Generating sets for polynomial rings as modules over the divided differential operator algebra $\mathcal{D}$

By

Xiao, Suo

Department of Mathematics

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Contents

Abstract

This work is about finding minimal generating sets for polynomial rings as modules over the divided differential operator algebra $\mathcal{D}$. This problem is referred to as the hit problem. We will discuss the hit problem of $\mathcal{D}$ on polynomial rings in 2, 3 and 4 variables over the rational field $\mathbb{Q}$ and the finite field $\mathbb{F}_2$. We will also give some results of the hit problem on polynomial rings of 2 and 3 variables over $\mathbb{F}_p$ for an odd prime p.

Declaration

No portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this and any other university or other institution of learning.

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1 Introduction

Let $M = \bigoplus_{d \geq 0} M^d$ be a graded left module over a graded ring $R = \bigoplus_{d \geq 0} R^d$. Let $R^+ = \bigoplus_{d \geq 1} R^d$ be the positively graded part of R. A *hit element* is an element $m \in M$ which can be expressed by $m = \sum_i r_i m_i$, where $r_i \in R^+$, $m_i \in M$ which have strictly lower grading than that of m. The hit elements form a submodule H of M. A closely related problem is to find an additive basis of the quotient module $C = M/H$. The set of representatives of such a basis is a *minimal generating set* of M as a R-module. We will call an element of M, which is not hit, a *non-hit* element. The problem of finding a basis of the quotient module C is referred to as the "hit problem" [23, Section 7].

Research on the hit problem for polynomial rings under the action of the *Steenrod algebra* over a finite field $\mathbb{F}_p$ has been going on for some years. The Steenrod algebra over $\mathbb{F}_2$, $\mathcal{A}_2$, is generated under composition by the Steenrod squares Sq^k. The Steenrod squares were introduced by N.E.Steenrod [19] as linear operators on ordinary cohomology $\mathbf{H}^*(X)$ for some topological space X over $\mathbb{F}_2$,

$$Sq^k : \mathbf{H}^n(X) \longrightarrow \mathbf{H}^{n+k}(X),$$

which have the following properties: for homogeneous $x,\ y \in \mathbf{H}^*(X)$,

(1) $Sq^0 = 1$.

(2) If $i = dim(x)$, $Sq^i(x) = x^2$.

(3) If $i > dim(x)$, $Sq^i(x) = 0$.

(4) Cartan formula:

$$Sq^n(xy) = \sum_{r+s=n} Sq^r(x) Sq^s(y).$$

(5) Adem relations:

$$Sq^iSq^j = \sum_{0\le k\le \lfloor \frac{i}{2}\rfloor} \binom{j-k-1}{i-2k} Sq^{i+j-k}Sq^k$$

for $0 < i < 2j$, where the binomial coefficients are taken modulo 2.

Let $X = \mathbb{R}P^\infty \times \cdots \times \mathbb{R}P^\infty$, the product of n copies of infinite real projective space. Over $\mathbb{F}_2$, $\mathbf{H}^*(X)$ is isomorphic to $\mathbb{F}_2[x_1, x_2, \cdots, x_n]$, the polynomial ring of n variables over $\mathbb{F}_2$.

The hit problem related to the Steenrod algebra was brought to attention by a conjecture made by F.Peterson [13]. The conjecture offers a set of monomials of $\mathbb{F}_2[x_1, x_2, \ldots, x_n]$ containing the generating elements under the action of the Steenrod algebra over $\mathbb{F}_2$. The conjecture has been proved [25]. Here we state a stronger form of the result.

Theorem 1.1 [3, Theorem 2.1] *Let $W_n = \mathbb{F}_2[x_1, x_2, \ldots, x_n]$ be the polynomial ring in n variables over $\mathbb{F}_2$. Let $f = x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}$ be a monomial in W_n with r exponents odd. Let $d = a_1 + a_2 + \ldots + a_n$ be the degree of f. If $\alpha(r+d) > r$ then f is in $\mathcal{A}_2^+W_n$, where $\alpha(t)$ is the number of 1's in the binary expansion of t.*

Let $\mathbb{N} = \{1, 2, \ldots\}$ be the set of natural numbers.

Example 1.2 [26, Section 2.3] *Under the action of $\mathcal{A}_2$, a minimal generating set of $\mathbb{F}_2[x, y]$ consists of monomials in the forms $x^{2^{a_1}-1}y^{2^{a_2}-1}$ and $x^{2^{b_1}-1}y^{2^{b_2}-2^{b_1-1}-1}$ where $a_1,\ a_2 = 0$ or $a_1,\ a_2 \in \mathbb{N}$, $b_1,\ b_2 \in \mathbb{N}$ and $b_2 > b_1$.*

A monomial $x_1^{2^{a_1}-1}x_2^{2^{a_2}-1}\ldots x_n^{2^{a_n}-1}$, which is called a *spike*, never appears in the image of any operation of $\mathcal{A}_2^+$ [25].

Over $\mathbb{F}_p$ for an odd prime p, the Steenrod reduced power operations are linear transformations $P^n : W_n^d \longrightarrow W_n^{d+n(p-1)}$ with the following properties [18, Chapter VI.1]:

(1) $P^0 = 1$.

(2) $P^k(f) = f^p$ if $deg(f) = k$ and $P^k(f) = 0$ if $deg(f) < k$.

(3) $P^k(x^d) = \binom{d}{k} x^{d+k(p-1)}$.

(4) Cartan formula:

$$P^k(fg) = \sum_i P^i(f)P^{k-i}(g),$$

where $f,\ g \in W_n$.

(5) Adem relations:

$$P^iP^j = \sum_{0\le k\le \lfloor \frac{i}{p} \rfloor} (-1)^{i+k} \binom{(p-1)(j-k)-1}{i-pk} P^{i+j-k}P^k,$$

for $0 < i < pj$, where the binomial coefficients are taken modulo p. When $p = 2$, P^k is Sq^k. By the Steenrod algebra over $\mathbb{F}_p$, $\mathcal{A}_p$, we mean the algebra generated by the Steenrod reduced power operations subject to the Adem relations [18, Chapter VI.4].

Let p be an arbitrary prime. The action of $\mathcal{A}_p$ commutes with the action of the general linear group $G = GL(n, \mathbb{F}_p)$ over $\mathbb{F}_p$. Hence the Steenrod algebra can be seen as a set of $\mathbb{F}_pG$−module homomorphisms on the polynomial ring $\mathbb{F}_p[x_1, x_2, \ldots, x_n]$. A theorem of Mitchell [12] shows that each irreducible $\mathbb{F}_pG$−module appears in the polynomial ring. By Schur's lemma, the first occurrence of an irreducible $\mathbb{F}_pG$−module is not hit. Hence the cokernel of the action of $\mathcal{A}_p$ on $\mathbb{F}_p[x_1, x_2, \ldots, x_n]$ includes among its composition factors a complete set of inequivalent irreducible $\mathbb{F}_pG$−modules.

Now we consider an algebra, the divided differential operator algebra $\mathcal{D}$ [23] [24]. $\mathcal{D}$ contains the Steenrod algebra as a subalgebra over a finite field. $\mathcal{D}$ is an algebra which is generated by the differential operators defined as follows:

$$D_k = \sum_{i=1}^{\infty} x_i^{k+1} \frac{\partial}{\partial x_i}$$

with respect to both wedge product and composition (Section 2.1). The natural coproduct $\psi(D_k) = 1 \otimes D_k + D_k \otimes 1$ makes $\mathcal{D}$ into a Hopf algebra with respect to both the composition and the wedge product. Under the composition $\mathcal{D}$ is isomorphic to the Landweber-Novikov algebra [24]. Let Γ_n be the semigroup consisting of all linear transformations on the polynomial ring of n variables which have the following property: when they act on the set $\{x_1, x_2, \ldots, x_n\}$, for $\phi \in \Gamma_n$, then either $\phi(x_i) = x_j$ or $\phi(x_i) = 0$. The action of $\mathcal{D}$ on polynomial rings commutes with the action of Γ_n. The symmetric group Σ_n is a subgroup of Γ_n [20]. Hence $\mathcal{D}$ can be seen as a set of Σ_n-module homomorphisms when it acts on a polynomial ring.

Let K be a field and $\mathcal{D}_K = \mathcal{D} \otimes_{\mathbb{Z}} K$. Let $M_n = K[x_1, \ldots, x_n]$, the polynomial ring of n variables over K. We may view M_n as a $\mathcal{D}_K-$module. A minimal generating set under the action of $\mathcal{D}_K$ on M_n is a set $N \subset M_n$ with $\mathcal{D}_K N = M_n$ and there is no proper subset U of N which satisfies $\mathcal{D}_K U = M_n$ over K. We say a polynomial $f \in M_n$ is hit under the action of $\mathcal{D}_K$ if $f = \sum_i \delta_i(g_i)$ for some $\delta_i \in \mathcal{D}_K^+$ and $g_i \in M_n$ which has lower degree than f. Let $S_n = x_1 \cdots x_n M_n$ which is a $\mathcal{D}_K-$submodule of M_n. Then the hit problem on M_n can be reduced to the hit problem on S_n, since M_n is the union of S_i for all $S_i \subset M_n$, where $1 \leq i \leq n$.

Suppose the field is $\mathbb{Q}$. We consider S_n as a $\mathcal{D}_{\mathbb{Q}}-$module, then we write $H_n = \mathcal{D}_{\mathbb{Q}}^+ S_n$, the set of hit elements in S_n under the action of $\mathcal{D}_{\mathbb{Q}}$. We write the quotient module $C_n = S_n/H_n$. We will prove that C_n is finite dimensional for all n in Theorem 3.9. We denote by $\{z_1, \ldots, z_t\}$ a set of representatives of a basis of C_n, then $\sum_{i=1}^{t} \mathcal{D}_{\mathbb{Q}} z_i = S_n$. Let Λ_n be the ring of symmetric polynomials in n variables. Λ_n is a subring of M_n. We denote by Λ_n^+ the positively graded part of Λ_n. We may view S_n as a Λ_n-module under the normal multiplication. Let $J_n = \Lambda_n^+ S_n$, then J_n is the set of hit elements in S_n under the action of Λ_n. A basis for the quotient module $A_n = S_n/J_n$ is known [16]. The basis is finite and is called the Artin basis which was found by E.Artin [2, II.G]. The Artin basis has a set of representatives:

$$\{a_i = x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \mid 0 \le i_j \le j \ for \ 1 \le j \le n\},$$

where the degree of $a_i \le \dfrac{n(n+1)}{2}$. We call the representatives the Artin elements. The Artin elements form a minimal generating set of S_n under the action of Λ_n and $S_n = \bigoplus_{i=1}^{r} \Lambda_n a_i$, where $r = n!$ [2, II.G]. Furthermore the Artin basis gives a regular representation of Σ_n [16].

Hence S_n is an additive group which carries two module structures. As an Λ_n−module, every element of S_n can be expressed uniquely by $\sum_{i=1}^{r} f_i a_i$, where $f_i \in \Lambda_n$. As a $\mathcal{D}_{\mathbb{Q}}$−module, every element of S_n can be expressed by $\sum_{i=1}^{t} \delta_i z_i$, where $\delta_i \in \mathcal{D}_{\mathbb{Q}}$. We think we can take the set of z_i to be the set of a_i.

Conjecture 1.3 [23, Conjecture 7.3] *Let H be the set of hit elements under the action of the divided differential operator algebra $\mathcal{D}_{\mathbb{Q}}$ on S_n and let $C = S_n/H$, then the Artin basis is a basis of C.*

We will verify Conjecture 1.3 up to $n = 4$. We will also look into the hit problem over a finite field $\mathbb{F}_p$. We denote $\mathcal{D} \otimes_{\mathbb{Z}} \mathbb{F}_p$ by $\mathcal{D}_p$. By Theorem 5.4 and Theorem 7.4, a minimal generating set for a polynomial ring of two variables under the action of $\mathcal{D}_p$ is an infinite set over $\mathbb{F}_p$. Hence we get that a minimal generating set for a polynomial ring of more than one variable under the action of $\mathcal{D}_p$ is an infinite set. Since $\mathcal{A}_p$ is a subalgebra of $\mathcal{D}_p$, the set of the hit elements for the action of $\mathcal{A}_p$ is a subset of the hit elements for the action of $\mathcal{D}_p$.

The action of $\mathcal{D}$ is a set of Σ_n− module homomorphisms. If L is an irreducible $K\Sigma_n$−module in a polynomial ring over a field K, then the action of $\mathcal{D}$ either maps L to 0 or maps L isomorphically to some higher degree of the polynomial ring. Hence results on the hit problem might be used in the study of the structure of irreducible $K\Sigma_n$-modules, especially over $\mathbb{F}_p$.

This thesis is concerned with the hit problem of $\mathcal{D} \otimes_{\mathbb{Z}} K$ on polynomial rings in a small number of variables over a field K.

In Section 2.1, we will state definitions related to the divided differential operator algebra $\mathcal{D}$. We will give generating sets of $\mathcal{D}_{\mathbb{Q}}$ and $\mathcal{D}_p$ under composition. Then we will discuss the action of $\mathcal{D}$ on polynomial rings. We will define the conjugation χ on $\mathcal{D}$ which is an anti-isomorphism of the Hopf algebra. We will give some formulas related to the conjugation. These formulas are very useful in calculations of the hit problem. In Section 2.2, we will define Young diagrams and tableaux. In Section 2.3, we will give the abstract definition of the Specht module. The Specht modules corresponding to all partitions of n form a complete set of irreducible representations of the symmetric group Σ_n over $\mathbb{Q}$. Then we will give a method to construct Specht modules in a polynomial ring. In Section 2.4, we will discuss symmetric functions and the algebraic Thom map. The algebraic Thom map gives an isomorphism of vector spaces between the divided differential operator algebra and the ring of symmetric functions.

In Section 3, we will give some combinatorial results and some general results on the hit problem. These results will be often used. The main result in this section is Theorem 3.9, due to G. Walker and R. M. W. Wood.

Theorem 3.9 *A minimal generating set of S_n under the action of $\mathcal{D}_{\mathbb{Q}}$ is finite for all n.*

In Section 4, we will find minimal generating sets for polynomial rings of 3 and 4 variables under the action of $\mathcal{D}_{\mathbb{Q}}$. These results show that Conjecture 1.3

is true for polynomial rings up to $n = 4$. The main methods we use to prove that a monomial is hit are Corollary 3.8 and Theorem 2.9, which is called the "χ trick". Let $x = x_1$, $y = x_2$, $z = x_3$ and $t = x_4$. The main results in this section are:

Theorem 4.1 *A minimal generating set of S_3 under the action of $\mathcal{D}_{\mathbb{Q}}$ is*

$$\{xyz,\ x^2yz,\ xy^2z,\ x^3yz,\ x^2y^2z,\ x^3y^2z\}.$$

Theorem 4.2 *A minimal generating set of S_4 under the action of $\mathcal{D}_{\mathbb{Q}}$ is*

$$\{x^iy^jz^kt \mid 1 \le i \le 4,\ 1 \le j \le 3,\ 1 \le k \le 2\}.$$

In Section 5, we will prove some results on the hit problem over $\mathbb{F}_2$. We have found minimal generating sets for 2 and 3 variable cases. The basic method we use to prove that a monomial is hit is to show the monomial is congruent to a monomial which is known to be hit, modulo the hit elements which we already know over $\mathbb{F}_2$.

The main results in this section are:

Theorem 5.4 $\{1,\ x,\ y,\ x^2y,\ x^{2^m-1}y \mid m \in \mathbb{N}\}$ *is a minimal generating set of $\mathbb{F}_2[x,y]$ under the action of $\mathcal{D}_2$.*

Theorem 5.5 *A minimal generating set of $\mathbb{F}_2[x,y,z]xyz$ under the action of $\mathcal{D}_2$ is:* $\{xyz,\ x^2yz,\ xy^2z,\ x^3yz,\ xy^3z,\ x^3y^2z,\ x^{2^k}y^{2^k-1}z,\ xy^{2^k-1}z^{2^k},\ x^{2^k+1}y^{2^k-1}z,\ x^{2^k+1}yz^{2^k-1},\ xy^{2^k+1}z^{2^k-1} \mid 2 \le k \in \mathbb{N}\}$.

In Section 6, we will give some results for the 4 variable case over $\mathbb{F}_2$. From these results we have a general view on the cokernel of the action of $\mathcal{D}_2$ on

$\mathbb{F}_2[x, y, z, t]$. Main results in this section are:

Theorem 6.6 *If a monomial f of n variables can be written as a product of non-hit monomials of 2 variables and there are no two monomials having common variables, then f is not hit under the action of $\mathcal{D}_2$.*

Theorem 6.16 *A monomial $x^a y^b z^c t^d$ with two exponents even is hit under the action of $\mathcal{D}_2$, if it is in a degree ≥ 11 which is not 2^m.*

Based on Theorem 6.6 and the related results we have got on the rational case which show that the product of two monomials in independent variables which are not hit is still not hit, we give a conjecture on the product of non-hit monomials with no common variables in general (Conjecture 6.30).

In Section 7, we will discuss the hit problem over $\mathbb{F}_p$ for an odd prime p. We have found a minimal generating set for the 2 variable case. The result is very similar to the result of the 2 variable case over $\mathbb{F}_2$. We believe that the minimal generating set for the 3 variable case over $\mathbb{F}_p$ is also close to the result over $\mathbb{F}_2$. We have also tried to get some results for the 3 variable case. But it seems much harder to find out hit elements over $\mathbb{F}_p$. It is difficult to follow methods we have used over $\mathbb{F}_2$. Since coefficients and signs all matter over $\mathbb{F}_p$, it is more difficult to cancel some terms in the image of an operation of $\mathcal{D}_p$ and to make the image simpler.

The main result in this subsection is:

Theorem 7.4 $\{1,\ x,\ y,\ xy,\ x^2y,\ x^{p^m-1}y \mid m \in \mathbb{N}\}$ *is a minimal generating set of $\mathbb{F}_p[x, y]$ under the action of $\mathcal{D}_p$.*

2 The background

In this section, we will give some related background. In Section 2.1, we will define the divided differential operator algebra $\mathcal{D}$ and products on $\mathcal{D}$ which are the composition and the wedge product. We will give the minimal generating sets of $\mathcal{D}$ under the composition over $\mathbb{Q}$ and $\mathbb{F}_p$. We will also discuss the action of the divided differential operator algebra on polynomial rings. Then we will introduce the conjugation χ of $\mathcal{D}$ and some formulas related to χ. These formulas will be useful in the later content. In Section 2.2, we will give definitions of Young diagrams and tableaux. In Section 2.3, we will give the abstract definition of the Specht module which is indexed by partitions, then give a method to construct Specht modules in polynomial rings. Finally we will introduce symmetric functions and the algebraic Thom map in Section 2.4.

2.1 The divided differential operator algebra

Let $X = \{x_1,\ x_2, \dots\}$ be an infinite sequence of variables. Let

$$M_n = \bigoplus_{d=0}^{\infty} M_n^d = K[x_1, x_2, \dots, x_n]$$

be the polynomial ring in n variables of X over the field K, where M_n^d consists of all the homogeneous polynomials of degree d. Hence M_n is a graded ring with the grading given by the degree. The differential operator $D_r:\ M_n \longrightarrow M_{n+r}$ for any $n \geq 1$ is defined by

$$D_r = \sum_{i=1}^{\infty} x_i^{r+1} \frac{\partial}{\partial x_i},$$

where $\frac{\partial}{\partial x_i}$ is the ordinary partial differential operator with respect to all $x_i \in X$ [23, Section 2].

The wedge product $\vee$ of D_r is defined in the following way: let the derivative of the first operator D_r pass the variable coefficients of the second operator D_s without acting. That is [23, Section 2]:

$$D_r \vee D_s = (\sum_{i=1}^{\infty} x_i^{r+1} \frac{\partial}{\partial x_i}) \vee (\sum_{i=1}^{\infty} x_i^{s+1} \frac{\partial}{\partial x_i}) = \sum_{i,j=1}^{\infty} x_i^{r+1} x_j^{s+1} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j}.$$

The composition of D_r and D_s is written as $D_r \circ D_s$ or simply $D_r D_s$ which is:

$$(\sum_{i=1}^{\infty} x_i^{r+1} \frac{\partial}{\partial x_i}) \circ (\sum_{j=1}^{\infty} x_j^{s+1} \frac{\partial}{\partial x_j}).$$

This is equal to [23, Section 2]:

$$(s+1)(\sum_{i=1}^{\infty} x_i^{r+s+1} \frac{\partial}{\partial x_i}) + \sum_{i,j=1}^{\infty} x_i^{r+1} x_j^{s+1} \frac{\partial^2}{\partial x_i \partial x_j} = (s+1) D_{s+r} + D_s \vee D_r. \quad (1)$$

So the composition and the wedge product are related. In general we have the following formula of the wedge product: [24, Lemma 3.3]:

$$D_{r_1} \vee D_{r_2} \vee \ldots \vee D_{r_k} = \sum_{i_1, i_2, \ldots, i_k} x_{i_1}^{r_1+1} \cdots x_{i_k}^{r_k+1} \frac{\partial^k}{\partial x_{i_1} \cdots \partial x_{i_k}},$$

where the sum is over all i_j for $i_j \in \mathbb{N}$. The $k-$fold wedge product of D_r, $D_r^{\vee k}$ can be derived from the above formula [24, Lemma 3.4]:

$$D_r^{\vee k} = \sum_{i_1, i_2, \ldots, i_k} (x_{i_1} \cdots x_{i_k})^{r+1} \frac{\partial^k}{\partial x_{i_1} \cdots \partial x_{i_k}},$$

where the sum is over all i_j for $i_j \in \mathbb{N}$. From the above formula, we can also see that the image of $D_r^{\vee k}$ is divisible by $k!$ when it acts on an arbitrary monomial. We call $\frac{D_r^{\vee k}}{k!}$ a *divided differential operator.*

Let d be a non-negative integer. We denote λ to be a *partition* of d. We write $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0$ where λ_i's are integers and $|\lambda| = \sum_{i=1}^{n} \lambda_i = d$. We will identify $(\lambda_1, \lambda_2, \ldots, \lambda_n)$, where $\lambda_i \neq 0$, with $(\lambda_1, \lambda_2, \ldots, \lambda_n, 0, \ldots, 0)$. We say a partition $\lambda = 0$ if $\lambda = (0, 0, \cdots)$. The *length* of a partition λ, $l(\lambda)$, is the number of the non-zero parts of λ.

We say λ is a *composition* of d if $\lambda = (\lambda_1,\ \lambda_2,\ \ldots,\ \lambda_n)$ and $|\lambda| = d$, where $0 \leq \lambda_i \in \mathbb{Z}$ for all i. Obviously, every partition of d is a composition of d as well. The *lexicographical* order $>$ on compositions is defined as follows: $\lambda \geq \mu$ if $\lambda = \mu$ or $\lambda_j > \mu_j$ for the least j for which $\lambda_j \neq \mu_j$. For two partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$, $\lambda + \mu$ is a partition with parts $\lambda_1, \lambda_2, \ldots, \lambda_n, \mu_1, \mu_2, \ldots, \mu_n$.

For a partition of a particular integer n, we will omit commas between parts of the partition. For example we write $(221) = (2, 2, 1)$. Let $\kappa = (\kappa_1^{r_1} \kappa_2^{r_2} \ldots \kappa_n^{r_n})$ which is a partition with r_i parts of distinct κ_i for $1 \leq i \leq n$. The *divided differential operator algebra* $\mathcal{D}$ over $\mathbb{Z}$ is defined as follows:

Definition 2.1 [23, Section 2.1] *The divided differential operator algebra $\mathcal{D}$ over $\mathbb{Z}$ is the algebra which is generated under the wedge product by the divided differential operators $\frac{D_r^{\vee k}}{k!}$.*

The elements $D(\kappa) = \frac{D_{r_1}^{\vee k_1}}{k_1!} \vee \frac{D_{r_2}^{\vee k_2}}{k_2!} \vee \ldots \vee \frac{D_{r_n}^{\vee k_n}}{k_n!}$ form an additive basis of $\mathcal{D}$, for all partitions κ.

Theorem 2.2 [24, Theorem 3.13] *The divided differential operator algebra $\mathcal{D}$ is closed under composition.*

Let f and g be two polynomials in n variables $\{x_1,\ x_2, \ldots,\ x_n\}$. The Leibniz formula for the wedge product applied the product of f and g can be derived from the formula of the $k-$fold wedge product [24, Lemma 3.5]:

$$D_r^{\vee k}(fg) = \sum_{i+j=k} \binom{k}{i} D_r^{\vee i}(f) D_r^{\vee j}(g). \tag{2}$$

In general we have:

$$D(\lambda)(fg) = \sum_{\mu+\kappa=\lambda} D(\mu)(f)D(\kappa)(g). \tag{3}$$

We write $E_r^k = \dfrac{D_r^{\vee k}}{k!}$. Then by the formula of the $k-$fold wedge product, we get:

Example 2.3 *For $m \geq k$,*

$$E_r^k(x^m) = \frac{1}{k!}m(m-1)\cdots(m-k+1)x^{rk+m} = \binom{m}{k}x^{rk+m},$$

and for $m < k$, $E_r^k(x^m) = 0$.

From (2), we get:

$$E_r^k(fg) = \sum_{i+j=k} E_r^i(f)E_r^j(g). \tag{4}$$

Let $x = x_1$ and $y = x_2$. From (4) and Example 2.3, we get:

$$E_r^k(x^a y^b) = \sum_{i+j=k} E_r^i(x^a)E_r^j(y^b) = \sum_{i+j=k} \binom{a}{i}\binom{b}{j} x^{a+ri}y^{b+rj}.$$

In general, we have:

$$E_r^k(x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}) = \sum_{\sum_i t_i=k} \binom{a_1}{t_1}\binom{a_2}{t_2}\cdots\binom{a_n}{t_n} x_1^{a_1+rt_1}x_2^{a_2+rt_2}\cdots x_n^{a_n+rt_n}. \tag{5}$$

The Lie bracket of D_k and D_l is defined as follows: $[D_k, D_l] = D_kD_l - D_kD_l$. From the definition of D_k we have:

Lemma 2.4 [23, Lemma 2.3] $[D_k, D_l] = (l-k)D_{k+l}$.

Let K be a field and $\mathcal{D}_K = \mathcal{D} \otimes_{\mathbb{Z}} K$. We have the following theorem.

Theorem 2.5 [23, Theorem 2.5] *$\mathcal{D}_{\mathbb{Q}}$ is generated by D_1 and D_2 under the composition.*

In [9], P.S. Landweber defined the cobordism operator B_k which acts on polynomial rings in the following way: $B_k(x_i) = x_i^{k+1}$. Let $\mu = \mu_1^{r_1}\mu_2^{r_2}\dots\mu_n^{r_n}$ be a partition. The cobordism operator B_μ is defined in the following way to act on polynomial rings:

$$B_\mu(fg) = \sum_{\lambda+\kappa=\mu} B_\lambda(f)B_\kappa(g), \tag{6}$$

where f, g are arbitrary monomials and $B_\mu(x_i) = 0$ for all i if μ has two or more non-zero parts.

The set of B_μ is called the set of the basic cobordism operations and it is a basis of the Landweber–Novikov algebra A^* with multiplication the composition [9]. A^* is isomorphic to $\mathcal{D}$ under the composition [24]. Comparing (6) with (3), we can see that B_μ corresponds to $D(\mu)$ under the isomorphism between A^* and $\mathcal{D}$.

Theorem 2.6 [9, Theorem 7.1] *A minimal set of generators of $A^* \otimes \mathbb{F}_p$ is:*

$$\{B_{1^k},\ B_{2^k} \mid k = 1 \text{ or } k = p^m,\ m \in \mathbb{N}\}.$$

If p is an odd prime, there is another minimal generating set besides the above one:

$$\{B_{1^k},\ B_{1^{2k}} \mid k = 1 \text{ or } k = p^m,\ m \in \mathbb{N}\}.$$

Since B_{1^k} is E_1^k and B_{2^k} is E_2^k, we get:

Theorem 2.7 *$\mathcal{D}_p$ has a minimal generating set under the composition:*

$$\{E_1^k,\ E_2^k \mid k = 1 \text{ or } k = p^m,\ m \in \mathbb{N}\}.$$

If p is a odd prime, there is another minimal generating set:

$$\{E_1^k,\ E_1^{2k} \mid k = 1 \text{ or } k = p^m,\ m \in \mathbb{N}\}.$$

From the definitions we have, E_1^k is the Steenrod square Sq^k over $\mathbb{F}_2$ and E_{p-1}^k is P^k over $\mathbb{F}_p$ when they act on polynomial rings. So $\mathcal{A}_p$ is a subalgebra of $\mathcal{D}_p$. When $r = 1$ and $p = 2$, the formula in (4) is the Cartan formula.

The natural coproduct $\psi(D_n) = 1 \otimes D_n + D_n \otimes 1$ makes $\mathcal{D}$ into a Hopf algebra with respect to both the composition and the wedge product. The operators D_n are primitives of the Hopf algebra [24, Section 3]. The conjugation χ of the Hopf algebra $\mathcal{D}$ with respect to the composition is defined as follows [23, Lemma 2.18]:

$$\chi D(\lambda) = -\sum_{\substack{\kappa+\mu=\lambda, \\ \kappa\neq 0}} D(\kappa)\chi D(\mu). \tag{7}$$

Example 2.8 *By the above formula, we can derive:*

$$-\chi(E_r^k) = \sum_{i=1}^{k} E_r^i \chi(E_r^{k-i}), \ \chi(D_r) = -D_r, \ \chi(E_r^2) = D_r D_r - E_r^2.$$

By (1), $D_r D_r = (r+1)D_{2r} + D_r \vee D_r = (r+1)D_{2r} + 2E_r^2$. Hence over $\mathbb{F}_2$, $D_r D_r = 0$ if r is odd and $\chi(E_r^2) = E_r^2$ for an odd r. Hence

$$\chi(E_1^3) = -E_1^3 + E_1^2 D_1 - D_1(\chi(E_1^2)) \equiv E_1^3 + E_1^2 D_1 + D_1 E_1^2 \ (mod\ 2).$$

From the Adem relations, we get $D_1 E_1^2 \equiv E_1^3$ (mod 2). Hence we get

$$\chi(E_1^3) \equiv E_1^2 D_1 \ (mod\ 2).$$

The conjugation is an anti-isomorphism on the Hopf algebra $\mathcal{D}$. Hence for δ_1, $\delta_2 \in \mathcal{D}$, $\chi(\delta_1\delta_2) = \chi(\delta_2)\chi(\delta_1)$ [4, Section 7]. Let f be a monomial in a polynomial ring with $n-$variables over an arbitrary field. Every monomial in $\chi(\delta)(f)$ is in the same degree as the degree of monomials in $\delta(f)$ for any $\delta \in \mathcal{D}$. Let f, g be two polynomials. we say $f \equiv g$ *(mod hit)* if $f - g$ is hit under the action of $\mathcal{D}$ over K.

Theorem 2.9 [23, Theorem 7.6] *For any $\delta \in \mathcal{D}$,*

$$f\delta(g) \equiv (\chi(\delta)(f))g \ (mod\ hit).$$

Let Γ_n be the semigroup consisting of all linear transformations on the set $X = \{x_1, x_2, \ldots, x_n\}$ which have the following property: for any $\phi \in \Gamma_n$ and $x_i, x_j \in X$, either $\phi(x_i) = x_j$ or $\phi(x_i) = 0$. We can identify Γ_n with the set of $n \times n$ matrices with at most one 1 in each row and all 0's elsewhere. The action of the divided differential operator algebra $\mathcal{D}$ on polynomial rings commutes with the action of Γ_n [20]. Let M be a polynomial ring, then we have the following commutative diagram:

$$\begin{array}{ccc} M & \xrightarrow{\mathcal{D}} & M \\ \downarrow \Gamma_n & & \downarrow \Gamma_n \\ M & \xrightarrow{\mathcal{D}} & M \end{array} .$$

2.2 Young diagrams and tableaux

Let λ be a partition. A *Young diagram* $[\lambda]$ corresponding to λ is a diagram:

$$[\lambda] = \begin{array}{ll} * \; * \; \ldots \; * \; * \; * & \lambda_1 \; nodes \\ * \; * \; \ldots \; * \; * & \lambda_2 \; nodes \\ \ldots & \\ \ldots & \\ * \; \ldots \; * & \lambda_n \; nodes \end{array} .$$

We write λ' to be the *transpose* of λ. It is a partition with the Young diagram obtained by the reflection through the diagonal of the Young diagram of λ.

Example 2.10 $\lambda = (211)$, $\lambda' = (31)$.

A $\lambda-$*tableau* is obtained from a Young diagram by replacing each node with an integer of the set $\{1, 2, \ldots, d\}$ in a square frame. We call an integer in a Young diagram an *entry* of the Young diagram. Let λ be a partition of d. If the sequence of entries counted down columns and along rows from the left to the right of a $\lambda-$tableau T is $i_1, i_2, \ldots, i_d$, we write $[i] = i_1, i_2, \ldots, i_d$. We denote by $T_{\lambda,[i]}$ a $\lambda-$tableau with entries $[i]$. The *content* of $T_{\lambda,[i]}$ is a sequence of numbers $\mu = \mu_1\mu_2\ldots\mu_l$, where μ_1 equals to the number of 1's in $[i]$, μ_2 equals to the number of 2's in $[i]$, and so on. Also $\sum_1^l \mu_i = d$, i.e. the content of $T_{\lambda,[i]}$ is a composition of d.

Example 2.11 $\lambda = (211)$, $[i] = 1, 3, 2, 1$. *Then*

$$T_{\lambda,[i]} = \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 3 & \\ \cline{1-1} 2 & \\ \cline{1-1} \end{array} .$$

The content of $T_{\lambda,[i]}$ *is* (211).

When we draw a Young diagram of $T_{\lambda,[i]}$, we may just denote the Young diagram by T_λ, since the content is shown in the diagram. We say $T_{\lambda,[i]}$ is *semi-standard* if entries of $T_{\lambda,[i]}$ are strictly increasing down columns and monotonic increasing along rows from the left to the right. $T_{\lambda,[i]}$ is *standard* if entries of $T_{\lambda,[i]}$ are strictly increasing down columns and along rows from the left to the right.

2.3 Specht modules and their construction in polynomial rings

Let G be a finite group, $\mathbb{C}$ be the field of complex numbers. $\mathbb{C}G$ is the group algebra of G over $\mathbb{C}$ which is the regular representation of G.

Theorem 2.12 [8, Section 11.9] *Suppose that* $\mathbb{C}G = \bigoplus_i k_i U_i$, *a direct sum of irreducible* $\mathbb{C}G-$*modules, then* $k_i = \dim\, U_i$, *where* $k_i U_i = U_i \bigoplus \cdots \bigoplus U_i$, *the direct sum of* k_i *copies of* U_i.

