THE FINE STRUCTURE OF ORBITS IN DYNAMICAL SYSTEMS

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Abstract

We study the ergodic sums $g^n(x) := \sum_{i=0}^{n-1} g(T^i x)$ for Hölder continuous functions g. We look at sets of points x for which the sums $g^n(x)$ have a specified behaviour as $n \to \infty$. For subshifts of finite type, Fan and Schmeling showed that many of these sets have the same Hausdorff dimension: for example, the set of points with bounded sums generally has the same dimension as the set of all points x for which $\frac{1}{n}g^n(x) \to 0$. We show how their method can be extended and applied to other dynamical systems (conformal expanding maps, and conformal hyperbolic diffeomorphisms and flows).

We also consider a problem concerning the homology classes of periodic orbits of Anosov flows. Our results give information about how the 'directions' of these homology classes are distributed.

Declaration

No portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

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Acknowledgement

The results in chapter 3 are joint work with my supervisor Richard Sharp; an account appeared in [CS]. I would like to thank Richard, not just for this part of the work, but for all the help he has given me over the last four years.

Chapter 1

Preliminaries

In this thesis we consider two separate problems. The first problem, which we look at in chapter 2, is to extend the results of Fan and Schmeling in their paper [FS]. Then in chapter 3 we look at the 'directions' of homology classes of periodic orbits for Anosov flows.

But first of all in this preliminary chapter we describe the dynamical systems that we will be interested in, and explain some of the standard methods for dealing with them.

1.1 Shift spaces

This section describes some very useful examples of dynamical systems – namely, subshifts of finite type. These turn out to have properties which make it possible to prove a wide variety of results. A large amount of theory is known, most of which will not be needed for our results. For the parts that we do need we will mostly be following the book by Parry and Pollicott [PP]. (Note that throughout this thesis, the references have been selected on the basis of convenience and are not intended to be historical.)

We start by considering a set of k 'symbols', $\{1, 2, \ldots, k\}$, and look at the set of doubly infinite sequences of such symbols, i.e.

$$S = \{1, 2, \dots, k\}^{\mathbb{Z}}.$$

For any $\theta \in (0, 1)$ we can define a metric on S by

$$d(x, y) = \theta^n$$

where n is the largest integer such that $x_i = y_i$ for all $|i| < n$. (Of course this only applies for $x \neq y$; when $x = y$ we have $d(x, y) = 0$.) With these definitions, S is a compact metric space.

The shift map $\sigma : S \to S$ is defined by

$$(\sigma x)_i = x_{i+1},$$

so we think of σ as shifting a sequence of symbols one place to the left. Then (S, σ) is called the *full shift* on k symbols.

To get further examples of maps, we want to consider the restriction of σ to certain closed subsets of S. Let A be a $k \times k$ matrix, with all its entries being either 0 or 1. Then we define

$$X_A = \left\{ x \in S : A_{x_i x_{i+1}} = 1 \ \forall i \in \mathbb{Z} \right\}.$$

Clearly σ maps X_A to itself, and $\sigma : X_A \to X_A$ is a homeomorphism; we call (X_A, σ) a *subshift of finite type.*

A finite sequence of symbols is called a *word.* Thus the definition of X_A says simply that the word pq is allowed to appear inside elements of X_A if and only if $A_{pq} = 1$. (We say that such a word is 'admissible'.) It follows (by an inductive argument) that $[A^n]_{pq}$ gives the number of admissible words of length $n+1$ which start with p and end with q. If for all p and q there exists some number n such that $[A^n]_{pq} > 0$, then we say that A is *irreducible.* If A satisfies the stronger condition that there exists some n for which $[A^n]_{pq} > 0$ for all p and q, then it is *aperiodic*. For most applications we will want to assume aperiodicity.

Having defined these 'two-sided' subshifts of finite type, we now go on to define the very similar *one-sided* subshifts of finite type. This means, instead of working with doubly infinite sequences of symbols $(x_i)_{i\in\mathbb{Z}}$, we look at sequences $(x_i)_{i\geq 0}$. (Note that indices will start at 0 here, whereas in [FS] (for example) they start at 1.) So we replace S by $S^+ := \{1, 2, \ldots, k\}^{\mathbb{N}_0}$. This new space can also be given a metric: for $x \neq y \in S^+$ we have $d(x, y) = \theta^n$, where n is the largest integer such that $x_i = y_i$ for all $0 \leq i < n$. Given a matrix A we have the closed subspaces

$$X_A^+ := \left\{x \in X^+ : A_{x_i x_{i+1}} = 1 \ \forall i \geq 0\right\}.$$

As before, we have a shift map σ defined by $(\sigma x)_i = x_{i+1}$, which maps X_A to itself; but in contrast to the situation for two-sided shifts, this σ is not invertible.

A *cylinder* in X_A^+ is a set

$$[s_0\, s_1\, \ldots\, s_{n-1}] := \left\{x \in X_A^+ : x_i = s_i \ \text{ for all } i < n\right\}.$$

We will say that this cylinder has 'length' n, and write $\mathrm{Cyl}(n)$ for the set of all cylinders of length n. (Note that our indices run from 0 to $n-1$, whereas a more common notation (e.g. [PP], [Pes]) is for indices to run from 0 to n. The former is more natural for our purposes.)

A *block* is an admissible word $b_0\, b_1\, \ldots\, b_{\ell-1}$; the block as a whole is denoted by a capital letter B. We tend to use the term 'block' rather than 'word' when we are interested in the places where B appears inside x, i.e. $x_{m+i} = b_i$ $(0 \leq i < \ell)$ for various x, m. Any block B of length ℓ defines a cylinder in X_A^+ (of length ℓ) which we will write as

$$[B] = [b_0\, b_1\, \ldots\, b_{\ell-1}].$$

When looking at functions $g : X_A \to \mathbb{R}$ or $g : X_A^+ \to \mathbb{R}$, we will often require them to be Hölder continuous, i.e. there exist constants $C > 0$, $\alpha \in (0, 1)$ such that

$$|g(x) - g(y)| \leq Cd(x,y)^\alpha.$$

(More generally, Hölder continuity can be defined for functions between any two metric spaces.) Notice that whether a function is Hölder continuous does not depend on the value of θ chosen in defining d, and in fact a function is Hölder continuous if and only if it is Lipschitz with respect to d for some θ.

Given two continuous functions g_1 and g_2 defined on X_A or X_A^+, we say that they are *cohomologous* if for some other continuous function h defined on the same space,

$$g_1 = g_2 + h \circ \sigma - h.$$

Cohomology of functions is a very important equivalence relation on the set of continuous functions (or the set of Hölder continuous functions). Clearly, for example, two functions which are cohomologous have the same integral with respect to any invariant measure on (X_A, σ) or (X_A^+, σ). It is also particularly important when we look at sums of the form

$$g^n(x) := \sum_{i=0}^{n-1} g(\sigma^i x).$$

(Here $g^n(x)$ is a standard notation used for this sum, which we will be adopting. These sums are the focus of chapter 2.) We see that if g_1 and g_2 are cohomologous then

$$g_1^n = g_2^n + h \circ \sigma^n - h,$$

and so $g_1^n - g_2^n$ is (uniformly) bounded in n. Furthermore, if x is a periodic point, say $\sigma^n x = x$, then we have $g_1^n(x) = g_2^n(x)$. And in fact for Hölder continuous functions the converse of this is also true, as expressed in the following theorem.

Theorem 1.1 (Livšic [Liv]) *Let $g_1, g_2 : X_A \to \mathbb{R}$ (or $X_A^+ \to \mathbb{R}$) be Hölder continuous. Then g_1 and g_2 are cohomologous if and only if $g_1^n(x) = g_2^n(x)$ whenever $\sigma^n x = x$ for $x \in X_A$, $n > 0$.*

The other result we will need concerning cohomology of functions is this:

Theorem 1.2 (Sinai [Sin]) *Let $g : X_A \to \mathbb{R}$ be Hölder continuous. Then we can find a Hölder continuous function $g^{(u)}$ which is cohomologous to g, such that $g^{(u)}(x) = g^{(u)}(y)$ whenever $x_i = y_i$ for all $i \geq 0$.*

The function $g^{(u)}$ produced by this theorem can be thought of as a function defined on X_A^+. So this theorem provides a way of deducing results about functions on X_A from those about X_A^+.

1.2 Maps modeled by subshifts

Subshifts of finite type are often studied not for their own sake, but because they are related to certain maps defined on manifolds. Results about these maps can be proved by considering the related subshift of finite type and making use of its relatively simple definition. Specifically, one-sided subshifts of finite type provide a model for *expanding maps*, whereas two-sided subshifts of finite type provide a model for *hyperbolic diffeomorphisms*. These are defined in this section. We follow the descriptions in [Pes] and [PP].

We look at a smooth Riemannian manifold M and a C^1 map $f : M \to M$. If J is a compact subset of M for which $f(J) = J$ then we say that the map f is *expanding* on J if there exist constants $C > 0$, $\lambda > 0$ such that

$$\|(df^n)_x v\| \geq Ce^{\lambda n}\|v\| \quad \text{for all } x \in M, v \in T_x M \text{ and } n \geq 1.$$

If in addition there is an open set $V \supseteq J$ such that $J = \{x \in V : f^n x \in V \text{ for all } n \geq 0\}$ then we say that J is a *repeller.*

We restrict attention to repellers of expanding maps for which $f : J \to J$ is topologically mixing.

A *Markov partition* for $f : J \to J$ is a finite collection of closed subsets $R_1, \ldots, R_k$ which cover J, and which have the following properties:

(i) Each set R_i is the closure of its interior;

(ii) If $i \neq j$ then $\mathrm{int} R_i \cap \mathrm{int} R_j = \varnothing$;

(iii) For each i the restriction of f to R_i is injective, and we have $f(R_i) = R_{j_1(i)} \cup \cdots \cup R_{j_{n(i)}(i)}$ for some $j_1(i), \ldots, j_{n(i)}(i)$.

For any $\delta > 0$, f has a Markov partition which consists of sets with diameter less than δ. Once we have a Markov partition we can consider the subshift of finite type $\sigma : X_A^+ \to X_A^+$, where the matrix A is defined by

$$A_{ij} = \begin{cases} 1 & \text{if } \mathrm{int}\, R_i \cap f^{-1}(\mathrm{int}\, R_j) \neq \varnothing \\ 0 & \text{if } \mathrm{int}\, R_i \cap f^{-1}(\mathrm{int}\, R_j) = \varnothing. \end{cases}$$

For each $x \in X_A^+$ there is a unique point $\chi(x) \in J$ such that $f^n(\chi(x)) \in R_{x_n}$ for all $n \geq 0$. Thus χ is a well-defined map from X_A^+ to J, which we call the *coding map*. The subshift $\sigma : X_A^+ \to X_A^+$ is then a symbolic model for $f : J \to J$ in the sense that the following diagram commutes:

$$\begin{array}{ccc} J & \xrightarrow{f} & J \\ \chi \big\uparrow & & \big\uparrow \chi \\ X_A^+ & \xrightarrow{\sigma} & X_A^+ \end{array}$$

The construction of the Markov partition is such that χ is Hölder continuous; if g is a Hölder continuous function defined on J then its pullback to X_A^+ is also Hölder continuous.

Cohomology of functions is defined in the same way as for subshifts: that is, we say that g_1, g_2 are cohomologous if there exists a function h such that $g_1 = g_2 + h \circ f - h$. Livšic's Theorem (1.1) holds, and we have that Hölder continuous functions g_1 and g_2 are cohomologous if and only if the corresponding (pullback) functions on the subshift are cohomologous.

The assumption that f is topologically mixing implies that A is aperiodic.

We now move on to consider invertible maps. We take M to be a smooth compact Riemannian manifold and $f : M \to M$ to be a C^1-diffeomorphism.

