0.62	1	1.38	2.1

the deformations of all the isotopes must be very nearly the same which seems rather remarkable in view of the probable large magnitude of these.

From the foregoing and in particular the example of the Eu isotopes, it must be considered as established, even in view of the rather large uncertainties

involved, that a difference in the deformations of two isotopes can have an effect on the I.S. comparable or even larger in magnitude than the usual volume dependent I.S. The possibility must also be strongly entertained that the large variations observed in the I.S. may be due to this deformation dependent effect. The I.S. data could then be used to give information about the variation of nuclear deformations, providing a valuable test for theories of nuclear structure which involve a deformed core. Of especial interest in this connection are the I.S. of a sequence of isotopes which have all rather large or small I.S. This must then be taken as indicating that the deformations continuously increase or decrease respectively from one isotope to the next, and not that merely the difference in deformation between any two isotopes remains approximately the same. Thus the large I.S. of the Sm isotopes would indicate that all the isotopes from Sm^{144} to Sm^{154} have successively larger deformations with an exceptionally large increase between Sm^{150} and Sm^{152}.

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