Lemma 2.13 (Schur's Lemma) [1, Section 5] *Let* R *be an arbitrary ring. Any non-zero homomorphism between simple* $R-$*modules is an* $R-$*isomorphism.*

Theorem 2.14 (Maschke's Theorem) [1, Section 5] *Let* G *be a finite group and* K *be a field. Suppose the characteristic of* K *is either zero or coprime to* $|G|$. *If* U *is a* $KG-$*module and* V *is a* $KG-$*submodule of* U, *then* V *is a direct summand of* U *as* $KG-$*modules.*

Definition 2.15 [6, Section 3.9] *A tabloid is an equivalence class of* $\lambda-$*tableaux. The equivalence relation is defined by* $T_{\lambda_1} \sim T_{\lambda_2}$ *if corresponding rows contain the same entries.*

We denote the equivalence class of T_λ by $\{T_\lambda\}$.

Example 2.16 *For a* $\lambda-$*tableau* $T_\lambda = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1}\end{array}$

we write the corresponding tabloid as follows:

$$\{T_\lambda\} = \begin{array}{ccc}\hline 1 & 3 & 4 \\ \hline 2 & & \\ \hline\end{array}.$$

We have:

$$\begin{array}{ccc}\hline 1 & 3 & 4 \\ \hline 2 & & \\ \hline\end{array} \sim \begin{array}{ccc}\hline 1 & 4 & 3 \\ \hline 2 & & \\ \hline\end{array} \sim \begin{array}{ccc}\hline 3 & 1 & 4 \\ \hline 2 & & \\ \hline\end{array} \sim \begin{array}{ccc}\hline 3 & 4 & 1 \\ \hline 2 & & \\ \hline\end{array} \sim \begin{array}{ccc}\hline 4 & 1 & 3 \\ \hline 2 & & \\ \hline\end{array} \sim \begin{array}{ccc}\hline 4 & 3 & 1 \\ \hline 2 & & \\ \hline\end{array}.$$

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ be a partition of n.

Definition 2.17 [6, Section 4.2] *Let K be an arbitrary field, M^λ is the vector space over K whose basis elements are the set of all $\lambda-$tabloids, each $\lambda-$tabloid corresponds to a $\lambda-$ tableau with distinct entries.*

Let Σ_n be the symmetric group on the set $\{1,\ 2,\ \ldots,\ n\}$. The permutation module M^λ is a cyclic Σ_n-module, generated by any one of the $\lambda-$tabloids. We have

$$dim\ M^\lambda = \frac{n!}{\lambda_1!\lambda_2!\cdots\lambda_n!}.$$

Let $Cl(\lambda)$ be the column permutation group. The action of $Cl(\lambda)$ on a $\lambda-$tableau is shown as follows: we mark positions of squares of the ith column of the $\lambda-$tableau $T_{\lambda,[j]}$ with $\{1, 2, \cdots, \lambda'_i\}$ for $1 \le i \le l(\lambda')$ from top to bottom. Let $Cl_i = \Sigma_{\lambda'_i}$, then we write $Cl(\lambda) = Cl_1 \times Cl_2 \times \cdots \times Cl_{l(\lambda')}$. For $\pi \in Cl_i$, $\pi T_{\lambda,[j]}$ is the tableau obtained by permuting marks of squares along with their entries in the ith column of $T_{\lambda,[j]}$ by π and keeping other columns unchanged.

Definition 2.18 [6, Section 4.3] *The polytabloid e_{T_λ} is defined as follows:*

$$e_{T_\lambda} = \sum_{\pi \in Cl_\lambda} (sign\ \pi)\{\pi(T_\lambda)\}.$$

Example 2.19 *Let* $T_\lambda = \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2} \end{array}$

$$\textit{Then } e_{T_\lambda} = \begin{array}{c}\hline 1\ 3\ 5 \\ \hline 2\ 4 \\ \hline \end{array} - \begin{array}{c}\hline 2\ 3\ 5 \\ \hline 1\ 4 \\ \hline \end{array} - \begin{array}{c}\hline 1\ 4\ 5 \\ \hline 2\ 3 \\ \hline \end{array} + \begin{array}{c}\hline 2\ 4\ 5 \\ \hline 1\ 3 \\ \hline \end{array}$$

Definition 2.20 [6, Section 4.5] *The Specht module Sp^λ is the Σ_n-module which is spanned by polytabloids generated by $\lambda-$tableaux.*

The following theorem shows that the dimension of Sp^λ is equal to the number of standard $\lambda-$tableaux.

Theorem 2.21 [6, Section 8.4] *The Specht module Sp^λ has a basis*

$$\{\ e_{T_\lambda}\ |\ T_\lambda\ is\ a\ standard\ \lambda - tableau\ \}.$$

Over $\mathbb{Q}$, the Specht modules indexed by all distinct partitions of n give a complete set of the ordinary irreducible representations of Σ_n.

Let $<,>$ be the bilinear form on M^λ where for tabloids $a,\ b$,

$$< a, b >= \begin{cases} 1 & if\ a = b \\ 0 & otherwise \end{cases}$$

Let $Sp^\mu \subseteq M^\lambda$. We denote by $Sp^{\mu\perp}$ the submodule of M^λ where for any $a' \in Sp^{\mu\perp}$, $< a', a >= 0$ for all $a \in Sp^\mu$.

Lemma 2.22 [6, Section 4.9] *Let K be an arbitrary field. $F^\lambda = \dfrac{Sp^\lambda}{Sp^\lambda \cap Sp^{\lambda\perp}}$ is zero or irreducible. If $Sp^\lambda \cap Sp^{\lambda\perp}$ is not zero then $Sp^\lambda \cap Sp^{\lambda\perp}$ is the unique maximal submodule of Sp^λ.*

A partition λ is $p-$row regular if $[\lambda]$ has no p rows which have the same number of nodes. Over a field of characteristic p, F^λ is non-zero if λ is $p-$row regular. This set of F^λ's gives a complete set of irreducible representations of Σ_n over any field K.

Theorem 2.23 (Young's rule) [6, Section 14.1] *Over $\mathbb{Q}$, the multiplicity of Sp^μ as a direct summand of M^λ equals the number of semistandard $\lambda-$tableaux of content μ.*

The multiplicity of Sp^{μ} in M^{λ} is called a Kostka number $K_{\lambda\ \mu}$. We list the tables of Kostka numbers for $n = 2,\ 3,\ 4$ in Appendix A.

By Theorem 2.12, Theorem 2.21 and Young' rule, we get that $M^{(1^n)}$ is a regular representation of Σ_n.

Let K be a field and $M_n = K[x_1, \ldots, x_n]$. Let $S_n = \oplus S_n^d = x_1 \ldots x_n M_n$, where S_n^d contains all the homogeneous polynomials of degree d divisible by $x_1 \ldots x_n$. S_n is a graded Σ_n- module. The module action is defined as follows: for $\delta \in \Sigma_n$ and $x_1^{d_1} x_2^{d_2} \ldots x_n^{d_n} \in S_n$, $\delta(x_1^{d_1} x_2^{d_2} \ldots x_n^{d_n}) = x_{\delta(1)}^{d_1} x_{\delta(2)}^{d_2} \cdots x_{\delta(n)}^{d_n}$. We denote by $\Sigma_{\{i+1,\ldots,n\}}$ the subgroup of Σ_n which permutes $\{i+1, \ldots, n\}$ and leaves $\{1, \ldots, i\}$ fixed and define $\Sigma_{\{1,\ldots,i\}}$ similarly.

Definition 2.24 *We call the set of monomials $\{\sigma(x_1^{a_1} \ldots x_n^{a_n}) \mid \sigma \in \Sigma_n\}$ the set of monomials in the form $[a_1 \cdots a_n]$. By $[\{a_1 \cdots a_i\}\{a_{i+1} \cdots a_n\}]$, we mean the set of monomials $\{\sigma_1(x_1^{a_1} \cdots x_i^{a_i})\sigma_2(x_{i+1}^{a_{i+1}} \cdots x_n^{a_n}) \mid \sigma_1 \in \Sigma_{\{1,\ldots,i\}},\ \sigma_2 \in \Sigma_{\{i+1,\ldots,n\}}\}$.*

Example 2.25 *We call all monomials obtained from permutations of $x^3y^2z^2t$ under the action of Σ_4 the monomials in the form* $[3221]$. *We call the following* 4 *monomials $x^3y^2z^2t$, $x^2y^3z^2t$, $x^3y^2zt^2$, $x^2y^3zt^2$ the monomials in the form* $[\{32\}\{21\}]$.

Definition 2.26 *For a monomial*

$$e = (x_1 \ldots x_{d_1})^{a_1}(x_{d_1+1} \ldots x_{d_2})^{a_2} \cdots (x_{d_{l-1}+1} \ldots x_{d_l})^{a_l}$$

where $a_1, a_2, \ldots, a_l$ are distinct, we define a composition $\mu = \mu_1\mu_2 \ldots \mu_l$, where $\mu_i = d_i - d_{i-1}$ with $d_0 = 0$. Let Σ_l be the symmetric group on $\{1, \ldots, l\}$ and $\lambda = \pi(\mu)$ be a partition for some $\pi \in \Sigma_l$. Then we say that e and the permutations of e under the action of Σ_n are monomials of the exponent type λ.

Example 2.27 *Since $x^3y^2z^2t = x^3(yz)^2t$, we get a composition* (121). *Hence all the monomials in the form* $[3221]$ *are monomials of the exponent type* (211).

Let λ be a partition of n. In S_n^d, all monomials of exponent type λ form a Σ_n−module which is isomorphic to M^λ. Hence S_n^d can be dicomposited into a direct sum of M^λ's.

We denote by $\Delta(n)$ in variables $x_1,\ x_2,\ \cdots,\ x_n$ the Van der Monde determinant:

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ \vdots & \vdots & & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{vmatrix} = \prod_{i<j}(x_j - x_i).$$

We can see that $\Delta(1) = 1$ from the above definition.

Definition 2.28 [17] *Let $\mu = (\mu_1,\ \mu_2,\ \ldots,\ \mu_l)$ be a partition of n. Let*

$$\Delta(\mu) = \Delta(\mu_1)\Delta(\mu_2)\cdots\Delta(\mu_l),$$

where $\Delta(\mu_1)$ is in variables $x_1,\ \ldots,\ x_{\mu_1}$, $\Delta(\mu_2)$ is in variables $x_{\mu_1+1},\ \ldots,\ x_{\mu_1+\mu_2}$ and so on.

This polynomial was firstly considered by W. Specht. It is now called the *Specht polynomial.*

Theorem 2.29 [17, Satz 1] *Over a field K of characteristic 0, the polynomial $x_1x_2\cdots x_n\Delta(\mu)$ generates an irreducible $K\Sigma_n$−submodule, which is isomorphic to the Specht module Sp^λ, in $S_n^{\sigma(\lambda)}$ where $\lambda = \mu'$ and $\sigma(\lambda) = \sum_i i\lambda_i$.*

This irreducible module generated by $\Delta(\mu)$ gives the first occurrence of Sp^λ in S_n. The following example shows how we decompose a permutation module into a direct sum of Specht modules in a set of monomials.

Example 2.30 *In S_3^6, $x_1^3x_2^2x_3, x_1^3x_2x_3^2, x_1^2x_2^3x_3, x_1^2x_2x_3^3, x_1x_2^3x_3^2, x_1x_2^2x_3^3$ have the exponent type* (111) *and any of them generates a $\mathbb{Q}\Sigma_3-$submodule $M^{(111)}$. It is the regular representation of Σ_3 over $\mathbb{Q}$. $Sp^{(3)}$ is generated by*

$$x_1^3x_2^2x_3 + x_1^3x_2x_3^2 + x_1^2x_2^3x_3 + x_1^2x_2x_3^3 + x_1x_2^3x_3^2 + x_1x_2^2x_3^3.$$

By Definition 2.26, the 1st occurrence of $Sp^{(111)}$ in S_3 is generated by:

$$xyz\Delta(3) = x_1x_2x_3 \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{vmatrix} = x_1x_2x_3 \prod_{i<j}(x_j - x_i).$$

$$= x_1^3x_2^2x_3 - x_1^3x_2x_3^2 + x_1^2x_2x_3^3 - x_1^2x_2^3x_3 + x_1x_2^3x_3^2 - x_1x_2^2x_3^3.$$

There are two semistandard $(21)-$*tableaux with content* (111)*. So the dimension of $Sp^{(21)}$ is two and there are two copies of $Sp^{(21)}$ in $M^{(111)}$. We give $\mathbb{Q}-$bases of the two $Sp^{(21)}$ which are direct summands of $M^{(111)}$ as follows:*

$$\{x_1x_2^2x_3^3 - x_1^2x_2x_3^3 + x_1x_2^3x_3^2 - x_1^2x_2^3x_3,\ x_1^2x_2x_3^3 - x_1x_2^2x_3^3 + x_1^3x_2x_3^2 - x_1^3x_2^2x_3\}$$

and

$$\{x_1x_2^2x_3^3 - x_1^2x_2^3x_3 + x_1^2x_2x_3^3 - x_1x_2^3x_3^2,\ x_1^2x_2x_3^3 - x_1^3x_2^2x_3 + x_1x_2^2x_3^3 - x_1^3x_2x_3^2\}.$$

We give a table of decomposition of S_4^d for $4 \le d \le 10$ in terms of M^λ and Sp^λ, where λ ranges over all partitions of 4, in Appendix B.

2.4 Symmetric functions and the algebraic Thom map

Let λ be a partition of d and $X^\lambda = x_1^{\lambda_1} \cdots x_n^{\lambda_n}$. Let $m_\lambda(x_1, \ldots, x_n) = \sum_\mu X^\mu$ where each μ is a permutation of λ and summed over all distinct permutations of λ. Let Λ_n be the ring of symmetric polynomials of n variables. The

$m_\lambda(x_1, \ldots, x_n)$ for all λ's where $l(\lambda) \leq n$ form a $\mathbb{Z}-$basis of Λ_n. The ring of symmetric functions is the inverse limit of the sequence of $\mathbb{Z}-$modules Λ_n:

$$\Lambda = \lim_{\leftarrow} \Lambda_n$$

where the homomorphism $\rho_{m,n} : \Lambda_m \to \Lambda_n$ sends x_i to 0 for all $m \geq i > n$ and the other x_i to themselves. Then we have a projection $\rho_n : \Lambda \to \Lambda_n$ by sending x_i to 0 for all $i > n$ and the other x_i's to themselves.

The *monomial symmetric function* m_λ is defined by $\rho_n(m_\lambda) = m_\lambda(x_1, \ldots, x_n)$ for each n. We denote P_d the *power sum* where $P_d = m_{(d)} = \sum x_i^d$. We denote e_d the *elementary symmetric function* where $e_d = m_{(1^d)}$. we define $e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_n}$. We denote h_d the *complete symmetric function* of homogeneous degree d where $h_d = \sum_{|\mu|=d} m_\mu$. Also we define $h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_n}$.

Let $s = x_1 x_2 \cdots x_n$, the product of n variables. We define a map $\Phi_n : \mathcal{D} \to \Lambda_n$ where

$$\Phi_n(D(\lambda)) = \frac{D(\lambda)(s)}{s} = m_\lambda(x_1, \ldots, x_n).$$

Note that for all λ where $l(\lambda) > n$, $\Phi_n(D(\lambda)) = 0$.

Example 2.31 *Let* $\lambda = (21)$ *and* $n = 3$, *then*

$$\Phi_3(D(21)) = \frac{1}{x_1 x_2 x_3}(D_2 \vee D_1)(x_1 x_2 x_3)$$

$$= \frac{1}{x_1 x_2 x_3}(x_1^3 x_2^2 x_3 + x_1^3 x_2 x_3^2 + x_1^2 x_2^3 x_3 + x_1 x_2^3 x_3^2 + x_1^2 x_2 x_3^3 + x_1 x_2^2 x_3^3)$$
$$= x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + x_2^2 x_3 + x_1 x_3^2 + x_2 x_3^2 = m_{(21)}(x_1, x_2, x_3).$$

The following diagram commutes:

$$\begin{tikzcd}[column sep=small] & \mathcal{D} \arrow[dl, "\Phi_{n+1}"'] \arrow[dr, "\Phi_n"] & \\ \Lambda_{n+1} \arrow[rr, "\rho_{n+1,n}"'] & & \Lambda_n \end{tikzcd} .$$

The algebraic Thom map Φ is defined by: $\Phi = \lim\limits_{\leftarrow} \Phi_n$. Then $\Phi(D(\lambda)) = m_\lambda$ and hence Φ is a group isomorphism between $\mathcal{D}$ and Λ. The following results show correspondences between Λ and $\mathcal{D}$ under Φ [23, Example 3.1]:

$$\Phi(D_k) = P_k,\ \Phi(E_1^k) = e_k,\ \Phi(D(\lambda)) = m_\lambda,\ \Phi(D_1^{\circ k}) = h_k.$$

We may consider $\Phi_n = \rho_n \circ \Phi$.

3 Some general results

In this section, we will give some results which will be used in Section 4 to Section 7. Firstly we will prove several combinatorial identities in Section 3.1. In Section 3.2 and Section 3.3, we will give some results on the hit problem. These results will often be used in the remaining sections. We will give a minimal generating set of $\mathbb{Q}[x,y]$ under the action of $\mathcal{D}_{\mathbb{Q}}$. From that we will generalize Proposition 3.7 which is an important result for the rational case. By Proposition 3.7, we get Theorem 3.9 which states that a minimal generating set for a polynomial ring of n variables is finite for any n under the action of $\mathcal{D}_{\mathbb{Q}}$. Theorem 3.10 has been proved in [23, Example 3.4]. Here we give a different proof. In the remaining content, when we write x^k where x is a variable, if we do not define the exponent k, then $k \in \mathbb{N}$.

3.1 Combinatorial lemmas

The following two results are standard combinatorial results. We list them since they will be applied in many places in the remaining content.

Theorem 3.1 [5, Theorem 3.4.1] *Let p be a prime number and let $a_n \dots a_1 a_0$ and $b_n \dots b_1 b_0$ be $p-$expansions of a, b. Then*

$$\binom{a}{b} \equiv \prod_{i=0}^{n} \binom{a_i}{b_i} \ (mod\ p)$$

Theorem 3.2 [5, Example 3.13] *Let k, m, n be non-negative integers. Then*

$$\sum_{i=0}^{k} \binom{m}{i}\binom{n}{k-i} = \binom{m+n}{k}.$$

Lemma 3.3 *Let t, k be non-negative integers and let p be a prime number. Suppose $u+v+tk=p^m$ where u, $v \in \mathbb{N}$. Then*

$$\sum_{i+j=k}(u+ti)\binom{u}{i}\binom{v}{j} \equiv \sum_{i+j=k}(v+tj)\binom{u}{i}\binom{v}{j} \equiv 0 \ (mod\ p).$$

Proof:

$$\sum_{i+j=k}(u+ti)\binom{u}{i}\binom{v}{j} = \sum_{i+j=k}u\binom{u}{i}\binom{v}{j} + t\sum_{i+j=k}i\binom{u}{i}\binom{v}{j}$$

$$= u\sum_{i+j=k}\binom{u}{i}\binom{v}{j} + tu\sum_{\substack{i+j=k,\\ i\geq 1}}\binom{u-1}{i-1}\binom{v}{j} = u\left(\binom{u+v}{k} + t\binom{u+v-1}{k-1}\right)$$

$$= \frac{u(u+v+tk)}{k}\binom{u+v-1}{k-1}.$$

But $u+v+tk=p^m$ and $k<p^m$, so

$$\frac{u(u+v+tk)}{k}\binom{u+v-1}{k-1} \equiv 0 \ (mod\ p).$$

Also because

$$\sum_{i+j=k}(u+ti)\binom{u}{i}\binom{v}{j} + \sum_{i+j=k}(v+tj)\binom{u}{i}\binom{v}{j}$$

$$= \sum_{i+j=k}(u+v+tk)\binom{u}{i}\binom{v}{j} \equiv (u+v+tk)\binom{u+v}{k} \equiv 0 \ (mod\ p),$$

we get

$$\sum_{i+j=k}(v+tj)\binom{u}{i}\binom{v}{j} \equiv 0 \ (mod\ p).$$

□

Let $y=x^2+x$ and we get $x=\dfrac{-1\pm(1+4y)^{1/2}}{2}$. If we choose the plus sign in the solution and expand the right hand side of the equation, the coefficient of y^n is called the nth Catalan number c_n, where $c_n=\dfrac{1}{n+1}\binom{2n}{n}$ [23, Example

3.4]. The equation $x = \dfrac{-1+(1+4y)^{1/2}}{2}$ is the generating function for Catalan numbers, which gives $x = \sum_{n\geq 0}(-1)^n c_n y^{n+1}$.

Example 3.4 $c_0 = 1,\ c_1 = 1,\ c_2 = 2,\ c_3 = 5,\ c_4 = 14,\ c_5 = 42,\ c_6 = 132.$

For $a \in \mathbb{Q}$, let $\lfloor a \rfloor$ be the greatest integer $\leq a$ and let $\lceil a \rceil$ be the least integer $\geq a$.

Lemma 3.5 $\displaystyle\sum_{i=0}^{\lfloor\frac{n+1}{2}\rfloor}(-1)^{n-k}\binom{n+1-k}{k}c_{n-k} = 0.$

Proof: Expand the right hand side of the equation $x = \dfrac{-1+(1+4y)^{1/2}}{2}$, we get $x = \sum_{i\geq 0}(-1)^i c_i y^{i+1}$. Then we substitute y by $x^2 + x$ and get

$$x = \sum_{i\geq 0}(-1)^i c_i (x+1)^{i+1} x^{i+1}. \tag{8}$$

The coefficient of x^i is 0 for all $i > 1$ in the right hand side of (8). We look at the coeficient of x^{n+1} for $n \geq 1$ in (8). The term x^{n+1} is in the expansion of $(-1)^i c_i (x+1)^{i+1} x^{i+1}$ only if $i \leq n$ and $2(i+1) \geq n+1$ which is $i \geq \lceil\frac{n-1}{2}\rceil$. This gives the range of i : $\lceil\frac{n-1}{2}\rceil \leq i \leq n$, which should be considered. Suppose some i is in the range, then $\binom{i+1}{n-i}$ is the coefficient of x^{n-i} in the expansion of $(x+1)^{i+1}$ and so the coefficient of x^{n+1} in the expansion of $(-1)^i c_i (x+1)^{i+1} x^{i+1}$ is $(-1)^i c_i \binom{i+1}{n-i}$.

Hence the coefficient of x^{n+1} in (8) is $\displaystyle\sum_{i=\lceil\frac{n-1}{2}\rceil}^{n}(-1)^i\binom{i+1}{n-i}c_i$ which is 0. So we get

$$\sum_{i=\lceil\frac{n-1}{2}\rceil}^{n}(-1)^i\binom{i+1}{n-i}c_i = 0. \tag{9}$$

Let $k = n - i$. When n is even, we have $n - \lceil\frac{n-1}{2}\rceil = \frac{n}{2}$, when n is odd, we have $n - \lceil\frac{n-1}{2}\rceil = \frac{n+1}{2}$. Hence $n - \lceil\frac{n-1}{2}\rceil = \lfloor\frac{n+1}{2}\rfloor$. By changing the index of (9) to k, we get

$$\sum_{k=0}^{\lfloor\frac{n+1}{2}\rfloor}(-1)^{n-k}\binom{n+1-k}{k}c_{n-k}=0.$$

$\square$

3.2 Some general results on the hit problem over $\mathbb{Q}$

The hit problem of $\mathcal{D}_K$ on $K[x]$ for an arbitrary field K is simple. For any x^a, we have $x^a = D_{a-1}(x)$. Hence x is the only generator.

Theorem 3.6 *A minimal generating set of $\mathbb{Q}[x,y]$ under the action of $\mathcal{D}_{\mathbb{Q}}$ is finite. One minimal generating set is $\{1,\ x,\ y,\ xy,\ x^2y\}$.*

Proof: For any monomial $x^ay^b \in \mathbb{Q}[x,y]$, $a,\ b \neq 0$, by Theorem 2.9,

$$x^ay^b = x^aD_{b-1}(y) \equiv \chi(D_{b-1})(x^a)y = -D_{b-1}(x^a)y = -ax^{a+b-1}y \ (mod\ hit).$$

So we need only to consider the monomials in the form x^ay. We have the following equations:

$$\begin{cases} E_1^1(x^{a-1}y) = (a-1)x^ay + x^{a-1}y^2 \\ E_1^2(x^{a-2}y) = \binom{a-2}{2}x^ay + (a-2)x^{a-1}y^2 \end{cases}$$

$\dfrac{(a-2)(a+1)}{2}$ is the determinant of the coefficient matrix of the above equations. It is not zero if $a > 2$. Hence x^ay is hit for any $a > 2$. When $a = 2$, we have only one equation: $D_1(xy) = x^2y + xy^2$, with two unknowns x^2y and xy^2. So we need one of them to be a generator, we choose x^2y. We already know that x generates x^k and y generates y^k for all k, hence a minimal generating set for $\mathbb{Q}[x,y]$ is $\{1,\ x,\ y,\ xy,\ x^2y\}$. $\square$

We can use the above method to generalize to a useful result. When we try to show a monomial $x_1^{a_1}x_2^{a_2}\cdots x_m^{a_m}$ for some $m \in \mathbb{N}$ is hit by using E_r^k operations over $\mathbb{Q}$, we write the monomial as $x_1^a f$, where $a = a_1$ and $f = x_2^{a_2}\cdots x_m^{a_m}$. By the formula (4) in Section 2.1, we have the following equations:

$$\begin{cases} E_r^1(x_1^{a-r}f) = (a-r)x_1^a f + x_1^{a-r}E_r^1(f) \\ E_r^2(x_1^{a-2r}f) = \binom{a-2r}{2}x_1^a f + (a-2r)x_1^{a-r}E_r^1(f) + x_1^{a-2r}E_r^2(f) \\ \qquad\vdots \\ E_r^i(x_1^{a-ir}f) = \binom{a-ir}{i}x_1^a f + \binom{a-ir}{i-1}x_1^{a-r}E_r^1(f) \\ \\ +\ldots + (a-ir)x_1^{a-(i-1)r}E_r^{i-1}(f) + x_1^{a-ir}E_r^i(f) \\ \qquad\vdots \end{cases}$$

$$Eq(1)$$

If the degree of f is less than i, then $E_r^t(f) = 0$ for all $t \geq i$. Hence every entry in the tth column for all $t \geq i$ is 0 in the coefficient matrix of the equation system $Eq(1)$. We denote by $\Delta_i^r(a)$ the determinant of the leading $i \times i$ submatrix of the coefficient matrix of the equation system $Eq(1)$, where r indicates we use E_r^k operations. We have:

$$\Delta_i^r(a) = \begin{vmatrix} a-r & 1 & 0 & \cdots & 0 \\ \binom{a-2r}{2} & a-2r & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ \binom{a-ir}{i} & \cdots & \binom{a-ir}{2} & & a-ir \end{vmatrix}$$

Proposition 3.7

$$\Delta_i^r(a) = \binom{a+i-1}{i} - r\binom{a+i-1}{i-1}$$

$$= \frac{a-ri}{i}\binom{a+i-1}{i-1} = \frac{a-ri}{a}\binom{a+i-1}{i}.$$

Proof: We expand $\Delta_i^r(a)$ by the last row to get:

$$\Delta_i^r(a) = (a-ir)\Delta_{i-1}^r(a) - \binom{a-ir}{2}\Delta_{i-2}^r(a)$$

$$+\ldots+(-1)^{i-2}\binom{a-ir}{i-1}\Delta_1^r(a) + (-1)^{i-1}\binom{a-ir}{i}. \tag{10}$$

It is convenient to define $\Delta_0^r(a) = 1$ which satisfies the formula

$$\Delta_0^r(a) = \binom{a-1}{0} - r\binom{a-1}{-1} = 1.$$

We argue by induction on i. Suppose up to $i-1$ the formula holds. We rewrite (10) as follows:

$$\sum_{t=0}^{i}(-1)^t\binom{a-ir}{t}\Delta_{i-t}^r(a) = 0.$$

If for i the formula $\Delta_i^r(a) = \binom{a+i-1}{i} - r\binom{a+i-1}{i-1}$ still holds, then we have:

$$\sum_{t=0}^{i}(-1)^t\binom{a-ir}{t}\binom{a+i-t-1}{i-t} = r\sum_{t=0}^{i-1}(-1)^t\binom{a-ir}{t}\binom{a+i-t-1}{i-t-1} \tag{11}$$

We now show that the equation (11) holds. The left hand side of (11) is the coefficient of x^i in the product of polynomials:

$$\sum_{t=0}^{i}(-1)^t\binom{a-ir}{t}x^t \cdot \sum_{t=0}^{i}\binom{a+i-t-1}{i-t}x^{i-t}$$

$$= \sum_{t=0}^{i}(-1)^t\binom{a-ir}{t}x^t \cdot \sum_{s=0}^{i}\binom{a+s-1}{s}x^s,$$

where $s = i - t$. This agrees with the expansion of the product of power series: $(1-x)^{a-ir} \cdot (1-x)^{-a} = (1-x)^{-ir}$ up to the term x^i. The coefficient of x^i is $\binom{(r+1)i-1}{i}$ in $(1-x)^{-ir}$. Hence

$$\sum_{t=0}^{i}(-1)^t\binom{a-ir}{t}\binom{a+i-t-1}{i-t} = \binom{(r+1)i-1}{i}.$$

The right hand side of (11) is the coefficient of x^i in the product of polynomials:

$$\begin{aligned} & r\sum_{t=0}^{i}(-1)^t\binom{a-ir}{t}x^t \cdot \sum_{t=0}^{i-1}\binom{a+i-t-1}{i-t-1}x^{i-t} \\ &= r\sum_{t=0}^{i}(-1)^t\binom{a-ir}{t}x^t \cdot x\sum_{s=0}^{i}\binom{a+s}{s}x^s, \end{aligned}$$

where $s = i - t - 1$. This agrees with the expansion of product of power series: $r(1-x)^{a-ir} \cdot x(1-x)^{-a-1} = rx(1-x)^{-ir-1}$ up to the term of x^i. The coefficient of x^{i-1} in $r(1-x)^{-ir-1}$ is $r\binom{i(r+1)-1}{i-1}$. This is the coefficient of x^i in the expansion of $rx(1-x)^{-ir-1}$. But

$$r\binom{i(r+1)-1}{i-1} = \frac{ir}{i}\binom{i(r+1)-1}{i-1} = \binom{i(r+1)-1}{i}$$

since

$$\frac{ir}{i}\binom{i(r+1)-1}{i-1} = \frac{ir}{i}\frac{(i(r+1)-1)\cdots(ir+1)}{(i-1)!} = \binom{i(r+1)-1}{i}.$$

Hence the equation (11) holds for i. This is equivalent to say that the formula $\Delta_i^r(a) = \binom{a+i-1}{i} - r\binom{a+i-1}{i-1}$ holds for i. By the induction hypothesis, the formula holds for all $n \in \mathbb{N}$. Finally

$$\binom{a+i-1}{i} - r\binom{a+i-1}{i-1} = (\frac{a}{i} - r)\binom{a+i-1}{i-1} = \frac{a-ri}{i}\binom{a+i-1}{i-1},$$

and

$$\binom{a+i-1}{i} - r\binom{a+i-1}{i-1} = (1 - r\frac{i}{a})\binom{a+i-1}{i} = \frac{a-ri}{a}\binom{a+i-1}{i}.$$

□

Corollary 3.8 *A monomial $x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}$ is hit under the action of $\mathcal{D}$ over $\mathbb{Q}$, if some $a_i > 1 + \sum_{1\le j\le n,\ j\ne i} a_j$.*

Proof: Let $k = 1 + \sum_{1\le j\le n,\ j\ne i} a_j$ and let $f = x_1^{a_1}\cdots x_{i-1}^{a_{i-1}}x_{i+1}^{a_{i+1}}\cdots x_n^{a_n}$, then $E_1^k(f) = 0$. Let $x_i^{a_i}f,\ x_i^{a_i-1}E_1^1(f),\ \ldots\ ,\ x_i^{a_i-k+1}E_1^{k-1}(f)$ be k unknowns. By Proposition 3.7, we have

$$\Delta_k^1(a_i) = \frac{a_i-k}{k}\binom{a_i+k-1}{k-1} \ne 0,$$

since $k < a_i$. Suppose we have the equations shown in $Eq(1)$ with $r = 1$, $i = k$ and $a = a_i$. Then because $\Delta_k^1(a_i) \ne 0$, we can solve those equations and show each of the above k unknowns is $\equiv 0$ $(mod\ hit)$, especially $x_i^{a_i}f \equiv 0$ $(mod\ hit)$. □

We have shown that a minimal generating set is finite for the 2 variable case. We now generalize this result:

Theorem 3.9 [21] *A minimal generating set of S_n under the action of $\mathcal{D}_{\mathbb{Q}}$ is finite for all n.*

Proof: If we can show for some d large enough, every monomial in degrees $\ge d$ in S_n is hit, then a minimal generating set for S_n is finite. We argue by induction on n. Suppose up to $n-1$, a minimal generating set of S_{n-1} is finite. Let $\{g_1,\ g_2,\ \ldots,\ g_k\}$ be a minimal generating set of S_{n-1} and let d be the highest degree of g_j for $1 \le j \le k$. Let $f = x_1^{a_1}\ \ldots\ x_{n-1}^{a_{n-1}}x_n^{a_n}$ be a monomial in S_n where $deg(f) > 2d+1$ and let $g = x_1^{a_1}\ \ldots\ x_{n-1}^{a_{n-1}}$. By the assumption, $g = \sum_{j=1}^{k}\delta_j(g_j)$ for some $\delta_j \in \mathcal{D}_{\mathbb{Q}}$. Then

$$f = x_n^{a_n}g = x_n^{a_n}\sum_{j=1}^{k}\delta_j(g_j) \equiv \sum_{j=1}^{k}(\chi(\delta_j)(x_n^{a_n}))g_j = \sum_{j=1}^{k} r_j x_n^{b_j} g_j\ (mod\ hit),$$

where $r_j x_n^{b_j} = \chi(\delta_j)(x_n^{a_n})$ for $r_j \in \mathbb{Q}$. Each term $r_j x_n^{b_j} g_j$ is in the same degree as the degree of f. Since $deg(f) > 2d+1$, every $b_j > deg(g_j)+1$, so each term $r_j x_n^{b_j} g_j$ is hit by Corollary 3.8. Hence f is hit and hence the minimal generating set for S_n is finite. By the induction hypothesis, the minimal generating set for S_n is finite for all n. $\square$

Lemma 3.10 [21] *$\chi(E_1^k)(x) = (-1)^k c_k x^{k+1}$ where c_k is the $k-$th Catalan number.*

Proof: We will prove the theorem by induction. We can easily calculate $\chi(D_1)(x) = -D_1(x) = -x^2$. Assume the result is true for all $k \le n-1$. Since $-\chi(E_1^n)(x) = \sum_{i=1}^{n} E_1^i \chi(E_1^{n-i})(x)$ and $\chi(E_1^{n-i})(x) = (-1)^{n-i} c_{n-i} x^{n-i+1}$, we get

$$\chi(E_1^n)(x) = \sum_{i=1}^{n} (-1)^{n-i+1} c_{n-i} E_1^i (x^{n-i+1}) = \sum_{i=1}^{n} (-1)^{n-i+1} c_{n-i} \binom{n-i+1}{i} x^{n+1}.$$

Note that $\binom{n-i+1}{i} = 0$ if $i > \lfloor \frac{n+1}{2} \rfloor$. So we can write the equation as follows

$$\chi E_1^n(x) = \sum_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} (-1)^{n-i+1} \binom{n-i+1}{i} c_{n-i} x^{n+1} = (-1)^n c_n x^{n+1}, \tag{12}$$

by Lemma 3.5. Hence we get the equation $\chi(E_1^n)(x) = (-1)^n c_n x^{n+1}$. By the induction hypothesis, the result holds for all $k \in \mathbb{N}$. $\square$

The following two results work over an arbitrary field.