Then a *hyperbolic set* Λ is an f-invariant subset of M with the property that the tangent bundle on the set Λ continuously splits as

$$T_\Lambda M = E^s \oplus E^u,$$

where E^s and E^u are df-invariant, and

- for $v \in E^s_x$ we have $\|(df^n)_x v\| \leq Ce^{-\lambda n}\|v\|$ for all $n \geq 0$;
- for $v \in E^u_x$ we have $\|(df^{-n})_x v\| \leq Ce^{-\lambda n}\|v\|$ for all $n \geq 0$,

for constants $C > 0$, $\lambda > 0$ which do not depend on x. Here E^s_x is the *stable subspace* and E^u_x is the *unstable subspace* for the point x. If M itself is a hyperbolic set then we say that the map f is *Anosov*.

We consider closed hyperbolic sets Λ which have the following properties:

(i) the periodic points of $f|_\Lambda$ are dense in Λ;

(ii) there exists a point $x \in \Lambda$ such that $\{f^n x : n \in \mathbb{Z}\}$ is dense in Λ;

(iii) we can find an open set $U \supseteq \Lambda$ with $\bigcup_{n\in\mathbb{Z}} f^n(U) = \Lambda$.

If Λ consists of a single periodic orbit then it may have these properties; but we want to exclude this possibility, for which everything becomes trivial. If we disallow the case of a single periodic orbit, the restriction of f to such a set Λ is called a *hyperbolic diffeomorphism*. Like with expanding maps, we will make the simplifying assumption that $f : \Lambda \to \Lambda$ is topologically mixing.

The link between hyperbolic diffeomorphisms and (two-sided) subshifts of finite type was described by Bowen in [Bow1]:

Theorem 1.3 ([Bow1]) *Let $f : \Lambda \to \Lambda$ be a hyperbolic diffeomorphism. Then we can find a subshift of finite type $\sigma : X_A \to X_A$, and a Hölder continuous, bounded-to-one surjection $\chi : X_A \to \Lambda$ such that $\chi \circ \sigma = f \circ \chi$.*

As with expanding maps, the assumption that f is topologically mixing ensures that A is aperiodic.

The coding is determined by a different type of Markov partition from the one used for expanding maps. As before we cover Λ by a finite number of closed sets $R_1, \ldots, R_k$; each set is the closure of its interior (in terms of the subset topology on $\Lambda \subseteq M$), and $\operatorname{int} R_i \cap \operatorname{int} R_j = \varnothing$ for $i \neq j$. The matrix A is again defined by $A_{ij} = 1$ iff $\operatorname{int} R_i \cap f^{-1}(\operatorname{int} R_j) \neq \varnothing$. The construction of the partition is done in such a way that for each $x \in X_A$ there is a unique $\chi(x)$ such that $f^n(\chi(x)) \in R_{x_n}$ for all $n \in \mathbb{Z}$; this defines the coding map χ.

In order for this construction to work, each of the sets R_i must have a particular structure, related to the way the tangent bundle splits into stable and unstable sub-bundles. Given a sufficiently small $\epsilon > 0$ and any point $x \in \Lambda$, there exist (local) *stable* and *unstable manifolds* $W^s_\epsilon(x)$, $W^u_\epsilon(x)$, which are tangent to E^s and E^u respectively at the point x, defined by

$$W^s_\epsilon(x) = \{y \in M : d(f^n y, f^n x) < \epsilon \ \forall n \geq 0\};$$
$$W^u_\epsilon(x) = \left\{y \in M : d(f^{-n} y, f^{-n} x) < \epsilon \ \forall n \geq 0\right\}.$$

These can alternatively be described (for sufficiently small ϵ) as being the sets of points $y \in B(x, \epsilon)$ for which $d(f^n y, f^n x)$ (respectively $d(f^{-n}y, f^{-n}x)$) goes to zero; furthermore for all such points the rate of convergence will be exponential.

For any sufficiently small δ (depending on ϵ) we have that if $x, y \in \Lambda$ with $d(x, y) < \delta$ then $W^s_\epsilon(x) \cap W^u_\epsilon(y)$ consists of a single point lying in Λ. This point is denoted by $[x, y]$.

We work with subsets $R \subseteq \Lambda$ for which $\operatorname{diam} R < \delta \ll \epsilon$; in particular we want $[x, y]$ to be defined whenever $x, y \in R$. The set R is a *rectangle* if $[x, y] \in R$ for all $x, y \in R$. In the Markov partitions for $f : \Lambda \to \Lambda$, each set R_i will be a rectangle. For $z \in \operatorname{int} R_i$ we can write

$$W^s_{R_i}(z) = W^s_\epsilon(z) \cap R_i,$$
$$W^u_{R_i}(z) = W^u_\epsilon(z) \cap R_i.$$

These sets have a simple interpretation in terms of the coding map: if $z = \chi(x)$

then we have

$$W^s_{R_i}(z) = \{\chi(y) : y \in X_A \text{ with } y_n = x_n \ \forall n \geq 0\}\,;$$
$$W^u_{R_i}(z) = \{\chi(y) : y \in X_A \text{ with } y_n = x_n \ \forall n \leq 0\}\,.$$

The rectangle R_i is then homeomorphic to the product $W^u_{R_i}(z) \times W^s_{R_i}(z)$, with the homeomorphism being given by the map $(x, y) \mapsto [x, y]$. We say that these rectangles have a *product structure.*

(For completeness, we finish by stating the condition that these rectangles must satisfy in order to be a Markov partition for $f : \Lambda \to \Lambda$. If $z \in \operatorname{int} R_i$ and $fz \in \operatorname{int} R_j$ then we insist that

$$f\left(W^s_{R_i}(z)\right) \subset W^s_{R_j}(fz);$$
$$f\left(W^u_{R_i}(z)\right) \supset W^u_{R_j}(fz).$$

This is the analogue of the third condition for Markov partitions for expanding maps. Given δ, a Markov partition can always be found whose sets are rectangles with diameter less than δ. A fuller description can be found in [PP].)

1.3 Suspended flows and hyperbolic flows

We start this section by giving the definition of a suspended flow on a (two-sided) subshift of finite type. We take a strictly positive, Hölder continuous function $r : X_A \to \mathbb{R}^+$, and define the set

$$X^r_A = \{(x, s) : x \in X_A, 0 \leq s \leq r(x)\}\,,$$

but with the point $(x, r(x))$ identified with $(\sigma x, 0)$ for each x. (So, formally X^r_A is defined as a quotient.) Like X_A itself, the space X^r_A can be given a metric (see [BS1]).

The flow σ^r_t is defined on X^r_A: for small t we define

$$\sigma^r_t(x, s) = (x, s + t).$$

Of course this only holds while $s+t \leq r(x)$; when $t = r(x) - s$ we have

$$\sigma_t^r(x,s) = (x, r(x)) = (\sigma x, 0),$$

and thus we can continue to define the flow by restarting 'vertically' from $(\sigma x, 0)$. That is, if we find the integer N such that $r^N(x) \leq s+t < r^{N+1}(x)$ then we have $\sigma_t^r(x,s) = (\sigma^N x, s+t-r^N(x))$.

In the same way that subshifts of finite type served as models for hyperbolic diffeomorphisms, these suspended flows are models for *hyperbolic flows*. Again we take M to be a smooth, compact Riemannian manifold; now let $\phi_t : M \to M$ be a C^1 flow. A *hyperbolic set* Λ for this flow is a ϕ_t-invariant subset of M such that the tangent bundle on the set Λ splits into $d\phi_t$-invariant subbundles as

$$T_\Lambda M = E \oplus E^s \oplus E^u,$$

where

- E is a one-dimensional subbundle, tangent to the flow;
- for $v \in E_x^s$ we have $\|(d\phi_t)_x v\| \leq Ce^{-\lambda t}\|v\|$ for all $t \geq 0$;
- for $v \in E_x^u$ we have $\|(d\phi_{-t})_x v\| \leq Ce^{-\lambda t}\|v\|$ for all $t \geq 0$,

for constants C, $\lambda > 0$. As with diffeomorphisms, if M itself is a hyperbolic set then we say that the flow ϕ is *Anosov*; this condition is satisfied by the geodesic flow on the unit tangent bundle of a negatively-curved manifold.

We consider closed hyperbolic sets $\Lambda \subseteq M$ such that

(i) the periodic orbits of $\phi|_\Lambda$ are dense in Λ;

(ii) there exists a point $x \in \Lambda$ with $\{\phi_t x : t \in \mathbb{R}\}$ dense in Λ;

(iii) we can find an open set $U \supseteq \Lambda$ with $\bigcap_{t\in\mathbb{R}} \phi_t(U) = \Lambda$.

(iv) Λ is not a single periodic orbit.

The restriction of ϕ_t to such a set Λ is a hyperbolic flow.

Again we have symbolic dynamics:

Theorem 1.4 ([Bow2]) *Let $\phi_t : \Lambda \to \Lambda$ be a hyperbolic flow. Then we can find a suspended flow $\sigma_t^r : X_A^r \to X_A^r$, with the matrix A being aperiodic, and a continuous, bounded-to-one surjection $\rho : X_A^r \to \Lambda$ such that $\rho \circ \sigma_t^r = \phi_t \circ \rho$.*

The construction of (X_A^r, σ_t^r) and ρ is somewhat more intricate than in the discrete-time case. The following is only a brief description of those facts that are important for our results. (More detailed descriptions are found in [PP] and [PS].)

The construction is based around finding disjoint closed sets $T_j \subset \Lambda$ ($1 \leq j \leq k$), called *Markov sections.* Each T_j is a local cross-section for the flow: that is, it is contained in a small C^1 submanifold $D_j \subset M$ of dimension $\dim M - 1$ which is transverse to the flow, and the set T_j is the closure of its interior (in the topology of $\Lambda \cap D_j$). Furthermore, if we let $\mathcal{T} = \bigcup T_i$ then we want to have $M = \bigcup_{t\in[0,a]} \phi_t(\mathcal{T})$ for some $a > 0$. This then ensures that each orbit of the flow intersects T at least once in any sufficiently large interval of time.

We can then define the Poincaré return map $P : \mathcal{T} \to \mathcal{T}$, which takes a point $x \in \mathcal{T}$ to $\phi_t x$ where t is the smallest positive real number such that $\phi_t x \in \mathcal{T}$. This map P is invertible. So every point $x \in \mathcal{T}$ defines a sequence $q(x) = \{q_i\}_{i\in\mathbb{Z}}$ such that $q_i = j$ if $P^i(x) \in T_j$. But there are problems at the boundaries of the sets T_i, where the Poincaré map may fail to be continuous. We look at the restriction of q to the dense subset

$$\mathcal{T}' := \left\{x \in \mathcal{T} : \nexists\, i, j \text{ s.t. } P^i(x) \in \partial T_j\right\}.$$

If A is defined by

$$A_{ij} = \begin{cases} 1 & \text{if } P(\operatorname{int} T_i) \cap \operatorname{int} T_j \neq \emptyset \\ 0 & \text{if } P(\operatorname{int} T_i) \cap \operatorname{int} T_j = \emptyset, \end{cases}$$

then the map $q : x \mapsto \{q_i\}$ takes $\mathcal{T}'$ to a subset of X_A injectively. Furthermore the construction is such that $q(\mathcal{T}')$ is dense in X_A and the inverse is continuous where defined. This inverse can then be extended by continuity to a function $p : X_A \to \mathcal{T}$ which is a bounded-to-one surjection.