Lemma 3.11 *$x_1^{a_1} x_2^{a_2} \dots x_n^{a_n}$ is hit if $a_1, a_2, \dots, a_n$ have a common factor $k > 1$.*

Proof: Let $d = a_1 + a_2 + \dots + a_n$. Suppose k is a common factor of $a_1,\ a_2, \dots,\ a_n$, then

$$E_{k-1}^{d/k}(x_1^{a_1/k}x_2^{a_2/k}\dots x_n^{a_n/k}) = E_{k-1}^{a_1/k}(x_1^{a_1/k})E_{k-1}^{a_2/k}(x_2^{a_2/k})\dots E_{k-1}^{a_n/k}(x_n^{a_n/k})$$

$$= x_1^{a_1}x_2^{a_2}\dots x_n^{a_n}.$$

□

Theorem 3.12 *Let f be a monomial in $K[x_1, x_2, \dots, x_n]$ where K is an arbitrary field. If f is not hit under the action of $\mathcal{D}$ over K then $x_{n+1}f$ is not hit in $K[x_1, x_2, \dots, x_{n+1}]$.*

Proof: By Theorem 2.5 and Theorem 2.7, $\{E_1^k,\ E_2^k \mid k \in \mathbb{N}\}$ form a generating set of $\mathcal{D}$ under the composition over any field, since a field either contains $\mathbb{F}_p$ or contains $\mathbb{Q}$. Suppose $x_{n+1}f$ is hit, then

$$x_{n+1}f = \sum_i \sum_j (r(i,j)E_1^i(g_{i,j}) + w(i,j)E_2^i(h_{i,j})), \tag{13}$$

where $g_{i,j},\ h_{i,j}$ are monomials $\in K[x_1, x_2, \dots, x_{n+1}]$ and $r(i,j),\ w(i,j) \in K$. Let $g_{i,j} = x_{n+1}^{a(i,j)}u_{i,j}$ and $h_{i,j} = x_{n+1}^{b(i,j)}v_{i,j}$, where $u_{i,j},\ v_{i,j}$ are monomials in $K[x_1, x_2, \dots, x_n]$. Then by (4) in Section 2.1, we have

$$E_1^i(g_{i,j}) = x_{n+1}^{a(i,j)}E_1^i(u_{i,j}) + \sum_{s+t=i,\ s\geq 1} E_1^s(x_{n+1}^{a(i,j)})E_1^t(u_{i,j}).$$

Similarly

$$E_2^i(h_{i,j}) = x_{n+1}^{b(i,j)}E_2^i(v_{i,j}) + \sum_{s+t=i,\ s\geq 1} E_2^s(x_{n+1}^{b(i,j)})E_2^t(v_{i,j}).$$

Some of $a(i,j)$ or $b(i,j)$ are equal to 1, since we must get $x_{n+1}f$ after cancellation in the right hand side of (13). We pick up all the terms with the exponent of x_{n+1} to be 1 and rewrite (13) as follows:

$$x_{n+1}f = x_{n+1} \sum_{i\geq 1,\ j} (r(i,j)E_1^i(u_{i,j}) + w(i,j)E_2^i(v_{i,j})) + S,$$

where S is a sum in which every term has the exponent of $x_{n+1} \geq 2$ and the summation takes on all $i \geq 1$ and j where $x_{n+1}E_1^i(u_{i,j})$ is in $\sum_i \sum_j r(i,j)E_1^i(g_{i,j})$ and $x_{n+1}E_2^i(v_{i,j})$ is in $\sum_i \sum_j w(i,j)E_2^i(h_{i,j})$ in (13).

Since S has to be cancelled out itself, we get

$$f = \sum_{i \geq 1, j} (r(i,j)E_1^i(u_{i,j}) + w(i,j)E_2^i(v_{i,j})).$$

But f is not hit. So the assumption that $x_{n+1}f$ is hit leads to a contradiction.

□

3.3 Some general results on the hit problem over $\mathbb{F}_p$

Lemma 3.13 *A monomial $x_1^{a_1}x_2^{a_2}\dots x_n^{a_n}$ is hit under the action of $\mathcal{D}_p$, if $n-1$ exponents are divisible by p and all $a_i > 1$ for $1 \leq i \leq n$.*

Proof: If all exponents of $f = x_1^{a_1}x_2^{a_2}\dots x_n^{a_n}$ are divisible by p, then f is hit by Lemma 3.11. Suppose a_i is not divisible by p, and all a_j's are divisible by p for $j \neq i$. Since $a_i > 1$, $D_{a_i-1}(x_i g) = x_i^{a_i}g + x_i D_{a_i-1}(g) \equiv f \ (mod\ p)$ where $g = f/x_i^{a_i}$, since $D_{a_i-1}(g) \equiv 0 \ (mod\ p)$. □

Lemma 3.14 *A monomial $f = x_1^{a_1}x_2^{a_2}\dots x_n^{a_n}$ is hit over $\mathbb{F}_p$, if the degree of $f \geq np+1$ and $n-1$ exponents of f are divisible by p with some $a_i = 1$.*

Proof: Suppose $a_i = 1$, and all a_j's are divisible by p for $j \neq i$. There exists an a_j with $p|a_j$ and $a_j \geq 2p$ for $1 \leq j \neq i \leq n$, since the degree of $f \geq np+1$ and every a_j is divisible by p for $j \neq i$. Let $t = a_j - p + 1$. Then we have

$$D_{p-1}(x_1^{a_1}x_2^{a_2}\dots x_i \dots x_j^t \dots x_n^{a_n}) \equiv tf + x_1^{a_1}x_2^{a_2}\dots x_i^p \dots x_j^t \dots x_n^{a_n} \ (mod\ p).$$

In the second term, all exponents are divisible by p except t. But now $t > 1$, hence $x_1^{a_1}x_2^{a_2}\dots x_i^p\dots x_j^t\dots x_n^{a_n}$ is hit by Lemma 3.13. So tf is hit and hence f is hit too, since $t \equiv 1\ (mod\ p)$. $\square$

The following result is well known in the hit problem of the Steenrod algebra. We also give a proof of it.

Lemma 3.15 *Let a be a non−negative integer. Suppose there are k 1's in the binary expansion of a. Then $x^a \equiv D(x^{2^k-1})\ (mod\ 2)$ where D is a composition of some E_1^i's for $i \geq 0$.*

Proof: If $a = 2^k - 1$, then $D = 1$. Suppose $a \neq 2^k - 1$ for any $k \geq 1$ and there are k 1's in the binary expansion of a. Let $a_n a_{n-1}\dots a_0$ be the binary expansion of a. We can write $a = 2^{d_1} + 2^{d_2} + \dots + 2^{d_k}$, where $n = d_1 > d_2 > \dots > d_k \geq 0$ and each 2^{d_j} corresponds to that $a_{d_j} = 1$. Observe that $2^k - 1$ has the same number of 1's in its binary expansion as the binary expansion of a. Starting with $E_1^{2^{k-1}}(x^{2^k-1}) \equiv x^{2^k+2^{k-1}-1}\ (mod\ 2)$, we can use a sequence of E_1^i on x^{2^k-1} where each i is twice of the previous one until $i = d_1 - 1$. Then we get

$$E_1^{2^{d_1-1}}\cdots E_1^{2^k}E_1^{2^{k-1}}(x^{2^k-1}) \equiv x^{2^{d_1}+2^{k-1}-1}\ (mod\ 2).$$

Let $\delta_1 = E_1^{2^{d_1-1}}\cdots E_1^{2^k}E_1^{2^{k-1}}$. Then δ_1 moves the 1st 1 counted from the left of the binary expansion of 2^k-1 to the d_1th position. Let $\delta_2 = E_1^{2^{d_2-1}}\cdots E_1^{2^{k-1}}E_1^{2^{k-2}}$. Then $\delta_2(x^{2^{d_1}+2^{k-1}-1}) \equiv x^{2^{d_1}+2^{d_2}+2^{k-2}-1}\ (mod\ 2)$ which moves the 2nd 1 counted from the left of the binary expansion of $2^k - 1$ to the d_2th position, ..., finally let $\delta_k = E_1^{2^{d_k-1}}\cdots E_1^2E_1$ which moves the kth 1 counted from the left of the binary expansion of $2^k - 1$ to the d_kth position. Note that if a is odd, then the last 1 in the binary expansion of a counted from left will not move. Then E_1^k where k is odd is not in any of δ_j's mensioned above for all j. Let $D = \delta_k\cdots\delta_2\delta_1$, then $x^a \equiv D(x^{2^k-1})\ (mod\ 2)$. $\square$

4 The hit problem on polynomial rings of 3 and 4 variables over $\mathbb{Q}$

In this section, we will give some results on the hit problem over $\mathbb{Q}$. We will find a minimal generating set for the action of $\mathcal{D}_{\mathbb{Q}}$ on polynomial ring of 3 variables in Section 4.1 and find a minimal generating set for the action of $\mathcal{D}_{\mathbb{Q}}$ on the polynomial ring of 4 variables in Section 4.2 to Section 4.4. The results we have got verify Conjecture 1.3 up to $n = 4$. We set $x = x_1$, $y = x_2$, $z = x_3$, $t = x_4$. Then Σ_4 acts on the set $\{x, y, z, t\}$ in the same way as it acts on the set $\{x_1, x_2, x_3, x_4\}$.

4.1 The hit problem on a polynomial ring of 3 variables over $\mathbb{Q}$

Theorem 4.1 *A minimal generating set of S_3 under the action of $\mathcal{D}_{\mathbb{Q}}$ is*

$$\{xyz,\ x^2yz,\ xy^2z,\ x^3yz,\ x^2y^2z,\ x^3y^2z\}.$$

Proof: By Theorem 3.6, $yz,\ y^2z$ generate all monomials of S_2 with variables $y,\ z$ under the action of $\mathcal{D}_{\mathbb{Q}}$. By Theorem 2.9, we have:

$$x^a y^b z^c = x^a(\delta_1(yz) + \delta_2(y^2z)) \equiv (\chi(\delta_1)(x^a))yz + (\chi(\delta_2)(x^a))y^2z\ (mod\ hit),$$

for some $\delta_1,\ \delta_2 \in \mathcal{D}^+_{\mathbb{Q}}$. $(\chi(\delta_1))(x^a)$ and $(\chi(\delta_2))(x^a)$ are monomials in x since δ_1, δ_2 are homogeneous elements of $\mathcal{D}$. We get: $x^a y^b z^c \equiv r_1 x^k yz + r_2 x^{k-1} y^2 z\ (mod\ hit)$ where $r_1,\ r_2 \in \mathbb{Q}$ and $k = a + b + c - 2$. So it is sufficient to consider whether monomials of forms $x^a yz$ and $x^a y^2 z$ are hit or not in each degree.

By Corollary 3.8, the monomial $x^a yz$ is hit if $a > 3$ and the monomial $x^a y^2 z$ is hit if $a > 4$. So all monomials in degrees ≥ 8 are hit.

In degree 7, monomials in the form [511] are hit by Corollary 3.8. We have

$$E_1^1(x^3y^2z) = 3x^4y^2z + 2x^3y^3z + x^3y^2z^2.$$

By Theorem 2.9,

$$x^3y^3z = E_2^2(xy)z \equiv xy(\chi(E_2^2)(z)) = 3xyz^5 \ (mod\ hit),$$

since $\chi(E_2^2)(z) = (D_2D_2 - E_2^2)(z) = 3z^5$. Now we have

$$x^3y^2z^2 = x^3E_1^2(yz) \equiv (\chi(E_1^2)(x^3))yz = 9x^5yz \ (mod\ hit),$$

since $\chi(E_1^2)(x^3) = (D_1D_1 - E_1^2)(x^3) = 12x^5 - 3x^5 = 9x^5$. Hence x^3y^3z, $x^3y^2z^2$ are hit and hence x^4y^2z is hit too. So all monomials in degree 7 are hit.

Now we check degrees ≤ 6. Obviously xyz is not hit. In degree 4, we have only one equation: $D_1(xyz) = x^2yz + xy^2z + xyz^2$ with three unknowns x^2yz, xy^2z and xyz^2. So we need any two generators. We choose x^2yz and xy^2z. Recall that $\mathcal{D}_{\mathbb{Q}}$ is generated under the composition by D_1, D_2 (Theorem 2.5), so it is enough to just check operations D_1 and D_2. In degree 5, we have four equations involving D_1 and D_2:

$$\begin{cases} D_1(x^2yz) = 2x^3yz + x^2y^2z + x^2yz^2 \\ D_1(xy^2z) = x^2y^2z + 2xy^3z + xy^2z^2 \\ D_1(xyz^2) = x^2yz^2 + xy^2z^2 + 2xyz^3 \\ D_2(xyz) = x^3yz + xy^3z + xyz^3 \end{cases}$$

We rewrite the above equations as follows:

$$\begin{cases} 2x^3yz + x^2y^2z + x^2yz^2 \equiv 0 \\ x^2y^2z + 2xy^3z + xy^2z^2 \equiv 0 \\ x^2yz^2 + xy^2z^2 + 2xyz^3 \equiv 0 \\ x^3yz + xy^3z + xyz^3 \equiv 0 \end{cases} \quad (mod\ hit)$$

The coefficient matrix is as follows:

$$\begin{pmatrix} x^3yz & xy^3z & xyz^3 & x^2y^2z & x^2yz^2 & xy^2z^2 \\ 2 & 0 & 0 & 1 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

The matrix has rank 4. There are 6 unknowns, so we need 2 generators. We choose x^3yz, x^2y^2z. After checking in the equation system, we see they are generating elements in this degree.

Finally in degree 6, the monomials in the form [222] are hit by Lemma 3.11, the monomials in the form [411] are hit by Corollary 3.8. From Example 2.30, we know monomials in the form [321] span $Sp^{(111)}$ which is the first occurrence of this irreducible module in S_3. Hence monomials in the form [321] are not hit by Schur's lemma. We have:

$$D_1(x^3yz) = 3x^4yz + x^3y^2z + x^3yz^2 \equiv x^3y^2z + x^3yz^2 \ (mod\ hit),$$

$$D_2(xy^2z) = 2xy^4z + x^3y^2z + xy^2z^3 \equiv x^3y^2z + xy^2z^3 \ (mod\ hit).$$

The first equation gives $x^3y^2z \equiv -(23)x^3y^2z$ $(mod\ hit)$ and the second equation gives $x^3y^2z \equiv -(13)x^3y^2z$ $(mod\ hit)$ for $(13), (23) \in \Sigma_3$. Since (13), (23) generate Σ_3, any monomial πx^3y^2z for $\pi \in \Sigma_3$ is congruent to $(mod\ hit)$ x^3y^2z up to sign. Hence x^3y^2z generate all monomials in the form [321]. So the minimal generating set of S_3 under the action of $\mathcal{D}_{\mathbb{Q}}$ is:

$$\{xyz, x^2yz, xy^2z, x^3yz, x^2y^2z, x^3y^2z\}.$$

□

4.2 The hit problem on a polynomial ring of 4 variables over $\mathbb{Q}$

In this section, we will prove the following theorem:

Theorem 4.2 *A minimal generating set of S_4 under the action of $\mathcal{D}_{\mathbb{Q}}$ is*

$$\{x^i y^j z^k t \mid 1 \le i \le 4,\ 1 \le j \le 3,\ 1 \le k \le 2\}.$$

By the result of Theorem 4.1 and a similar argument as we used in the proof of Theorem 4.1, for an arbitrary monomial in S_4, we have

$$x^a y^b z^c t^d \equiv r_1 x^k yzt + r_2 x^{k-1} y^2 zt + r_3 x^{k-1} yz^2 t + r_4 x^{k-2} y^3 zt$$

$$+ r_5 x^{k-2} y^2 z^2 t + r_6 x^{k-3} y^3 z^2 t \ (mod\ hit),$$

where $r_i \in \mathbb{Q}$ and $k = a + b + c + d - 3$. Hence we only need to check monomials of the forms: $[a111]$, $[a211]$, $[a221]$, $[a311]$, $[a321]$.

Lemma 4.3

$$x^{ia} y^{ja} z^a t^b \equiv r_1 x^i y^j z t^{b+(i+j+1)(a-1)} \ (mod\ hit),$$

$$x^a y^a z^b t^c \equiv r_2 xyF \ (mod\ hit),$$

where $a,\ b,\ c,\ k \in \mathbb{N}$, $r_1,\ r_2 \in \mathbb{Q}$ and F is a polynomial in the variables z and t.

Proof: By Theorem 2.9, we get:

$$x^{ia} y^{ja} z^a t^b = E_{a-1}^{i+j+1}(x^i y^j z) t^b \equiv x^i y^j z (\chi(E_{a-1}^{i+j+1})(t^b))$$

$$\equiv r_1 x^i y^j z t^{b+(i+j+1)(a-1)} \ (mod\ hit),$$

$$x^a y^a z^b t^c = (E_{a-1}^2(xy)) z^b t^c \equiv xy(\chi(E_{a-1}^2)(z^b t^c)) \ (mod\ hit),$$

where $\chi(E_{a-1}^2)(z^b t^c)$ is a polynomial in z and t. □

4.3 Monomials in degrees ≥ 11

Proposition 4.4 *Every monomial of S_4 in degrees ≥ 11 is hit under the action of $\mathcal{D}_{\mathbb{Q}}$.*

Proof: By Corollary 3.8, monomials in the form $[a111]$ are hit if $a > 4$, monomials in the form $[a211]$ are hit if $a > 5$, monomials of the forms $[a311]$ and $[a221]$ are hit if $a > 6$ and monomials in the form $[a321]$ are hit if $a > 7$. So all monomials in degrees > 13 are hit.

(1) Degree 13:

In degree 13, there are monomials of the following forms: $[10, 111]$, $[9211]$, $[8311]$, $[8221]$, $[7411]$, $[7321]$, $[7222]$, $[6511]$, $[6421]$, $[6331]$, $[6322]$, $[5521]]$, $[5431]$, $[5422]$, $[5332]$, $[4441]$, $[4432]$, $[4333]$.

Monomials in the forms [10,111], [9211], [8311], [8221] are hit by Corollary 3.8. By Lemma 4.3, we get:

$$x^7y^2z^2t^2 \equiv r_1x^{10}yzt \ (mod\ hit), \quad x^4y^3z^3t^3 \equiv r_2x^{10}yzt \ (mod\ hit),$$

$$x^4y^4z^4t \equiv r_3xyzt^{10} \ (mod\ hit), \quad x^6y^3z^3t \equiv r_4x^2yzt^9 \ (mod\ hit),$$

$$x^6y^3z^2t^2 \equiv r_5x^3y^8zt \ (mod\ hit), \quad x^5y^4z^2t^2 \equiv r_6x^9y^2zt \ (mod\ hit),$$

$$x^4y^4z^3t^2 \equiv r_7x^2y^2z^8t \ (mod\ hit), \quad x^7y^4zt \equiv r_8xy^8z^2t^2 \ (mod\ hit),$$

for $r_i \in \mathbb{Q}$.

Hence monomials in the forms [7222], [4333], [4441], [6331], [6322], [5422], [4432] and [7411] are hit as well.

We need to show monomials in the forms [7321], [6511], [6421], [5521], [5431], [5332] are hit. Let c_i be the ith Catalan number. By Theorem 2.9 and Lemma

3.10, we have

$$c_4x^6y^5zt = (\chi(E_1^4)(y))x^6zt \equiv yE_1^4(x^6zt) \ (mod\ hit)$$
$$= \binom{6}{4}x^{10}yzt + \binom{6}{3}x^9y(z^2t+zt^2) + \binom{6}{2}x^8yz^2t^2.$$

The 3 terms are hit so x^6y^5zt is hit. Hence *all monomials which have two exponents 1 are hit.* By Lemma 4.3, we get that *all monomials with two equal exponents are hit.* By Theorem 2.9 and Lemma 3.10,

$$-c_5x^6y^4z^2t = (\chi(E_1^5)(x))y^4z^2t \equiv xE_1^5(y^4z^2t) \equiv 8xy^7z^3t^2 + 6xy^6z^4t^2 \ (mod\ hit),$$

$$c_6xy^7z^3t^2 = (\chi(E_1^6)(y))xz^3t^2 \equiv yE_1^6(xz^3t^2) = x^2yz^6t^4 \ (mod\ hit), \qquad (14)$$

we get

$$-c_5c_6x^6y^4z^2t \equiv 8x^2yz^6t^4 + 6c_6xy^6z^4t^2 \ (mod\ hit). \qquad (15)$$

Since a monomial with two equal exponents is hit, we get:

$$D_4(x^2yz^2t^4) \equiv 2x^6yz^2t^4 + 2x^2yz^6t^4 \ (mod\ hit),$$

$$D_3(x^6yz^2t) \equiv x^6y^4z^2t + x^6yz^2t^4 \ (mod\ hit).$$

Hence

$$x^2yz^6t^4 \equiv -x^6yz^2t^4 \equiv x^6y^4z^2t \ (mod\ hit). \qquad (16)$$

Also

$$D_5(xyz^4t^2) \equiv x^6yz^4t^2 + xy^6z^4t^2 \ (mod\ hit),$$

$$D_3(x^6yzt^2) \equiv x^6y^4zt^2 + x^6yz^4t^2 \ (mod\ hit),$$

$$D_1(x^6y^4zt) \equiv x^6y^4zt^2 + x^6y^4z^2t \ (mod\ hit).$$

Hence

$$xy^6z^4t^2 \equiv -x^6yz^4t^2 \equiv x^6y^4zt^2 \equiv -x^6y^4z^2t \ (mod\ hit). \qquad (17)$$

By (15), (16), (17) and Example 3.4, we get

$$(c_5c_6 - 6c_6 + 8)x^6y^4z^2t = 4760x^6y^4z^2t \equiv 0 \ (mod\ hit).$$

So $x^6y^4z^2t$ is hit. By (14), monomials in the form [7321] are hit as well.

Finally by Theorem 2.9, we get:

$$2x^5y^4z^3t = (D_2(y^2))x^5z^3t \equiv y^2(\chi(D_2)(x^5z^3t)) \ (mod\ hit)$$

$$= -y^2D_2(x^5z^3t) = -5x^7y^2z^3t - 3x^5y^2z^5t - x^5y^2z^3t^3.$$

The last three monomials in the above equation are hit by the previous argument, so $x^5y^4z^3t$ is hit. Hence every monomial in degree 13 is hit.

(2) Degree 12:

In degree 12, there are monomials in the following forms: [9111], [8211], [7311], [7221], [6411], [6321], [6222], [5511], [5421], [5331], [5322], [4431], [4422], [4332], [3333]. Monomials in the forms [9111], [8211], [7311], [7221] are hit by Corollary 3.8 and monomials in the forms [6222], [4422], [3333] are hit by Lemma 3.11.

$$-c_5x^6y^4zt = (\chi(E_1^5)(x))y^4zt \equiv xE_1^5(y^4zt) = xy^8(z^2t + zt^2) + 4xy^7z^2t^2 \ (mod\ hit),$$

the last three monomials in the above equation are hit so x^6y^4zt is hit. Hence monomials in the form [6411] are hit.

By Theorem 2.9 and Lemma 3.10,

$$c_4x^5y^5zt = (\chi(E_1^4)(x))y^5zt \equiv xE_1^4(y^5zt) \ (mod\ hit)$$

$$= 5xy^9zt + 10xy^8(z^2t + zt^2) + 10xy^7(z^2t^2),$$

the last three monomials in the above equation are hit so x^5y^5zt is hit. Hence monomials in the form [5511] are hit. Hence *all monomials which have two* *exponents 1 are hit.* By Lemma 4.3, *monomials with two equal exponents are hit as well.* So monomials in the forms [5331], [5322], [4431], [4332] are hit. Hence

$$D_1(xy^5z^3t^2) \equiv 5xy^6z^3t^2 + 3xy^5z^4t^2 \ (mod\ hit), \tag{18}$$

$$-c_5x^6y^3z^2t = ((\chi(E_1^5))(x))y^3z^2t \equiv xE_1^5(y^3z^2t) \equiv 2xy^6z^3t^2 + 3xy^5z^4t^2 \ (mod\ hit),$$

we get by subtraction $c_5x^6y^3z^2t \equiv 3xy^6z^3t^2$ $(mod\ hit)$.

Note that in the above equation, the monomial in the right hand side is the monomial of the left hand side with all exponents shifted cyclically one position to the right. By symmetry, $c_5xy^6z^3t^2 \equiv 3x^2yz^6t^3$ $(mod\ hit)$. Hence

$$c_5{}^2x^6y^3z^2t \equiv 3c_5xy^6z^3t^2 \equiv 9x^2yz^6t^3 \ (mod\ hit).$$

For the same reason, we get: $c_5{}^2x^2yz^6t^3 \equiv 9x^6y^3z^2t$. Hence

$$(c_5{}^4 - 81)x^6y^3z^2t \equiv 0 \ (mod\ hit).$$

Since $c_5 = 42$, the coefficient of $x^6y^3z^2t$ is obviously not 0, $x^6y^3z^2t$ is hit. From (18) $xy^5z^4t^2$ is hit. Hence all monomials in degree 12 are hit.

(3) Degree 11:

In degree 11, there are monomials in the following forms: [8111], [7211], [6311], [6221], [5411], [5321], [5222], [4421], [4331], [4322], [3332]. Monomials in the forms [8111], [7211] are hit by Corollary 3.8. By Lemma 4.3, we get

$$x^5y^2z^2t^2 \equiv r_1x^8yzt \ (mod\ hit) \quad x^3y^3z^3t^2 \equiv r_2xyzt^7 \ (mod\ hit)$$

for some $r_1,\ r_2 \in \mathbb{Q}$. So monomials in the forms [5222], [3332] are hit as well . Hence by Theorem 2.9 and Lemma 3.10, we have:

$$-c_5x^6y^3zt = (\chi(E_1^5)(x))y^3zt \equiv xE_1^5(y^3zt) = xy^6z^2t^2 = xy^6E_1^2(zt)$$

$$\equiv (\chi(E_1^2)(xy^6))zt = ((D_1D_1 - E_1^2)(xy^6))zt \equiv 2x^3y^6zt \ (mod\ hit), \qquad (19)$$

since the other terms in $((D_1D_1 - E_1^2)(xy^6))zt$ are x^2y^7zt and xy^8zt which are hit.

Note that in the above equation, the monomial of the right hand side is the monomial of the left hand side with exponents 6 and 3 interchanged. By repeating the procedure once more on x^3y^6zt, we get:

$$(c_5^2 - 4)x^6y^3zt = 1760x^6y^3zt \equiv 0 \ (mod\ hit).$$

Hence x^6y^3zt is hit and hence monomials in the form [6311] are hit. By (19), monomials in the form of [6221] are hit too. By Theorem 2.9 and Lemma 3.10,

$$c_4x^5y^4zt = (\chi(E_1^4)(x))y^4zt \equiv xE_1^4(y^4zt)$$

$$= xy^8zt + 4xy^7(z^2t + zt^2) + 6xy^6z^2t^2 \equiv 0 \ (mod\ hit),$$

the last four terms are hit so x^5y^4z is hit. Hence monomials in the form [5411] are hit. Since *all monomials in this degree which have two exponents 1 are hit*, by Lemma 4.3, monomials in the forms [4421], [4331], [4322] are hit as well. Hence

$$D_1(x^4y^3z^2t) = 4x^5y^3z^2t + 3x^4y^4z^2t + 2x^4y^3z^3t + x^4y^3z^2t^2 \equiv 4x^5y^3z^2t \ (mod\ hit).$$

Now we have shown every monomial whose total degree ≥ 11 is hit. □

4.4 Monomials in degrees ≤ 10

Let H_4 be the set of hit elements and let $C_4 = S_4/H_4$ be the cokernel. By the discussion of section 1, the set of representatives of a basis of C_4 is a minimal generating set of S_4 as an $\mathcal{D}_{\mathbb{Q}}$–module. Suppose, for $i \in \mathbb{N}$, $\{f_i\}$ is a finite set of monomials which span a $\mathbb{Q}$–subspace F of S_4. If the set of f_i and hit elements span S_4, i.e. $F + H_4 = S_4$, then $F/(F \cap H_4) \approx C_4$. Then $\{f_i\}$ contains a set of representatives of a basis of C_4. Hence $\{f_i + H\}$ is a $\mathbb{Q}$–spanning set of C_4. Equivalently we may also say $\{f_i\}$ *(mod hit)* is a $\mathbb{Q}$–spanning set of C_4. We grade the cokernel $C_4 = C_4^d$ by $C_4^d = \{\ f + H \in C_4 |\ deg(f) = d\ \}$.

Proof of Theorem 4.2:

Now we consider monomials in degrees ≤ 10.

(1) Degree 4 and degree 5:

In degree 4, $xyzt$ is obviously not hit. In degree 5, there is only one equation $D_1(xyzt) = x^2yzt + xy^2zt + xyz^2t + xyzt^2$ with 4 unknowns x^2yzt, xy^2zt, xyz^2t and $xyzt^2$. Hence we need three generators. We choose $x^2yzt,\ xy^2zt,\ xyz^2t.$

(2) Degree 6:

In degree 6, there are monomials in the forms [3111] and [2211]. By Appendix A and Appendix B, monomials in the form [2211] span $M^{(22)}$. In $M^{(22)}$, a submodule $Sp^{(22)}$ which is two dimensional is the first occurrence of $Sp^{(22)}$ in S_4. Hence the submodule $Sp^{(22)}$ is not hit by Schur's lemma. So monomials in the form [2211] are not hit. There is only one $Sp^{(31)}$ in the degrees lower than 6 and there are two $Sp^{(31)}$'s in degree 6, so one copy of $Sp^{(31)}$ which is three dimensional is not hit. Hence C_4^6 is at least five dimensional. We choose the following 5 monomials:

$$x^3yzt,\ xy^3zt,\ x^2y^2zt,\ x^2yz^2t,\ xy^2z^2t$$

and we will show that they are the representatives of a basis of C_4^6. Consider the

$\mathbb{Q}$−subspace N of S_4^6 spanned by the above 5 monomials and the hit elements in S_4^6. We will show that the subspace is equal to S_4^6. Hence the 5 monomials (*mod hit*) span C_4^6. Since C_4^6 has dimension ≥ 5, the 5 monomials (*mod hit*) form a basis of C_4^6. We have:

$$D_1(x^2yzt) = 2x^3yzt + x^2y^2zt + x^2yz^2t + x^2yzt^2 \equiv x^2yzt^2 \ (mod\ N) \qquad (20)$$

$$D_1(xy^2zt) = x^2y^2zt + 2xy^3zt + xy^2z^2t + xy^2zt^2 \equiv xy^2zt^2 \ (mod\ N).$$

We have now accounted for 5 monomials in the form [2211]. We can have the remaining one by

$$E_1^2(xyzt) = x^2y^2zt + x^2yz^2t + x^2yzt^2 + xy^2z^2t + xy^2zt^2 + xyz^2t^2.$$

Hence monomials in the form [2211] are hit (*mod* N). Also monomials in the form [3111] are hit (*mod* N) as well by (20) and its permutations.

(3) Degree 7:

In degree 7, there are monomials in the following forms [4111], [3211], [2221]. From Appendix A and Appendix B, the three dimensional $Sp^{(211)}$ in $M^{(211)}$ which is spanned by monomials in the form [3211] is the first occurrence in S_4. Hence monomials in the form [3211] are not hit.

Since D_1 and D_2 form a generating set of $\mathcal{D}_{\mathbb{Q}}$ (Theorem 2.5), if an irreducible Σ_4−module is hit in S_4^7, then it has a preimage in degree 5 or degree 6. By Appendix B, there are four copies of $Sp^{(31)}$ in S_4^7 and there are three copies of $Sp^{(31)}$ in S_4^5 and S_4^6. Hence at least one copy of $Sp^{(31)}$ is not hit. So C_4^7 is at least 6-dimensional. We choose the following 6 monomials:

$$x^4yzt,\ x^3y^2zt,\ x^3yz^2t,\ x^2y^3zt,\ xy^3z^2t,\ x^2y^2z^2t$$

as the generating set. We will show that the $\mathbb{Q}$–subspace N of S_4^7 spanned by the above 6 monomials and the hit elements in S_4^7 is equal to S_4^7. Hence the 6 monomials (*mod hit*) span C_4^7 so they form a minimal generating set. By Theorem 2.9 and Lemma 3.10,

$$xy^2z^2t^2 = xE_1^3(yzt) \equiv (\chi(E_1^3)(x))yzt = -c_3x^4yzt \equiv 0 \ (mod\ N), \qquad (21)$$

$$D_1(x^2y^2zt) = 2x^3y^2zt + 2x^2y^3zt + x^2y^2z^2t + x^2y^2zt^2 \equiv x^2y^2zt^2 \ (mod\ N), \quad (22)$$

$$E_1^3(xyzt) = x^2y^2z^2t + x^2y^2zt^2 + x^2yz^2t^2 + xy^2z^2t^2.$$

We get that monomials in the form [2221] are hit (*mod* N), so monomials in the form [4111] are hit (*mod* N) by (21) and symmetry. By applying suitable permutations on (22) and since monomials in the form [2221] are hit (*mod* N), we get that every monomial in the form [3211] is congruent (*mod* N) to a monomial of the same form with opposite sign by interchanging exponents 2 and 3. Hence we only need to consider the following 6 monomials: x^3y^2zt, x^3yz^2t, x^3yzt^2, xy^3z^2t, xy^3zt^2, xyz^3t^2, after taking modulo N. Because x^3y^2zt, x^3yz^2t, xy^3z^2t are in N, the problem is reduced to x^3yzt^2, xy^3zt^2 and xyz^3t^2 (mod N). By

$$D_1(xy^3zt) = x^2y^3zt + 3xy^4zt + xy^3z^2t + xy^3zt^2 \equiv xy^3zt^2 \ (mod\ N),$$

$$D_1(x^3yzt) = 3x^4yzt + x^3y^2zt + x^3yz^2t + x^3yzt^2 \equiv x^3yzt^2 \ (mod\ N),$$

$$D_2(xyzt^2) = x^3yzt^2 + xy^3zt^2 + xyz^3t^2 + 2xyzt^4 \equiv xyz^3t^2 \ (mod\ N).$$

We get that monomials in the form [3211] are hit (mod N). So $N = S_4^7$.