Next the function r is defined, initially only on $q(\mathcal{T}')$, by $r(q(x)) = \min\{t > 0 : \phi_t x \in \mathcal{T}\}$. Again this function can be extended by continuity, giving a Hölder continuous function $r : X_A \to \mathbb{R}^+$. If x is any point in X_A, then for any $n \in \mathbb{Z}$ we have $\phi_{r^n(x)}(p(x)) \in T_{x_n}$ (where we define $r^n(x) = -\sum_{j=n}^{-1} r(\sigma^j x)$ for $n < 0$). This r can be used to define a suspended flow (X_A^r, σ_t^r). Finally, the function $\rho : X_A^r \to \Lambda$ that we want is

$$\rho(x, s) = \phi_s p(x).$$

As with the rectangles for hyperbolic diffeomorphisms, the sets T_j have a product structure. Firstly we have that for any sufficiently small τ, there is a diffeomorphism

$$\bigcup_{t \in (-\tau,\tau)} \phi_t(D_j) \to D_j \times (-\tau, \tau),$$

obtained by travelling along the flow. We define π_j to be the projection map $\bigcup_{t\in(-\tau,\tau)} \phi_t(D_j) \to D_j$. The sets T_j can be chosen to be sufficiently close together that Λ is covered by the sets $\bigcup_{t\in(-\tau,\tau)} \phi_t(T_j)$.

For any point $x \in \Lambda$ we have a stable manifold $W_\epsilon^u(x)$ and an unstable manifold $W_\epsilon^u(x)$, analogous to those for hyperbolic diffeomorphisms. Given two points $x, y \in \Lambda$ which are sufficiently close together, there is a unique t with $|t| < \tau$ such that $W_\epsilon^s(\phi_t x) \cap W_\epsilon^u(y) \neq \varnothing$, and for this value of t the intersection is a single point which lies in Λ. Now if $x, y \in T_j \subset D_j$ this point might not lie inside D_j, however we can project it to D_j using the map π_j, giving a point which we define to be $[x, y] \in D_j \cap \Lambda$. (In order for this to work, the sets T_j are chosen to be not too close to the boundary of D_j, and small enough that $[x, y]$ is defined.) The set T_j is a *rectangle* if $[x, y] \in T_j$ whenever $x, y \in T_j$. In the construction of the symbolic dynamics, each of the Markov sections T_j is a rectangle.

For $z \in \operatorname{int} T_j$ we look at the projections of $W^s_\epsilon(z)$ and $W^u_\epsilon(z)$ onto T_j, i.e.

$$W^s_{T_j}(z) := \left\{y \in T_j : \pi_j^{-1}(\{y\}) \cap W^s_\epsilon(z) \neq \varnothing\right\};$$
$$W^u_{T_j}(z) := \left\{y \in T_j : \pi_j^{-1}(\{y\}) \cap W^u_\epsilon(z) \neq \varnothing\right\}.$$

Again these sets can be described in terms of the coding map: if $z = \rho(x, 0)$ then

$$W^s_{T_j}(z) = \{\rho(y, 0) : y \in X_A \text{ with } y_n = x_n \ \forall n \geq 0\};$$
$$W^u_{T_j}(z) = \{\rho(y, 0) : y \in X_A \text{ with } y_n = x_n \ \forall n \leq 0\}.$$

The rectangle T_j then has a product structure, $T_j \to W^u_{T_j}(z) \times W^s_{T_j}(z)$.

In chapter 3 we will consider questions involving periodic orbits of hyperbolic flows. For this set-up we have the problem that periodic orbits in Λ do not have a one-to-one correspondence with periodic orbits in X^r_A. Results which rely on counting periodic orbits in subshifts need to include corrections which take this into account. The methods used are explained in [Bow2], and draw on work by Manning in [Man] for discrete time. We will not need to know the details here.

1.4 Equilibrium states

Suppose we have a transformation $T : X \to X$, where X may be any compact metric space and T a continuous transformation. Then for any probability measure μ on X which is invariant for this transformation, we have the entropy $h_\mu(T)$. If we let $M(X, T)$ be the set of all invariant probability measures, then the supremum of $h_\mu(T)$ over all of $M(X, T)$ gives the topological entropy $h(T)$. This is the 'variational principle' (see, for example, [Wal]).

A measure μ for which the supremum is attained is called a measure of maximal entropy. In the case of a subshift of finite type, it is guaranteed that there exists a unique measure of maximal entropy, and this is often denoted by μ_0.

This is a special case of the definition of *pressure*. Given a continuous function $\psi : X \to \mathbb{R}$, its pressure $P(\psi)$ is a generalisation of the topological entropy; it is possible to give a definition which makes no mention of invariant measures, but the simplest definition is by a variational principle:

$$P(\psi) = \sup_{\mu \in M(X,T)} \left\{ h_\mu(T) + \int \psi \, d\mu \right\}.$$

A measure μ for which $h_\mu(T) + \int \psi \, d\mu = P(\psi)$ is called an *equilibrium state* for ψ. Just like in the case of measures of maximal entropy (which this reduces to by taking $\psi = 0$), we have the following:

Theorem 1.5 ([PP]) *Let $\psi : X_A \to \mathbb{R}$ (or $X_A^+ \to \mathbb{R}$) be Hölder continuous. Then ψ has a unique equilibrium state.*

This equilibrium state is guaranteed to be ergodic and fully supported. And when looking at Hölder continuous functions, equilibrium states are linked to the equivalence classes for cohomology of functions, by the following result.

Proposition 1.6 ([PP]) *Suppose $\psi_1, \psi_2 : X_A \to \mathbb{R}$ (or $X_A^+ \to \mathbb{R}$) are Hölder continuous. Then if ψ_1 and ψ_2 are cohomologous, or more generally if $\psi_1 - \psi_2$ is cohomologous to a constant function, then ψ_1 and ψ_2 have the same equilibrium state. Conversely, if ψ_1 and ψ_2 have the same equilibrium state then $\psi_1 - \psi_2$ must be cohomologous to a constant function.*

Suppose μ is the equilibrium state for a function ψ on X_A^+; then the map $\sigma^n : X_A^+ \to X_A^+$ also has μ as an equilibrium state: it is the equilibrium state for the function ψ^n (where $\psi^n(x) := \sum_{i=0}^{n-1} \psi(\sigma^i x)$ as usual). Indeed, we could define a new subshift of finite type whose symbols are words of length n in X_A^+, i.e. a point $x \in X_A^+$ corresponds to the point

$$(x_0 x_1 \dots x_{n-1} \ , \ x_n x_{n+1} \dots x_{2n-1} \ , \ x_{2n} x_{2n+1} \dots x_{3n-1} \ , \ \dots)$$

in the new subshift. (Clearly the assumption that A is aperiodic is important here.) Functions and measures carry across from one space to the other, and the map σ^n on X_A^+ corresponds to the shift map on the new space. μ is then an equilibrium state for this subshift.

We also look at equilibrium states for expanding maps and hyperbolic diffeomorphisms. By transferring the results of Theorem 1.4 for the related subshifts of finite type, it can be shown that Hölder continuous functions on these maps also have unique equilibrium states. Indeed, suppose ψ_J is a Hölder continuous function defined on a repeller J for some expanding map. Then by using the coding map χ we can pull this function back to a Hölder continuous function ψ_X on X_A^+. Suppose μ is the unique equilibrium state for ψ_X. Then we get a measure ν on J which is the pushforward of μ by the coding map. As we might have hoped, the measure ν turns out to be the unique equilibrium state for ψ_J. Equilibrium states for hyperbolic diffeomorphisms are related to those for two-sided subshifts in the same way.

The definitions of pressure and equilibrium states still make sense if, instead of a transformation $T : X \to X$, we have a flow $\phi_t : X \to X$. Again, in the cases we are interested in (hyperbolic flows, and suspended flows on subshifts of finite type), if $\psi : X \to X$ is a Hölder continuous function, then it has a unique equilibrium state. And the equilibrium states for hyperbolic flows are related to those for suspended flows via the map $\rho : X_A^r \to \Lambda$.

We can also relate equilibrium states for suspended flows to those for the underlying subshifts of finite type. First we look at the difference between a one-sided subshift (X_A^+, σ) and the corresponding two-sided subshift (X_A, σ). We can consider a map between the two spaces,

$$\pi_+ : X_A \to X_A^+,$$

which 'forgets' the negative co-ordinates in X_A. (That is, $(\pi_+ x)_i = x_i$ for $i \geq 0$.) Then for any σ-invariant measure μ on X_A, we can define the pushforward measure μ^+ on X_A^+ by

$$\mu^+(S) = \mu\left(\pi_+^{-1}(S)\right).$$

Now, suppose we have a Hölder continuous function $\psi : X_A \longrightarrow \mathbb{R}$, with equilibrium state μ. We know from Theorem 1.2 that we can find a cohomologous function $\psi^{(u)}$ such that $\psi^{(u)}(x) = \psi^{(u)}(y)$ whenever $x_i = y_i$ for all $i \geq 0$. And from Proposition 1.6, $\psi^{(u)}$ also has equilibrium state μ. But $\psi^{(u)}$ can also be regarded as a function on X_A^+, and the corresponding equilibrium state on X_A^+ turns out to be μ^+, defined as above.

Now, suppose that we have a suspended flow (X_A^r, σ_t^r); we want to compare its equilibrium states to those for (X_A, σ). Note that for any (Hölder) continuous function $g : X_A^r \longrightarrow \mathbb{R}$, there is a natural way of producing a continuous function on X_A, which we will call $\mathcal{I}g$, given by

$$\mathcal{I}g(x) = \int_0^{r(x)} g(x, s)\, ds.$$

If g (and also r) is Hölder continuous, then so is $\mathcal{I}g$. Two Hölder continuous functions g_1, g_2 on X_A^r are cohomologous if and only if $\mathcal{I}g_1$ and $\mathcal{I}g_2$ are cohomologous on X_A ([BS1]). (For a general flow $\phi_t : X \longrightarrow X$ we say that functions g_1, g_2 on X are cohomologous if there exists some bounded function q on X such that

$$g_1(x) - g_2(x) = \lim_{t \to 0} \frac{q(\phi_t x) - q(x)}{t}$$

for every $x \in X$.)

Any invariant probability measure for (X_A^r, σ_t^r) must be of the form $(\nu \times l)/(\int r\, d\nu)$, where l is Lebesgue measure on $\mathbb{R}$ and ν is an invariant probability measure for (X_A, σ). In particular, the equilibrium state for a function $\psi : X_A^r \longrightarrow \mathbb{R}$ can be written in this form, and ν turns out to be an equilibrium state for (X_A, σ):

Proposition 1.7 ([Sha2]) *If $\psi : X_A^r \longrightarrow \mathbb{R}$ is Hölder continuous, then its equilibrium state is*

$$\frac{\mu \times l}{\int r\, d\mu},$$

where μ is the measure on X_A which is the equilibrium state for the function $-P(\psi)r + \mathcal{I}\psi$.

1.5 Dimension and conformality

We work with the *Hausdorff dimension* for subsets of a metric space (X, d). That is, for a set $S \subseteq X$ we look at the ways of covering S by a finite or countable collection of sets; say we let $\mathrm{Cover}(S, \epsilon)$ be the set of all finite or countable covers of S by open sets each with diameter at most ϵ. Then for any non-negative real number α we define

$$m_H(S, \alpha) = \lim_{\epsilon \to 0} \inf_{\mathcal{U} \in \mathrm{Cover}(S,\epsilon)} \left(\sum_{U \in \mathcal{U}} (\mathrm{diam}\, U)^{\alpha} \right).$$

The Hausdorff dimension of S is then defined to be the unique number $\dim_H S \geq 0$ such that

$$m_H(S, \alpha) = \infty \quad \text{for all } \alpha < \dim_H S,$$
$$m_H(S, \alpha) = 0 \quad \text{for all } \alpha > \dim_H S.$$

(In all our examples the Hausdorff dimension will be finite, but there are some metric spaces for which it can be infinite.)

An equivalent definition is to take instead $\mathrm{Cover}(S, \epsilon)$ to be the set of all finite or countable covers of S by *closed* sets (or even by *general* sets) of diameter at most ϵ. This does not change $m_H(S, \alpha)$. Alternatively, we can take $\mathrm{Cover}(S, \epsilon)$ to be the set of all finite or countable covers of S by *balls* of diameter at most ϵ; in this case the values of $m_H(S, \alpha)$ may change but we still get the same answer for $\mathrm{diam}_H S$.

We will need to make use of the following properties of Hausdorff dimension (see, for example, [Fal], [Pes]):

(a) If $S_1 \subseteq S_2$ then $\dim_H S_1 \leq \dim_H S_2$.

(b) If $\{S_i : i \in I\}$ is a finite or countable collection of sets, then $\dim_H \bigcup_{i \in I} S_i = \sup_{i \in I} \dim_H S_i$.

(c) Suppose we have metric spaces (X, d_X) and (Y, d_Y) with $S \subseteq X$ and $T \subseteq Y$. If $F : S \longrightarrow T$ is a Lipschitz continuous surjection, then $\dim_H S \geq \dim_H T$.

(d) If $F : S \longrightarrow T$ is such that for any $x_1, x_2 \in S$ we have $d_X(F(x_1), F(x_2)) \geq c(d_Y(x_1, x_2))^{1+\epsilon}$, then $\dim_H T \geq (1+\epsilon)^{-1} \dim_H S$.

If we look instead at

$$\overline{m}_B(S, \alpha) = \limsup_{\epsilon \longrightarrow 0} \inf_{\mathcal{U} \in \mathrm{Cover}(S, \epsilon)} \left(\sum_{U \in \mathcal{U}} \epsilon^{\alpha} \right)$$

then we can define the upper box dimension $\overline{\dim}_B S = \inf\{\alpha \geq 0 : \overline{m}_B(S, \alpha) = 0\}$. (And similarly by replacing the limsup with a liminf we get the lower box dimension $\underline{\dim}_B S$.) We always have $\dim_H S \leq \underline{\dim}_B S \leq \overline{\dim}_B S$. While we will not look at box dimensions in their own right, they are useful for one further property of Hausdorff dimension:

(e) If $\dim_H T = \overline{\dim}_B T$ then $\dim_H(S \times T) = \dim_H S + \dim_H T$.

(In fact for any S and T we have $\dim_H S + \dim_H T \leq \dim_H(S \times T) \leq \dim_H S + \overline{\dim}_B T$.)

There is also a concept of Hausdorff dimension for *measures*. If μ is a Borel probability measure on X, we define

$$\dim_H \mu = \inf\{\dim_H Z : \mu(Z) = 1\}.$$

When looking at subshifts of finite type, the simple metric makes it relatively easy to get some results about the dimensions of subsets. However these results cannot be readily transferred to general expanding maps, hyperbolic diffeomorphisms or hyperbolic flows. (The coding map certainly does not preserve dimension.) In order to be able to make use of the symbolic dynamics, we will have to require that our dynamical systems are *conformal*. We also need better than C^1 differentiability:

- Let J be a repeller for a $C^{1+\alpha}$ map $f : M \longrightarrow M$. Then we say f is conformal on J if there is a continuous function $a : J \longrightarrow \mathbb{R}^+$ such that

$$(df)_x = a(x)\, I_x,$$

where $I_x : T_xM \longrightarrow T_{fx}M$ is an isometry. The function a must then be Hölder continuous and $a(x) > 1$ for all x.

The set J is then a *conformal repeller*.

- Let $f : M \longrightarrow M$ be a $C^{1+\alpha}$ diffeomorphism and $\Lambda \subseteq M$ a subset for which $f|_\Lambda$ is a hyperbolic diffeomorphism. Then we say that $f : \Lambda \longrightarrow \Lambda$ is conformal if there exist continuous functions $a^{(u)}, a^{(s)} : \Lambda \longrightarrow \mathbb{R}^+$ such that

$$(df)_x|_{E^u_x} = a^{(u)}(x)\, I^u_x, \quad (df)_x|_{E^s_x} = a^{(s)}(x)\, I^s_x,$$

for isometries $I^u_x : E^u_x \longrightarrow E^u_{fx}$, $I^s_x : E^s_x \longrightarrow E^s_{fx}$. The functions $a^{(u)}$, $a^{(s)}$ are both Hölder continuous, and we have $a^{(u)}(x) > 1$, $0 < a^{(s)}(x) < 1$ for all $x \in \Lambda$.

- Let $f : M \longrightarrow M$ be a C^2 flow and $\Lambda \subseteq M$ a subset for which $f|_\Lambda$ is a hyperbolic flow. Then we say that $f : \Lambda \longrightarrow \Lambda$ is conformal if there exist continuous functions $a^{(u)}, a^{(s)} : \Lambda \times \mathbb{R} \longrightarrow \mathbb{R}^+$ such that

$$(d\phi_t)_x|_{E^u_x} = a^{(u)}(x,t)\, I^u_{x,t}, \quad (d\phi_t)_x|_{E^s_x} = a^{(s)}(x,t)\, I^s_{x,t},$$

for isometries $I^u_{x,t} : E^u_x \longrightarrow E^u_{\phi_t x}$, $I^u_{x,t} : E^u_x \longrightarrow E^u_{\phi_t x}$. Now if we let

$$v^{(u)} = \frac{\partial}{\partial t} \log a^{(u)}(x,t)\Big|_{t=0}, \quad v^{(s)} = \frac{\partial}{\partial t} \log a^{(s)}(x,t)\Big|_{t=0},$$

then $v^{(u)}$ and $v^{(s)}$ are Hölder continuous, and $v^{(u)}(x) > 0$, $v^{(s)}(x) < 0$ for all $x \in \Lambda$.

Examples:

(i) If M is 1-dimensional then any repeller for a $C^{1+\alpha}$ map on M is necessarily conformal. Similarly, a hyperbolic diffeomorphism on a 2-dimensional manifold is conformal, and a hyperbolic flow on a 3-dimensional manifold is conformal.

(ii) As a particular case of (i), the geodesic flow on (the unit tangent bundle of) a 2-dimensional negatively–curved manifold is conformal.

(iii) For the geodesic flow on a manifold N with $\dim N \geq 3$, the conformality condition is equivalent to N having constant curvature ([Kan]).

One reason why the conformality condition is particularly important for hyperbolic diffeomorphisms and flows is the following:

Proposition 1.8 ([Pes], [PS] after [Has]) *(a) Suppose $f : \Lambda \longrightarrow \Lambda$ is a conformal hyperbolic diffeomorphism. Then for any rectangle R_i in the Markov partition for f, and any $z \in \operatorname{int} R_i$, the product structure $R_i \longrightarrow W^u_{R_i}(z) \times W^s_{R_i}(z)$ is a bi-Lipschitz homeomorphism.*

(b) Suppose $\phi_t : \Lambda \longrightarrow \Lambda$ is a conformal hyperbolic flow. Then if T_j is one of the Markov sections used to construct the symbolic dynamics for ϕ_t, and we take any $z \in \operatorname{int} T_j$, the product structure $T_j \longrightarrow W^u_{T_j}(z) \times W^s_{T_j}(z)$ is a bi-Lipschitz homeomorphism.

This will allow us to relate the dimension of subsets of Λ to dimensions of subsets of stable and unstable manifolds, using properties (c) and (e) of Hausdorff dimension.

Other techniques for dealing with conformal maps and flows are explained in section 2.2.

Note that subshifts of finite type can be thought of as satisfying a conformal-like condition, with the function a being constant.

1.6 Periodic orbits and homology

In this section we describe some preliminaries needed for chapter 3, where we look at the periodic orbits of a transitive Anosov flow $\phi_t : M \to M$. A reference for everything in this section is the survey [Sha2].

A basic result is that an Anosov flow has an infinite but countable number of periodic orbits. (Indeed this follows from the existence of a symbolic model for the flow as described in section 1.3.) Furthermore, for any $T > 0$ there are only finitely many periodic orbits with period at most T. So it makes sense to 'count' periodic orbits: for example, if we write $l(\gamma)$ to mean the least period of the periodic orbit γ, we can define a function

$$\pi(T) = \#\{\gamma : \; l(\gamma) \le T\}.$$

We might then ask how this function grows with T. A famous result ([PP]) for weak-mixing flows is that

$$\pi(T) \sim \frac{e^{hT}}{hT} \quad \text{as } T \to \infty,$$

where h is the topological entropy of the flow.

A variation on this theme is to count those periodic orbits which satisfy certain conditions. One condition that has been studied has to do with the homology of the manifold M. A periodic orbit γ for the flow can be regarded as simply being a closed curve in M, and as such we can look at its homology class, which we write as $[\gamma] \in H_1(M, \mathbb{Z})$. We then have 'counting' results for the number of periodic orbits in a fixed homology class. Given $\alpha \in H_1(M, \mathbb{Z})$ we define

$$\pi(T, \alpha) = \#\{\gamma : \; l(\gamma) \le T, \; [\gamma] = \alpha\}.$$

To be able to state a result about the behaviour of $\pi(T, \alpha)$ we need to understand the structure of the homology group $H_1(M, \mathbb{Z})$. (We will follow the description in [Sha2].) We have that $H_1(M, \mathbb{Z})$ is isomorphic to $\mathbb{Z}^b \oplus \mathrm{Tor}$, where Tor is a finite abelian group (the 'torsion subgroup') and b is the first Betti number of M.

The behaviour of $\pi(T,\alpha)$ is determined by the torsion-free part of α, which can be represented as a point in $\mathbb{Z}^b$ by making use of the isomorphism. In fact for our purposes we will generally ignore the torsion component of $H_1(M,\mathbb{Z})$, and (with a slight abuse of notation) we will also use $[\gamma]$ to denote the point in $\mathbb{Z}^b$ which represents the torsion-free part of the homology class of γ, after fixing a choice of the isomorphism. We will assume that b is strictly positive, otherwise this becomes trivial.

Since $\mathbb{Z}^b$ is a lattice inside $\mathbb{R}^b$, we may also think of $[\gamma]$ as being a point in $\mathbb{R}^b$. Indeed we can choose to look at the real homology group $H_1(M,\mathbb{R}) = H_1(M,\mathbb{Z}) \otimes \mathbb{R}$, which is isomorphic to $\mathbb{R}^b$. By fixing an isomorphism we are effectively choosing a basis for $H_1(M,\mathbb{R})$.

We now have the following result of Sharp, which generalises the work of Katsuda and Sunada in [KS]:

Theorem 1.9 ([Sha1]) *Suppose that each homology class in $H_1(M,\mathbb{Z})$ contains at least one periodic orbit. Then there exist positive constants C and h^*, and a vector $\xi \in \mathbb{R}^b$, such that for all $\alpha \in H_1(M,\mathbb{Z})$,*

$$\pi(T,\alpha) \sim Ce^{-\langle \xi,\alpha' \rangle} \frac{e^{h^*T}}{T^{b/2+1}} \quad \text{as } T \to \infty,$$

where α' is the torsion-free part of α.

(Here $\langle , \rangle$ is the usual inner product on $\mathbb{R}^b$.) If every homology class contains a periodic orbit then we say that the flow is *homologically full*. Not every transitive Anosov flow has this property (but it does hold in some important cases, as we will explain in chapter 3). Indeed, if the flow has a global cross-section ([Sch]) then the homology classes of periodic orbits are restricted to an open half-space in $\mathbb{R}^b$ which does not include the origin, and there are only a finite number of periodic orbits in any homology class.

1.7 Main results of this thesis

What follows is a brief summary. The definitions and results here will all be explained again in the main sections of the thesis.

In chapter 2 our starting point is the paper [FS] by Fan and Schmeling. They look at the behaviour of the ergodic sums $g^n(x)$, where g is a Hölder continuous function defined on a (one-sided) subshift of finite type $\sigma : X_A^+ \to X_A^+$. The aim is to describe the sets of points $x \in X_A^+$ for which the sums $g^n(x)$ have a specified behaviour as $n \to \infty$. For example, we can start by looking at the sets of points which have a particular ergodic average:

$$\mathrm{Ave}_{X_A^+}(g, \alpha) := \left\{ x \in X_A^+ : \tfrac{1}{n} g^n(x) \to \alpha \ \text{ as } n \to \infty \right\}.$$

The ergodic theorem tells us that this set has full μ–measure, for any ergodic probability measure μ such that $\int g \, d\mu = \alpha$.