(4) Degree 8:

In degree 8, there are monomials in the following forms: [5111], [4211], [3311], [3221], [2222]. Monomials in the form [2222] are hit by Lemma 3.11. Monomials in the form [5111] are hit by Corollary 3.8. By Appendix B, there are two copies of $Sp^{(211)}$ in S_4^8 and there is only one copy of $Sp^{(211)}$ in lower degrees, there are three copies of $Sp^{(22)}$ in S_4^8 and there are only two copies of $Sp^{(22)}$ in lower degrees. So there are one copy of $Sp^{(22)}$ and one copy of $Sp^{(211)}$ in S_4^8 which are not hit, and hence C_4^8 is at least 5 dimensional.

We have the following equations involving monomials in the form [4211]:

$$\left\{\begin{array}{l}
D_3(xy^2zt) \equiv x^4y^2zt + xy^2z^4t + xy^2zt^4 \\
D_3(xyz^2t) \equiv x^4yz^2t + xy^4z^2t + xyz^2t^4 \\
D_3(xyzt^2) \equiv x^4yzt^2 + xy^4zt^2 + xyz^4t^2 \\
D_3(x^2yzt) \equiv x^2y^4zt + x^2yz^4t + x^2yzt^4 \\
D_1(x^4yzt) \equiv x^4y^2zt + x^4yz^2t + x^4yzt^2 \\
D_1(xy^4zt) \equiv x^2y^4zt + xy^4z^2t + xy^4zt^2 \\
D_1(xyz^4t) \equiv x^2yz^4t + xy^2z^4t + xyz^4t^2 \\
D_1(xyzt^4) \equiv x^2yzt^4 + xy^2zt^4 + xyz^2t^4
\end{array}\right. \quad (mod\ hit)$$

The coefficient matrix of the above equations is:

$$\begin{array}{c}
\begin{array}{cccccccccccc}
4211 & 4121 & 4112 & 2411 & 1421 & 1412 & 2141 & 1241 & 1142 & 2114 & 1214 & 1124
\end{array} \\
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1
\end{pmatrix}
\end{array}$$

M_1

M_1 has rank 7. We will show the subspace N spanned by the following 5 monomials: x^4y^2zt, x^4yz^2t, x^3y^3zt, $x^3y^2z^2t$, $x^2y^3z^2t$ and the hit elements in S_4^8 is equal to S_4^8. By Theorem 2.9, Lemma 3.10 and by writing $c_3 = 5$,

$$-5x^2yzt^4 = (\chi(E_1^3)(t))x^2yz \equiv (E_1^3(x^2yz))t \equiv x^4y^2zt + x^4yz^2t + 2x^3y^2z^2t$$

$$\equiv -x^4yzt^2 + 2x^3y^2z^2t \ (mod\ hit),$$

since $D_1(x^4yzt) \equiv x^4y^2zt + x^4yz^2t + x^4yzt^2$ $(mod\ hit)$. Hence we get:

$$2x^3y^2z^2t \equiv x^4yzt^2 - 5x^2yzt^4 \ (mod\ hit). \tag{23}$$

By interchanging x and y, we get

$$2x^2y^3z^2t \equiv xy^4zt^2 - 5xy^2zt^4 \ (mod\ hit). \tag{24}$$

By interchanging y and t, we get

$$2x^3yz^2t^2 - x^4y^2zt + 5x^2y^4zt \equiv 0 \ (mod\ hit). \tag{25}$$

By symmetry *every monomial in the form* [3221] *is congruent* (*mod hit*) *to a linear combination of two monomials in the form* [4211].

By Theorem 2.9, Lemma 3.10 and by writing $c_2 = 2$,

$$2x^3y^3zt = (\chi(E_1^2)(y))x^3zt \equiv E_1^2(x^3zt)y \equiv 3(x^4yz^2t + x^4yzt^2) + x^3yz^2t^2 \ (mod\ hit),$$

we get:

$$2x^3y^3zt + 3x^4y^2zt - x^3yz^2t^2 \equiv 0 \ (mod\ hit). \tag{26}$$

By (25)+2(26), we get:

$$4x^3y^3zt \equiv -5x^4y^2zt - 5x^2y^4zt \ (mod\ hit). \tag{27}$$

By symmetry *every monomial in the form* [3311] *is congruent* (*mod hit*) *to a linear combination of two monomials in the form* [4211].

Now we add 5 rows to M_1 corresponding to coefficients of monomials in the form [4211] in the following equations:

$$x^4y^2zt \equiv 0 \ (mod\ N), \quad x^4yz^2t \equiv 0 \ (mod\ N), \quad (23) \equiv 0 \ (mod\ N),$$

$$(24) \equiv 0 \ (mod\ N), \quad (27) \equiv 0 \ (mod\ N).$$

We get the following matrix:

$$\begin{array}{c}
\begin{matrix} 4211 & 4121 & 4112 & 2411 & 1421 & 1412 & 2141 & 1241 & 1142 & 2114 & 1214 & 1124 \end{matrix} \\
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -5 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -5 & 0 \\
-5 & 0 & 0 & -5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\end{array}$$

M_2

M_2 has rank 12, hence every monomial in the form [4211] is hit $(mod\ N)$. By the above discussion, the monomials in the forms [3311] and [3221] are hit $(mod\ N)$ as well. So x^4y^2zt, x^4yz^2t, x^3y^3zt, $x^3y^2z^2t$, $x^2y^3z^2t$ span $(mod\ hit)$ S_4^8.

(5) Degree 9:

In degree 9, there are monomials of the following forms: [6111], [5211], [4311], [4221], [3321], [3222]. x^6yzt is hit by Corollary 3.8. $x^3y^2z^2t^2 \equiv rx^6yzt \ (mod\ hit)$ by Lemma 4.3 for some $r \in \mathbb{Q}$. Hence monomials in the form [3222] are hit as well. By Appendix B, there are four copies of $Sp^{(211)}$ in S_4^9 and there are only three $Sp^{(211)}$'s in degree 7 and degree 8. Hence one $Sp^{(211)}$ is not hit. Since $Sp^{(211)}$ is 3 dimensional, C_4^d is at least 3 dimensional. We have the following equations involving monomials of the form [5211]:

$$\begin{cases} D_1(x^5yzt) \equiv x^5y^2zt + x^5yz^2t + x^5yzt^2 \\ D_1(xy^5zt) \equiv x^2y^5zt + xy^5z^2t + xy^5zt^2 \\ D_1(xyz^5t) \equiv x^2yz^5t + xy^2z^5t + xyz^5t^2 \\ D_1(xyzt^5) \equiv x^2yzt^5 + xy^2zt^5 + xyz^2t^5 \\ D_4(x^2yzt) \equiv x^2y^5zt + x^2yz^5t + x^2yzt^5 \\ D_4(xy^2zt) \equiv x^5y^2zt + xy^2z^5t + xy^2zt^5 \\ D_4(xyz^2t) \equiv x^5yz^2t + xy^5z^2t + xyz^2t^5 \\ D_4(xyzt^2) \equiv x^5yzt^2 + xy^5zt^2 + xyz^5t^2 \end{cases} \quad (mod\ hit) \qquad Eq(2)$$

By Theorem 2.9, Lemma 3.10 and by writing $c_4 = 14$,

$$14x^2yzt^5 = x^2yz(\chi(E_1^4)(t)) \equiv tE_1^4(x^2yz) = x^4y^2z^2t\ (mod\ hit),$$

we get

$$x^4y^2z^2t \equiv 14x^2yzt^5\ (mod\ hit). \tag{28}$$

By applying suitable permutations on (28), we get

$$x^2y^2z^4t \equiv 14xyz^2t^5\ (mod\ hit), \tag{29}$$

$$x^2y^2zt^4 \equiv 14xyz^5t^2\ (mod\ hit), \tag{30}$$

$$x^2y^4z^2t \equiv 14xy^2zt^5\ (mod\ hit), \tag{31}$$

$$x^2yz^2t^4 \equiv 14xy^5zt^2\ (mod\ hit), \tag{32}$$

we also get that *any monomial in the form* [4221] *is congruent (mod hit) to a monomial in the form* [5211]. We add two more equations:

$$D_3(x^2y^2zt) = 2x^5y^2zt + 2x^2y^5zt + x^2y^2z^4t + x^2y^2zt^4, \tag{33}$$

$$D_3(x^2yz^2t) = 2x^5yz^2t + x^2y^4z^2t + 2x^2yz^5t + x^2yz^2t^4, \tag{34}$$

Putting (29) to (32) into (33) and (34) we get:

$$2x^5y^2zt + 2x^2y^5zt + 14xyz^2t^5 + 14xyz^5t^2 \equiv 0 \ (mod\ hit), \tag{35}$$

$$2x^5yz^2t + 14xy^2zt^5 + 2x^2yz^5t + 14xy^5zt^2 \equiv 0 \ (mod\ hit). \tag{36}$$

Together with the equations of $Eq(2)$, we get the following coefficient matrix:

$$\begin{array}{c}
\begin{matrix} 5211 & 5121 & 5112 & 2511 & 1521 & 1512 & 2151 & 1251 & 1152 & 2115 & 1215 & 1125 \end{matrix} \\
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 7 & 0 & 0 & 7 \\
0 & 1 & 0 & 0 & 0 & 7 & 1 & 0 & 0 & 0 & 7 & 0
\end{pmatrix}
\end{array}$$

M_3

M_3 has rank 9. Let N be a subspace spanned by x^4y^3zt, $x^4y^2z^2t$, $x^3y^3z^2t$ and the hit elements in S_4^9. We will show $N = S_4^9$.

By Theorem 2.9, Lemma 3.10 and by writing $c_3 = 5$,

$$-5x^4y^3zt = (\chi(E_1^3)(x))y^3zt \equiv xE_1^3(y^3zt) \equiv 3xy^5z^2t + 3xy^5zt^2 + 3xy^4z^2t^2 \ (mod\ hit),$$

$$D_1(xy^5zt) \equiv xy^5z^2t + xy^5zt^2 + x^2y^5zt \ (mod\ hit)$$

and by applying a suitable permutation to (28), we get:

$$5x^4y^3zt \equiv 3x^2y^5zt - 42x^5y^2zt \ (mod\ hit). \tag{37}$$

Hence, by symmetry, *every monomial in the form* [4311] *is a linear combination of two monomials in the form* [5211]. By Theorem 2.9,

$$x^3y^3z^2t = (E_2^2(xy))z^2t \equiv xy(\chi(E_2^2)(z^2t))$$

$$\equiv xy(D_2D_2 - E_2^2)(z^2t) \equiv xy(2z^4t^3 + 3z^2t^5)\ (mod\ hit),$$

we get

$$x^3y^3z^2t - 2xyz^4t^3 - 3xyz^2t^5 \equiv 0\ (mod\ hit). \tag{38}$$

By applying a suitable permutation on (37), we get

$$5xyz^4t^3 \equiv 3xyz^2t^5 - 42xyz^5t^2\ (mod\ hit). \tag{39}$$

By putting 5 (38) + 2(39), we get

$$5x^3y^3z^2t \equiv 21xyz^2t^5 - 84xyz^5t^2\ (mod\ hit). \tag{40}$$

Hence *every monomial in the form* [3321] *is congruent* (*mod hit*) *to a linear combination of two monomials in the form* [5211]. Now we add 3 rows corresponding to the coefficients of monomials in the form [5211] in the right hand sides of equations (28), (37) and (40) to M_3 and get a 12×12 matrix as follows:

$$\begin{array}{c}
\begin{array}{cccccccccccc} 5211 & 5121 & 5112 & 2511 & 1521 & 1512 & 2151 & 1251 & 1152 & 2115 & 1215 & 1125 \end{array} \\
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 7 & 0 & 0 & 7 \\
0 & 1 & 0 & 0 & 0 & 7 & 1 & 0 & 0 & 0 & 7 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 14 & 0 & 0 \\
-42 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -84 & 0 & 0 & 21
\end{pmatrix}
\end{array}.$$

M_4

M_4 has rank 12. Hence x^4y^3zt, $x^4y^2z^2t$ and $x^3y^3z^2t$ span (*mod hit*) monomials in the form [5211] and hence span (*mod hit*) S_4^9.

(6) Degree 10

In degree 10, there are monomials in the following forms: [7111], [6211], [5311], [5221], [4411], [4321], [4222], [3331], [3322]. Monomials in forms [7111] and [6211] are hit by Corollary 3.8. $x^4y^2z^2t^2$ is hit by Lemma 3.11. Also by Lemma 4.3, $x^3y^3z^3t \equiv rxyzt^7$ (*mod hit*) for some $r \in \mathbb{Q}$, hence monomials in the forms [4222] and [3331] are hit as well. By Appendix B, $Sp^{(1^4)}$ is the first occurrence in S_4, so monomials in the form [4321] are not hit. By Theorem 2.9, Lemma 3.10 and by writing $c_4 = 14$,

$$14x^5y^3zt = (\chi(E_1^4(x))y^3zt \equiv xE_1^4(y^3zt) = 3xy^5z^2t^2 \ (mod\ hit) \qquad (41)$$

$$xy^5z^2t^2 = xy^5E_1^2(zt) \equiv \chi(E_1^2)(xy^5)zt \equiv 2x^3y^5zt \ (mod\ hit),$$

we get

$$14x^5y^3zt \equiv 6x^3y^5zt \ (mod\ hit).$$

Repeating the procedure once more to x^3y^5zt in the above equation, we get $160x^5y^3zt \equiv 0$ (*mod hit*). Hence monomials in the form [5311] are hit, and monomials in the form [5221] are hit as well by (41). Also by writing $c_3 = 5$,

$$-5x^4y^4zt = (\chi(E_1^3)(x))y^4zt \equiv xE_1^3(y^4zt) \equiv 4xy^5z^2t^2 \equiv 0 \ (mod\ hit),$$

so monomials in the form [4411] are hit. Since every monomial with two exponents 1 is hit, by Lemma 4.3, monomials in the form [3322] are hit as well. We will show that $x^4y^3z^2t$ is a generator of S_4^{10}. Let (12), (13), (14) be elements in Σ_4. From

$$D_1(x^3y^3z^2t) \equiv 3x^4y^3z^2t + 3x^3y^4z^2t \ (mod\ hit),$$

$$D_2(x^2y^3z^2t) \equiv 2x^4y^3z^2t + 2x^2y^3z^4t \ (mod\ hit),$$

$$D_3(xy^3z^2t) \equiv x^4y^3z^2t + xy^3z^2t^4 \ (mod\ hit),$$

we get:

$$x^4y^3z^2t \equiv -(12)x^4y^3z^2t \ (mod\ hit),$$

$$x^4y^3z^2t \equiv -(13)x^4y^3z^2t \ (mod\ hit),$$

$$x^4y^3z^2t \equiv -(14)x^4y^3z^2t \ (mod\ hit).$$

As (12), (13), (14) generate Σ_4, for any $\pi \in \Sigma_4$,

$$x^4y^3z^2t \equiv sign(\pi)x^4y^3z^2t \ (mod\ hit).$$

Hence $x^4y^3z^2t$ generates all monomials in the form [4321] under the action of $\mathcal{D}_{\mathbb{Q}}$ so it generates S_4^{10}. Putting the above results together, we have proved the theorem 4.2. $\square$

5 The hit problem on polynomial rings of 2 and 3 variables over $\mathbb{F}_2$

In this section, we will give the minimal generating sets under the action of $\mathcal{D}_2$ on polynomial rings of 2 and 3 variables over $\mathbb{F}_2$. The results we have got show that a generating set under the action of $\mathcal{D}_2$ on $\mathbb{F}_2[x_1, x_2, \ldots, x_n]$ is an infinite set for all $n \geq 2$. We have proved some general results for the n variable case, for example: a monomial of any number of variables with 2 odd exponents is not hit if it is in degree 2^m for $m \in \mathbb{N}$. In Section 5 and Section 6, when we write 2^m, if we do not specify m, then $m \in \mathbb{N}$. When we write an integer k into its binary expansion, $k = k_n k_{n-1} \ldots k_0$, we call k_0 the 0th digit of the binary expansion of k, k_1 the 1st digit of the binary expansion of k and so on. Let p be any prime number. In the remaining sections, if we work over $\mathbb{F}_p$, when we write "*mod hit*" we mean "*mod p*" as well.

5.1 The hit problem on a polynomial ring of 2 variables over $\mathbb{F}_2$

Proposition 5.1 *Over $\mathbb{F}_2$, $x^a y^b$ is hit if any of a, b is divisible by 2 and $a+b > 3$.*

Proof: By Lemma 3.13 and Lemma 3.14. □

Proposition 5.2 *In degree 2^m with $m \in \mathbb{N}$, a monomial $x^a y^b$ is not hit under the action of $\mathcal{D}_2$ if a, b are odd.*

Proof: Let N be the number of monomials which have two odd exponents in the equation $E_r^k(x^u y^v) = \sum_{i+j=k} \binom{u}{i}\binom{v}{j} x^{u+ri} y^{v+rj}$, where $u + v + rk = 2^m$ and $r = 1,\ 2$. Then

$$N \equiv \sum_{i+j=k} (u+ri)\binom{u}{i}\binom{v}{j} \equiv \sum_{i+j=k} (v+rj)\binom{u}{i}\binom{v}{j} \ (mod\ 2).$$

This is because $u + ri$ and $v + rj$ are either both even or both odd. Hence all terms with even exponents will be cancelled out by the coefficient $u + ri$ or $v + rj$. By Lemma 3.3, $N \equiv 0\ (mod\ 2)$.

Recall that $\{E_1^{2^k},\ E_2^{2^k} \mid k = 0 \text{ or } k \in \mathbb{N}\}$ is a generating set of $\mathcal{D}_2$ (Theorem 2.7). Suppose an arbitrary monomial f in degree 2^m is hit under the action of $\mathcal{D}_2$. Then we can write

$$f \equiv \sum_{i\geq 0}\sum_{j} E_1^{2^i}(f_{i,j}) + \sum_{i\geq 0}\sum_{j} E_2^{2^i}(g_{i,j}) \ (mod\ 2),$$

where $f_{i,j},\ g_{i,j} \in \mathbb{F}_2[x,y]xy$. Then the number of non-zero terms with 2 odd exponents in f must be even, since each $E_1^{2^i}(f_{i,j})$ or $E_2^{2^i}(g_{i,j})$ in the right hand side of the above equation has an even number of non-zero terms with odd exponents in it.

Hence if $dx^a y^b$ with $a,\ b$ odd is hit under the action of $\mathcal{D}_2$, then $d \equiv 0\ (mod\ 2)$. Hence $x^a y^b$ is not hit in degree 2^m. □

Proposition 5.3 *In degree 2^m, a monomial $x_1^{a_1} x_2^{a_2} \dots x_n^{a_n}$, with at least two exponents odd, is not hit under the action of $\mathcal{D}_2$.*

Proof: Suppose a_i is not divisible by 2. We take an operator $\pi \in \Gamma_n$ which maps x_i to x_i and maps the other variables to x_j where $j \neq i$ (Section 1). Then π commutes with the action of $\mathcal{D}_2$. Suppose $x_1^{a_1} x_2^{a_2} \dots x_n^{a_n}$ is hit over $\mathbb{F}_2$, then we have

$$\sum_t \delta_t(f_t) \equiv x_1^{a_1} x_2^{a_2} \dots x_n^{a_n} \ (mod\ 2) \tag{42}$$

where each f_t is a monomial in $\mathbb{F}_2[x_1, x_2, \ldots, x_n]$ and each $\delta_t \in \mathcal{D}_2^+$. Then we apply π to the left hand side of (42) and get

$$\pi \sum_t \delta_t(f_t) = \sum_t \delta_t \pi(f_t) = \sum_t \delta_t(g_t),$$

where each g_t is a monomial in $\mathbb{F}_2[x_i, x_j]$. We apply π to the right hand side of (42) and get $\pi(x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n}) = x_i^{a_i} x_j^{2^m - a_i}$. Hence we get $x_i^{a_i} x_j^{2^m - a_i} \equiv \sum_t \delta_t(g_t)\ (mod\, 2)$. But $x_i^{a_i} x_j^{2^m - a_i}$ is not hit under the action of $\mathcal{D}_2$, since it is in degree 2^m and a_i is odd. So the assumption that $x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n}$ is hit leads to a contradiction. $\square$

Theorem 5.4 $\{1,\ x,\ y,\ x^2y,\ x^{2^m-1}y \mid m \in \mathbb{N}\}$ *is a minimal generating set of* $\mathbb{F}_2[x, y]$ *under the action of* $\mathcal{D}_2$.

Proof: Since we know the 1 variable case, we only need to find a minimal generating set for $\mathbb{F}_2[x, y]xy$. Suppose $b > 1$, then $D_{b-1}(x^a y) = x^a y^b + a x^{a+b-1} y$. So we get $x^a y^b \equiv a x^{a+b-1} y$ $(mod\ hit)$. Hence $\{x^a y \mid a \in \mathbb{N}\}$ is a generating set of $\mathbb{F}_2[x, y]xy$ over $\mathbb{F}_2$.

In degree 3, there is only one equation $D_1(xy) = x^2y + xy^2$ with two unknowns $x^2y,\ xy^2$. So we need x^2y in the generating set. Obviously xy is not hit. Let $a > 2$. By Proposition 5.2, $x^a y$ is not hit in degree 2^m. By Proposition 5.1, a monomial $x^a y$ is hit if a is even where $a > 2$. Hence we only need to consider monomials in the form $[a1]$ where $2 \nmid a$ and $a \neq 2^m - 1$. We write the binary expansion of a as $a_n a_{n-1} \ldots a_0$ with $a_n \neq 0$. As $a + 1 \neq 2^m$, some $a_i \neq 1$ for $1 \leq i \leq n$. Let a_i, for $1 \leq i \leq n - 1$, be the first digit of the binary expansion of a which is equal to 0 counted from right. As $i > 0$, we have

$$E_1^{2^i}(x^{a-2^i} y) = \binom{a - 2^i}{2^i} x^a y + \binom{a - 2^i}{2^i - 1} x^{a-1} y^2.$$

Since $a_i = 0$, the ith digit of the binary expansion of $(a - 2^i)$ is 1, we have $\binom{a - 2^i}{2^i} \not\equiv 0\ (mod\ 2)$. By Proposition 5.1 $x^{a-1}y^2$ is hit, so $x^a y$ is hit as well.

Hence $\{x^2y,\ x^{2^m-1}y \mid m \in \mathbb{N}\}$ is a minimal generating set of $\mathbb{F}_2[x,y]xy$ under the action of $\mathcal{D}_2$. $\square$

5.2 The hit problem on a polynomial ring of 3 variables over $\mathbb{F}_2$

We will prove the following theorem:

Theorem 5.5 *A minimal generating set of $\mathbb{F}_2[x,y,z]xyz$ under the action of $\mathcal{D}_2$ is: $\{xyz,\ x^2yz,\ xy^2z,\ x^3yz,\ xy^3z,\ x^3y^2z,\ x^{2^k}y^{2^k-1}z,\ xy^{2^k-1}z^{2^k},\ x^{2^k+1}y^{2^k-1}z,\ x^{2^k+1}yz^{2^k-1},\ xy^{2^k+1}z^{2^k-1} \mid 2 \leq k \in \mathbb{N}\}$.*

5.3 Some general results for the 3 variable case

For a monomial $x^ay^bz^c$, choosing a largest exponent among a, b and c, we may assume $a \geq b,\ c$. We can express it by:

$$x^ay^bz^c \equiv x^a\left(\sum_k \delta_k(y^{2^k-1}z) + \delta(y^2z)\right)$$

$$\equiv \sum_k(\chi(\delta_k)(x^a))y^{2^k-1}z + (\chi(\delta)(x^a))y^2z\ (mod\ hit),$$

where $a > 2^k-1$, $k \in \mathbb{N}$ and $\delta_k,\ \delta \in \mathcal{D}_2$. So when we want to know which monomial is hit in $\mathbb{F}_2[x,y,z]xyz$, it is enough to check monomials in the forms $[a(2^k-1)1]$ and $[a21]$.

Lemma 5.6 *A monomial $x^ay^bz^c$ in degrees ≥ 7 is hit over $\mathbb{F}_2$, if two of a, b, c are even and the remaining one is odd.*

Proof: By Lemma 3.13 and Lemma 3.14. $\square$

This lemma will often be used in the remaining part of this section. If a monomial in degrees ≥ 7 is a term of the image of any operation of $\mathcal{D}_2$ on $\mathbb{F}_2[x,y,z]xyz$ and has two even exponents, we will consider it hit and omit it (*mod hit*).

Lemma 5.7 *If k is even and $a+b+k \geq 6$, then*

$$E_1^k(x^a y^b z) \equiv \sum_{i+j=k} \binom{a}{i}\binom{b}{j} x^{a+i}y^{b+j}z \ (mod\ hit)$$

over $\mathbb{F}_2$.

Proof: From the formula (5) of Section 2.1,

$$E_1^k(x^a y^b z) = \sum_{i+j=k} \binom{a}{i}\binom{b}{j} x^{a+i}y^{b+j}z + \sum_{i+j=k-1} \binom{a}{i}\binom{b}{j} x^{a+i}y^{b+j}z^2. \tag{43}$$

We look at the second sum in (43). Since k is even, $i+j=k-1$ is odd. Hence one of i, j is odd. Suppose $a+b$ is even, then either both a and b are odd or both a and b are even. So one of $a+i$ and $b+j$ is even. By Lemma 5.6, $x^{a+i}y^{b+j}z^2$ has two even exponents so it is hit since the total degree ≥ 7. Suppose $a+b$ is odd. If both $a+i$ and $b+j$ are odd, then we must have either: a is even, i is odd and b is odd, j is even or: a is odd, i is even and b is even, j is odd. But in both cases $\binom{a}{i}\binom{b}{j} \equiv 0 \ (mod\ 2)$. Hence the second sum in (43) $\equiv 0 \ (mod\ 2)$. $\square$

We can generalize the above result as follows:

Lemma 5.8 *In $E_1^k(x^u y^v z^w)$ where k is even and $u+v+w+k \geq 7$, a term $x^{u+i}y^{v+j}z^{w+t}$ with nonzero coefficient is hit under the action of $\mathcal{D}_2$, if any of $u+i$, $v+j$ or $w+t$ is even with corresponding u, v or w odd.*

Proof: From the formula (5) of Section 2.1, we have

$$E_1^k(x^u y^v z^w) = \sum_{i+j+t=k} \binom{u}{i}\binom{v}{j}\binom{w}{t} x^{u+i}y^{v+j}z^{w+t}. \tag{44}$$

Since k is even, either i, j, t are all even or exactly two of them are odd. If i, j, t are all even, then there is no odd u, v or w which will change to even in the right hand side of (44). We may assume that i, j are odd and t is even. Suppose a term $\binom{u}{i}\binom{v}{j}\binom{w}{t}x^{u+i}y^{v+j}z^{w+t} \not\equiv 0$ (mod 2). Then u, v must be odd, otherwise either $\binom{u}{i} \equiv 0$ (mod 2) or $\binom{v}{j} \equiv 0$ (mod 2). Hence $u+i$ and $v+j$ are even. So the term has two even exponents and it is hit by Lemma 5.6.

$\square$

This lemma will often be used. In the remaining part of this section, when we write out $E_1^k(x^u y^v z^w)$ where k is even and $u+v+w+k \geq 7$, we will use Lemma 5.8 without further comment and omit (mod hit) every term $x^a y^b z^c$ where a, b or c is even with corresponding u, v or w odd. By the above lemma, we can use E_1^k with k even to change the number and position of 1's in the binary expansions of a, b and c and get a monomial which is congruent (mod hit) to the original monomial. In the following examples, we repeatedly use Lemma 5.7 and Lemma 5.8.

Example 5.9 *From*

$$E_1^4(x^{20}y^7z) \equiv x^{24}y^7z + x^{20}y^{11}z \ (mod\ hit),$$

$$E_1^2(x^{22}y^7z) \equiv x^{24}y^7z + x^{22}y^9z \ (mod\ hit),$$

$$E_1^8(x^{14}y^9z) \equiv x^{22}y^9z + x^{14}y^{17}z \ (mod\ hit),$$

we get:

$$x^{24}y^7z \equiv x^{20}y^{11}z \equiv x^{22}y^9z \equiv x^{14}y^{17}z \ (mod\ hit).$$

We write $a,\ b,\ c$ into binary expansions: $a = a_n a_{n-1} \cdots a_0$, $b = b_m b_{m-1} \cdots b_0$, $c = c_k c_{k-1} \cdots c_0$. We may represent $x^a y^b z^c$ by the following diagram:

$$\begin{array}{c} a_n a_{n-1} \cdots a_0 \\ b_m b_{m-1} \cdots b_0 \\ c_k c_{k-1} \cdots c_0 \end{array},$$

with $a_n,\ b_m,\ c_k \neq 0$.

5.4 Generating elements for degrees ≤ 6

Proposition 5.10 *A minimal generating set of monomials in degrees $\leq\ 6$ in $\mathbb{F}_2[x,y,z]xyz$ is $\{xyz, x^2yz, xy^2z, x^3yz, xy^3z, x^3y^2z\}$.*

Proof: By Theorem 2.7, we only need to check the following operations: $E_1^{2^k}$, $E_2^{2^k}$ for $k = 0$ or $k \in \mathbb{N}$.

In degree 4, there is only one equation

$$D_1(xyz) = x^2yz + xy^2z + xyz^2$$

with 3 unknowns x^2yz, xy^2z and xyz^2. We choose x^2yz and xy^2z to be generators. Notice that this is the degree of 2^2. We list all operations whose images are in degree 5:

$$\begin{cases} D_1(x^2yz) \equiv x^2y^2z + x^2yz^2 \\ D_1(xy^2z) \equiv x^2y^2z + xy^2z^2 \\ D_1(xyz^2) \equiv x^2yz^2 + xy^2z^2 & \qquad (mod\ 2) \\ D_2(xyz) = x^3yz + xy^3z + xyz^3 \\ E_1^2(xyz) = x^2y^2z + x^2yz^2 + xy^2z^2 \end{cases}$$

From the above equations, we get that monomials in the form [221] are hit. We have only one equation which involves x^3yz, xy^3z and xyz^3. So we choose two generators x^3yz and xy^3z. Note that not are Artin elements contained in the cokernel of the $\mathcal{D}_2$ action, since the Artin element x^2y^2z is hit.

In degree 6, since $E_1^2(x^2yz) \equiv x^4yz$ (*mod hit*) by Lemma 5.8, monomials in the form [411] are hit. By Lemma 3.11 $x^2y^2z^2$ is hit. We have the following equations which involve monomials in the form [321]:

$$\left\{\begin{aligned} D_1(x^3yz) &\equiv x^3y^2z + x^3yz^2 \\ D_1(xy^3z) &\equiv x^2y^3z + xy^3z^2 \\ D_1(xyz^3) &\equiv x^2yz^3 + xy^2z^3 \\ D_2(x^2yz) &\equiv x^2y^3z + x^2yz^3 \\ D_2(xy^2z) &\equiv x^3y^2z + xy^2z^3 \\ D_2(xyz^2) &\equiv x^3yz^2 + xy^3z^2 \end{aligned}\right. \quad (\mathit{mod\ hit})$$

The coefficient matrix of the above equations is as follows:

$$\begin{pmatrix} x^3y^2z & x^3yz^2 & x^2y^3z & xy^3z^2 & x^2yz^3 & xy^2z^3 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

The matrix has rank 5. If we add one row to the matrix with 1 in the last column and 0 elsewhere, then we get a matrix with determinant $\not\equiv 0$ (*mod* 2). Hence xy^2z^3 generates the monomials in the form [321] and so does x^3y^2z by symmetry. Summing up the results of this section, we get the generating set. □

5.5 Monomials in degrees 2^m and 2^m+1 for $m \geq 2$

For the remaining content of this section, we may assume degrees are ≥ 7.

Proposition 5.11 *In degrees 2^m, 2^m+1, a monomial $x^a y^b z$ with at least one of a, b odd is not hit under the action of $\mathcal{D}_2$.*

Proof: By Proposition 5.3 and Theorem 3.12. $\square$

Proposition 5.12 *Any monomial $x^a y^b z$ in degree 2^m or 2^m+1, where b is odd and has k 1's in its binary expansion, is congruent (mod hit) to a monomial in the form $x^u y^v z$ where $u = 2^m - 2^k + r$ for $r = 0$ or 1 and $v = 2^k - 1$.*

Proof: Suppose $b \neq 2^n - 1$ for any n. By Lemma 3.15, $y^b \equiv D(y^{2^k-1})$ $(mod\ 2)$ where D is a composition of some E_1^i's and k is the number of 1's in the binary expansion of b. Then by Theorem 2.9,

$$x^a y^b z \equiv x^a z D(y^{2^k-1}) \equiv (\chi(D)(x^a z)) y^{2^k-1}\ (mod\ hit).$$

$\chi(D)$ is a sum of some compositions of E_1^i's, since χ is the anti–isomorphism on the Steenrod algebra generated by all E_1^i's.

In $(\chi(D)(x^a z)) y^{2^k-1}$, suppose the exponent of z of a term changes, it can only change to 2 firstly and then remains even. Suppose $x^a y^b z$ is in degree 2^m. Then a is even and the exponent of x is always even in every term of $(\chi(D)(x^a z)) y^{2^k-1}$. If the exponent of z of a term changes to even, then the term is hit by Lemma 5.6. Suppose $x^a y^b z$ is in degree 2^m+1, then a is odd. If the exponent of z of a term changes to even, a has to change to even as well since total degree is odd. Again the term is hit by Lemma 5.6. Hence $x^a y^b z \equiv l x^{a'} y^{2^k-1} z$ $(mod\ hit)$, where $l = 0$ or 1 and $a' = 2^m - 2^k + r$ for $r = 0$ or 1 since the degree of $x^a y^b z$ is 2^m or 2^m+1. We have $l = 1$ since $x^a y^b z$ is not hit by Proposition 5.11. $\square$

Proposition 5.13 *Suppose* $a = 2^m - 2^k + r$ *where* $r = 0$ *or* 1 *and* $m > k$, $b = 2^k - 1$, *then* $x^a y^b z \equiv x^{2^{m-1}+r} y^{2^{m-1}-1} z$ *(mod hit)*.