Fan and Schmeling ask whether it is possible to find points for which $\frac{1}{n} g^n(x)$ converges to α at a particular rate. For simplicity we set $\alpha = 0$. The basic result in [FS] concerns the set of points with *bounded* sums:

$$\mathrm{Bdd}_{X_A^+}(g) := \left\{ x \in X_A^+ : g^n(x) \ \text{ is bounded} \right\}.$$

Clearly $\mathrm{Bdd}_{X_A^+}(g) \subseteq \mathrm{Ave}_{X_A^+}(g, 0)$. But while $\mathrm{Bdd}_{X_A^+}(g)$ might appear to be a much smaller set (if g is not cohomologous to a constant then the set has zero measure with respect to any equilibrium state), Fan and Schmeling show that it has the same Hausdorff dimension as the whole of $\mathrm{Ave}_{X_A^+}(g, 0)$, provided that there exists an equilibrium state μ such that $\int g \, d\mu = 0$:

Theorem A1 (Fan, Schmeling [FS]) *Let $g : X_A^+ \to \mathbb{R}$ be Hölder continuous, and suppose there exists an equilibrium state μ such that $\int g \, d\mu = 0$. Then*

$$\dim_H \mathrm{Bdd}_{X_A^+}(g) = \dim_H \mathrm{Ave}_{X_A^+}(g, 0).$$

Results from [BS2] tell us that the existence of μ is not too strong a condition: if g is not cohomologous to a constant, then the possible values of $\int g\,d\mu$ (taken over all equilibrium states μ) are an open interval $(\underline{\alpha}, \overline{\alpha})$. Thus the two sets are shown to have the same dimension provided that this interval contains zero. On the other hand if zero lies outside the *closed* interval $[\underline{\alpha}, \overline{\alpha}]$ then both sets are empty. So it is only when zero is an endpoint of the interval that we cannot say the dimensions are equal.

Fan and Schmeling use Theorem A1 to find other subsets of $\dim_H \mathrm{Ave}_{X_A^+}(g, 0)$ which have the same dimension:

Theorem A2 (Fan, Schmeling [FS]) *Let $g : X_A^+ \to \mathbb{R}$ be a Hölder continuous function not cohomologous to a constant, and suppose there exists an equilibrium state μ such that $\int g\,d\mu = 0$. Then for any $a \in \mathbb{R}$ and $0 < \beta < 1$,*

$$\dim_H \left\{ x \in X_A^+ : \lim_{n\to\infty} \frac{g^n(x)}{n^\beta} = a \right\} = \dim_H \mathrm{Ave}_{X_A^+}(g, 0).$$

The aim of chapter 2 is to extend the results of Theorems A1 and A2. Our first new result is a stronger version of Theorem A2:

Theorem A3 *Let $g : X_A^+ \to \mathbb{R}$ be a Hölder continuous function not cohomologous to a constant, and suppose there exists an equilibrium state μ such that $\int g\,d\mu = 0$. Now let $r : X_A^+ \to \mathbb{R}^+$ be a strictly positive Hölder continuous function, and let $F : \mathbb{R}^+ \to \mathbb{R}$ be a continuous function with the property that $\sup_{\tau\in[0,1]} |F(t+\tau) - F(t)| \to 0$ as $t \to \infty$. Then we have*

$$\dim_H \left\{ x \in X_A^+ : g^n(x) = F(r^n(x)) + O(1) \ \text{ as } \ n \to \infty \right\} = \dim_H \mathrm{Ave}_{X_A^+}(g, 0).$$

We can recover Theorem A2 from this by taking $F(t) = at^\beta$ and $r \equiv 1$. But our version allows for more general functions F. For example we can take F to be any differentiable function for which $F'(t) \to 0$ as $t \to \infty$. We have also introduced a new function r which is helpful in adapting this theorem to work for flows. But the main improvement over Theorem A2 is that we are requiring the error term $g^n(x) - F(r^n(x))$ to be bounded, whereas in Theorem A2 the points satisfy the much weaker condition that $g^n(x) \sim F(r^n(x))$. (Though in light of Theorem A1 it should perhaps not be too surprising that we can ask for a bounded error term.)

After this we show how the methods used for subshifts of finite type can be adapted to give analogues of Theorems A1 and A3 for other dynamical systems. We state here the most general versions of our results, in which we look at the ergodic sums of an $\mathbb{R}^d$–valued Hölder continuous function $\mathbf{g}$. (Multi-dimensional results like these were proved for subshifts of finite type in [FS].) For a general map $T : X \to X$, with Hölder continuous functions $\mathbf{g} : X \to \mathbb{R}^d$ and $r : X \to \mathbb{R}^+$, and a continuous function $\mathbf{F} : \mathbb{R}^+ \to \mathbb{R}$, we define

$$\begin{aligned} \mathrm{Ave}_X(\mathbf{g}, \boldsymbol{\alpha}) &= \left\{ x \in X : \tfrac{1}{n}\mathbf{g}^n(x) \to \boldsymbol{\alpha} \ \text{ as } n \to \infty \right\}; \\ \mathrm{Bdd}_X(\mathbf{g}) &= \{ x \in X : \mathbf{g}^n(x) \ \text{ is bounded} \}; \\ L_X(\mathbf{g}, \mathbf{F}, r) &= \{ x \in X : \mathbf{g}^n(x) = \mathbf{F}\left(r^n(x)\right) + O(1) \}. \end{aligned}$$

As in Theorem A3 we need some control over $\mathbf{F}$: we require $\sup_{\tau \in [0,1]} \|\mathbf{F}(t+\tau) - \mathbf{F}(t)\| \to 0$ as $t \to \infty$. (Here we use $\| \cdot \|$ to mean the usual Euclidean norm on $\mathbb{R}^d$.)

We then have results for conformal expanding maps, and conformal hyperbolic diffeomorphisms:

Theorem A4 *Let J be a conformal repeller for a $C^{1+\alpha}$ map $f : M \to M$, and let $\mathbf{g} : J \to \mathbb{R}^d$ be a Hölder continuous function whose components are cohomologously independent. Suppose there exists an equilibrium state ν on J such that $\int_J \mathbf{g}\, d\nu = \mathbf{0}$. Then*

$$\dim_H \mathrm{Bdd}_J(\mathbf{g}) = \dim_H L_J(\mathbf{g}, \mathbf{F}, r) = \dim_H \mathrm{Ave}_J(\mathbf{g}, \mathbf{0}),$$

whenever $\mathbf{F}$, r are as above.

Theorem A5 *Let $f : \Lambda \to \Lambda$ be a conformal hyperbolic diffeomorphism, and let*

$\mathbf{g} : \Lambda \to \mathbb{R}^d$ *be a Hölder continuous function whose components are cohomologously independent. Suppose there exists an equilibrium state* ν *on* Λ *such that* $\int_\Lambda \mathbf{g}\, d\nu = \mathbf{0}$. *Then*

$$\dim_H \mathrm{Bdd}_\Lambda(\mathbf{g}) = \dim_H L_\Lambda(\mathbf{g}, \mathbf{F}, r) = \dim_H \mathrm{Ave}_\Lambda(\mathbf{g}, \mathbf{0}),$$

whenever $\mathbf{F}$, r *are as above.*

Finally, we can get similar results for conformal hyperbolic *flows* $\phi_t : \Lambda \to \Lambda$, if instead of the ergodic sum $\mathbf{g}^n(x)$ we look at the integral $\int_0^t \mathbf{g}\,(\phi_\tau x)\ d\tau$. We can define

$$\begin{aligned}
\mathrm{Ave}_\Lambda(\mathbf{g}, \boldsymbol{\alpha}) &:= \left\{ x \in \Lambda : \frac{1}{t} \int_0^t \mathbf{g}\,(\phi_\tau x)\ d\tau \to \boldsymbol{\alpha} \ \text{ as } t \to \infty. \right\};\\
\mathrm{Bdd}_\Lambda(\mathbf{g}) &:= \left\{ x \in \Lambda : \int_0^t \mathbf{g}\,(\phi_\tau x)\ d\tau \text{ is bounded} \right\};\\
L_\Lambda(\mathbf{g}, \mathbf{F}) &:= \left\{ x \in \Lambda : \int_0^t \mathbf{g}\,(\phi_\tau x)\ d\tau = \mathbf{F}(t) + O(1) \right\}.
\end{aligned}$$

We will show the following:

Theorem A6 *Let* $\phi_t : \Lambda \to \Lambda$ *be a conformal hyperbolic flow, and let* $\mathbf{g} : \Lambda \to \mathbb{R}^d$ *be a Hölder continuous function whose components are cohomologously independent. Suppose there exists an equilibrium state* ν *on* Λ *such that* $\int_\Lambda \mathbf{g}\, d\nu = \mathbf{0}$. *Then*

$$\dim_H \mathrm{Bdd}_\Lambda(\mathbf{g}) = \dim_H L_\Lambda(\mathbf{g}, \mathbf{F}) = \dim_H \mathrm{Ave}_\Lambda(\mathbf{g}, \mathbf{0}),$$

whenever $\mathbf{F}$ *satisfies the usual condition.*

In the rather shorter chapter 3 we look at a problem which was originally suggested to Richard Sharp by François Ledrappier, concerning the periodic orbits of a transitive Anosov flow $\phi_t : M \to M$.

Given a periodic orbit γ, we look at its homology class $[\gamma] \in H_1(M, \mathbb{Z})$. As explained in section 1.6, if we ignore the torsion part of $H_1(M, \mathbb{Z})$ then we can think of $[\gamma]$ as being represented by a point in $\mathbb{Z}^b$. If this point is non-zero we can consider its projection onto the (Euclidean) unit sphere in $\mathbb{R}^b$ (using the projection map $p_S : \mathbb{R}^b \setminus \{0\} \to S^{b-1}$ defined by $p_S(\mathbf{v}) = \mathbf{v}/\|\mathbf{v}\|_2$, where $\|\cdot\|_2$ is the usual Euclidean norm). This gives us a point $\theta(\gamma) = p_S([\gamma]) \in S^{b-1}$ which represents the 'direction' of the homology class of γ.

Now given $T > 0$ we can define a measure ν_T on S^{b-1} by

$$\nu_T = \frac{1}{\pi(T)} \sum_{l(\gamma) \le T, [\gamma] \ne 0} \delta_{\theta(\gamma)},$$

where $\delta_{\theta(\gamma)}$ is the Dirac measure at $\theta(\gamma)$.

Our main result in chapter 3 is that the measures ν_T have a (weak*) limit as $T \to \infty$, and we are are able to describe this limit ν_∞.

We find that the nature of ν_∞ depends on the asymptotic cycle Φ_0 associated to the measure of maximal entropy. (We define this object properly in chapter 3; for now it is sufficient to know that it can be represented by a point in $\mathbb{R}^b$.)

Theorem B1 *The measures ν_T have a weak* limit ν_∞ as $T \to \infty$:*

(i) *If $\Phi_0 \ne 0$ then ν_∞ is the Dirac measure at $p_S(\Phi_0)$.*

(ii) *If $\Phi_0 = 0$ then ν_∞ is fully supported on S^{b-1}. Indeed there is a norm $\|\cdot\|$ on $\mathbb{R}^b$ with the property that for any open set $D \subseteq S^{b-1}$ we have*

$$\nu_\infty(D) = \frac{\mathrm{Vol}(p_S^{-1}(D) \cap B_{\|.\|})}{\mathrm{Vol}(B_{\|.\|})},$$

where $B_{\|.\|}$ is the unit ball for the norm $\|\cdot\|$.