Proof: By Lemma 5.8, $E_1^2(x^{a-2}y^b z) \equiv x^a y^b z + x^{a-2} y^{b+2} z$ (*mod hit*), we get $x^a y^b z \equiv x^{a-2} y^{2^k+1} z$ (*mod hit*) and

$$x^{a-2}y^{2^k+1}z = \begin{array}{r} 1\cdots 101\cdots 1r \\ 10\cdots 01 \\ 1 \end{array},$$

where $a - 2 = 2^m - 2^k - 2 + r$.

We claim that $x^{a-2}y^{2^k+1}z \equiv x^{2^{m-1}-2+r}y^{2^{m-1}+1}z$ (*mod hit*). We argue by induction on the exponent of y. There is nothing to prove in the case $k = m - 1$. So we may assume $k < m - 1$. By Lemma 5.8 we have,

$$E_1^{2^k}(x^{2^m-2^{k+1}-2+r}y^{2^k+1}z) \equiv x^{a-2}y^{2^k+1}z + x^{2^m-2^{k+1}-2+r}y^{2^{k+1}+1}z \text{ (mod hit)}, \quad (45)$$

because for j even, $\binom{2^k+1}{j} \equiv 1$ (*mod* 2) only if $j = 0,\ 2^k$ and if $j = 0$, $\binom{2^m - 2^{k+1} - 2 + r}{2^k} \equiv 1$ (*mod* 2). Hence

$$x^{a-2}y^{2^k+1}z \equiv x^{2^m-2^{k+1}-2+r}y^{2^{k+1}+1}z \text{ (mod hit)}.$$

By the induction hypothesis, we must reach

$$x^{a-2}y^{2^k+1}z \equiv x^{2^{m-1}-2+r}y^{2^{m-1}+1}z \text{ (mod hit)}.$$

$x^{2^{m-1}-2+r}y^{2^{m-1}+1}z$ is a monomial as follows:

$$\begin{array}{r} 1\cdots 1\cdots 1r \\ 10\cdots 0\cdots 01 \\ 1 \end{array}.$$

Finally by Lemma 5.8,

$$E_1^2(x^{2^{m-1}-2+r}y^{2^{m-1}-1}z) \equiv x^{2^{m-1}+r}y^{2^{m-1}-1}z + x^{2^{m-1}-2+r}y^{2^{m-1}+1}z \ (mod\ hit).$$

Hence

$$x^a y^b z \equiv x^{2^{m-1}-2+r}y^{2^{m-1}+1}z \equiv x^{2^{m-1}+r}y^{2^{m-1}-1}z \ (mod\ hit).$$

$\square$

Proposition 5.14 *The dimension of the cokernel of the action of $\mathcal{D}_2$ in the degree 2^m for $m \geq 3$ of $\mathbb{F}_2[x,y,z]$ is 2.*

Proof: By Proposition 5.12 and Proposition 5.13, any monomial $x^a y^b z$ with $1 < b$ odd in degree 2^m is congruent $(mod\ hit)$ to a monomial $x^{2^{m-1}}y^{2^{m-1}-1}z$. There are 6 monomials in the form $[(2^{m-1})(2^{m-1}-1)1]$. Because

$$D_{2^{m-1}-2}(x^{2^{m-1}}yz) \equiv x^{2^{m-1}}y^{2^{m-1}-1}z + x^{2^{m-1}}yz^{2^{m-1}-1} \ (mod\ 2),$$

we get

$$x^{2^{m-1}}y^{2^{m-1}-1}z \equiv x^{2^{m-1}}yz^{2^{m-1}-1} \ (mod\ hit).$$

Hence there are three monomials in this form which generate all the $x^a y^b z$ with $1 < b$ odd in degree 2^m under operations of $\mathcal{D}_2$. They are:

$$(1)\ x^{2^{m-1}}y^{2^{m-1}-1}z, \qquad (2)\ x^{2^{m-1}-1}y^{2^{m-1}}z, \qquad (3)\ x^{2^{m-1}-1}yz^{2^{m-1}}.$$

If $b = 1$, there are three monomials in the forms: $x^{2^m-2}yz$, $xy^{2^m-2}z$, xyz^{2^m-2}. They are also generated by (1), (2) and (3). For example:

$$D_{2^{m-1}-1}(x^{2^{m-1}-1}yz) \equiv x^{2^m-2}yz + x^{2^{m-1}-1}y^{2^{m-1}}z + x^{2^{m-1}-1}yz^{2^{m-1}} \ (mod\ 2).$$

So $x^{2^m-2}yz \equiv$ (2)+(3) $(mod\ hit)$. We will have similar results for $xy^{2^m-2}z$ and xyz^{2^m-2}. Hence every monomial in degree 2^m is generated by (1), (2) and (3) over $\mathbb{F}_2$. So the dimension of the cokernel in degree 2^m is ≤ 3.

Any single one of (1), (2) and (3) is not hit by Proposition 5.3. Also a sum of any two of them is not hit. For example: writing $k = 2^{m-1}$,

$$D_1(xy^{k-1}z^{k-1}) \equiv x^2y^{k-1}z^{k-1} + xy^kz^{k-1} + xy^{k-1}z^k \ (mod\ 2).$$

Since $D_{k-2}(xy^kz) \equiv x^{k-1}y^kz + xy^kz^{k-1}$ $(mod\ 2)$, $xy^kz^{k-1} \equiv (2)$ $(mod\ hit)$. Similarly $xy^{k-1}z^k \equiv (3)$ $(mod\ hit)$. So $(2) + (3) \equiv x^2y^{k-1}z^{k-1}$ $(mod\ hit)$, which is not hit by Proposition 5.3. By a similar argument, $(1) + (2)$ and $(1) + (3)$ are not hit. Because

$$D_{k-2}(x^2y^{k-1}z) \equiv x^2y^{2k-3}z + x^2y^{k-1}z^{k-1} \ (mod\ 2),$$

and also by Proposition 5.12 and Proposition 5.13, $x^2y^{2k-3}z \equiv x^ky^{k-1}z$ $(mod\ hit)$. We get

$$(2) + (3) \equiv (1) \ (mod\ hit).$$

Hence (1), (2), (3) are linearly dependent $(mod\ hit)$. So the cokernel is two dimensional. □

Proposition 5.15 *In degree $2^m + 1$ with $m \geq 2$, the number of monomials in the form x^ay^bz, where a, b are odd, is even in the image of any operation of $\mathcal{D}_2$.*

Proof: By $E_r^k(x^uy^vz^w) = \sum_{i+j=k} E_r^i(x^uy^v)E_r^j(z^w)$, a monomial in the form x^ay^bz only appears in the situation when $i = k$, $j = 0$ and $w = 1$. Hence the number of monomials in the form x^ay^bz with a, b odd in $E_r^k(x^uy^vz)$ is equal to the number of monomials in the form $[ab]$ with a, b odd in $E_r^k(x^uy^v)$. Because $u+v+k = 2^m$, by the proof of Proposition 5.2, the number of monomials in the form $[ab]$, where a, b are odd, is even in $E_r^k(x^uy^vz)$. Since $\{E_1^k,\ E_2^k \mid k \in \mathbb{N}\}$ form a generating set of $\mathcal{D}_2$ (Theorem 2.7), we always get an even number of monomials in the form $[ab1]$ with a, b odd in the image of any operation of $\mathcal{D}_2$. □

Theorem 5.16 *A monomial $x_1^{a_1}x_2^{a_2}\dots x_n^{a_n}$ in degree 2^m+1 is not hit under the action of $\mathcal{D}_2$, if it has at least three odd exponents and some $a_i = 1$.*

Proof: Let $x_1^{a_1}x_2^{a_2}\dots x_n^{a_n}$ be such a monomial. Suppose $a_1 = 1$, a_2, a_3 are odd, we have a map $\pi \in \Gamma_n$ (Section 1) which maps x_1 to x_1, x_2 to x_2 and all the other x_i to x_3 for $i = 3,\ 4,\ \dots,\ n$. Then π commutes with the action of $\mathcal{D}_2$. If $x_1x_2^{a_2}\dots x_n^{a_n}$ is hit under the action of $\mathcal{D}_2$, then we have

$$x_1x_2^{a_2}\dots x_n^{a_n} \equiv \sum_i \delta_i(f_i)\ (mod\ 2), \tag{46}$$

where each δ_i is in $\mathcal{D}_2^+$ and each f_i is a monomial in $\mathbb{F}_2[x_1, x_2, \dots, x_n]$.

Then we apply π to the right hand side of (46) and get

$$\pi\sum_i \delta_i(f_i) = \sum_i \delta_i\pi(f_i) = \sum_i \delta_i(g_i),$$

where $g_i \in \mathbb{F}_2[x_1, x_2, x_3]$. Let $k = \sum_{i=3}^{n} a_i$. We apply π to the left hand side of (46) and get

$$\pi(x_1x_2^{a_2}\dots x_n^{a_n}) = x_1x_2^{a_2}x_3^k.$$

Hence we get $x_1x_2^{a_2}x_3^k \equiv \sum_i \delta_i(g_i)\ (mod\ 2)$. But $x_1x_2^{a_2}x_3^k$ is not hit under the action of $\mathcal{D}_2$ by Proposition 5.11. So the assumption that $x_1x_2^{a_2}\dots x_n^{a_n}$ is hit leads to a contradiction. □

Proposition 5.17 *In degree 2^m+1 with $m \geq 3$, the dimension of the cokernel of $\mathbb{F}_2[x, y, z]xyz$ under the action of $\mathcal{D}_2$ is 3.*

Proof: For a monomial $x^ay^bz^c$ in degree 2^m+1 with $m \geq 3$, if one of the exponents is even, there must be another even exponent. So all monomials with any even exponents are hit by Lemma 5.6. We only need to consider monomials with all odd exponents. By Proposition 5.12 and Proposition 5.13, any monomial

$x^a y^b z$ with a, b odd and $b > 1$ in this degree is congruent (*mod hit*) to the monomial $x^{k-1}y^{k+1}z$ with $k = 2^{m-1}$. Also by Lemma 5.8,

$$E_1^2(x^{k-1}y^{k-1}z) \equiv x^{k-1}y^{k+1}z + x^{k+1}y^{k-1}z \ (mod\ hit).$$

So there are 3 monomials in the form $[(k+1)(k-1)1]$ which span all $x^a y^b z$ with a, b odd and $b > 1$ under operations of $\mathcal{D}_2$ and none of them is hit by Proposition 5.11. They are:

$$(1)\ x^{k-1}y^{k+1}z, \qquad (2)\ x^{k-1}yz^{k+1}, \qquad (3)\ xy^{k-1}z^{k+1}.$$

If $b = 1$ then we have:

$$D_k(x^{k-1}yz) \equiv x^{2k-1}yz + x^{k-1}y^{k+1}z + x^{k-1}yz^{k+1} \ (mod\ 2).$$

Hence $x^{2k-1}yz \equiv (1) + (2)$ (*mod hit*). Similarly we get

$$xy^{2k-1}z \equiv (1) + (3) \ (mod\ hit), \qquad xyz^{2k-1} \equiv (2) + (3) \ (mod\ hit).$$

Hence every monomial with all three exponents odd in degree $2^m + 1$ is congruent (*mod hit*) to a linear combination of (1), (2) and (3) over $\mathbb{F}_2$. So the dimension of the cokernel is ≤ 3. We claim that a sum of two or three of the monomials (1), (2) and (3) is not hit. By Proposition 5.15, there is always an even number of monomials in the form $x^a y^b z$ with a, b odd in the image of any operation in $\mathcal{D}_2$. So $x^{k-1}y^{k+1}z$ always appears in the image of any operation of $\mathcal{D}_2$ with another monomial $x^a y^b z$ where $1 < a$, b are odd. Hence (1)+(2) and (1)+(3) are not hit since there is only one monomial in each of the sums which has the exponent of z to be 1. It is similar for the monomials (2) or (3). Hence the sum of any two of (1), (2) and (3) is not hit. The sum of three of them is not hit for the same reason. Hence the cokernel is three dimensional. $\square$

Proposition 5.18 *A monomial $x^a y^b z^c$ with $a,\ b,\ c > 1$ in degree $2^m + 1$ is hit under the action of $\mathcal{D}_2$.*

Proof: If there exists an even exponent in $x^a y^b z^c$, then there must be two even exponents since the total degree of $x^a y^b z^c$ is odd. Then the monomial is hit by Lemma 5.6. Hence we only need to consider the case where $a,\ b,\ c$ are odd.

$$D_{c-1}(x^a y^b z) \equiv x^a y^b z^c + x^{a+c-1} y^b z + x^a y^{b+c-1} z \ (mod\ 2).$$

By Proposition 5.12 and Proposition 5.13, both $x^{a+c-1}y^b z$ and $x^a y^{b+c-1} z$ are congruent to $x^{2^{m-1}+1} y^{2^{m-1}-1} z$ $(mod\ hit)$. So

$$x^a y^b z^c \equiv 2x^{2^{m-1}+1} y^{2^{m-1}-1} z \equiv 0\ (mod\ hit).$$

Hence $x^a y^b z^c$ is hit. $\square$

5.6 Monomials in the form $[a(2^k)1]$

We need to check monomials in the form $[a21]$. Also we will check monomials in the form $[a(2^k)1]$ for some $k > 1$, since the result will be used in Section 5.7.

Proposition 5.19 *A monomial in the form $[a11]$ with $a \geq 5$ is hit over $\mathbb{F}_2$, if it is not in degree 2^m or $2^m + 1$ for some $m \geq 3$.*

Proof: Suppose we have a monomial $x^a yz$ with $a \geq 5$, we write the binary expansion of $a = a_n a_{n-1} \cdots a_0$ with $a_n \neq 0$. If $x^a yz$ is not in degrees 2^m and 2^m+1, then there is an $a_i = 0$ for $1 \leq i < n$. Let $u = a - 2^i$, then $\binom{u}{2^i} \not\equiv 0\ (mod\ 2)$. We have $E_1^{2^i}(x^u yz) \equiv x^a yz\ (mod\ hit)$ by Lemma 5.8. If all $a_i = 1$ for $1 \leq i \leq n$, then the monomial is either in degree 2^m if $a_0 = 0$ or in degree $2^m + 1$ if $a_0 = 1$.

$\square$

Proposition 5.20 *A monomial in the form* $[a21]$ *is hit under the action of* $\mathcal{D}_2$, *if* $a \geq 7$ *and* $a \equiv 3$ $(mod\ 4)$.

Proof: Let $a \geq 7$ and $a \equiv 3$ $(mod\ 4)$, we have:

$$D_4(x^{a-4}y^2z) \equiv x^ay^2z + x^{a-4}y^2z^5 \ (mod\ 2), \tag{47}$$

$$D_2(x^{a-2}yz^2) \equiv x^ayz^2 + x^{a-2}y^3z^2 \ (mod\ 2),$$

$$E_2^2(x^{a-4}yz^2) \equiv x^ayz^2 + x^{a-2}y^3z^2 + x^{a-4}yz^6 \ (mod\ 2),$$

since $a_0 = a_1 = 1$. This gives that $x^{a-4}yz^6$ is hit. Also:

$$D_1(x^{a-4}yz^5) \equiv x^{a-4}yz^6 + x^{a-3}yz^5 + x^{a-4}y^2z^5 \ (mod\ 2).$$

So $x^{a-3}yz^5 \equiv x^{a-4}y^2z^5$ $(mod\ hit)$. But

$$E_1^2(x^{a-5}yz^5) \equiv x^{a-3}yz^5 + x^{a-5}y^2z^6 \ (mod\ 2)$$

and $x^{a-5}y^2z^6$ is hit by Lemma 3.11.

Hence $x^{a-3}yz^5$ is hit so $x^{a-4}y^2z^5$ is hit and hence x^ay^2z is hit by (47). □

Proposition 5.21 : *A monomial in the form* $[a21]$ *with* $a \geq 4$ *is hit under the action of* $\mathcal{D}_2$ *if it is not in degree* 2^m.

Proof: Let $a_n \cdots a_1a_0$ be the binary expansion of a. By Lemma 5.6, x^ay^2z is hit if a is even. Let a be odd. By Proposition 5.20, x^ay^2z is hit if $a_1 = 1$.

Let $a_0 = 1$, and $a_1 = 0$. Suppose there is an $a_i = 0$ for $2 \leq i < n$. By Lemma 5.8,

$$E_1^{2^i}(x^{a-2^i}y^2z) \equiv \binom{a-2^i}{2^i}x^ay^2z + \binom{a-2^i}{2^i-2}x^{a-2}y^4z \equiv x^ay^2z \ (mod\ hit),$$

since $\binom{a-2^i}{2^i-2} \equiv 0$ $(mod\ 2)$ as the 1st digit of the binary expansion of $2^i - 2$ is 1 and the 1st digit of the binary expansion of $a - 2^i$ is 0. Hence we only need to consider an a which has $a_1 = 0$ and $a_j = 1$ for $j \neq 1$. But this case is in degree 2^m. So monomials in the form $[a21]$ with $a \geq 4$ are hit except in degree 2^m. $\square$

Proposition 5.22 *A monomial in the form $[a(2^k)1]$ in degrees ≥ 7 with $k \geq 1$ is hit under the action of $\mathcal{D}_2$, if it is not in degree 2^m.*

Proof: We claim that $x^a y^{2^k} z$ is either hit or $\equiv x^{a+2^k-2} y^2 z$ $(mod\ hit)$. We argue by induction on the exponent of y. By Lemma 5.8, for $k > 1$,

$$E_1^{2^{k-1}}(x^a y^{2^{k-1}} z) \equiv \binom{a}{2^{k-1}} x^{a+2^{k-1}} y^{2^{k-1}} z + x^a y^{2^k} z \ (mod\ hit),$$

since $\binom{2^{k-1}}{i} \equiv 0$ except $i = 0,\ 2^{k-1}$.

If $\binom{a}{2^{k-1}} = 0$ $(mod\ 2)$, then $x^a y^{2^k} z$ is hit, otherwise

$$x^a y^{2^k} z \equiv x^{a+2^{k-1}} y^{2^{k-1}} z \ (mod\ hit).$$

Hence $x^a y^{2^k} z$ is either hit or $\equiv x^{a+2^{k-1}} y^{2^{k-1}} z$ $(mod\ hit)$. By the induction hypothesis, we get that $x^a y^{2^k} z$ is either hit or $\equiv x^{a+2^k-2} y^2 z$ $(mod\ hit)$. But $x^{a+2^k-2} y^2 z$ is hit by Proposition 5.21, hence $x^a y^{2^k} z$ is hit. $\square$

5.7 Monomials in the form $[a(2^k-1)1]$, $k \geq 2$

Proposition 5.23 *: A monomial in the form $[ab1]$ with $a \geq b$ is hit under the action of $\mathcal{D}_2$, if $a \equiv 3$ $(mod\ 4)$, $b = 2^k - 1$ and $k > 1$.*

Proof: We have the following equations:

$$D_{b-1}(x^{a-b+1} y^b z) \equiv x^a y^b z + x^{a-b+1} y^{2b-1} z + x^{a-b+1} y^b z^b \ (mod\ 2),$$

$$E_{b-1}^2(x^{a-b+1}yz) = \binom{a-b+1}{2} x^{a+b-1}yz + (a-b+1)(x^a y^b z + x^a y z^b) + x^{a-b+1}y^b z^b$$

$$\equiv x^a y^b z + x^a y z^b + x^{a-b+1} y^b z^b \ (mod\ 2),$$

since $a - b + 1 \equiv 1\ (mod\ 4)$ and $\binom{a-b+1}{2} \equiv 0\ (mod\ 2)$. Hence we get: $x^a y z^b \equiv x^{a-b+1}y^{2b-1}z\ (mod\ hit)$. But:

$$E_1^2(x^{a-b-1}y^{2b-1}z) \equiv x^{a-b+1}y^{2b-1}z\ (mod\ hit),$$

by Lemma 5.8 and $\binom{2b-1}{2} \equiv 0\ (mod\ 2)$ since the 1st digit of the binary expansion of $2b-1$ is 0. Hence $x^{a-b+1}y^{2b-1}z$ is hit and hence $x^a y z^b$ is hit. $\square$

By the above proposition, a spike in the form $[(2^l-1)(2^k-1)1]$ where $l \geq k > 1$ is hit under the action of $\mathcal{D}_2$.

Lemma 5.24 *A monomial in the form $[ab1]$ with a even, $a+b \geq 7$ and $b = 2^k-1$ where $k > 1$, which is not in degree 2^m, is hit over $\mathbb{F}_2$.*

Proof: We have

$$D_1(x^{a-1}y^b z) = (a-1)x^a y^b z + b x^{a-1}y^{b+1}z + x^{a-1}y^b z^2 \equiv x^a y^b z + x^{a-1} y^b z^2\ (mod\ hit),$$

since $b+1 = 2^k$ and $x^{a-1}y^{b+1}z$ is hit by Proposition 5.22. Now

$$D_{b-1}(x^{a-1}yz^2) \equiv x^{a-1}y^b z^2 + x^{a+b-2}yz^2\ (mod\ 2),$$

and by Proposition 5.21 $x^{a+b-2}yz^2$ is hit. Hence $x^{a-1}y^b z^2$ is hit and hence $x^a y^b z$ is hit as well. $\square$

Proposition 5.25 *Let $a_n \cdots a_1 a_0$ be the binary expansion of a. A monomial in the form $x^a y^b z$ with a odd, $b = 2^k - 1$ and $k > 1$ is hit over $\mathbb{F}_2$, if there are two 0's between a_k and a_{n-1}.*

Proof: Let $a = a_n \cdots a_1 a_0$. Suppose $a_i = a_j = 0$ for $k \le i < j \le n-1$. Then by Lemma 5.8,

$$E_1^{2^j}(x^{a-2^j}y^b z) \equiv x^a y^b z + \sum_{0 \le t \le 2^i} \binom{a-2^j}{2^j - t}\binom{b}{t} x^{a-t}y^{b+t}z \ (mod\ 2).$$

But each $\binom{a-2^j}{2^j-t} \equiv 0\ (mod\ 2)$ since the ith digit of the binary expansion of $a - 2^j$ is 0 and the ith digit of the binary expansion of $2^j - t$ is 1. □

Proposition 5.26 *A monomial in the form [a31] with a odd is hit under the action of $\mathcal{D}_2$, if it is not in degree $2^m + 1$ where $m \ge 1$.*

Proof: By Proposition 5.23, $x^a y^3 z$ is hit except for the a where $a_1 = 0$. We may assume that $a \equiv 1\ (mod\ 4)$. By Proposition 5.25, $x^a y^3 z$ is hit except the a where there is at most one 0 between a_2 and a_{n-1}. Let $a_1 = a_j = 0$ for $2 \le j \le n-1$. Then

$$E_1^{2^j}(x^{a-2^j}y^3 z) \equiv \binom{a-2^j}{2^j} x^a y^3 z + \binom{a-2^j}{2^j-2} x^{a-2}y^5 z \equiv x^a y^3 z\ (mod\ hit),$$

by Lemma 5.8 and $\binom{a-2^j}{2^j-2} \equiv 0\ (mod\ 2)$ since the 1st digit of the binary expansion of $a - 2^j$ is equal to $a_1 = 0$ as $j \ge 2$ and the 1st digit of the binary expansion of $2^j - 2$ is 1. The only case left is that $a_i = 1$ for $i \ne 1$ and $a_1 = 0$, but then $x^a y^3 z$ is in degree $2^m + 1$. □

Proposition 5.27 *A monomial in the form [ab1] with $b = 2^k - 1$, a odd and $k > 1$ is hit under the action of $\mathcal{D}_2$, if it is not in degree $2^m + 1$ where $m \ge 1$.*

Proof: Let $a = a_n \cdots a_1 a_0$. If $a_1 = 1$ then $x^a y^b z$ is hit by Proposition 5.23. So we only consider an a where $a_1 = 0$. We have

$$E_1^2(x^{a-2}y^b z) \equiv x^a y^b z + x^{a-2}y^{2^k+1}z\ (mod\ hit).$$

We claim that $x^{a-2}y^{2^k+1}z$ is either hit or $\equiv x^{a+b-3}y^3z$ $(mod\ hit)$. We argue by induction on the exponent of y.

Suppose $k>1$. By Lemma 5.8,

$$E_1^{2^{k-1}}(x^{a-2}y^{2^{k-1}+1}z) \equiv x^{a-2}y^{2^k+1}z + rx^{a+2^{k-1}-2}y^{2^{k-1}+1}z\ (mod\ hit),$$

since for j even, $\binom{2^{k-1}+1}{j} \equiv 0\ (mod\ 2)$ except $j=0$ or $j=2^{k-1}$. Here $r=\binom{a-2}{2^{k-1}}$. If $r\equiv 0\ (mod\ 2)$ then $x^{a-2}y^{2^k+1}z$ is hit otherwise $r\equiv 1\ (mod\ 2)$ and $x^{a-2}y^{2^k+1}z \equiv x^{a-2+2^{k-1}}y^{2^{k-1}+1}z\ (mod\ hit)$. Hence $x^{a-2}y^{2^k+1}z$ is either hit or $\equiv x^{a-2+2^{k-1}}y^{2^{k-1}+1}z\ (mod\ hit)$. By the induction hypothesis, we finally must reach either $x^{a-2}y^{2^k+1}z$ is hit or $\equiv x^{a+b-3}y^3z\ (mod\ hit)$. By Proposition 5.26, $x^{a+b-3}y^3z$ is hit since it is not in degree 2^m+1. Hence x^ay^bz is hit. $\square$

Now putting the results which we have got together we have proved Theorem 5.5.

5.8 Representations of the cokernels for the 2 and 3 variable cases

In the following argument, we will use the results in the decomposition matrices of Specht modules over $\mathbb{F}_2$ in [7].

There is only one irreducible $\mathbb{F}_2\Sigma_2$–module, $F^{(2)} \approx Sp^{(2)}$. For the 2 variable case, since the cokernel in degree 2^m is 1–dimensional, the cokernel has to be one copy of $F^{(2)}$. Also from $D_{2^{m-1}}(xy) = x^{2^{m-1}}y + xy^{2^{m-1}}$, we get

$$(12)(x^{2^{m-1}}y) = xy^{2^{m-1}} \equiv x^{2^{m-1}}y\ (mod\ hit),$$

where $(12)\in\Sigma_2$. Hence $x^{2^{m-1}}y$ generates $(mod\ hit)$ a $\mathbb{F}_2\Sigma_2$–module $F^{(2)}$.

There are two isomorphism classes of irreducible $\mathbb{F}_2\Sigma_3-$modules, $F^{(3)} \approx Sp^{(3)}$ and $F^{(21)} \approx Sp^{(21)}$ which is 2−dimensional.

Lemma 5.28 *The monomials in the form* $[a11]$ *span (mod hit) a* $\mathbb{F}_2\Sigma_3-$*modules which is isomorphic to* $F^{(21)}$ *in the cokernel.*

Proof: Obviously a single monomial in the form $[a11]$ is not hit since it is in the cokernel. From $D_{a-1}(xyz) = x^a yz + xy^a z + xyz^a$, we get that the sum of any two monomials in the form $[a11]$ is not hit as well. The monomials in the form $[a11]$ span a $\mathbb{F}_2\Sigma_3-$module which is isomorphic to $F^{(21)}$ where $x^a yz$ and $xy^a z$ form a basis. We may check this by,

$$(12)(x^a yz) = xy^a z, \qquad (23)(x^a yz) = x^a yz,$$

$$(13)(x^a yz) = xyz^a \equiv x^a yz + xy^a z \ (mod\ hit),$$

since (12), (13) and (23) generate Σ_3. From the above relations we can see that $x^a yz$ and $xy^a z$ generate an irreducible two dimensional $\mathbb{F}_2\Sigma_3-$module. Since the module is 2−dimensional, it has to be isomorphic to $F^{(21)}$. □

The module structure of the cokernel for the 3 variable case is as follows.

In degree 4, there are 2 generators of the cokernel which are x^2yz and xy^2z. By Lemma 5.28, the cokernel in this degree is isomorphic to $F^{(21)}$ where x^2yz and xy^2z are basis vectors.

In degree 5, there are 2 generators of the cokernel which are x^3yz and xy^3z. Hence the cokernel in this degree is also isomorphic to $F^{(21)}$ where x^3yz and xy^3z are basis vectors.

In degree 6, since the cokernel is 1−dimensional, it has to be isomorphic to $F^{(3)}$ where a generator can be any monomial in the form $[321]$.

By the results of Section 5.6, the cokernel is 2−dimensional in degree 2^m. A monomial in the form $[(2^m-2)11]$ is not hit. By Lemma 5.28, the cokernel in degree 2^m is isomorphic to $F^{(21)}$ where $x^{2^m-2}yz$ and $xy^{2^m-2}z$ are basis vectors.

In degree 2^m+1, the cokernel is 3−dimensional. Again by Lemma 5.28, the monomials $x^{2^m-1}yz$ and $xy^{2^m-1}z$ form basis of a submodule $F^{(21)}$. Recall the proof of Proposition 5.14, there are 3 generators for the cokernel in degree 2^m+1,

$$(1)\ \ x^{2^k-1}y^{2^k+1}z, \quad (2)\ \ x^{2^k-1}yz^{2^k+1}, \quad (3)\ \ xy^{2^k-1}z^{2^k+1}.$$

The submodule which is isomorphic to $F^{(3)}$ is generated by (1)+(2)+(3). Hence the cokernel in degree 2^m+1 is a direct sum of one copy of $F^{(21)}$ and one copy of $F^{(3)}$.

6 The hit problem on a polynomial ring of 4 variables over $\mathbb{F}_2$

In this section, we will explore the hit problem of the 4 variable case over $\mathbb{F}_2$. By Theorem 5.5, there is a generating set of $\mathbb{F}_2[x, y, z, t]xyzt$ under the action of $\mathcal{D}_2$ whose elements are in the following forms: $[a111]$, $[a211]$, $[a311]$, $[a321]$, $[a(2^k)(2^k-1)1]$, $[a(2^k+1)(2^k-1)1]$ for $k \geq 2$. We have determined the hit elements of the above forms except some cases for the monomials in the form $[a(2^k+1)(2^k-1)1]$. The results we have got give a general view of the location of the hit elements for the 4 variable case over $\mathbb{F}_2$. From our results, we can see that the cokernel of the 4 variable case is much more complex compared with the 2 variable case and the 3 variable case. This is because in the 4 variable case, the product of two non-hit monomials with no common variables is still a non-hit monomial, which can be in any degree $2^n + 2^m$.

6.1 Some general results for the 4 variable case

Lemma 6.1 *In degrees ≥ 8, a monomial $x^a y^b z^c t^d$ with at least three exponents even is hit under the action of $\mathcal{D}_2$.*

Proof: By Lemma 3.13 and Lemma 3.14. □

Again this result will often be used. If a monomial is a term of the image of any operation of $\mathcal{D}_2$ on $\mathbb{F}_2[x, y, z]xyz$ and has three even exponents in degrees ≥ 8, we will consider it hit and omit it (*mod hit*).

Lemma 6.2 *A monomial $x^a y^b z^c t^d$ is not hit over $\mathbb{F}_2$, if the monomial is in degree 2^m where at least two of a, b, c, d are odd, in degree 2^m+1 where three of a, b, c, d are odd and one of them is 1 or in degree 2^m+2 where all a, b, c, d are odd and at least two of them are 1's.*

Proof: By Proposition 5.3, Theorem 5.16 and Theorem 3.12. $\square$

Lemma 6.3 *In $E_1^k(x^u y^v z^w t^s)$ where k is even and one of u, v, w, s is even and $u+v+w+s+k \geq 8$, a nonzero term $x^a y^b z^c t^d$ is hit under the action of $\mathcal{D}_2$, if any a, b, c or d is even where the corresponding u, v, w or s is odd.*

Proof: By (5) of Section 2.1, we have

$$E_1^k(x^u y^v z^w t^s) = \sum_{i+j+l+m=k} \binom{u}{i}\binom{v}{j}\binom{w}{l}\binom{s}{m} x^{u+i} y^{v+j} z^{w+l} t^{s+m}. \tag{48}$$

Let $\binom{u}{i}\binom{v}{j}\binom{w}{l}\binom{s}{m} x^{u+i} y^{v+j} z^{w+l} t^{s+m}$ be a term in (48) which has a nonzero coefficient. We can never change an even exponent of $x^u y^v z^w t^s$ to odd in the image of E_1^k, since, for example, if u is even and i is odd, then $\binom{u}{i} \equiv 0 \ (mod\ 2)$. Hence if three of u, v, w and s are even then every term in (48) is hit by Lemma 6.1. Suppose at least two of u, v, w and s are odd. Assume that u is odd where $u+i$ is even. Then i is odd and hence one of j, l, m is odd, since k is even. Suppose j is odd, then v has to be odd in order that the coefficient is not 0. Because at least one of w, s is even, l, m have to be both even. So one of $w+l$ and $s+m$ is even. Hence the term has three even exponents so it is hit by Lemma 6.1. $\square$

Lemma 6.4 *A monomial $x^a y^b z^2 t$ is not hit under the action of $\mathcal{D}_2$, if $a+b=2^m$ for $m \geq 1$ and both a, b are odd. In particular, $x^u y^2 zt$ and $x^v y^3 z^2 t$ are not hit over $\mathbb{F}_2$ if $u+1=2^m$ and $v+3=2^m$.*

Proof: Suppose we want $x^a y^b z^2 t$, where $a,\ b$ are odd and $a+b=2^m$, in the image of the action of $\mathcal{D}_2$. We either use $E_r^k(x^u y^v z^2 t)$ for $r=1,\ 2$ or $E_1^k(x^u y^v zt)$ by Theorem 2.7. Suppose we use

$$E_r^k(x^u y^v z^2 t) \equiv \sum_{i+j=k} \binom{u}{i}\binom{v}{j} x^{u+i} y^{v+j} z^2 t + S \ (mod\ hit),$$

where S is a sum whose terms do not have $z^2 t$.

The number of monomials in the form $x^a y^b z^2 t$ with $a,\ b$ odd in $E_r^k(x^u y^v z^2 t)$ is congruent $(mod\ 2)$ to $\sum_{i+j=k} (u+i)\binom{u}{i}\binom{v}{j}$, since $u+i$ and $v+j$ are both even or both odd. By Lemma 3.3, $\sum_{i+j=k} (u+i)\binom{u}{i}\binom{v}{j} \equiv 0\ (mod\ 2)$, since $u+v+k=2^m$. Also monomials in the form $x^a y^b z t^2$ do not appear in $E_r^k(x^u y^v z^2 t)$. So the number of monomials in the form $[\{ab\}\{21\}]$ with $a,\ b$ odd in $E_r^k(x^u y^v z^2 t)$ is even.