Part (ii) of this theorem follows from a more general result. Suppose we are given a set $A \subseteq \mathbb{Z}^b$. Then we can look at

$$\pi(T, A) := \#\{\gamma : l(\gamma) \le T, [\gamma] \in A\}.$$

We can also define a quantity

$$d_{\|.\|}(A) = \lim_{r \to \infty} \frac{\#\{\alpha \in A : \|\alpha\| \le r\}}{\#\{\alpha \in \mathbb{Z}^b : \|\alpha\| \le r\}},$$

if this limit exists. This is the *density* of the set A with respect to the norm $\|\cdot\|$. We find that these are linked in the following way:

Theorem B2 *Suppose $\Phi_0 = 0$. Then if $A \subseteq \mathbb{Z}^b$ is a set for which the density $d_{\|\cdot\|}(A)$ exists, we have*

$$\lim_{T\to\infty} \frac{\pi(T, A)}{\pi(T)} = d_{\|\cdot\|}(A).$$

Chapter 2

The pointwise behaviour of ergodic sums

Suppose we have a transformation $T : X \to X$ and a Hölder continuous function $g : X \to \mathbb{R}$. We then have the sums

$$g^n(x) := \sum_{i=0}^{n-1} g(T^i x).$$

We are interested in the sets of points $x \in X$ for which $g^n(x)$ has a specified behaviour as $n \to \infty$.

The simplest result of this type comes from the ergodic theorem: suppose we define

$$\mathrm{Ave}_X(g, \alpha) = \left\{ x \in X : \tfrac{1}{n} g^n(x) \to \alpha \text{ as } n \to \infty \right\}.$$

Then for any ergodic measure μ on X, the set $\mathrm{Ave}_X(g, \mathbb{E}_\mu g)$ has full measure. (Here $\mathbb{E}_\mu g$ is the mean value of g with respect to the measure μ, i.e. equal to $\int g\, d\mu$ when μ is a probability measure.)

But we will look at subsets of $\mathrm{Ave}_X(g, \alpha)$ on which the sums $g^n(x)$ are more tightly controlled. In the case of (one-sided) subshifts of finite type, results about such subsets were obtained by Fan and Schmeling in [FS]. In particular, they prove the existence of points $x \in X_A^+$ such that

$$g^n(x) - n\mathbb{E}_\mu g \sim F(n),$$

for various functions F. Indeed, for suitable F they show that the Hausdorff dimension of the set of points with this property is actually equal to the dimension of the set $\mathrm{Ave}_X(g, \mathbb{E}_\mu g)$.

Our aim is show that similar results hold if, instead of a subshift of finite type, we have a conformal expanding map or hyperbolic diffeomorphism. We do this by developing the methods used in [FS]. In fact we will prove that we can actually ask for

$$g^n(x) - n\mathbb{E}_\mu g = F(n) + O(1),$$

provided that F is sufficiently well-behaved, and this set of points still has the same dimension.

We also have analogous results for flows $\phi_t : X \to X$, if in place of $g^n(x)$ we look at $\int_0^t g\,(\phi_\tau x)\; d\tau$. We will be able to find points such that

$$\int_0^t g\,(\phi_\tau x)\; d\tau - t\mathbb{E}_\mu g = F(t) + O(1).$$

Note that if g is cohomologous to a constant function then we do not get any interesting behaviour of ergodic sums: if $g = h \circ T - h + c$ for a constant c then $g^n = h \circ T^n - h + nc$, and so for *every* point x we have that $g^n(x) - nc$ is bounded by $2\|h\|_\infty$. The ergodic theorem then implies that $\mathbb{E}_\mu g = c$ for every ergodic μ, and the only behaviour we will get is

$$g^n(x) - n\mathbb{E}_\mu g = O(1).$$

So we will always assume that g is not cohomologous to a constant. In this case Theorem 1.1 guarantees that there are some points with non-trivial behaviour: we can at least find a periodic point x (with period m, say) such that

$$g^m(x) - m\mathbb{E}_\mu g \neq 0,$$

which implies that $g^n(x)$ grows at a linear rate. We also have the following (see [PP]):

Theorem 2.1 (Central Limit Theorem) *Let $\sigma : X_A^+ \to X_A^+$ be a subshift of finite type, with $g : X_A^+ \to \mathbb{R}$ a Hölder continuous function. Suppose that μ is an equilibrium state for some Hölder continuous function ψ on X_A^+. Then if g is not cohomologous to a constant, we have*

$$\mu\left(\left\{x : \frac{g^n(x) - n\mathbb{E}_\mu g}{\sqrt{n}} < t\right\}\right) \to N(t) \quad \text{as } n \to \infty,$$

where N is a normal distribution with mean zero and variance depending on g and μ.

This will be an important tool for producing points whose behaviour is controlled. (Note that while we have stated the result for one-sided subshifts, it can also be immediately transferred to the other maps we are interested in.)

There is also a Central Limit Theorem for flows ([Rat]), but we will not need to use this explicitly.

2.1 Results for subshifts of finite type

We start by looking at a one-sided subshift of finite type $\sigma : X_A^+ \to X_A^+$. Let $\mathcal{M}(X_A^+)$ be the set of invariant probability measures on X_A^+. This is a compact convex set in the weak* topology (see [Wal]). Thus for any Hölder continuous function g on X_A which is not cohomologous to a constant, the set

$$\left\{\int g\,d\mu : \mu \in \mathcal{M}(X_A^+)\right\}$$

is a closed interval $[\underline{\alpha}, \overline{\alpha}]$. $\left\{\int g\,d\mu : \mu \in \mathcal{M}(X_A^+)\right\}$ologous to a constant then this set is just a single point. When g is not cohomologous to a constant, Theorem 1.1 guarantees that $\underline{\alpha} < \overline{\alpha}$.)

Now, by the definition of Hausdorff dimension for a measure, we have that if $\int g\,d\mu = \alpha$ for some $\mu \in \mathcal{M}(X_A^+)$ then $\dim_H \mathrm{Ave}_{X_A^+}(g, \alpha) \geq \dim_H \mu$. The following theorem shows that if α lies in the open interval $(\underline{\alpha}, \overline{\alpha})$, there actually exists a measure μ for which we have equality:

Theorem 2.2 **([BS2])**

1. *If* $\alpha \notin [\underline{\alpha}, \overline{\alpha}]$ *then* $\mathrm{Ave}_{X_A^+}(g, \alpha) = \varnothing$.

2. *If* $\alpha \in (\underline{\alpha}, \overline{\alpha})$ *then* $\mathrm{Ave}_{X_A^+}(g, \alpha) \neq \varnothing$*, and*

$$\dim_H \mathrm{Ave}_{X_A^+}(g, \alpha) = \sup \left\{ \dim_H \mu : \mu \in \mathcal{M}(X_A^+) \text{ and } \int g \, d\mu = \alpha \right\}.$$

Furthermore the supremum is attained for a measure μ *which is an equilibrium state for some Hölder continuous function.*

Conversely, suppose that μ is an equilibrium state for a Hölder continuous function, and write $\alpha = \int g \, d\mu$. Then it follows from the Central Limit Theorem that that there exists a periodic point u^+ with period p^+ such that $g^{p^+}(u^+) > p^+\alpha$. (We will show how to construct such a point in the proof of Theorem 2.18.) By considering the invariant probability measure that is supported on this periodic orbit, we have that $\frac{1}{p^+} g^{p^+}(u^+) \in [\underline{\alpha}, \overline{\alpha}]$, and so $\alpha < \overline{\alpha}$. Similarly we can show that $\alpha > \underline{\alpha}$.

Thus $\alpha \in (\underline{\alpha}, \overline{\alpha})$ if and only if there exists an equilibrium state μ such that $\int g d\mu = \alpha$.

2.1.1 Points with bounded sums

We define

$$\mathrm{Bdd}_{X_A^+}(g) = \left\{ x \in X_A^+ : g^n(x) \ \text{ is bounded} \right\}.$$

Since $\mathrm{Bdd}_{X_A^+}(g) \subseteq \mathrm{Ave}_{X_A^+}(g, 0)$, we know from Theorem 2.2 that if $0 \notin [\underline{\alpha}, \overline{\alpha}]$ then $\mathrm{Bdd}_{X_A^+}(g)$ is empty.

On the other hand, the basic result in Fan and Schmeling's paper [FS] tells us that if $0 \in (\underline{\alpha}, \overline{\alpha})$ then $\mathrm{Bdd}_{X_A^+}(g)$ is non-empty, and indeed we have a lower bound for the dimension:

Theorem 2.3 (Fan, Schmeling [FS]) *Let $g : X_A^+ \to \mathbb{R}$ be Hölder continuous, and let μ be an equilibrium state for some Hölder continuous function. Suppose that $\int g\, d\mu = 0$. Then*

$$\dim_H \mathrm{Bdd}_{X_A^+}(g) \geq \dim_H \mu.$$

(Note that this still holds if g is cohomologous to a constant – then the condition $\int g\, d\mu = 0$ implies that the constant is zero, which in turn implies that $\mathrm{Bdd}_{X_A^+}(g)$ is the whole of X_A^+.)

In particular we can apply Theorem 2.3 to the measure μ for which the supremum is attained in Theorem 2.2 (for $\alpha = 0$). This gives $\dim_H \mathrm{Bdd}_{X_A^+}(g) \geq \dim_H \mathrm{Ave}_{X_A^+}(g, 0)$. But $\mathrm{Bdd}_{X_A^+}(g) \subseteq \mathrm{Ave}_{X_A^+}(g, 0)$ and so we have:

Theorem 2.4 ([FS]) *Suppose that $0 \in (\underline{\alpha}, \overline{\alpha})$, then*

$$\dim_H \mathrm{Bdd}_{X_A^+}(g) = \dim_H \mathrm{Ave}_{X_A^+}(g, 0).$$

(Fan and Schmeling prove this directly from their main result without quoting Theorem 2.2.)

Thus we have a seemingly 'small' subset of $\mathrm{Ave}_{X_A^+}(g, 0)$ which nevertheless has the same dimension as the whole of $\mathrm{Ave}_{X_A^+}(g, 0)$. We will prove many more results of this type.

2.1.2 The method of adding blocks

Fan and Schmeling use their result about points with bounded ergodic sums to prove the following:

Theorem 2.5 (Fan, Schmeling [FS]) *Let $g : X_A^+ \to \mathbb{R}$ be a Hölder continuous function not cohomologous to a constant, and let μ be an equilibrium state for some Hölder continuous function. Suppose that $\int g\, d\mu = 0$. Then for any $a \in \mathbb{R}$ and $0 < \beta < 1$,*

$$\dim_H \left\{ x \in X_A^+ : \lim_{n \to \infty} \frac{g^n(x)}{n^\beta} = a \right\} \geq \dim_H \mu.$$

Their method of proof is something that we will develop and generalise to give us our results.

The strategy is to start with a set $S \subset X_A^+$ on which we have good control over the ergodic sums. Then for any $x \in S$ we aim to construct a point $\xi(x) \in X_A^+$ which has the desired property, i.e. in the theorem above we want $g^n(\xi(x)) \sim an^\beta$ as $n \to \infty$. This process produces a new set S' (the image of S under ξ) and we aim to relate the dimension of S' to the dimension of the original set S.

In [FS] the proof takes $S = S_K$ where

$$S_K = \left\{x \in X_A^+ : |g^n(x)| < K \quad \forall n\right\}.$$

Using Theorem 2.3, we know that for any $\epsilon > 0$ we can find such a set S_K with $\dim_H S_K > \dim_H \mu - \epsilon$. The construction of the function ξ is done in such a way that we have bounds on $d(\xi(x), \xi(y))$ in terms of $d(x, y)$ (for general points $x, y \in S_K$) which are good enough to show that dimension of S'_K is equal to the dimension of S_K.

The construction of $\xi(x)$ from x is the main part of the proof. It is based on a procedure which Fan and Schmeling refer to as "inserting blocks" into x. As the name suggests, this works as follows: given our point x, we use the definition of X_A^+ as a subset of $\{1, 2, \ldots, k\}^{\mathbb{N}_0}$ to think of x as the sequence of symbols

$$x_0\, x_1\, x_2\, x_3 \ldots.$$

Then, "inserting a block B (of length ℓ) behind x_i" means replacing x by

$$x_0\, x_1 \ldots x_{i-1}\, x_i\, b_0\, b_1 \ldots b_{\ell-1}\, x_{i+1}\, x_{i+2} \ldots,$$

which for simplicity we will write in an abbreviated form as

$$x_0\, x_1 \ldots x_{i-1}\, x_i\, B\, x_{i+1}\, x_{i+2} \ldots.$$

In order to produce the point $\xi(x)$, this process is done repeatedly, inserting blocks at various different positions in x. (We never insert a block inside a previously-inserted block.) In the construction used to prove Theorem 2.5 in [FS], the blocks that are inserted, and the positions they are inserted at, do not depend on the point x. (The blocks only depend on the symbols which come immediately before and after it: this is necessary to ensure that the blocks are admissible.) But we will want more freedom than this in our constructions.

We now define some notation for a general form of the block-adding construction. Each point $x \in S$ is written as a sequence of symbols $x_0\, x_1\, x_2 \ldots$. If a block is to be inserted behind x_i, then we will call this block B_i and say that it has length ℓ_i. If no block is to be inserted behind x_i then we will define $\ell_i = 0$ and B_i to be an empty sequence. So if $i_1 < i_2 < i_3 < \cdots$ are the positions where blocks are to be inserted, the point $\xi(x)$ has the form

$$x_0\, x_1\, \ldots\, x_{i_1}\, B_{i_1}\, x_{i_1+1}\, \ldots\, x_{i_2}\, B_{i_2}\, x_{i_2+1}\, \ldots .$$

The blocks B_i and their lengths ℓ_i are allowed to depend on the point x. (We will write $B_i(x)$, $\ell_i(x)$ when necessary to compare different points.) We will say that a function $\xi : S \longrightarrow S'$ defined in this way is a 'block-adding process'.

It sometimes helps to say that the symbol x_i is 'shifted' to the new position i', where

$$i' = i + \sum_{j<i} \ell_j .$$

Again we can write $i'(x)$ if necessary to emphasise that this depends on the point x.

In order to ensure that we have some control over $d(\xi(x), \xi(y))$, we will require our block-adding processes to satisfy certain conditions.

Definition 2.6 *We say that the block-adding process $\xi : S \longrightarrow S'$ is **defined on cylinders** if, whenever x and y are points in S with $x_k = y_k$ for all $k \leq i$, we have*

(i) $\ell_i(x) = \ell_i(y)$;

(ii) If in addition $x_{i+1} = y_{i+1}$, then the blocks $B_i(x)$ and $B_i(y)$ are identical.

In particular, the decision of whether to add a block behind x_i depends only on the symbols $x_0, x_1, \ldots, x_i$.

These conditions are sufficient to guarantee certain simple properties of the function ξ:

Proposition 2.7 *If the block-adding process $\xi : S \to S'$ is defined on cylinders then:*

(i) ξ is injective, so that there is a well-defined 'block-removing function' $\xi^{-1} : S' \to S$.

(ii) ξ is Lipschitz continuous.

(iii) If $C \in \mathrm{Cyl}(i+1)$ is a cylinder with $C \cap S \neq \varnothing$, then there is a unique i' and $C' \in \mathrm{Cyl}(i'+1)$ such that

(a) If $x \in C \cap S$ then $\xi(x) \in C'$;

(b) If $x \in C \cap S$ then the symbol x_i is shifted to position i' in $\xi(x)$.

(iv) If $Q \in \mathrm{Cyl}(j+1)$ is a cylinder with $Q \cap S' \neq \varnothing$, then there is a unique i and $\widehat{Q} \in \mathrm{Cyl}(i+1)$ such that

(a) If $\xi(x) \in Q \cap S'$ then $x \in \widehat{Q}$;

(b) If $\xi(x) \in Q \cap S'$ and the symbols x_i, x_{i+1} are shifted to positions i', $(i+1)'$ respectively, then $i' \leq j < (i+1)'$.

Proof: We first observe that if x and y are two points in S with $x_k = y_k$ for all $k \leq i$ then the symbols x_k and y_k are shifted to the same positions (in $\xi(x)$ and $\xi(y)$ respectively) for any $k \leq i+1$. This is an easy induction on k, with the case $k = 0$ being trivial and the inductive step being immediate from part (i) of the definition. Each of the properties of ξ follows from this:

(i) : If $x \neq y$ then we can find i such that $x_k = y_k$ for all $k \leq i$ but $x_{k+1} \neq y_{k+1}$. But then the symbols x_{i+1} and y_{i+1} are shifted to the same positions so we must have $\xi(x) \neq \xi(y)$.

(iii) : This follows immediately from the observation above plus part (b) of the definition.

(ii) : This now follows from (iii) – in fact we see that $d(\xi(x), \xi(y)) \leq d(x, y)$.

(iv) : Pick any point y with $\xi(y) \in Q \cap S'$. Then choose the largest i such that $i'(y) \leq j$. There is then a unique cylinder $\widehat{Q} \in \mathrm{Cyl}(i+1)$ with $y \in \widehat{Q}$. We want to show that if x is any other point with $\xi(x) \in Q \cap S'$, we have $x \in \widehat{Q}$. Choose the smallest k such that $x_k \neq y_k$. Then from the observation above we know that the symbols x_k are y_k shifted to the same position k', and since $\xi(x), \xi(y) \in Q \in \mathrm{Cyl}(j+1)$ we must have $k' > j$. But then $k' > i'$ and so $k > i$. So we have shown that $x_l = y_l$ for all $l \leq i$, i.e. $x \in \widehat{Q}$. One further use of the original observation shows that we also have property (b). $\square$

Thus, both the block-adding function ξ and the block-removing function ξ^{-1} can be regarded as acting on cylinders.

Finally in this section we explain why the procedure of adding blocks produces points for which we have some information about the sums $g^n(\xi(x))$.

For a Hölder continuous function $g : X_A^+ \to \mathbb{R}$, we can define

$$\mathrm{var}_n^+ g = \sup \left\{ |g(x) - g(y)| : x_i = y_i \quad \forall i < n \right\}.$$

Then, by the Hölder continuity of g, the sequence $\{\mathrm{var}_n^+ g\}_{n \geq 1}$ is bounded by a geometric progression, so we can define

$$V(g) = \sum_{n=0}^{\infty} \mathrm{var}_n^+ g.$$

And this constant has the property that for any $n \geq 1$, if we have two points $x, y \in X_A^+$ for which $x_i = y_i$ for all $i < n$, then

$$\left| g^k(x) - g^k(y) \right| \leq V(g) \quad \text{for all } k \leq n.$$

Now, suppose we have a point $x \in X_A^+$ and insert a single block B behind x_{i-1}, producing a new point y. Say that B consists of the first ℓ symbols of a point $b \in X_A^+$. Then we know

- $g^n(x) - V(g) \le g^n(y) \le g^n(x) + V(g)$ for $n \le i$;
- $g^i(x)+g^{n-i}(b)-2V(g) \le g^n(y) \le g^i(x)+g^{n-i}(b)+2V(g)$ for $i < n \le i+\ell$;
- $g^{n-\ell}(x) + g^\ell(b) - 2V(g) \le g^n(y) \le g^{n-\ell}(x) + g^\ell(b) + 2V(g)$ for $n > i + \ell$.

In particular, if $x \in S_K$ then

- $-K - V(g) \le g^n(y) \le K + V(g)$ for $n \le i$;
- $g^\ell(b) - K - 2V(g) \le g^n(y) \le g^\ell(b) + K + 2V(g)$ for $n > i + \ell$.

So if b was chosen so that $g^\ell(b) > K + 2V(g)$, we have produced a point such that all the partial sums after the inserted block are bounded away from zero.

Similar calculations show that if we have a block-adding procedure that inserts blocks in infinitely many places in x, we can get bounds on the rate of growth of $g^n(\xi(x))$ as $n \to \infty$.

2.1.3 Points with sums which grow at a specified rate

We now prove a stronger version of Theorem 2.5.

Theorem 2.8 *Let $g : X_A^+ \to \mathbb{R}$ be a Hölder continuous function not cohomologous to a constant, and let μ be an equilibrium state for some Hölder continuous function on X_A^+, with $\int g \, d\mu = 0$. Now let $r : X_A^+ \to \mathbb{R}^+$ be a strictly positive Hölder continuous function, and let $F : \mathbb{R}^+ \to \mathbb{R}$ be a continuous function with the property that* $\sup_{\tau \in [0,1]} |F(t+\tau) - F(t)| \to 0$ *as $t \to \infty$. Then we have*

$$\dim_H \left\{ x \in X_A^+ : g^n(x) = F(r^n(x)) + O(1) \quad as \quad n \to \infty \right\} \ge \dim_H \mu.$$

Note that the main improvement over Theorem 2.5 is that we are requiring the error term to be bounded. We have also introduced a new variable r: this would arise naturally if we were dealing with flows. Theorem 2.5 can be deduced by taking r to be a constant function.

The conditions on F are clearly stronger than necessary (for example we could add any bounded function to F without changing the set we are looking at), but still weak enough to allow F to be any differentiable function whose derivative tends to zero; this covers most of the functions we might normally be interested in, including all of the functions studied in [FS]. We can even take F to be a slowly oscillating function such as $F(t) = (\log t)\sin(\sqrt{t})$. On the other hand, it is certainly necessary to have better control on F than merely taking $F(t) = o(t)$, because if for example we take $F(t) = \sqrt{t}\sin t$, and r is small, then there can be no point x with $g^n(x) = F(r^n(x)) + O(1)$.

Proof of Theorem 2.8: First of all, because r is continuous and strictly positive, we can set $r_{min} = \inf r$ and $r_{max} = \sup r$. For most of the proof, the bounds $0 < r_{min} \leq r \leq r_{max}$ will be the only properties of r that we need to use.

As in the proof of Theorem 2.5, we consider sets

$$S_K = \left\{x \in X_A^+ : |g^n(x)| < K \quad \forall n\right\}.$$

We have $\mathrm{Bdd}_{X_A^+}(g) = \bigcup_{K\in\mathbb{N}} S_K$. So by Theorem 2.3, given any $\epsilon > 0$ we can find a K such that

$$\dim_H S_K \geq \dim_H \mu - \epsilon.$$

We aim to construct a block-adding process such that for each point $x \in S_K$, the ergodic sums $g^n(\xi(x))$ have the desired behaviour. The set $S_K' = \{\xi(x) : x \in S_K\}$ will be shown to have the same dimension as S_K. This implies that the dimension of the set of *all* points with the required property is at least $\dim_H \mu - \epsilon$, and so the theorem follows because ϵ was arbitrary.