Suppose we use

$$E_1^k(x^u y^v zt) = \sum_{i+j=k} E_1^i(x^u y^v) E_1^j(zt)$$

$$= E_1^k(x^u y^v)zt + (E_1^{k-1}(x^u y^v))(z^2 t + z t^2) + (E_1^{k-2}(x^u y^v))z^2 t^2,$$

where $u+v+k=2^m-1$. A monomial in the form $x^a y^b z^2 t$ always appears with a monomial in the form $x^a y^b z t^2$. Hence the number of monomials in the form $[\{ab\}\{21\}]$ with $a,\ b$ odd in $E_1^k(x^u y^v zt)$ is even.

Hence the number of monomials in the form $[\{ab\}\{21\}]$ is always even in the image of any operation of $\mathcal{D}_2$. Hence $x^a y^b z^2 t$ is not hit. □

Proposition 6.5 *Suppose a monomial $fg \in \mathbb{F}_2[x,y,z,t]xyzt$ satisfies that $f,\ g$ have no common variables and both are not hit under the action of $\mathcal{D}_2$, then fg is not hit.*

Proof: If one of $f,\ g$ has 3 variables, then the other one is a single variable of degree 1. Hence fg is not hit by Theorem 3.12. Suppose $f,\ g$ both have two distinct variables and f and g are not hit. Let f be in the form $[a_1b_1]$ and g be in the form $[a_2b_2]$.

If one of $f,\ g$ is in the form [11], fg is not hit by applying Theorem 3.12 twice. If both $f,\ g$ are monomials of the form [12], then fg is not hit. The detail can be seen in Section 6.2.

If f is a monomial in the form [21], g is in a degree ≥ 4 and is not hit, i.e. g is in degree 2^m for $m \geq 2$ and has two odd exponents by Theorem 5.4. By Lemma 6.4 fg is not hit. Similarly for the case in which g is a monomial in the form [21] and f has two odd exponents and is in degree 2^m for $m \geq 2$. Hence fg is not hit.

Suppose both $f,\ g$ are in degree 2^m for $m \geq 2$. By the proof of Proposition 5.2, monomials in the form $[a_1b_1]$ and $[a_2b_2]$ appear in even numbers under operations of E_r^k where $r = 1,\ 2$.

Suppose we try to have fg in $E_r^k(f_kg_k)$ where $r = 1,\ 2$ and $f_k,\ g_k$ have 2 distinct variables. Then there are an even number of monomials in the form $[\{a_1b_1\}\{a_2b_2\}]$ in each $E_r^k(f_kg_k) = \sum_{i+j=k} E_r^i(f_k)E_r^j(g_k)$ where $r = 1,\ 2$. Because at least one of $i,\ j \neq 0$, so we have either there are an even number of monomials in the form $[a_1b_1]$ in any $E_r^i(f_k)$ or there are an even number of monomials of form $[a_2b_2]$ in any $E_r^j(g_k)$. Hence there are an even number of monomials in the form $[\{a_1b_1\}\{a_2b_2\}]$ in the image of E_r^k for $r = 1,\ 2$. So there are an even number of monomials in the form $[\{a_1b_1\}\{a_2b_2\}]$ in the image of any operation of $\mathcal{D}_2$, since the set $\{E_r^k \mid k \geq 1,\ r = 1,\ 2\}$ generates $\mathcal{D}_2$ (Theorem 2.7). Hence fg is not hit.

□

Theorem 6.6 *If a monomial f of n variables can be written as a product of non-hit monomials of 2 variables and there are no two monomials having common variables, then f is not hit under the action of $\mathcal{D}_2$.*

Proof: Let $m = n/2$. We write $f = \prod_{t=1}^{m} g_t$ where each g_t is a monomial in two variables and g_i, g_j have no common variables if $i \neq j$. Let $f_s = \prod_{t=1}^{s} g_t$ where $1 \leq s \leq m$ and $f_m = f$. We will prove f is not hit by induction on s. Suppose f_s is a monomial in the form of $[\{a_1a_2\}\{\cdots\cdots\}\{a_{2s-1}a_{2s}\}]$. By the proof of Proposition 5.2, we assume up to s, f_s is not hit and the monomials in the form $[\{a_1a_2\}\{\cdots\cdots\}\{a_{2s-1}a_{2s}\}]$ appear in even numbers under operations of E_r^k where $r = 1,\ 2$. If $s < m$ and g_{s+1} is a monomial in the form $[a_{2s+1}a_{2s+2}]$, then $f_{s+1} = f_s g_{s+1}$ is a monomial in the form $[\{a_1a_2\}\{\cdots\cdots\}\{a_{2s-1}a_{2s}\}\{a_{2s+1}a_{2s+2}\}]$.

Suppose we try to have f_{s+1} in

$$\sum_k E_r^k(u_kv_k) = \sum_k \sum_{i+j=k} E_r^i(u_k)E_r^j(v_k),$$

where $r = 1,\ 2$. Here u_k has the same variables as f_s and v_k has the same variables as g_{s+1}. Since $k > 0$, we have that either there is an even number of monomials in the form $[\{a_1a_2\}\{\cdots\cdots\}\{a_{2s-1}a_{2s}\}]$ in $E_r^i(u_k)$ or there are an even number of monomials in the form $[a_{2s+1}a_{2s+2}]$ in $E_r^i(v_k)$. Hence there are an even number of monomials of the form $[\{a_1a_2\}\{\cdots\cdots\}\{a_{2s-1}a_{2s}\}\{a_{2s+1}a_{2s+2}\}]$ in each $E_r^k(u_kv_k)$, hence there must be an even number of monomials in this form in $\sum_k E_r^k(u_kv_k)$. Hence f_{s+1} is not hit. By the induction hypothesis and by Theorem 2.7, f is not hit under the action of $\mathcal{D}_2$. $\square$

6.2 Monomials in degrees ≤ 10

By Theorem 2.7, we only need to check the operations $E_1^{2^k}$ and $E_2^{2^k}$ for $k = 0$ or $k \in \mathbb{N}$.

Degree 4 is trivial. In degree 5, there are 4 unknowns in the form [2111] and only one operation $D_1(xyzt)$ which has image in this degree. Hence three of the monomials in the form [2111] need to be generators.

In degree 6, we need to consider the following equations which involve the monomials in the form [2211]:

$$\begin{cases} D_1(x^2yzt) \equiv x^2y^2zt + x^2yz^2t + x^2yzt^2 \\ D_1(xy^2zt) \equiv x^2y^2zt + xy^2z^2t + xy^2zt^2 \\ D_1(xyz^2t) \equiv x^2yz^2t + xy^2z^2t + xyz^2t^2 \\ D_1(xyzt^2) \equiv x^2yzt^2 + xy^2zt^2 + xyz^2t^2 \\ E_1^2(xyzt) = x^2y^2zt + x^2yz^2t + x^2yzt^2 + xy^2z^2t + xy^2zt^2 + xyz^2t^2 \end{cases} \qquad (mod\ 2)$$

The coefficient matrix of the above equations has rank 4, hence 2 of the monomials in the form [2211] need to be generators. Note that there are 3 Artin elements in the form [2211]. There is only one equation involving the monomials in the form [3111], which is

$$D_2(xyzt) = x^3yzt + xy^3zt + xyz^3zt + xyzt^3.$$

Hence we need 3 of the monomials in the form [3111] to be generators. Note that there are only 2 Artin elements in the form [3111].

In degree 7, by Theorem 2.9, we have

$$x^2y^2z^2t = (E_1^3(xyz))t \equiv xyz(\chi(E_1^3))(t) \equiv xyzt^4 \ (mod\ hit). \qquad (49)$$

We need to consider the following equations which involve the monomials in the form [4111] and [2221]:

$$
\left\{
\begin{array}{l}
D_1(x^2y^2zt) \equiv x^2y^2z^2t + x^2y^2zt^2 \\
D_1(x^2yz^2t) \equiv x^2y^2z^2t + x^2yz^2t^2 \\
D_1(x^2yzt^2) \equiv x^2y^2zt^2 + x^2yz^2t^2 \\
D_1(xy^2z^2t) \equiv x^2y^2z^2t + xy^2z^2t^2 \\
D_1(xy^2zt^2) \equiv x^2y^2zt^2 + xy^2z^2t^2 \\
D_1(xyz^2t^2) \equiv x^2yz^2t^2 + xy^2z^2t^2 \\
E_1^2(x^2yzt) = x^4yzt + x^2y^2z^2t + x^2y^2zt^2 + x^2yz^2t^2 \\
E_1^2(xy^2zt) = xy^4zt + x^2y^2z^2t + x^2y^2zt^2 + xy^2z^2t^2 \\
E_1^2(xyz^2t) = xyz^4t + x^2y^2z^2t + x^2yz^2t^2 + xy^2z^2t^2 \\
E_1^2(xyzt^2) = xyzt^4 + x^2y^2zt^2 + x^2yz^2t^2 + xy^2z^2t^2
\end{array}
\right.
\qquad (mod\ 2)
$$

In the last 4 equations, we can substitute the monomials in the form [4111] by the monomials in the form [2221], based on some suitable permutations on (49). Then we get that the right hand sides of the last 4 equations of the above equations are identical, so we only include one of them. Hence the coefficient matrix of the above equation system is:

$$
\begin{array}{cccc}
x^2y^2z^2t & x^2y^2zt^2 & x^2yz^2t^2 & xy^2z^2t^2
\end{array}
$$
$$
\begin{pmatrix}
1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 \\
1 & 1 & 1 & 1
\end{pmatrix}.
$$

The matrix has rank 3, so we need one monomial in the form [2221] to be a generator.

We have the following equations involving monomials in the form [3211],

$$\left\{\begin{array}{l} D_1(x^3yzt) \equiv x^4yzt + x^3y^2zt + x^3yz^2t + x^3yzt^2 \\ D_1(xy^3zt) \equiv x^2y^3zt + xy^4zt + xy^3z^2t + xy^3zt^2 \\ D_1(xyz^3t) \equiv x^2yz^3t + xy^2z^3t + xyz^4t + xyz^3t^2 \\ D_1(xyzt^3) \equiv x^2yzt^3 + xy^2zt^3 + xyz^2t^3 + xyzt^4 \\ D_2(x^2yzt) \equiv x^2y^3zt + x^2yz^3t + x^2yzt^3 \\ D_2(xy^2zt) \equiv x^3y^2zt + xy^2z^3t + xy^2zt^3 \\ D_2(xyz^2t) \equiv x^3yz^2t + xy^3z^2t + xyz^2t^3 \\ D_2(xyzt^2) \equiv x^3yzt^2 + xy^3zt^2 + xyz^3t^2 \end{array}\right. \quad (mod\ 2)$$

If the monomials in the form [4111] are given, then the coefficient matrix indexed by the monomials in the form [3211] of the above equation system has rank 7. Hence 5 monomials in the form [3211] need to be generators. So the cokernel is 6 dimensional in this degree. Note that there is no monomial to be hit in degrees ≤ 7.

In degree 8, by Lemma 3.11 and Lemma 6.2, monomials in the form [2222] are the only monomials which are hit. In degree 9, monomials in the forms [6111], [5211], [4311] and [3321] are not hit by Lemma 6.2. Monomials in the forms [4221] and [3222] are hit by Lemma 6.1. In degree 10, by Lemma 6.2, monomials in the forms [7111] and [5311] are not hit. Monomials in the form [4222] are hit by Lemma 3.11. By Lemma 6.3, $E_1^2(x^2y^4zt) \equiv x^4y^4zt\ (mod\ hit)$. Hence monomials in the form [4411] are hit. By Theorem 2.9,

$$x^3y^3z^3t = E_2^3(xyz)t \equiv xyz(\chi(E_2^3)(t)) \equiv xyz(E_2^3 + E_2^2D_2 + D_2\chi(E_2^2))(t)$$

$$\equiv xyz(E_2^2D_2 + D_2D_2D_2)(t) \equiv 0\ (mod\ hit),$$

hence monomials in the form [3331] are hit. From

$$D_2(x^3yz^2t^2) \equiv x^5yz^2t^2 + x^3y^3z^2t^2 \ (mod\ 2),$$

$$D_2(xy^3z^2t^2) \equiv x^3y^3z^2t^2 + xy^5z^2t^2 \ (mod\ 2),$$

we get

$$x^5yz^2t^2 \equiv x^3y^3z^2t^2 \equiv xy^5z^2t^2 \ (mod\ hit). \tag{50}$$

Also

$$E_2^2(x^2y^2zt) \equiv x^6y^2zt + x^2y^6zt + x^2y^2z^3t^3 \ (mod\ hit), \tag{51}$$

$$x^5yz^2t^2 = x^5yE_1^2(zt) \equiv ((\chi(E_1^2))(x^5y))zt$$

$$\equiv (E_1^2(x^5y))zt \equiv x^6y^2zt \ (mod\ hit), \tag{52}$$

by Theorem 2.9. By applying a suitable permutation on (52), we get

$$xy^5z^2t^2 \equiv x^2y^6zt \ (mod\ hit), \tag{53}$$

By (50), (51), (52) and (53), we get $x^2y^2z^3t^3$ is hit. So the monomials in the form [3322] are hit and monomials in the forms [6211] and [5221] are hit as well by the above relations.

We have carried out some calculation with << Maple >>. By our calculation result, the dimension of the cokernel in degree 8 and the dimension of the cokernel in degree 9 are 6, the dimension of the cokernel in degree 10 is 7. Monomials in the form [4321] are not hit. Any monomial in the form [4321] generates all monomials in the form [4321] and 6 monomials in the form [5311] generate the other non−hit monomials. We will give a proof for the general case of the cokernel in degree $2^m + 2$ in Section 6.8.

6.3 Monomials in the form $[a(2^l)(2^k)1]$ for $l \geq k \geq 1$

We need to check the monomials in the form $[a221]$. We will firstly check monomials in the form $[a421]$, the result is needed for the $[a221]$ case. We will also check a general case for the monomials in the form $[a(2^l)(2^k)1]$, the result is needed in the proofs of some other results in the remaining content of this section.

Proposition 6.7 *A monomial in the form $[a421]$ is hit under the action of $\mathcal{D}_2$, if $a \geq 7$ and $a \equiv 3$ (mod 4).*

Proof: Look into the proof of Proposition 5.20, the operations used in the proof are D_k's, E_1^2 and E_2^2, they have no effect on y^4. So we have:

$$D_2(x^{a-2}y^4zt^2) \equiv x^ay^4zt^2 + x^{a-2}y^4z^3t^2 \ (mod\ 2).$$

$$E_2^2(x^{a-4}y^4zt^2) \equiv x^ay^4zt^2 + x^{a-2}y^4z^3t^2 + x^{a-4}y^4zt^6 \ (mod\ 2).$$

We have that $x^{a-4}y^4zt^6$ is hit. Also

$$D_4(x^{a-4}y^4z^2t) \equiv x^ay^4z^2t + x^{a-4}y^4z^2t^5 \ (mod\ 2).$$

$$D_1(x^{a-4}y^4zt^5) \equiv x^{a-3}y^4zt^5 + x^{a-4}y^4z^2t^5 + x^{a-4}y^4zt^6 \ (mod\ 2).$$

We get $x^ay^4z^2t + x^{a-3}y^4zt^5 \equiv 0$ (*mod hit*). By Lemma 6.3,

$$E_1^2(x^{a-5}y^4zt^5) \equiv \binom{5}{2}x^{a-3}y^4zt^7 + \binom{a-5}{2}x^{a-3}y^4zt^5 \equiv x^{a-3}y^4zt^5 \ (mod\ hit),$$

since the 1st digit of the binary expansion of $a-5$ is 1 and $\binom{5}{2} \equiv 0$ (*mod* 2). Hence $x^{a-3}y^4zt^5$ is hit, and so $x^ay^4z^2t$ is hit. □

Note that $x^ay^4z^2t$ in degree $2^m + 2$ for $m \geq 4$ is hit by Proposition 6.7.

Proposition 6.8 *A monomial in the form $[a421]$ is hit under the action of $\mathcal{D}_2$ if it is not in degree 2^m and $a \geq 4$.*

Proof: By Lemma 6.1 and Proposition 6.7, we only consider an a where $a_1 = 0$, $a_0 = 1$, i.e. $a \equiv 1\ (mod\ 4)$.

(1) Suppose there is an $a_i = 0$ for $3 \leq i \leq n-1$. By Lemma 6.3,

$$E_1^{2^i}(x^{a-2^i}y^4z^2t) \equiv x^ay^4z^2t + \binom{a-2^i}{2^i-4}x^{a-4}y^8z^2t\ (mod\ hit), \tag{54}$$

since $a_1 = 0$, $\binom{a-2^i}{2^i-2} \equiv 0\ (mod\ 2)$ and hence if z^2 changes to z^4 then the two odd exponents must change to even as well, and then the term is hit by Lemma 6.1. Also $\binom{a-2^i}{2^i-4} \equiv 0\ (mod\ 2)$ unless $a_2 = 1$. Then the 2nd digit of the binary expansion of $a-4$ is 0. So by Lemma 6.3,

$$\begin{aligned} E_1^4(x^{a-4}y^4z^2t) &\equiv \binom{a-4}{4}x^ay^4z^2t + \binom{a-4}{2}x^{a-2}y^4z^4t + x^{a-4}y^8z^2t \\ &\equiv x^{a-4}y^8z^2t\ (mod\ hit), \end{aligned}$$

since the 1st and the 2nd digits of the binary expansion of $a-4$ are 0's, we have $\binom{a-4}{2} \equiv \binom{a-4}{4} \equiv 0\ (mod\ 2)$. Hence $x^{a-4}y^8z^2t$ is hit and by (54) $x^ay^4z^2t$ is hit as well.

(2) Suppose $a_1 = 0$ and $a_i = 1$ for $i \neq 1$, i.e. $a = 2^m - 3$ for $m \geq 3$. Then by Lemma 6.3,

$$E_1^2(x^{a-2}y^4z^2t) \equiv x^ay^4z^2t + x^{a-2}y^4z^4t\ (mod\ hit), \tag{55}$$

and

$$\begin{aligned} x^{a-2}y^4z^4t = E_3^2(yz)x^{a-2}t &\equiv (\chi(E_3^2)(x^{a-2}t))yz \\ &\equiv x^{a+4}yzt + x^{a+1}yzt^4\ (mod\ hit), \end{aligned} \tag{56}$$

since $\chi(E_3^2) \equiv E_3^2\ (mod\ 2)$ by Example 2.8.

By Lemma 6.1 and since $\binom{a}{2} \equiv \binom{a}{3} \equiv 0\ (mod\ 2)$, we get

$$E_1^4(x^ayzt) \equiv x^{a+1}y^2z^2t^2 + x^{a+4}yzt \equiv x^{a+4}yzt\ (mod\ hit),$$

hence $x^{a+4}yzt$ is hit. Also

$$E_1^2(x^{a+1}yzt^2) \equiv x^{a+1}yzt^4 + x^{a+3}yzt^2 \ (mod\ hit). \tag{57}$$

Now $a+3 = 2^m$ for $m \geq 3$, so

$$E_1^4(x^{a-1}yzt^2) \equiv \binom{a-1}{2} x^{a+1}yzt^4 + x^{a+3}yzt^2 \equiv x^{a+3}yzt^2 \ (mod\ hit),$$

by Lemma 6.3 and $\binom{a-1}{2} \equiv 0 \ (mod\ 2)$. Hence $x^{a+1}yzt^4$ is hit by (57) and $x^{a-2}y^4z^4t$ is hit by (56). Finally $x^ay^4z^2t$ is hit by (55).

(3) The only case left is $a_1 = a_2 = 0$ and $a_i = 1$ for $i \neq 1, 2$, i.e. $a = 2^m - 7$ for $m \geq 4$. But then $x^ay^4z^2t$ is in degree 2^m. □

Proposition 6.9 *A monomial in the form [a221] is hit under the action of $\mathcal{D}_2$ if $a \geq 6$ and it is not in degree 2^m.*

Proof: If a is even, then $x^ay^2z^2t$ is hit by Lemma 6.1. Let $7 \leq a$ be odd.

(1) Suppose there is an $a_i = 0$ for $2 \neq i \geq 1$. Let a_i be the first 0 counted from the right for $2 \neq i \geq 1$, then by Lemma 6.3 and Proposition 6.8,

$$E_1^{2^i}(x^{a-2^i}y^2z^2t) \equiv x^ay^2z^2t + x^{a-2}y^4z^2t + x^{a-2}y^2z^4t + rx^{a-4}y^4z^4t$$

$$\equiv x^ay^2z^2t + rx^{a-4}y^4z^4t \ (mod\ hit),$$

where $r = \binom{a-4}{2^i-4}$. If $r = 0$ then $x^ay^2z^2t$ is hit, otherwise by Lemma 6.3 and Proposition 6.8,

$$E_1^2(x^{a-4}y^4z^2t) \equiv x^{a-4}y^4z^4t + \binom{a-4}{2} x^{a-2}y^4z^2t \equiv x^{a-4}y^4z^4t \ (mod\ hit).$$

Hence $x^ay^2z^2t$ is hit in this case.

(2) If $a = 2^m - 1$ for $m \geq 3$, then by Theorem 2.9 and Lemma 6.3 we have the following equations,

$$x^a y^2 z^2 t = x^a t E_1^2(yz) \equiv \chi(E_1^2)(x^a t) yz \equiv x^{a+2} yzt + x^{a+1} yzt^2 \ (mod\ hit), \tag{58}$$

$$E_1^4(x^{a-2} yzt) \equiv x^{a+2} yzt \ (mod\ hit), \tag{59}$$

$$E_1^4(x^{a-3} yzt^2) \equiv x^{a+1} yzt^2 + \binom{a-3}{2} x^{a-1} yzt^4 \equiv x^{a+1} yzt^2 \ (mod\ 2). \tag{60}$$

For (59), $\binom{a-2}{i} \equiv 0\ (mod\ 2)$ for $1 \leq i \leq 4$, unless $i = 1$ or 4. For (60), $\binom{a-3}{2} \equiv 0\ (mod\ 2)$ since the 1st digit of the binary expansion of $a-3$ is 0. By (58), (59) and (60) $x^a y^2 z^2 t$ is hit.

(3) The only case left is $a_2 = 0$ and $a_j = 1$ for $j \neq 2$. But then the monomial is in degree 2^m. □

Proposition 6.10 *A monomial in the form $[a(2^i)(2^j)1]$ for $i,\ j \geq 1$ is hit under the action of $\mathcal{D}_2$, if it is in a degree ≥ 11 which is not 2^m.*

Proof: By the previous discussion, we may assume that $i+j \geq 4$. If a is even, then $x^a y^{2^i} z^{2^j} t$ is hit by Lemma 6.1. Let a be odd. We may assume that $j \geq i$. If $i = 1$, we go to (2). Suppose $i \geq 2$.

(1) We claim that $x^a y^{2^i} z^{2^j} t$ is either hit or $\equiv x^{a+2^i-4} y^4 z^{2^j} t\ (mod\ hit)$. We argue by induction on the exponent of y. If $i > 2$, by Lemma 6.3 and since $\binom{2^{i-1}}{s} \equiv 0\ (mod\ 2)$ unless $s = 0$, or 2^{i-1},

$$E_1^{2^{i-1}}(x^a y^{2^{i-1}} z^{2^j} t) \equiv x^a y^{2^i} z^{2^j} t + r x^{a+2^{i-1}} y^{2^{i-1}} z^{2^j} t\ (mod\ hit),$$

where $r = \binom{a}{2^{i-1}}$, note that the operation has no effect on the exponent of z since $i \leq j$.

We get that if $r \equiv 0 \ (mod\ 2)$ $x^a y^{2^i} z^{2^j} t$ is hit, otherwise

$$x^a y^{2^i} z^{2^j} t \equiv x^{a+2^{i-1}} y^{2^{i-1}} z^{2^j} t \ (mod\ hit).$$

By the induction hypothesis, we must get $x^a y^{2^i} z^{2^j} t$ is either hit or

$$x^a y^{2^i} z^{2^j} t \equiv x^{a+2^i-4} y^4 z^{2^j} t \ (mod\ hit). \tag{61}$$

Then also by Lemma 6.3

$$E_1^2(x^{a+2^i-4} y^2 z^{2^j} t) \equiv x^{a+2^i-4} y^4 z^{2^j} t + \binom{a+2^i-4}{2} x^{a+2^i-2} y^2 z^{2^j} t \ (mod\ hit).$$

If $\binom{a+2^i-4}{2} \equiv 0 \ (mod\ 2)$ then $x^{a+2^i-4} y^4 z^{2^j} t$ is hit and hence $x^a y^{2^i} z^{2^j} t$ is hit by (61). Otherwise

$$x^a y^{2^i} z^{2^j} t \equiv x^{a+2^i-4} y^4 z^{2^j} t \equiv x^{a+2^i-2} y^2 z^{2^j} t \ (mod\ hit)$$

and the 1st digit of the binary expansion of $a+2^i-4$ is 1. So the 1st digit of the binary expansion of $a+2^i-2$ is 0.

(2) Let $a' = a+2^i-2$. We claim that either $x^{a'} y^2 z^{2^j} t$ is hit or

$$x^{a'} y^2 z^{2^j} t \equiv x^{a+2^j+2^i-6} y^2 z^4 t \ (mod\ hit).$$

We argue by induction on the exponent of z. By Lemma 6.3 and since $\binom{2^{j-1}}{s} \equiv 0 \ (mod\ 2)$ unless $s = 0$ or 2^{j-1},

$$E_1^{2^{j-1}}(x^{a'} y^2 z^{2^{j-1}} t) \equiv \binom{a'}{2^{j-1}-2} x^{a'+2^{j-1}-2} y^4 z^{2^{j-1}} t + x^{a'} y^2 z^{2^j} t + r x^{a'+2^{j-1}} y^2 z^{2^{j-1}} t$$
$$\equiv x^{a'} y^2 z^{2^j} t + r x^{a'+2^{j-1}} y^2 z^{2^{j-1}} t \ (mod\ hit),$$

where $r = \binom{a'}{2^{j-1}} \cdot \binom{a'}{2^{j-1}-2} \equiv 0 \ (mod\ 2)$ since the 1st digit of the binary expansion of a' is 0. Hence $x^{a'} y^2 z^{2^j} t$ is hit if $r \equiv 0 \ (mod\ 2)$, otherwise

$$x^{a'} y^2 z^{2^j} t \equiv x^{a'+2^{j-1}} y^2 z^{2^{j-1}} t \ (mod\ hit).$$

Note the 1st digit of the binary expansion of the exponent of $a' + 2^{j-1}$ is still 0 for $j > 2$, we can repeat again if the exponent of $z > 4$. By the induction hypothesis, we must get either $x^{a'}y^2z^{2^j}t$ is hit or

$$x^{a'}y^2z^{2^j}t \equiv x^{a''}y^2z^4t \ (mod\ hit), \tag{62}$$

where $a'' = a + 2^j + 2^i - 6$. $x^{a''}y^2z^4t$ is hit by Proposition 6.8, since it is not in degree 2^m. Hence $x^ay^{2^i}z^{2^j}t$ is hit by (61) and (62). $\square$

Corollary 6.11 *A monomial in the form $[ab(2^i)(2^j)]$ for $i,\ j \geq 1$ is hit under the action of $\mathcal{D}_2$ if it is in a degree ≥ 11 which is not 2^m.*

Proof: If any of $a,\ b$ is even then the monomial is hit by Lemma 6.1. If one of $a,\ b = 1$, then $x^ay^bz^{2^i}t^{2^j}$ is hit by Proposition 6.10. Suppose $1 < a,\ b$ are odd. $D_{b-1}(x^ayz^{2^i}t^{2^j}) \equiv x^ay^bz^{2^i}t^{2^j} + x^{a+b-1}yz^{2^i}t^{2^j}\ (mod\ 2)$. $x^{a+b-1}yz^{2^i}t^{2^j}$ is hit by Proposition 6.10, hence $x^ay^bz^{2^i}t^{2^j}$ is hit. $\square$

6.4 Monomials in the forms $[a211]$ and $[a111]$

Proposition 6.12 *A monomial in the form $[a211]$ is hit under the action of $\mathcal{D}_2$, if $a \geq 8$ and there is an $a_i = 0$ for $i \geq 2$ in the binary expansion of a.*

Proof: Suppose $a_i = 0$ for some $i \geq 2$, then

$$E_1^{2^i}(x^{a-2^i}y^2zt) \equiv x^ay^2zt + \binom{a-2^i}{2^i-2}x^{a-2}y^4zt\ (mod\ hit),$$

by Lemma 6.3. Hence x^ay^2zt is hit if $\binom{a-2^i}{2^i-2} \equiv 0\ (mod\ 2)$.

Suppose $\binom{a-2^i}{2^i-2} \not\equiv 0\ (mod\ 2)$ then $a_j = 1$ for $1 \leq j \leq i-1$. Hence the 1st digit of the binary expansion of $a-2$ is 0. So

$$E_1^2(x^{a-2}y^2zt) \equiv \binom{a-2}{2}x^ay^2zt + x^{a-2}y^4zt \equiv x^{a-2}y^4zt\ (mod\ hit),$$

by Lemma 6.3 and $\binom{a-2}{2} \equiv 0 \ (mod\ 2)$. Hence $x^a y^2 zt$ is hit. $\square$

Proposition 6.13 *A monomial in the form [a211] is hit under the action of $\mathcal{D}_2$, if $8 \leq a$ is even and it is not in degree 2^m.*

Proof: $D_1(x^{a-1}y^2zt) \equiv x^a y^2 zt + x^{a-1}(y^2z^2t + y^2zt^2 + yz^2t^2) \ (mod\ 2)$. But monomials in the form $[(a-1)221]$ are hit by Proposition 6.9. $\square$

Proposition 6.14 *A monomial in the form [a211] is hit under the action of $\mathcal{D}_2$ if $a \geq 8$ and it is not in degrees 2^m, $2^m + 1$ and $2^m + 3$.*

Proof: A monomial in the form $[a211]$ is not hit if it is in degrees 2^m, $2^m + 1$ and $2^m + 3$ by Lemma 6.2 and Lemma 6.4. By Proposition 6.12 and Proposition 6.13, the only case which has not been discussed is $a_1 = 0$ and $a_i = 1$ for $i \neq 1$ in the binary expansion of a, but then the degree is $2^m + 1$. $\square$

Proposition 6.15 *A monomial in the form [a111] for $a \geq 8$ is hit under the action of $\mathcal{D}_2$, if it is not in degrees 2^m, $2^m + 1$ and $2^m + 2$.*

Proof: If $x^a yzt$ is in degrees 2^m, 2^m+1 and 2^m+2, it is not hit by Lemma 6.2. Let $a_n a_{n-1} \ldots a_0$ be the binary expansion of a. Suppose $x^a yzt$ is not in degrees 2^m, $2^m + 1$ and $2^m + 2$, then there is an $a_i = 0$ for $i \geq 2$ or $a_0 = a_1 = 0$ in the binary expansion of a.

(1) Suppose $a_0 = a_1 = 0$, then

$$E_1^2(x^{a-2}yzt) \equiv x^a yzt + x^{a-2}(y^2z^2t + y^2zt^2 + yz^2t^2) \ (mod\ hit).$$

But monomials in the form $x^{a-2}y^2z^2t$ are hit by Proposition 6.9. Hence $x^a yzt$ is hit in this case.

(2) Suppose $a_i = 0$ for some $i \geq 2$ and if $i > 2$ then $a_j = 1$ for $2 \leq j < i$.

a) If $a_0 = 0$, then

$$E_1^{2^i}(x^{a-2^i}yzt) \equiv x^a yzt + x^{a-2}(y^2z^2t + y^2zt^2 + yz^2t^2) \equiv x^a yzt \ (mod\ hit),$$

again by Proposition 6.9.

b) If $a_0 = 1$, then

$$E_1^{2^i}(x^{a-2^i}yzt) \equiv x^a yzt + x^{a-1}(y^2zt + yz^2t + yzt^2) + x^{a-2}(y^2z^2t + y^2zt^2 + yz^2t^2)$$

$$\equiv x^a yzt + x^{a-1}(y^2zt + yz^2t + yzt^2) \equiv x^a yzt \ (mod\ hit),$$

by Proposition 6.9 and since $D_1(x^{a-1}yzt) \equiv x^{a-1}(y^2zt + yz^2t + yzt^2)$ $(mod\ 2)$.

Hence $x^a yzt$ is hit and so monomials in the form $[a111]$ are hit if they satisfy the condition. □

Theorem 6.16 *A monomial $x^a y^b z^c t^d$ with two exponents even is hit under the action of $\mathcal{D}_2$, if it is in a degree ≥ 11 which is not 2^m.*

Proof: If three or four of a, b, c, d are even then $x^a y^b z^c t^d$ is hit by Lemma 6.1. Let two of a, b, c, d be even and the other two be odd. We may assume that a, b are even and c, d are odd. Since

$$D_{d-1}(x^a y^b z^c t) \equiv x^a y^b z^c t^d + x^a y^b z^{d+c-1} t \ (mod\ 2), \tag{63}$$

we may only consider the monomials in the form $[abc1]$ where a, b are even and c is odd.

(1) If $c = 2^k - 1$, we move to (2). Suppose $c \neq 2^n - 1$ for any $n \geq 1$. Let k be the number of 1's in the binary expansion of c. Then by Lemma 3.15, $z^c \equiv D(z^{2^k-1})$ $(mod\ 2)$ where D is a composition of some E_1^r's. Hence by Theorem 2.9,

$$x^a y^b z^c t \equiv (D(z^{2^k-1}))x^a y^b t \equiv z^{2^k-1}(\chi(D))(x^a y^b t) \ (mod\ hit). \tag{64}$$

Let $D' = \chi(D)$ which is a sum of some compositions of E_1^r's. We may see this by considering χ to be the conjugate of the Steenrod algebra since there are only E_1^r operations involved in D. We have

$$z^{2^k-1}(D'(x^a y^b t)) \equiv z^{2^k-1}t(D'(x^a y^b)) \equiv \sum_i f_i z^{2^k-1}t \ (mod\ hit), \qquad (65)$$

where each f_i is a monomial in x, y which has two even exponents. This is because that a, b are even and D' is a sum of some compositions of E_1^r's, if the exponent of t of a term in $z^{2^k-1}D'(x^a y^b t)$ changes, it can only change to 2 firstly. Then the term is hit by Lemma 6.1. So we shall only consider the terms which have the exponent of t remaining unchanged. Also the exponents of x and y will remain even in the process. So we only consider monomials in the form $x^u y^v z^{2^k-1}t$ where u, v are even in $z^{2^k-1}D'(x^a y^b t)$.

(2) Now

$$D_1(x^{u-1}y^v z^{2^k-1}t) \equiv x^u y^v z^{2^k-1}t + x^{u-1}y^v z^{2^k}t + x^{u-1}y^v z^{2^k-1}t^2 \ (mod\ hit). \qquad (66)$$

Let s be the number of 1's in the binary expansion of $u-1$. By Lemma 3.15, there is a D which is a composition of E_1^r's and $D(x^{2^s-1}) \equiv x^{u-1}$ $(mod\ 2)$. Hence by a similar argument as what is used in part (1),

$$x^{u-1}y^v z^{2^k}t \equiv (D(x^{2^s-1}))y^v z^{2^k}t \equiv x^{2^s-1}t(\chi(D))(y^v z^{2^k}) \ (mod\ hit).$$

Since $\chi(D)$ is a compositions of E_1^r's, so each E_1^r acts on $y^v z^{2^k}$, z^{2^k} can either remain unchanged or be squared, and the exponent of y of each term in $(\chi(D))(y^v z^{2^k})$ remains even. Hence we get

$$x^{u-1}y^v z^{2^k}t \equiv x^{2^s-1}t(\sum_i y^{v_i}z^{2^i}) \ (mod\ hit), \qquad (67)$$

where all v_i's are even. Then for each term $x^{2^s-1}y^{v_i}z^{2^i}t$, we have

$$D_1(x^{2^s-1}y^{v_i-1}z^{2^i}t) \equiv x^{2^s-1}y^{v_i}z^{2^i}t + x^{2^s}y^{v_i-1}z^{2^i}t + x^{2^s-1}y^{v_i-1}z^{2^i}t^2 \ (mod\ 2).$$

But the last 2 terms are hit by Corollary 6.11. Hence every $x^{2^s-1}y^{v_i}z^{2^i}t$ in (67) is hit and so $x^{u-1}y^v z^{2^k}t$ is hit. For $x^{u-1}y^v z^{2^k-1}t^2$, we have

$$D_{2^k-2}(x^{u-1}y^v zt^2) \equiv x^{u+2^k-3}y^v zt^2 + x^{u-1}y^v z^{2^k-1}t^2 \ (mod\ 2). \tag{68}$$

By Lemma 3.15, we may write $x^{u+2^k-3} \equiv D(x^{2^s-1})$ $(mod\ 2)$ for some D which is a composition of E_1^r's and s is the number of 1's in the binary expansion of $u+2^k-3$. Then by a similar argument,

$$x^{u+2^k-3}y^v zt^2 \equiv (D(x^{2^s-1}))y^v zt^2 \equiv x^{2^s-1}z(\chi(D))(y^v t^2)$$

$$\equiv x^{2^s-1}z(\sum_i y^{v_i}t^{2^i}) \ (mod\ hit). \tag{69}$$

Also by a similar argument, every term in the form $x^{2^s-1}zy^{v_i}t^{2^i}$ in (69) is hit. Hence $x^{u+2^k-3}y^v zt^2$ is hit and $x^{u-1}y^v z^{2^k-1}t^2$ is hit by (68). So every monomial in the form $x^u y^v z^{2^k-1}t$ in $z^{2^{k+2}-1}D'(x^u y^v t)$ is hit by (66). By (63), (64), (65), $x^a y^b z^c t^d$ is hit. □

6.5 Monomials in the form $[a321]$

Before we prove the $[a321]$ case, we will firstly prove the following lemma which is needed for the proof of the $[a321]$ case and for some of remaining part of this section.

Lemma 6.17 *A monomial in the form* $[u521]$ *is hit under the action of* $\mathcal{D}_2$*, if* $4 \le u$*,* $u \equiv 0$ *or* 1 $(mod\ 4)$ *and the monomial is not in degree* 2^m *or* 2^m+1*.*

Proof: Let $u_n \dots u_1 u_0$ be the binary expansion of u and $u_1 = 0$. Since $u_1 = 0$, $u \neq 2^m - 5$. $x^u y^5 z^2 t$ is not hit if $u = 2^m - 5$ by Lemma 6.4. $x^u y^5 z^2 t$ is not hit if it is in degree 2^m or 2^m+1 by Lemma 6.2. If u is even then $x^u y^5 z^2 t$ is hit by Theorem 6.16. Hence we only consider the case where u is odd.

Suppose $x^u y^5 z^2 t$ is not in degrees 2^m, 2^m+1 and $u_1 = 0$.

a) Suppose $u_2 = 1$, then $u \geq 5$. By Theorem 2.9,

$$x^u y^5 z^2 t \equiv E_4^2(x^{u-4}y)z^2 t \equiv x^{u-4}y(\chi(E_4^2))(z^2 t)$$

$$\equiv x^{u-4}y(E_4^2 + D_4 D_4)(z^2 t) \equiv x^{u-4}yz^{10}t + x^{u-4}yz^2t^9 \ (mod\ hit). \tag{70}$$

By Lemma 6.3,

$$E_1^4(x^{u-4}yz^6t) \equiv x^{u-4}yz^{10}t \ (mod\ hit),$$

and

$$E_1^4(x^{u-4}yz^2t^5) \equiv x^{u-4}yz^2t^9 \ (mod\ hit),$$

since the 1st and the 2nd digits of the binary expansion of $u-4$ are 0's and $\binom{5}{2} \equiv 0 \ (mod\ 2)$, the terms contain x^{u-4} are the only terms having nonzero coefficients in $E_1^4(x^{u-4}yz^6t)$. Hence $x^u y^5 z^2 t$ is hit by (70).

b) Suppose $u_2 = 0$, then $u \geq 9$. If there is an $u_j = 0$ for some $j > 2$, then by Lemma 6.3,

$$E_1^{2^j}(x^{u-2^j}y^5z^2t) \equiv \binom{u-2^j}{2^j-2}x^{u-2}y^5z^4t + \binom{u-2^j}{2^j-4}x^{u-4}y^9z^2t$$

$$+\binom{u-2^j}{2^j-6}x^{u-6}y^9z^4t + x^u y^5 z^2 t \equiv x^u y^5 z^2 t \ (mod\ hit),$$

because $\binom{u-2^j}{2^j-s} \equiv 0 \ (mod\ 2)$ for $s = 2,\ 4,\ 6$ since the 1st and the 2nd digits of the binary expansion of $u - 2^j$ are 0's. Hence $x^u y^5 z^2 t$ is hit in this case.

c) If $u_1 = u_2 = 0$ and all $u_j = 1$ except for $j = 1,\ 2$, then the monomial is in degree $2^m + 1$.

Hence we have proved the result. □

Proposition 6.18 *A monomial in the form [a321] with $a \geq 6$ is hit over $\mathbb{F}_2$, if it is not in degrees 2^m, $2^m + 1$ and $2^m + 3$.*

Proof: $x^a y^3 z^2 t$ is in degree 2^m if $a = 2^m - 6$. $x^a y^3 z^2 t$ is in degree $2^m + 1$ if $a = 2^m - 5$. They are not hit by Lemma 6.2. $x^a y^3 z^2 t$ with $a = 2^m - 3$ in degree $2^m + 3$ is not hit by Proposition 6.5.

Suppose $x^a y^3 z^2 t$ is not in degrees 2^m, $2^m + 1$ and $2^m + 3$. Let $a_n a_{n-1} \dots a_0$ be the binary expansion of a. If a is even, then $x^a y^3 z^2 t$ is hit by Theorem 6.16. Suppose a is odd, we have the following cases:

(1) Suppose $a_i = 0$ for some $i \geq 3$, then $a \geq 16$. By Lemma 6.3,

$$E_1^{2^i}(x^{a-2^i} y^3 z^2 t) \equiv x^a y^3 z^2 t + \binom{a - 2^i}{2^i - 2}(x^{a-2} y^5 z^2 t + x^{a-2} y^3 z^4 t)$$
$$+ \binom{a - 2^i}{2^i - 4} x^{a-4} y^5 z^4 t \ (mod\ hit). \tag{71}$$

Suppose $\binom{a - 2^i}{2^i - 2} \not\equiv 0 \ (mod\ 2)$ then $a_j = 1$ for $1 \leq j \leq i - 1$. Since $a_1 = 1$, the 1st digit of the binary expansion of $a - 2$ is 0. Then by Lemma 6.3,

$$E_1^2(x^{a-2} y^3 z^2 t) \equiv x^{a-2} y^5 z^2 t + x^{a-2} y^3 z^4 t \ (mod\ hit). \tag{72}$$

By (71) and (72), we get:

$$x^a y^3 z^2 t + \binom{a - 2^i}{2^i - 4} x^{a-4} y^5 z^4 t \equiv 0 \ (mod\ hit). \tag{73}$$

If $\binom{a - 2^i}{2^i - 4} \not\equiv 0 \ (mod\ 2)$ then $a_j = 1$ for $2 \leq j \leq i - 1$. Let $a_2 = 1$, then the 2nd digit of the binary expansion of $a - 4 = 0$. Then

$$E_1^{2^i}(x^{a-4-2^i} y^5 z^4 t) \equiv x^{a-4} y^5 z^4 t + \binom{a - 4 - 2^i}{2^i - 4}(x^{a-8} y^9 z^4 t + x^{a-8} y^5 z^8 t)$$
$$+ \binom{a - 4 - 2^i}{2^i - 8} x^{a-12} y^9 z^8 t \equiv x^{a-4} y^5 z^4 t + x^{a-12} y^9 z^8 t \ (mod\ hit), \tag{74}$$

by Lemma 6.3 and $\binom{a-4-2^i}{2^i-4} \equiv 0 \ (mod\ 2)$, since the 2nd digit of the binary expansion of 2^i-4 is 1 and the 2nd digit of the binary expansion of $a-4-2^i$ is 0. Also $\binom{a-4-2^i}{2^i-8} \equiv 1 \ (mod\ 2)$ since the digits between the third to $(i-1)$th of the binary expansion of $a-4-2^i$ are 1's. Then

$$E_1^4(x^{a-12}y^9z^4t) \equiv \binom{a-12}{4}x^{a-8}y^9z^4t + \binom{9}{4}x^{a-12}y^{13}z^4t$$
$$+x^{a-12}y^9z^8t \equiv x^{a-12}y^9z^8t \ (mod\ hit), \qquad (75)$$

by Lemma 6.3 and $\binom{a-12}{4} \equiv \binom{9}{4} \equiv 0 \ (mod\ 2)$. By (74) and (75), $x^{a-4}y^5z^4t$ is hit, hence $x^ay^3z^2t$ is hit by (73).

(2) Let $a = 2^m - 1$ for $m \geq 3$, then $a \geq 7$. By the equations

$$D_2(x^{a-2}y^3z^2t) \equiv x^ay^3z^2t + x^{a-2}y^5z^2t + x^{a-2}y^3z^2t^3 \ (mod\ 2),$$

$$E_2^2(x^{a-2}yz^2t) \equiv x^ay^3z^2t + x^ayz^2t^3 + x^{a-2}y^3z^2t^3 + x^{a-2}yz^6t \ (mod\ hit),$$

we get:

$$x^ayz^2t^3 \equiv x^{a-2}y^5z^2t + x^{a-2}yz^6t \equiv x^{a-2}yz^6t \ (mod\ hit), \qquad (76)$$

by Lemma 6.17, since the 1st digit of the binary expansion of $a-2$ is 0. For $x^{a-2}yz^6t$, we have:

$$D_1(x^{a-2}yz^5t) \equiv x^{a-1}yz^5t + x^{a-2}yz^6t + x^{a-2}z^5(y^2t + yt^2)$$
$$\equiv x^{a-1}yz^5t + x^{a-2}yz^6t \ (mod\ hit), \qquad (77)$$

again by Lemma 6.17.

$$E_1^2(x^{a-1}yz^3t) \equiv x^{a-1}yz^5t + x^{a+1}yz^3t \ (mod\ hit). \qquad (78)$$

$a + 1 = 2^m$ for $m \geq 3$, then

$$E_1^4(x^{a-3}yz^3t) \equiv x^{a+1}yz^3t + \binom{a-3}{2}x^{a-1}yz^5t \equiv x^{a+1}yz^3t \ (mod\ hit),$$

by Lemma 6.3 and the 0th and the 1st digits of the binary expansion of $a-3$ are 0's so $\binom{a-3}{2} \equiv 0$ $(mod\ 2)$. Hence $x^{a-1}yz^5t$ is hit by (78) and $x^{a-2}yz^6t$ is hit by (77). So $x^ayz^2t^3$ is hit by (76) and $x^ay^3z^2t$ is hit as well by symmetry.

(3) If an odd $a \neq 2^m-1,\ 2^m-3,\ 2^m-5$ and all the $a_i = 1$ for $i \geq 3$, then $a_0 = 1$ and $a_1 = a_2 = 0$, i.e. $a = 2^m - 7$ for $m \geq 4$.

$$E_1^4(x^{a-4}y^3z^2t) \equiv x^ay^3z^2t + \binom{a-4}{2}x^{a-2}(y^5z^2t + y^3z^4t) + x^{a-4}y^5z^4t$$
$$\equiv x^ay^3z^2t + x^{a-4}y^5z^4t\ (mod\ hit),$$

by Lemma 6.3 and $\binom{a-4}{2} \equiv 0$ $(mod\ 2)$. Then

$$E_1^2(x^{a-4}y^5z^2t) \equiv x^{a-4}y^5z^4t + \binom{a-4}{2}x^{a-2}y^5z^2t + \binom{5}{2}x^{a-4}y^7z^2t$$
$$\equiv x^{a-4}y^5z^4t\ (mod\ hit),$$

by Lemma 6.3 and $\binom{a-4}{2} \equiv \binom{5}{2} \equiv 0$ $(mod\ 2)$. Hence $x^{a-4}y^5z^4t$ is hit and so $x^ay^3z^2t$ is hit in this case. □

6.6 Monomials in the form $[a311]$

Corollary 6.19 *A monomial in the form $[ab11]$, where $a = 2^l - 1$, $b = 2^k - 1$, is not hit under the action of $\mathcal{D}_2$.*

Proof: By Proposition 6.5. □

Proposition 6.20 *A monomial in the form $[a431]$ is hit under the action of $\mathcal{D}_2$, if it is not in degree 2^m or $2^m + 1$.*

Proof: Let $a_n \dots a_0$ be the binary expansion of a. A monomial $x^a y^3 z^4 t$ is hit if a is even and the degree is not 2^m by Theorem 6.16. We may assume that a is odd.

(1) Suppose $a_1 = 1$. By Theorem 2.9,

$$x^a y^3 z^4 t = (E_2^2(x^{a-2}y))z^4 t \equiv x^{a-2} y \chi(E_2^2)(z^4 t)$$

$$= x^{a-2} y(E_2^2 + D_2 D_2)(z^4 t) \equiv x^{a-2} y z^4 t^5 \ (mod\ hit).$$

By Lemma 6.3,

$$E_1^2(x^{a-2} y z^2 t^5) \equiv x^{a-2} y z^4 t^5 \ (mod\ hit),$$

since $\binom{a-2}{2} \equiv \binom{5}{2} \equiv 0 \ (mod\ 2)$ as the 1st digit of the binary expansion of $a-2$ is 0. Hence the monomial $x^a y^3 z^4 t$ is hit in this case.

(2) Suppose $a_1 = 0$. Then by Lemma 6.3,

$$E_1^2(x^a y^3 z^2 t) \equiv x^a y^3 z^4 t + x^a y^5 z^2 t \equiv x^a y^3 z^4 t \ (mod\ hit),$$

since $x^a y^5 z^2 t$ is hit by Lemma 6.17 as the degree is not $2^m + 1$.

Hence monomials in the form [a431] are hit, if it is not in degree 2^m or $2^m + 1$.

□

Proposition 6.21 *A monomial in the form* [a311] *is hit under the action of* $\mathcal{D}_2$, *if it is not in degrees* 2^m, $2^m + 1$, $2^m + 2$ *and* $2^m + 4$.

Proof: $x^a y^3 z t$ is in degree 2^m if $a = 2^m - 5$. $x^a y^3 z t$ is in degree $2^m + 1$ if $a = 2^m - 4$. $x^a y^3 z t$ with $a = 2^m - 3$ is in degree $2^m + 2$. They are not hit by Lemma 6.2. $x^a y^3 z t$ with $a = 2^m - 1$ in degree $2^m + 4$ is not hit by Proposition 6.5.

We only need to decide those cases in which $a \neq 2^m - 1,\ 2^m - 3,\ 2^m - 4,\ 2^m - 5$. Let $a_n \dots a_1 a_0$ be the binary expansion of a.

(1) Suppose a is even. $a \neq 2^m - 4$.

a) If there is an $a_i = 0$ for $i \geq 2$, then

$$E_1^{2^i}(x^{a-2^i} y^3 zt) \equiv x^a y^3 zt + \binom{a-2^i}{2^i-2} x^{a-2} y^5 zt \ (mod\ hit),$$

by Lemma 6.3. If $\binom{a-2^i}{2^i-2} \not\equiv 0\ (mod\ 2)$ then $a_j = 1$ for $1 \leq j \leq i-1$. So the 1st digit of the binary expansion of $a-2$ is 0, so

$$E_1^2(x^{a-2} y^3 zt) \equiv \binom{a-2}{2} x^a y^3 zt + x^{a-2} y^5 zt \equiv x^{a-2} y^5 zt \ (mod\ hit),$$

by Lemma 6.3 and $\binom{a-2}{2} \equiv 0\ (mod\ 2)$. Hence $x^{a-2} y^5 zt$ is hit and so $x^a y^3 zt$ is hit as well.

b) If $a_0 = 0$ and all $a_i = 1$ for $i \geq 1$, i.e. $a = 2^m - 2$. By Theorem 2.9 and Lemma 6.3, we get

$$E_1^3(x^{2^m-5} y^3 z)t \equiv x^{2^m-2} y^3 zt + x^{2^m-3} y^4 zt + x^{2^m-3} y^3 z^2 t$$

$$+ x^{2^m-4} y^5 zt + x^{2^m-5} y^6 zt + x^{2^m-5} y^5 z^2 t \ (mod\ hit),$$

$$E_1^3(x^{2^m-5} y^3 z)t \equiv x^{2^m-5} y^3 z(\chi(E_1^3))(t)$$

$$\equiv x^{2^m-5} y^3 z(E_1^2 D_1(t)) \equiv x^{2^m-5} y^3 zt^4 \equiv 0 \ (mod\ hit),$$

by Proposition 6.20. Hence

$$x^{2^m-2} y^3 zt + x^{2^m-3} y^4 zt + x^{2^m-3} y^3 z^2 t + x^{2^m-4} y^5 zt$$

$$+ x^{2^m-5} y^6 zt + x^{2^m-5} y^5 z^2 t \equiv 0 \ (mod\ hit). \tag{79}$$

By Lemma 6.3 and since $\binom{5}{2} \equiv \binom{2^m-3}{2} \equiv 0$ $(mod\ hit)$,

$$E_1^2(x^{2^m-6}y^5zt) \equiv x^{2^m-4}y^5zt\ (mod\ hit), \tag{80}$$

$$E_1^2(x^{2^m-3}y^2zt) \equiv x^{2^m-3}y^4zt\ (mod\ hit). \tag{81}$$

By Theorem 2.9 and Proposition 6.20,

$$(x^{2^m-3}y^3 + x^{2^m-5}y^5)z^2t \equiv (E_1^2(x^{2^m-5}y^3))z^2t \equiv x^{2^m-5}y^3(\chi(E_1^2))(z^2t)$$

$$\equiv x^{2^m-5}y^3(E_1^2)(z^2t) \equiv x^{2^m-5}y^3z^4t \equiv 0\ (mod\ hit), \tag{82}$$

Hence by (79), (80), (81) and (82), we get:

$$x^{2^m-2}y^3zt + x^{2^m-5}y^6zt \equiv 0\ (mod\ hit). \tag{83}$$

By

$$E_2^2(x^{2^m-5}yz^2t) \equiv x^{2^m-1}yz^2t + x^{2^m-3}y^3z^2t$$

$$+x^{2^m-3}yz^2t^3 + x^{2^m-5}yz^6t + x^{2^m-5}y^3z^2t^3\ (mod\ 2),$$

$$D_2(x^{2^m-3}yz^2t) \equiv x^{2^m-1}yz^2t + x^{2^m-3}y^3z^2t + x^{2^m-3}yz^2t^3\ (mod\ 2),$$

we get,

$$x^{2^m-5}yz^6t \equiv x^{2^m-5}y^3z^2t^3\ (mod\ hit). \tag{84}$$

From

$$D_2(x^{2^m-5}y^3z^2t) \equiv x^{2^m-3}y^3z^2t + x^{2^m-5}y^5z^2t + x^{2^m-5}y^3z^2t^3\ (mod\ 2),$$

and (82), we get

$$x^{2^m-5}y^3z^2t^3 \equiv 0\ (mod\ hit). \tag{85}$$

By (83), (84) and (85) $x^{2^m-2}y^3zt$ is hit in this case.

(2) Suppose a is odd and $a \neq 2^m-1,\ 2^m-3,\ 2^m-5$. We have the following cases:

c) If there is an $a_i = 0$ for $i \geq 3$, then by Theorem 6.16,

$$E_1^{2^i}(x^{a-2^i}y^3zt) \equiv x^ay^3zt + \binom{a-2^i}{2^i-2}x^{a-2}y^5zt \ (mod\ hit). \tag{86}$$

If $\binom{a-2^i}{2^i-2} \equiv 0\ (mod\ 2)$, then x^ay^3zt is hit. If $\binom{a-2^i}{2^i-2} \not\equiv 0\ (mod\ 2)$, then $a_1 = 1$, so the 1st digit of the binary expansion of $a-2$ is 0. Then

$$E_1^2(x^{a-2}y^3zt) \equiv x^{a-2}y^5zt \ (mod\ hit), \tag{87}$$

we truncate the remaining terms by Theorem 6.16. Hence $x^{a-2}y^5zt$ is hit and hence x^ay^3zt is hit by (86) in this case.

d) If all $a_i = 1$ for $i \geq 3$ and $a \neq 2^m-1,\ 2^m-3,\ 2^m-5$, then $a_1 = a_2 = 0$, i.e. $a = 2^m-7$. For this case,

$$E_1^4(x^{a-4}y^3zt) \equiv x^ay^3zt + \binom{a-4}{2}x^{a-2}y^5zt \equiv x^ay^3zt \ (mod\ hit),$$

we truncate the remaining terms by Theorem 6.16 and $\binom{a-4}{2} \equiv 0\ (mod\ 2)$. So x^ay^3zt is hit.

Hence monomials in the form x^ay^3zt are hit if they satisfy those conditions.

□

6.7 Monomials in the forms $[a(2^k-1)(2^k)1]$ and $[a(2^k-1)(2^k+1)1]$

Proposition 6.22 *A monomial in the form $[a(2^k-1)(2^k)1]$ for $k \geq 2$ is hit under the action of $\mathcal{D}_2$, if it is not in degree 2^m or 2^m+1.*

Proof: A monomial $x^ay^{2^k}z^{2^k-1}t$ is hit, if a is even and the degree is not 2^m by Theorem 6.16. Suppose a is odd. Let $a_na_{n-1}\dots a_0$ be the binary expansion of a. We have the following cases:

(1) If $a_0 = a_1 = 1$ then because operations D_{2^k-2} and $E^2_{2^k-2}$ have no effect on z^{2^k}, we get:

$$D_{2^k-2}(x^{a-2^k+2}y^{2^k-1}z^{2^k}t) \equiv x^a y^{2^k-1}z^{2^k}t + x^{a-2^k+2}y^{2^{k+1}-3}z^{2^k}t$$

$$+x^{a-2^k+2}y^{2^k-1}z^{2^k}t^{2^k-1} \ (mod\ 2), \tag{88}$$

$$E^2_{2^k-2}(x^{a-2^k+2}yz^{2^k}t) \equiv \binom{a-2^k+2}{2} x^{a+2^k-2}yz^{2^k}t$$

$$+x^a y^{2^k-1}z^{2^k}t + x^a yz^{2^k}t^{2^k-1} + x^{a-2^k+2}y^{2^k-1}z^{2^k}t^{2^k-1}$$

$$\equiv x^a y^{2^k-1}z^{2^k}t + x^a yz^{2^k}t^{2^k-1} + x^{a-2^k+2}y^{2^k-1}z^{2^k}t^{2^k-1} \ (mod\ 2), \tag{89}$$

by Lemma 6.3 and $\binom{2^k}{j} \equiv 0$ unless $j = 0$ for $j \leq 2$. Also since $a_1 = 1$ and $k \geq 2$, $\binom{a-2^k+2}{2} \equiv 0 \ (mod\ 2)$.

By (88) and (89), we get: $x^a yz^{2^k}t^{2^k-1} \equiv x^{a-2^k+2}y^{2^{k+1}-3}z^{2^k}t \ (mod\ hit)$. Then

$$E^2_1(x^{a-2^k}y^{2^{k+1}-3}z^{2^k}t) \equiv x^{a-2^k+2}y^{2^{k+1}-3}z^{2^k}t + \binom{2^{k+1}-3}{2} x^{a-2^k}y^{2^{k+1}-1}z^{2^k}t$$

$$\equiv x^{a-2^k+2}y^{2^{k+1}-3}z^{2^k}t \ (mod\ hit),$$

by Lemma 6.3 and $\binom{2^{k+1}-3}{2} \equiv 0 \ (mod\ 2)$. Hence $x^a yz^{2^k}t^{2^k-1}$ is hit, and by symmetry, $x^a y^{2^k-1}z^{2^k}t$ is hit too.

(2) Suppose $a_0 = 1$ and $a_1 = 0$. Note that the total degree is not $2^m + 3$, since $a_1 = 0$. By Lemma 6.3,

$$E^2_1(x^{a-2}y^{2^k-1}z^{2^k}t) \equiv x^a y^{2^k-1}z^{2^k}t + x^{a-2}y^{2^k+1}z^{2^k}t. \ (mod\ hit) \tag{90}$$

We claim that either $x^{a-2}y^{2^k+1}z^{2^k}t$ is hit or $\equiv x^{a+2^k-4}y^3z^{2^k}t \ (mod\ hit)$. We argue by induction on the exponent of y. By Lemma 6.3

$$E^{2^{k-1}}_1(x^{a-2}y^{2^{k-1}+1}z^{2^k}t) \equiv x^{a-2}y^{2^k+1}z^{2^k}t + rx^{a+2^{k-1}-2}y^{2^{k-1}+1}z^{2^k}t \ (mod\ hit),$$

where $r = \binom{a-2}{2^{k-1}}$, since $\binom{2^{k-1}+1}{s} \equiv 0\ (mod\ 2)$ unless $s = 0,\ 2^{k-1}$ for s even. If $r \equiv 0\ (mod\ 2)$, then $x^{a-2}y^{2^k+1}z^{2^k}t$ as well as $x^a y^{2^k-1}z^{2^k}t$ is hit. If $r \equiv 1\ (mod\ 2)$,

$$x^{a-2}y^{2^k+1}z^{2^k}t \equiv x^{a+2^{k-1}-2}y^{2^{k-1}+1}z^{2^k}t\ (mod\ hit).$$

We may repeat the procedure on $x^{a+2^{k-1}-2}y^{2^{k-1}+1}z^{2^k}t$ if $k-1>1$. By the induction hypothesis, we finally will get either $x^{a-2}y^{2^k+1}z^{2^k}t$ is hit or

$$x^{a-2}y^{2^k+1}z^{2^k}t \equiv x^{a+2^k-4}y^3z^{2^k}t\ (mod\ hit). \tag{91}$$

Let $a' = a+2^k-4$. Note that the 1st digit of the binary expansion of a' is 0, since $a_1 = 0$ and $k \geq 2$.

We claim that either $x^{a'}y^3z^{2^k}t$ is hit or

$$x^{a'}y^3z^{2^k}t \equiv x^u y^3 z^4 t\ (mod\ hit)$$

where $u = a'+2^k-4 = a+2^{k+1}-8$. We argue this by induction on the exponent of z. Suppose $k>2$, by Lemma 6.3,

$$E_1^{2^{k-1}}(x^{a'}y^3z^{2^{k-1}}t) \equiv x^{a'}y^3z^{2^k}t + \binom{a'}{2^{k-1}-2}x^{a'+2^{k-1}-2}y^5z^{2^{k-1}}t$$

$$+\binom{a'}{2^{k-1}}x^{a'+2^{k-1}}y^3z^{2^{k-1}}t \equiv x^{a'}y^3z^{2^k}t + \binom{a'}{2^{k-1}}x^{a'+2^{k-1}}y^3z^{2^{k-1}}t\ (mod\ hit).$$

This is because that the 1st digit of the binary expansion of $a' = 0$, so $\binom{a'}{2^{k-1}-2} \equiv 0\ (mod\ 2)$ and $\binom{2^{k-1}}{j} \equiv 0\ (mod\ 2)$ unless $j = 0,\ 2^{k-1}$. If $\binom{a'}{2^{k-1}} \equiv 0\ (mod\ 2)$, $x^{a'}y^3z^{2^k}t$ is hit otherwise

$$x^{a'}y^3z^{2^k}t \equiv x^{a'+2^{k-1}}y^3z^{2^{k-1}}t\ (mod\ hit).$$

We may repeat the procedure on $x^{a'+2^{k-1}}y^3z^{2^{k-1}}t$ with $E_1^{2^{k-2}}$, if $k-1>2$. By the induction hypothesis, we finally will get either $x^{a'}y^3z^{2^k}t$ is hit or

$$x^{a'}y^3z^{2^k}t \equiv x^{a+2^{k+1}-8}y^3z^4t \equiv 0\ (mod\ hit), \tag{92}$$

by Proposition 6.20.

Hence by (90), (91) and (92), $x^a y^{2^k-1} z^{2^k} t$ is hit and hence the monomials in the form $[a(2^k-1)(2^k)1]$ are hit. $\square$

We have got a partial result for the monomials in the form $[a(2^k-1)(2^k+1)1]$. For the cases we have not proved, we have a conjecture on them in Section 6.8.

Proposition 6.23 *A monomial $x^a y^{2^k+1} z^{2^k-1} t$ in degrees ≥ 11, where $k \geq 2$ and $4 \leq a \equiv 0$ or 1 (mod 4), is hit under the action of $\mathcal{D}_2$, if the monomial is not in degree 2^m or 2^m+1.*

Proof: By Theorem 6.16,

$$E_1^2(x^{a-2} y^{2^k+1} z^{2^k-1} t) \equiv x^a y^{2^k+1} z^{2^k-1} t + x^{a-2} y^{2^k+1} z^{2^k+1} t \ (mod\ hit). \tag{93}$$

Note that the 1st digit of the binary expansion of $a-2$ is 1. We now consider the monomial $x^{a-2} y^{2^k+1} z^{2^k+1} t$.

(1) Suppose a is odd. Then by Theorem 2.9,

$$x^{a-2} y^{2^k+1} z^{2^k+1} t = (E_{2^k}^2(yz)) x^{a-2} t$$

$$\equiv yz(\chi(E_{2^k}^2))(x^{a-2}t) \equiv yz(E_{2^k}^2 + D_{2^k} D_{2^k})(x^{a-2}t)$$

$$\equiv yz(x^{a+2^{k+1}-2} t + x^{a+2^k-2} t^{2^k+1} + D_{2^k}(x^{a+2^k-2} t + x^{a-2} t^{2^k+1}))$$

$$\equiv yz(x^{a+2^k-2} t^{2^k+1} + x^{a-2} t^{2^{k+1}+1}) \ (mod\ hit). \tag{94}$$

By Theorem 6.16 and $\binom{2^k+1}{r} \not\equiv 0 \ (mod\ 2)$ unless $r = 0$ or 2^k for r even,

$$E_1^{2^k}(x^{a-2} yzt^{2^k+1}) \equiv \binom{a-2}{2^k} x^{a+2^k-2} yzt^{2^k+1} + x^{a-2} yzt^{2^{k+1}+1} \ (mod\ hit). \tag{95}$$

If $\binom{a-2}{2^k} \not\equiv 0$ $(mod\ 2)$, then by (93) and (94), $x^a y^{2^k+1} z^{2^k-1} t$ is hit. In particular, $x^a y^{2^k+1} z^{2^k-1} t$ is hit when $a = 2^m - 2^{k+1} + 1$ for $m > k+1$, i.e. the monomial is in degree $2^m + 2$.

Suppose $\binom{a-2}{2^k} \equiv 0$ $(mod\ 2)$ and so the degree is not $2^m + 2$. Then $x^{a-2} yzt^{2^{k+1}+1}$ is hit by (95), so from (93) and (94),

$$x^a y^{2^k+1} z^{2^k-1} t \equiv x^{a+2^k-2} yzt^{2^k+1} \ (mod\ hit). \tag{96}$$

We claim that $x^{a+2^k-2} yzt^{2^k+1}$ is either hit or $\equiv x^{a+2^{k+1}-4} yzt^3$ $(mod\ hit)$. We argue by induction on the exponent of t.

By Theorem 6.16 and $\binom{2^{k-1}+1}{s} \not\equiv 0$ $(mod\ 2)$ unless $s = 0$ or 2^{k-1} for s even,

$$E_1^{2^{k-1}}(x^{a+2^k-2} yzt^{2^{k-1}+1}) \equiv rx^{a+2^k+2^{k-1}-2} yzt^{2^{k-1}+1} + x^{a+2^k-2} yzt^{2^k+1} \ (mod\ hit),$$

where $r = \binom{a+2^k-2}{2^{k-1}}$. Hence if $r \equiv 0$ $(mod\ 2)$, $x^{a+2^k-2} yzt^{2^k+1}$ is hit, otherwise

$$x^{a+2^k-2} yzt^{2^k+1} \equiv x^{a+2^k+2^{k-1}-2} yzt^{2^{k-1}+1} \ (mod\ hit).$$

If $k > 2$, we may repeat the procedure by applying $E_1^{2^{k-2}}$ on $x^{a+2^k+2^{k-1}-2} yzt^{2^{k-1}+1}$. By the induction hypothesis we get either $x^{a+2^k-2} yzt^{2^k+1}$ is hit or

$$x^{a+2^k-2} yzt^{2^k+1} \equiv x^{a+2^{k+1}-4} yzt^3 \ (mod\ hit).$$

Since the 1st digit of the binary expansion of a is 0, the 1st digit of the binary expansion of $a + 2^{k+1} - 4$ is 0. Hence the monomial $x^{a+2^{k+1}-4} yzt^3$ is in degree $a + 2^{k+1} + 1$ which is $\not\equiv 0$ $(mod\ 4)$, hence it is not in degree $2^m + 4$. By Proposition 6.21, $x^{a+2^{k+1}-4} yzt^3$ is hit if it is not in degrees 2^m, $2^m + 1$, $2^m + 2$ and $2^m + 4$. We know that $x^{a+2^{k+1}-4} yzt^3$ is not in degrees 2^m, $2^m + 1$ and $2^m + 2$ as well. Hence $x^{a+2^{k+1}-4} yzt^3$ is hit and so $x^{a+2^k-2} yzt^{2^k+1}$ is hit. Then finally by (96), $x^a y^{2^k+1} z^{2^k-1} t$ is hit in this case.

(2) Suppose a is even, i.e. $a \equiv 0 \ (mod\ 4)$. Then by Theorem 2.9,

$$x^{a-2}y^{2^k+1}z^{2^k+1}t = (E^2_{2^k}(yz))x^{a-2}t \equiv yz(\chi(E^2_{2^k}))(x^{a-2}t)$$

$$\equiv yz(E^2_{2^k} + D_{2^k}D_{2^k})(x^{a-2}t) \equiv yz(x^{a+2^{k+1}-2}t + D_{2^k}(x^{a-2}t^{2^k+1}))$$

$$\equiv yz(x^{a+2^{k+1}-2}t + x^{a-2}t^{2^{k+1}+1}) \ (mod\ hit). \qquad (97)$$

By Lemma 6.15, $x^{a+2^{k+1}-2}yzt$ is hit since it is not in degrees 2^m, 2^m+1, 2^m+2. By a similar argument as the part (1) of this proof, we get $x^{a-2}yzt^{2^{k+1}+1}$ is either hit or $\equiv x^{a+2^{k+1}-4}yzt^3 \ (mod\ hit)$. Also the monomial is not in the degree 2^m+4 since the degree of the monomial is odd. Hence $x^{a-2}y^{2^k+1}z^{2^k+1}t$ is hit and $x^a y^{2^k+1}z^{2^k-1}t$ is hit for this case. □

Proposition 6.24 *A monomial in the form* $x^a y^{2^k-1}z^{2^k+1}t$ *for* $k \geq 2$ *is not hit under the action of* $\mathcal{D}_2$*, if a is one of the following cases:*

(1) $2^m - 2^{k+1}$, *where* $m > k+1$.

(2) $2^m - 2^{k+1} - 1$, *where* $m > k+1$.

(3) $2^m - 1$, *where* $m \geq 1$.

(4) $2^m - 2^k - 1$, *where* $m > k$.

Proof: For case (1), the monomial is in degree $2^m + 1$. For case (2), the monomial is in degree 2^m. They are not hit by Lemma 6.2. For case (3), $x^a t$ and $y^{2^k-1}z^{2^k+1}$ are not hit. For case (4), $x^a z^{2^k+1}$ and $y^{2^k-1}t$ are not hit. Hence they are not hit by Proposition 6.5. □

6.8 Some results on the cokernel and some conjectures

In this subsection, we will analyse the cokernel in degree $2^m + 2$ for $m \geq 4$. This is the simplest case of the degrees which contains a set of representatives of a subspace of the cokernel. Then we will give some conjecture for the monomials which we have not proved.

Proposition 6.25 *In degree $2^m + 2$, every monomial $x^a y^b zt$ where a, b are odd is congruent (mod hit) to a monomial in the form of $x^{2^m - 2^k + 1} y^{2^k - 1} zt$ where k is the number of 1's in the binary expansion of b.*

Proof: $x^a y^b zt$ is not hit by Lemma 6.2. By Lemma 3.15, there is a D which is a composition of some E_1^k's, such that $D(y^{2^k - 1}) \equiv y^b \ (mod\ 2)$ where k is the number of 1's in the binary expansion of b. By Theorem 2.9,

$$x^a y^b zt \equiv (D(y^{2^k-1}))x^a zt \equiv y^{2^k-1}(\chi(D))(x^a zt) \ (mod\ hit),$$

where $\chi(D)$ is a sum of compositions of E_1^i operations. Since the degree of $(\chi(D))(x^a zt)$ is odd, in $(\chi(D))(x^a zt)$, if an exponent of a term f is even then there are two even exponents in f. Then fy^{2^k-1} is hit by Theorem 6.16. By a similar argument as the proof of Proposition 5.12,

$$y^{2^k-1}(\chi(D))(x^a zt) \equiv y^{2^k-1} zt(\chi(D))(x^a) \ (mod\ hit).$$

Then $(\chi(D))(x^a)$ has to be $x^{2^m - 2^k + 1}$ since the degree is $2^m + 2$ and the monomial is not hit. □

Proposition 6.26 *In degree $2^m + 2$, a monomial $x^{2^m - 2^k + 1} y^{2^k - 1} zt$ is congruent (mod hit) to $x^{2^{m-1}+1} y^{2^{m-1}-1} zt$.*

Proof: The argument is similar with the proof of Proposition 5.13, since the operations in that proof are E_1^i's for $i \geq 2$. Hence the exponents of z and t of a term in the image of an operation in the process will keep to be 1's, since otherwise the term is hit by Theorem 6.16. $\square$

Proposition 6.27 *In degree* $2^m + 2$ *for* $m \geq 4$, *the cokernel is* 6 *dimensional and a basis is given by the following* 6 *monomials:*

$$x^{2^{m-1}+1}y^{2^{m-1}-1}zt, \quad x^{2^{m-1}+1}yz^{2^{m-1}-1}t, \quad x^{2^{m-1}+1}yzt^{2^{m-1}-1},$$

$$xy^{2^{m-1}+1}z^{2^{m-1}-1}t, \quad xy^{2^{m-1}+1}zt^{2^{m-1}-1}, \quad xyz^{2^m+1}t^{2^{m-1}-1}.$$

Proof: By the previous results, monomials in degree $2^m + 2$ are hit, except the monomials in the form $[ab11]$ where a, b are odd. By Proposition 6.25 and Proposition 6.26, every monomial in the form $[ab11]$ is congruent (*mod hit*) to a monomial in the form $[(2^{m-1}+1)(2^{m-1}-1)11]$.

By Theorem 6.16,

$$E_1^2(x^{2^{m-1}-1}y^{2^{m-1}-1}zt) \equiv x^{2^{m-1}+1}y^{2^{m-1}-1}zt + x^{2^{m-1}-1}y^{2^{m-1}+1}zt \ (mod\ hit). \quad (98)$$

Since $m \geq 4$, we are able to apply Theorem 6.16 and truncate the remaining terms in (98). If $m < 4$, the result is not true. For example, suppose $m = 3$,

$$E_1^2(x^3y^3zt) \equiv x^5y^3zt + x^3y^5zt + x^4y^3z^2t + x^4y^3zt^2 \ (mod\ hit).$$

But a monomial in the form $[4321]$ is not hit.

By applying suitable permutations to (98), we get

$$x^{2^{m-1}+1}yz^{2^{m-1}-1}t \equiv x^{2^{m-1}-1}yz^{2^{m-1}+1}t \ (mod\ hit),$$

$$x^{2^{m-1}+1}yzt^{2^{m-1}-1} \equiv x^{2^{m-1}-1}yzt^{2^{m-1}+1} \ (mod\ hit),$$

$$xy^{2^{m-1}+1}z^{2^{m-1}-1}t \equiv xy^{2^{m-1}-1}z^{2^{m-1}+1}t \ (mod\ hit),$$

$$xy^{2^{m-1}+1}zt^{2^{m-1}-1} \equiv xy^{2^{m-1}-1}zt^{2^{m-1}+1} \ (mod\ hit),$$

$$xyz^{2^{m-1}+1}t^{2^{m-1}-1} \equiv xyz^{2^{m-1}-1}t^{2^{m-1}+1} \ (mod\ hit).$$

Hence the 6 monomials mentioned above generate all monomials in the form $[(2^{m-1}+1)(2^{m-1}-1)11]$ and hence generate all monomials in degree 2^m+2 by Proposition 6.25 and Proposition 6.26.

Any of the 6 monomials is not hit by Lemma 6.2. A sum of any combination of the 6 monomials is not hit too. For example, suppose $x^{2^{m-1}+1}yz^{2^{m-1}-1}t$ is a term in f which is a sum of some of the 6 monomials. $x^{2^{m-1}+1}y^{2^{m-1}-1}zt$ is the only monomial in the form x^ay^bzt where $1 < a,\ b$ are odd and $a+b=2^m$ in f. But the monomials in the form x^ay^bzt where $1 < a,\ b$ odd and $a+b=2^m$ always appear in even numbers in any image of the action of $\mathcal{D}_2$. Suppose we want x^ay^bzt where $1 < a,\ b$ odd and $a+b=2^m$ in $E_r^k(x^uy^vzt) = \sum_{i+j=k} E_r^i(x^uy^v)E_r^j(zt)$ for $r=1$ or 2 and $k \geq 1$. Then x^ay^bzt appears only in the term $E_r^k(x^uy^v)zt$. By the proof of Proposition 5.2, the number of monomials in the form x^ay^b with $a,\ b$ odd and $a+b=2^m$ in $E_r^k(x^uy^v)$ is even. So the number of monomials in the form x^ay^bzt in the image of E_r^k operations is even.

Since $x^{2^{m-1}+1}y^{2^{m-1}-1}zt$ is the only monomial in the form x^ay^bzt in f, f is not hit under any E_r^k operations for $r=1,\ 2$ and hence is not hit under any operation of $\mathcal{D}_2$ by Theorem 2.7. □

In the 3 variable case, the cokernel in degree 2^m for $m \geq 2$ or in degree 2^m+1 for $m \geq 3$ is a constant. We think that the same thing happens in the 4 variable case. By the calculation in Section 6.2, we pose the following conjecture,

Conjecture 6.28 *The cokernel in degree 2^m or 2^m+1 for $m \geq 3$ is 6 dimensional.*

By the results we have got, we have a conjecture as follows:

Conjecture 6.29 *The non-hit elements in degrees ≥ 8 of $\mathbb{F}_2[x,y,z,t]xyzt$ lie in odd degrees 2^m+1, 2^m+3 and in even degree 2^m, 2^m+2^k where $m > k \geq 1$.*

Over $\mathbb{Q}$, the product of two Artin elements in sets of distinct variables is obviously an Artin element. For the product of two non-hit monomials which are not Artin elements in cases we have explored, we can rearrange the exponents and make it into an Artin element. Over $\mathbb{F}_2$, from the results we have got, the non-hit monomials always appear in even numbers in the image of any operation of $\mathcal{D}_2$. So the product of two non-hit monomials is not hit since it does not appear alone in the image of any operation of $\mathcal{D}_p$. Hence we pose the following conjecture:

Conjecture 6.30 *Let $X = \{x_1, x_2, \ldots x_n\}$ and $Y = \{y_1, y_2, \ldots y_m\}$ be two disjoint sets of variables where $X \cap Y = \{\}$. If $x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}$ and $y_1^{b_1}y_2^{b_2}\cdots y_m^{b_m}$ are not hit under the action of $\mathcal{D}$ over an arbitrary field K, where $a_i, b_i \in \mathbb{N}$, then $x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}y_1^{b_1}y_2^{b_2}\cdots y_m^{b_m}$ is not hit as well.*

7 The hit problem on polynomial rings of 2 and 3 variables over $\mathbb{F}_p$

In this section we will investigate the action of $\mathcal{D}_p$ on polynomial rings over an odd prime field $\mathbb{F}_p$. We will give a minimal generating set for the 2 variable case (Theorem 7.4) and a partial result for the 3 variable case. In this section, if we write p^m without defining m, then we mean $m \in \mathbb{N}$. We write the $p-$expansion of a as $a_n a_{n-1} \cdots a_0$ with $a_n \neq 0$, we call a_i the ith digit of the $p-$expansion of a. The problem seems harder for the odd prime case. Because there are more elements in the field in $\mathbb{F}_p$ for an odd prime p, when we apply an operation of $\mathcal{D}_p$ to a monomial, there are more terms in the image in general. So it is difficult to follow the method we used over $\mathbb{F}_2$.

7.1 The hit problem on a polynomial ring of 2 variables over $\mathbb{F}_p$

Over a finite field $\mathbb{F}_p$, a generating set under the action of $\mathcal{D}_p$ on $\mathbb{F}_p[x, y]$ is an infinite set. From $D_{b-1}(x^a y) = x^a y^b + a x^{a+b-1} y$, we get that the set of monomials in the form $[a1]$ is a generating set of $\mathbb{F}_p[x, y]xy$.

Lemma 7.1 *Over $\mathbb{F}_p$, $x^a y^b$ is hit if either of a, b is divisible by p and $a+b > p+1$.*

Proof: By Lemma 3.13 and Lemma 3.14. $\square$

Lemma 7.2 *In degree p^m, a monomial $x^a y^b$ is not hit under the action of $\mathcal{D}_p$, if a, b are not divisible by p.*

Proof: Recall $\{E_1^k \mid k \in \mathbb{N}\}$ is a generating set of $\mathcal{D}_p$ (Theorem 2.7).

Suppose an arbitrary polynomial f in degree p^m of $\mathbb{F}_p[x,y]xy$ is hit. Then we can write

$$f \equiv \sum_i \sum_j r(i,j) E_1^i(g_{i,j}) \ (mod\ p), \tag{99}$$

where every $r(i,j) \in \mathbb{F}_p$ and every $g_{i,j}$ is a monomial in $\mathbb{F}_p[x,y]xy$. For each $E_1^i(g_{i,j})$ in (99), let $g_{i,j} = x^u y^v$. Then we have

$$E_1^i(x^u y^v) = \sum_{s+t=i} \binom{u}{s}\binom{v}{t} x^{u+s} y^{v+t},$$

where $u+v+i = p^m$. Note that $\sum_{s+t=i}(u+s)\binom{u}{s}\binom{v}{t} \equiv 0\ (mod\, p)$, by Lemma 3.3. Hence the sum of the products of the coefficient of a monomial in (99) and the exponent of x in the monomial $\equiv 0\ (mod\ p)$ where sum takes over all monomials in (99).

So if any polynomial f in degree p^m of $\mathbb{F}_p[x,y]xy$ is hit under the action of $\mathcal{D}_p$, then the sum, which takes over all monomials in f, of the products of the coefficient of a monomial in f and the exponent of x in every monomial in the monomial is $\equiv 0\ (mod\ p)$. Suppose $dx^a y^b$ is hit for some $d \in \mathbb{F}_p$, where a, b are $\not\equiv 0\ (mod\ p)$ and $a + b = p^m$. Then we must have $ad \equiv 0\ (mod\ p)$ and hence $d \equiv 0\ (mod\ p)$ since $a \not\equiv 0\ (mod\ p)$. This completes the proof. In particular $x^{p^m-1}y$ is not hit. $\square$

Lemma 7.3 *Monomials in the form $x^a y$ where $a = kp^m - 1$, $2 \leq k \leq p-1$ are hit over $\mathbb{F}_p$ for an odd p.*

Proof: We have the following equations:

$$\begin{cases} E_1^{p^m}(x^{a-p^m}y) = \binom{a-p^m}{p^m} x^a y + \binom{a-p^m}{p^m-1} x^{a-1}y^2 \\ D_1(x^{a-1}y) = (a-1)x^a y + x^{a-1}y^2 \end{cases}$$

The determinant of the coefficient matrix is:

$$\begin{vmatrix} \binom{a-p^m}{p^m} & \binom{a-p^m}{p^m-1} \\ a-1 & 1 \end{vmatrix} = \binom{a-p^m}{p^m} - \binom{a-p^m}{p^m-1}(a-1)$$

$$\equiv (k-2)-(a-1) \equiv k \not\equiv 0 \ (mod\ p),$$

since $a = kp^m - 1$ for $k \geq 2$. So we can solve those equations to get that $x^a y$ is hit. □

Theorem 7.4 *$\{1,\ x,\ y,\ xy,\ x^2y,\ x^{p^m-1}y \mid m \in \mathbb{N}\}$ is a minimal generating set of $\mathbb{F}_p[x,y]$ under the action of $\mathcal{D}_p$.*

Proof: We only need to find a minimal generating set for $\mathbb{F}_p[x,y]xy$, since we know $\{1,x\}$ or $\{1,y\}$ is minimal generating set in the 1 variable cases for variable x or y. Now consider $x^a y$ of degree $d \leq p+1$, if $a = p-1$, then $d = p$ and $x^a y$ is not hit by Lemma 7.2. Consider the following equations:

$$\begin{cases} E_1^1(x^{a-1}y) = (a-1)x^a y + x^{a-1}y^2 \\ E_1^2(x^{a-2}y) = \binom{a-2}{2} x^a y + (a-2)x^{a-1}y^2 \end{cases}$$

The determinant of the coefficient matrix is $(a-2)(a+1)/2$. By Lemma 7.1, a monomial $x^a y$ is hit if $p|a$ where $a > p$. By Lemma 7.2, $x^a y$ is not hit in degree p^m. We only need to consider monomials in the form $x^a y$ where $p \nmid a$ and $a+1 \neq p^m$. For $a = p$ or $2 < a < p-1$, the determinant is not congruent to 0 $(mod\ p)$. Hence $x^a y$ is hit in these cases. When $a = 2$, there is only

So $x^a y^b z^c \equiv -ax^{a+c-1}y^b z - bx^a y^{b+c-1}z$ *(mod hit)*. □

Lemma 7.6 *In degree* $\geq 2p+2$*, a monomial* $x^a y^b z^c$ *with two of* a, b, c *divisible by* p *is hit over* $\mathbb{F}_p$.

Proof: By Lemma 3.13 and Lemma 3.14. □

Proposition 7.7 *In degree* p^m*, a monomial* $x_1^{a_1}x_2^{a_2}\ldots x_n^{a_n}$*, which has at least two exponents not divisible by* p*, is not hit over* $\mathbb{F}_p$.

Proof: The proof is similar to the proof of Proposition 5.3 and by Lemma 7.2. □

Hence in degree p^m, a monomial $x^a y^b z^c$, where two of a, b, c are not divisible by p, is not hit over $\mathbb{F}_p$.

Lemma 7.8 *In degree* p^m+1*, a monomial* $x^a y^b z$ *with* $p \nmid a, b$ *is not hit over* $\mathbb{F}_p$.

Proof: By Theorem 3.12 and Theorem 7.2. □

Lemma 7.9 *A monomial* $x^a y^p z$*, where* $a = kp^2 - 1$ *for* $k \geq 2$*, is hit over* $\mathbb{F}_p$.

Proof: Let $a_n \cdots a_1 a_0$ be the $p-$expansion of a and let $t = (p-1)/2$. We have

$$D_t(x^a y^{t+1} z) = ax^{a+t}y^{t+1}z + (t+1)x^a y^p z + x^a y^{t+1} z^{t+1}. \qquad (100)$$

We will show that the 1st and the 3rd terms of the above equation are hit and hence $x^a y^p z$ is hit since $t+1 \not\equiv 0$ *(mod p)*.

(1) For the monomial $x^{a+t}y^{t+1}z$, we have the following cases:

(a) Suppose the 2nd digit of the $p-$expansion of $a+t \neq 1$. Then

$$E_1^{p^2}(x^{a+t-p^2}y^{t+1}z) \equiv \binom{a+t-p^2}{p^2} x^{a+t}y^{t+1}z$$
$$+ \sum_{1\leq s\leq t+1} \binom{a+t-p^2}{p^2-s}\binom{t+1}{s} x^{a+t-s}y^{t+1+s}z$$
$$+ \sum_{0\leq s\leq t+1} \binom{a+t-p^2}{p^2-s-1}\binom{t+1}{s} x^{a+t-s-1}y^{t+1+s}z^2 \ (mod\ p).$$

But

$$\binom{a+t-p^2}{p^2-s} \equiv 0 \ (mod\ p)$$

for $1 \leq s \leq t+1$ and

$$\binom{a+t-p^2}{p^2-s-1} \equiv 0 \ (mod\ p)$$

for $0 \leq s \leq t+1$. Since the 1st digits of the $p-$expansions of p^2-s and p^2-s-1 are $p-1$ and the 1st digit of the $p-$expansion of $a+t-p^2$ is 0. Also since the 2nd digit of the $p-$expansion of $a+t \neq 1$ and hence the 2nd digit of the $p-$expansion of $a+t-p^2 \neq 0$, we get $\binom{a+t-p^2}{p^2} \not\equiv 0 \ (mod\ p)$. So $x^{a+t}y^{t+1}z$ is hit for this case.

(b) Suppose the 2nd digit of the $p-$expansion of $a+t=1$. Then

$$E_1^{2p^2}(x^{a+t-2p^2}y^{t+1}z) \equiv \binom{a+t-2p^2}{2p^2} x^{a+t}y^{t+1}z$$
$$+ \sum_{1\leq s\leq t+1} \binom{a+t-2p^2}{2p^2-s}\binom{t+1}{s} x^{a+t-s}y^{t+1+s}z$$
$$+ \sum_{0\leq s\leq t+1} \binom{a+t-2p^2}{2p^2-s-1}\binom{t+1}{s} x^{a+t-s-1}y^{t+1+s}z^2 \ (mod\ p).$$

Again

$$\binom{a+t-2p^2}{2p^2-s} \equiv 0 \ (mod\ p)$$

for $1 \leq s \leq t+1$ and

$$\binom{a+t-2p^2}{2p^2-s-1} \equiv 0 \ (mod\ p)$$

for $0 \leq s \leq t+1$, since the 1st digits of the $p-$expansions of $2p^2 - s$ and $2p^2 - s - 1$ are $p-1$ and the 1st digit of the $p-$expansion of $a+t-2p^2$ is 0. Also $\binom{a+t-2p^2}{2p^2} \not\equiv 0 \ (mod\ p)$, since the 2nd digit of the $p-$expansion of $a+t=1$ and hence the 2nd digit of the $p-$expansion of $a+t-2p^2 = p-1 \geq 2$ for any odd prime p.

Hence $x^{a+t}y^{t+1}z$ in (100) is hit.

(2) For the monomial $x^a y^{t+1} z^{t+1}$, by Theorem 2.9,

$$x^a y^{t+1} z^{t+1} = x^a E_t^2(yz) \equiv (\chi(E_t^2)(x^a))yz$$

$$= ((D_t D_t - E_t^2)(x^a))yz \equiv (a(a+t) - \binom{a}{2})x^{a+2t}yz \equiv x^{a+2t}yz \ (mod\ hit),$$

since $a(a+t) - \binom{a}{2} \not\equiv 0 \ (mod\ p)$.

For the monomial $x^{a+2t}yz$, we have the following cases:

(c) Suppose the 2nd digit of the $p-$expansion of $a+2t \neq 1$. Then

$$E_1^{p^2}(x^{a+2t-p^2}yz) \equiv \binom{a+2t-p^2}{p^2} x^{a+2t}yz$$

$$+\binom{a+2t-p^2}{p^2-1} x^{a+2t-1}(y^2z+yz^2) + \binom{a+2t-p^2}{p^2-2} x^{a+2t-2}y^2z^2 \ (mod\ p).$$

But

$$\binom{a+2t-p^2}{p^2-1} \equiv \binom{a+2t-p^2}{p^2-2} \equiv 0 \ (mod\ p),$$

since the 1st digit of the $p-$expansion of $a+2t-p^2$ is 0 and the 1st digits of the $p-$expansions of p^2-1 and p^2-2 are $p-1$. Also $\binom{a+2t-p^2}{p^2} \not\equiv 0 \ (mod\ p)$ since the 2nd digit of the $p-$expansion of $a+2t-p^2 \neq 0$. Hence $x^{a+2t}yz$ is hit for this case.

(d) Finally the 2nd digit of the $p-$expansion of $a+2t=1$, then

$$E_1^{2p^2}(x^{a+2t-2p^2}yz) \equiv \binom{a+2t-2p^2}{2p^2} x^{a+2t}yz$$

$$+\binom{a+2t-2p^2}{2p^2-1} x^{a+2t-1}(y^2z+yz^2) + \binom{a+2t-2p^2}{2p^2-2} x^{a+t-2}y^2z^2$$

$$\equiv x^{a+t-2}y^2z^2 \ (mod\ p).$$

We see that $\binom{a+2t-2p^2}{2p^2-1} \equiv \binom{a+2t-2p^2}{2p^2-2} \equiv 0 \ (mod\ p)$ since the 1st digit of the $p-$expansion of $a+2t-2p^2$ is 0 and the 1st digit of the $p-$expansion of $2p^2-1$ or $2p^2-1$ is $p-1$. Also $\binom{a+2t-2p^2}{2p^2} \not\equiv 0 \ (mod\ p)$ since the 2nd digit of the $p-$expansion of $a+2t-2p^2$ is $p-1 \geq 2$. Hence $x^a y^{t+1} z^{t+1}$ in (100) is hit.

By the results of (1), (2) and the equation (100), $x^a y^p z$ is hit. □

Lemma 7.10 *Let $a_n a_{n-1} \cdots a_0$ be the $p-$expansion of a. A monomial $x^a y^p z$ is hit over $\mathbb{F}_p$ if $a \geq 2p^2-1$ and either $a_1 \leq p-3$ or $a_0 < p-1$.*

Proof: If $a_0 \neq p-1$, then

$$D_{p-1}(x^{a-p+1}y^p z) \equiv (a-p+1)x^a y^p z + x^{a-p+1}y^p z^p \equiv (a-p+1)x^a y^p z \ (mod\ hit),$$

by Lemma 7.6. Hence $x^a y^p z$ is hit in this case. Suppose $a_0 = p-1$ and $a_1 \leq p-3$. We have the following cases:

(1) If $a_n > 1$ then we set $k = p^n$.

(2) If $a_n = 1$, then $n \geq 3$ since $a \geq 2p^2-1$. For this case, if $a_{n-1} > 1$ we set $k = p^{n-1}$ and if $a_{n-1} = 1$ we set $k = 2p^{n-1}$.

Because $\binom{p}{s} \equiv 0 \ (mod\ p)$ unless $s = 0$ or p, we get

$$E_1^k(x^{a-k}y^p z) \equiv \binom{a-k}{k} x^a y^p z + \binom{a-k}{k-1} x^{a-1} y^p z^2$$

$$+\binom{a-k}{k-p} x^{a-p} y^{2p} z + \binom{a-k}{k-p-1} x^{a-p-1} y^{2p} z^2 \ (mod\ p).$$

The 1st digit of the $p-$expansion of $a-k=a_1\leq p-3$ and the 1st digits of the $p-$expansions of $k-1,\ k-p,\ k-p-1$ are $\geq (p-2)$, so

$$\binom{a-k}{k-1}\equiv\binom{a-k}{k-p}\equiv\binom{a-k}{k-p-1}\equiv 0\ (mod\ p).$$

Also $\binom{a-k}{k}\not\equiv 0\ (mod\ p)$. It is obviously true for case (1) and case (2) where $a_{n-1}>1$. For case (2) where $a_{n-1}=1$, $\binom{a-2p^{n-1}}{2p^{n-1}}\not\equiv 0\ (mod\ p)$, since the $(n-1)$th digit of the $p-$expansion of $a-2p^{n-1}$ is $p-1$ which is ≥ 2. So $x^a y^p z$ is hit. $\square$

Proposition 7.11 *A monomial $x^a y^p z$ where $a\geq 2p^2-1$ is hit over $\mathbb{F}_p$ if it is not in degree kp^m for $1\leq k\leq p-1$.*

Proof: By the last two lemmas, we only consider an a where $a_0=p-1$ and $a_1=p-2$. Because the monomial is not in degree kp^m, for some i where $2\leq i\leq n-1$, there is an $a_i<p-1$. Again because $\binom{p}{s}\equiv 0\ (mod\ p)$ unless $s=0$ or p, we have:

$$E^{p^i}_{p-1}(x^{a-(p-1)p^i}y^p z)\equiv\binom{a-(p-1)p^i}{p^i}x^a y^p z+\binom{a-p^i(p-1)}{p^i-1}x^{a-p+1}y^p z^p$$
$$+\binom{a-(p-1)p^i}{p^i-p}x^{a-p^2+p}y^{p^2}z+\binom{a-(p-1)p^i}{p^i-p-1}x^{a-p^2+1}y^{p^2}z^p\ (mod\ p)$$
$$\equiv\binom{a-(p-1)p^i}{p^i}x^a y^p z\ (mod\ hit),$$

by Lemma 7.6 and since the 1st digit of the $p-$expansion of $a-p^i(p-1)$ is $p-2$ and the 1st digit of the $p-$expansion of p^i-p is $p-1$,

$$\binom{a-p^i(p-1)}{p^i-p}\equiv 0\ (mod\ p).$$

Also $\binom{a-(p-1)p^i}{p^i}\not\equiv 0\ (mod\ p)$, since $a_i\neq p-1$, the ith digit of the $p-$expansion of $a-p^i(p-1)\geq 1$. $\square$

Proposition 7.12 *A monomial $x^a y^{p^t} z$, where $a \geq 2p^2 - 1$ and $t > 1$, is hit over $\mathbb{F}_p$ if it is not in degree kp^m with $1 \leq k \leq p-1$.*

Proof: We argue by induction on the exponent of y. Suppose $t > 1$. Because $\binom{p^{t-1}}{s} \equiv 0$ $(mod\ p)$ unless $s = 0$ and p^{t-1}, we have

$$E_{p-1}^{p^{t-1}}(x^a y^{p^{t-1}} z) \equiv x^a y^{p^t} z + \binom{a}{p^{t-1}} x^{a+(p-1)p^{t-1}} y^{p^{t-1}} z$$

$$+\binom{a}{p^{t-1}-1} x^{a+(p-1)(p^{t-1}-1)} y^{p^{t-1}} z^p \equiv x^a y^{p^t} z + \binom{a}{p^{t-1}} x^{a+(p-1)p^{t-1}} y^{p^{t-1}} z\ (mod\ hit),$$

by Lemma 7.6. This gives

$$x^a y^{p^t} z \equiv -\binom{a}{p^{t-1}} x^{a+(p-1)p^{t-1}} y^{p^{t-1}} z\ (mod\ hit).$$

If $\binom{a}{p^t} \not\equiv 0$ $(mod\ p)$ and $t - 1 > 1$, then we can repeat the procedure again. By the induction hypothesis, we must get that $x^a y^{p^t} z$ is either hit or $\equiv r x^{a+p^t-p} y^p z$ $(mod\ hit)$ for some $r \in \mathbb{F}_p$, which is hit by Proposition 7.11.

□

Theorem 7.13 *A monomial $x^a yz$, where $a \geq 2p^2 + p - 1$, is hit over $\mathbb{F}_p$ if it is not in degree kp^m and $kp^m + 1$ with $1 \leq k \leq p-1$.*

Proof: Because $x^a yz$ is not in degree $kp^m + 1$, some $a_i \neq p-1$ for $0 \leq i \leq n-1$ in the $p-$expansion of $a = a_n a_{n-1} \ldots a_0$ where $a_n \neq 0$. Suppose a_i is the first digit counted from right which is not equal to $p-1$. Since $x^a yz$ is not in degree kp^m, by Lemma 7.6 and Proposition 7.11,

$$E_{p-1}^{p^i}(x^{a-(p-1)p^i} yz) = \binom{a-(p-1)p^i}{p^i} x^a yz + \binom{a-(p-1)p^i}{p^i - 1} x^a (y^p z + yz^p)$$

$$+\binom{a-(p-1)p^i}{p^i - 2} x^a y^p z^p \equiv \binom{a-(p-1)p^i}{p^i} x^a yz\ (mod\ p),$$

but $\binom{a-(p-1)p^i}{p^i} \not\equiv 0 \ (mod\ p)$, since $a_i \neq p-1$ and hence the ith digit of the $p-$expansion of $a-(p-1)p^i \geq 1$. $\square$

We have not found results for some monomials in the form $[a11]$, monomials in the form $[a21]$ and monomials in the form $[a(p^m-1)1]$ for the 3 variable case over $\mathbb{F}_p$. Over $\mathbb{F}_2$, we solved monomials in the form $[a(2^m)1]$ in Section 5.6, then we used the result in Section 5.7 to solve monomials in the form $[a(2^m-1)1]$. Because for a monomial $x^a y^{2^m-1} z$, we may either change the exponent of y to 2^m or change the exponent of x to 2 by using some E_1^k operations. But over $\mathbb{F}_p$, it is difficult to do similar thing to $x^a y^{p^m-1} z$, because there is no such an operation which either increases p^m-1 to p^m or increases the exponent of z to p, since $p-1$ is no longer equal to 1. We have got a partial result for the monomials in the form $[a(p^m)1]$ for $m \geq 0$ and we can only use the result to get a partial result for the monomials in the form $[a11]$.

8 Appendix

A Table of Kostka numbers for $n = 2,\ 3,\ 4$

$n = 2:$

$\mu \setminus \lambda$	(11)	(2)
(11)	1	1
(2)	0	1

$n = 3:$

$\mu \setminus \lambda$	(111)	(21)	(3)
(111)	1	2	1
(21)	0	1	1
(3)	0	0	1

$n = 4:$

$\mu \setminus \lambda$	(1111)	(211)	(22)	(31)	(4)
(1111)	1	3	2	3	1
(211)	0	1	1	2	1
(22)	0	0	1	1	1
(31)	0	0	0	1	1
(4)	0	0	0	0	1

Note that the numbers in the first row corresponding to each column index λ of a table are also numbers of standard λ–tableaux.

B Decomposition table of S_4^d in terms of M^λ and Sp^λ

We give the decomposition table of S_4^d in terms of M^λ and Sp^λ for $d \leq 10$.

d	*The decomposition of* S_n^d *into* M^λ	*The decomposition of* S_n^d *into* Sp^λ
4	$M^{(4)}$	$Sp^{(4)}$
5	$M^{(31)}$	$Sp^{(4)} \oplus Sp^{(31)}$
6	$M^{(31)} \oplus M^{(22)}$	$2Sp^{(4)} \oplus 2Sp^{(31)} \oplus Sp^{(22)}$
7	$2M^{(31)} \oplus M^{(211)}$	$3Sp^{(4)} \oplus 4Sp^{(31)} \oplus Sp^{(22)} \oplus Sp^{(211)}$
8	$M^{(4)} \oplus M^{(31)} \oplus 2M^{(211)} \oplus M^{(22)}$	$5Sp^{(4)} \oplus 6Sp^{(31)} \oplus 3Sp^{(22)} \oplus 2Sp^{(211)}$
9	$2M^{(31)} \oplus 4M^{(211)}$	$6Sp^{(4)} \oplus 10Sp^{(31)} \oplus 4Sp^{(22)} \oplus 4Sp^{(211)}$
10	$3M^{(31)} \oplus 2M^{(22)} \oplus 3M^{(211)} \oplus M^{(1111)}$	$9Sp^{(4)} \oplus 14Sp^{(31)} \oplus 7Sp^{(22)} \oplus 6Sp^{(211)} \oplus Sp^{(1111)}$

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