We first specify which blocks are to be used in the construction. Using the aperiodicity of the subshift we can find N such that $A^{N+1} > 0$, i.e. for any two symbols $s, t \in \{1, 2, \ldots, k\}$ we can find at least one block $W(s,t)$ of length N such that $s\,W(s,t)\,t$ is an admissible sequence of symbols. We fix some choice of $W(s,t)$ for each pair s, t. Now let K_B be a positive constant, chosen to be greater than $10(V(g) + M_g + K + 1)$, where $M_g = \sup_{x \in X_A^+} |g(x)|$. Then, because g is not cohomologous to a constant, we can find points $u^+, u^- \in X_A^+$ and an integer ℓ_u such that

$$g^{\ell_u}(u^+) \geq K_B + V(g) + 2NM_g;$$
$$g^{\ell_u}(u^-) \leq -K_B - V(g) - 2NM_g.$$

(This follows from the Central Limit Theorem.) Now for $s, t \in \{1, 2, \ldots, k\}$ we define the block $B^+(s,t)$ to have the form

$$W(s, u_0^+)\, u_0^+\, u_1^+\, \ldots\, u_{\ell_u-1}^+\, W(u_{\ell_u-1}^+, t).$$

Thus $B^+(s,t)$ is a block of length $\ell = \ell_u + 2N$ for which $s\,B^+(s,t)\,t$ is admissible, and if x is any point in the cylinder $[B^+(s,t)]$, then

$$g^\ell(x) \geq K_B.$$

Similarly we define the block $B^-(s,t)$ as

$$W(s, u_0^-)\, u_0^-\, u_1^-\, \ldots\, u_{\ell_u-1}^-\, W(u_{\ell_u-1}^-, t),$$

and for any $x \in [B^-(s,t)]$ we have

$$g^\ell(x) \leq -K_B.$$

These $2k^2$ blocks will be the only blocks used in the construction (though for convenience we will sometimes also speak about 'adding' blocks of zero length, when no block is needed). If we say that a non-trivial block is inserted after x_i, that block must be either $B^+(x_i, x_{i+1})$ or $B^-(x_i, x_{i+1})$. Note that these blocks all have the same length ℓ.

Now we diverge from the proof of Theorem 2.5 in [FS]. Our choice of places to insert the blocks will have an inductive definition, and these places will depend on the point x (but in such a way that the process is defined on cylinders as per definition 2.6).

Given our point $x \in S_K$ we construct a sequence $(x^{(j)})_{j \geq -1}$, starting with $x^{(-1)} = x$. This sequence will take the following form:

$$\begin{aligned} x^{(-1)} &\text{ is } \; x_0\, x_1\, x_2\, x_3\, x_4\, \ldots \\ x^{(0)} &\text{ is } \; x_0\, B_0\, x_1\, x_2\, x_3\, x_4\, \ldots \\ x^{(1)} &\text{ is } \; x_0\, B_0\, x_1\, B_1\, x_2\, x_3\, x_4\, \ldots \\ x^{(2)} &\text{ is } \; x_0\, B_0\, x_1\, B_1\, x_2\, B_2\, x_3\, x_4\, \ldots \\ &\ldots \text{ and so on,} \end{aligned}$$

where $(B_j)_{j \geq 0}$ is a sequence of blocks depending on x; however, most of these blocks will have zero length. So an alternative description is that $x^{(j)}$ is defined by *either*

- $x^{(j)} = x^{(j-1)}$; *or*
- $x^{(j)}$ is the same as $x^{(j-1)}$ except for a block (of length ℓ) inserted behind $x_{j'}^{(j-1)}$, where j' is the position that the symbol x_j has been shifted to in $x^{(j-1)}$.

For fixed i we see that the sequence $(x_i^{(j)})_{j \geq -1}$ is eventually constant. So the limit $\lim_{j \to \infty} x^{(j)}$ exists, and we will define

$$\xi(x) = \lim_{j \to \infty} x^{(j)}.$$

Clearly this definition makes ξ a block-adding process as defined in section 2.1.2, whose blocks are these B_i.

Now, the sequences $(x^{(j)})_{j \geq -1}$ will be defined inductively, and simultaneously for all points x: that is, when deciding what the block $B_j(x)$ is to be, we will assume we have already defined $y^{(j-1)}$ for all $y \in S_K$.

Furthermore we will insist that *each stage* of the construction is defined on cylinders, by which we mean that the function $x \mapsto x^{(j)}$ is itself a block-adding process which is defined on cylinders. This ensures that ξ itself is defined on cylinders.

From the definition of F we can find N_F such that whenever $t \geq N_F r_{min}$,

$$|F(t+\tau) - F(t)| < 1 \quad \forall 0 \leq \tau \leq \max\{V(r), (\ell+1)r_{max}\}.$$

In particular, suppose $x, y \in X_A^+$ with $x_i = y_i$ for all $i < n$. Then $|r^n(x) - r^n(y)| \leq V(r)$ and so if $n \geq N_F$ we have $|F(r^n(x)) - F(r^n(y))| \leq 1$.

We now pick some large constant K_0 (much larger than K_B). Because the functions g, r and F are continuous, we can choose K_0 sufficiently large that for all $n \leq N_F + \ell$ and all $z \in X_A^+$ we have

$$|g^n(z) - F(r^n(z))| < K_0.$$

It will be convenient to have another notation for cylinders: for $n \geq 0$ we write

$$C_n(x; S_K) := \{y \in S_K : y_i = x_i \quad \forall i < n\} = [x_0\, x_1\, \ldots\, x_{n-1}] \cap S_K.$$

Now take $j \geq 0$; we want to define $x^{(j)}$ in terms of $x^{(j-1)}$, which means defining the block $B_j(x)$.

Say that the symbol x_j is shifted to position j' in $x^{(j-1)}$. (The nature of the construction then means that the symbol x_j is shifted to this same position j' in $x^{(i)}$ for all $i \geq j-1$, and hence also in $\xi(x)$.) Since we assume that the function $y \mapsto y^{(j-1)}$ is defined on cylinders, we have that if $y \in C_{j+1}(x; S_K)$ then the symbol y_j is shifted to the same position j' in $y^{(j-1)}$. So it is natural to look at the values of $g^{j'+1}(y^{(j-1)})$ and $r^{j'+1}(y^{(j-1)})$.

Since g is Hölder continuous and $y \mapsto y^{(j-1)}$ is defined on cylinders, we have for any $y, z \in C_{j+1}(x; S_K)$:

$$|g^{j'+1}(y^{(j-1)}) - g^{j'+1}(z^{(j-1)})| \leq V(g);$$

and if $j' \geq N_F$ we also have

$$\left| F\left(r^{j'+1}(y^{(j-1)})\right) - F\left(r^{j'+1}(z^{(j-1)})\right)\right| \leq 1.$$

Thus for $j' \geq N_F$,

$$\begin{aligned} &\Big|\left(g^{j'+1}(y^{(j-1)}) - F(r^{j'+1}(y^{(j-1)}))\right) \\ &\qquad - \left(g^{j'+1}(z^{(j-1)}) - F(r^{j'+1}(z^{(j-1)}))\right)\Big| \leq V(g) + 1, \quad (2.1) \end{aligned}$$

whenever $y, z \in C_{j+1}(x; S_K)$.

Now we can define $x^{(j)}$ as follows:

- If $j' < N_F$, or if $-K_0 \leq g^{j'+1}(y^{(j-1)}) - F\left(r^{j'+1}(y^{(j-1)})\right) \leq K_0$ for all $y \in C_j(x; S_K)$ then we take $x^{(j)} = x^{(j-1)}$;

- If $j' \geq N_F$ and there is some $y \in C_j(x; S_K)$ such that $g^{j'+1}(y^{(j-1)}) - F\left(r^{j'+1}(y^{(j-1)})\right) > K_0$ then we take $B_j(x) = B^-(x_j, x_{j+1})$. (And thus $x^{(j)}$ is the same as $x^{(j-1)}$ but with this block $B_j(x)$ inserted behind $x_{j'}^{(j-1)}$.)

- If $j' \geq N_F$ and there is some $y \in C_j(x; S_K)$ such that $g^{j'+1}(y^{(j-1)}) - F\left(r^{j'+1}(y^{(j-1)})\right) < -K_0$ then we take $B_j(x) = B^+(x_j, x_{j+1})$.

We took K_0 to be large, certainly large enough that $2K_0 > V(g) + 1$, and so the second and third possibilities cannot happen simultaneously.

With this definition, the function $x \mapsto x^{(j)}$ is clearly defined on cylinders as required. This completes the inductive definition of the sequence $\left(x^{(j)}\right)_{j \geq -1}$.

As stated above, we take $\xi(x) = \lim_{j\to\infty} x^{(j)}$, and this block-adding function is defined on cylinders. We now want to show that the point $\xi(x)$ has sums $g^n(\xi(x))$ with the desired asymptotic property. Let us define

$$\Delta_n(x) = g^n(\xi(x)) - F\left(r^n(\xi(x))\right).$$

As above, for each $j \in \mathbb{N}_0$, let j' be the position that the symbol x_j is shifted to in $\xi(x)$. Then $[\xi(x)]_i = x_i^{(j-1)}$ for all $i \leq j'$, and so if $j' \geq N_F$ we have two useful inequalities:

$$\left|\Delta_{j'}(x) - \left(g^{j'}(x^{(j-1)}) - F\left(r^{j'}(x^{(j-1)})\right)\right)\right| \leq V(g) + 1. \tag{2.2}$$

$$\left|\Delta_{j'+1}(x) - \left(g^{j'+1}(x^{(j-1)}) - F\left(r^{j'+1}(x^{(j-1)})\right)\right)\right| \leq V(g) + 1. \tag{2.3}$$

In particular, from (2.3) we see that if $\Delta_{j'+1}(x) > K_0 + V(g) + 1$ then $B_j(x) = B^-(x_j, x_{j+1})$; conversely (also using inequality (2.1)), if $B_j(x) = B^-(x_j, x_{j+1})$ then we must have $\Delta_{j'+1}(x) > K_0 - 2V(g) - 2$. Similar inequalities hold for $B^+(x_j, x_{j+1})$.

Claim 1: If $j' \geq N_F$ and $B_j(x) = B^-(x_j, x_{j+1})$ then

$$\Delta_{(j+1)'}(x) \leq \Delta_{j'}(x) - D,$$

for a constant $D \geq 3V(g) + 4 + 2K$.

Proof of Claim 1: We have

$$g^{(j+1)'}(x) = g^{j'}(x) + g(\sigma^{j'}x) + g^{\ell}(\sigma^{j'+1}x),$$

and $\sigma^{j'+1}x \in [B^-(x_j, x_{j+1})]$ so $g^{\ell}(\sigma^{j'+1}x) \leq -K_B$, so

$$g^{(j+1)'}(x) \leq g^{j'}(x) + M_g - K_B.$$

Also $r^{(j+1)'}(x) - r^{j'}(x) \leq (\ell + 1)r_{max}$ and so

$$\left|F\left(r^{(j+1)'}(x)\right) - F\left(r^{j'}(x)\right)\right| \leq 1.$$

Combining these inequalities gives

$$\Delta_{(j+1)'}(x) \leq \Delta_{j'}(x) + M_g + 1 - K_B.$$

Take $D = K_B - M_g - 1$. Then since we chose $K_B \geq 10(V(g) + M_g + K + 1)$ we have $D \geq 3V(g) + 4 + 2K$ as required.

Claim 2: $\Delta_{j'}(x) \leq K_0 + V(g) + 1$ for all $x \in S_K$ and all $j \geq 0$.

Proof of Claim 2: We first show that if $\Delta_{j'}(x) \leq K_0 + V(g) + 1$ for some $j' \geq N_F$ then $\Delta_{(j+1)'}(x) \leq K_0 + V(g) + 1$. We split into three cases:

- If $B_j(x)$ is empty (i.e. no block was inserted behind x_j) then we must have $\Delta_{j'+1}(x) \leq K_0 + V(g) + 1$. Furthermore in this case we have $(j+1)' = j'+1$, and so $\Delta_{(j+1)'}(x) \leq K_0 + V(g) + 1$ as desired.

- If $B_j(x) = B^-(x_j, x_{j+1})$ then Claim 1 shows that $\Delta_{(j+1)'}(x) < \Delta_{j'}(x)$.

- If $B_j(x) = B^+(x_j, x_{j+1})$ then $\Delta_{j'+1}(x) < -K_0 + 2V(g) + 2$. Now we have

$$g^{(j+1)'}(x) = g^{j'+1}(x) + g^{\ell}(\sigma^{j'+1}x) \leq g^{j'+1}(x) + \ell M_g;$$
$$\left|F\left(r^{(j+1)'}(x)\right) - F\left(r^{j'+1}(x)\right)\right| \leq 1;$$

and so $\Delta_{(j+1)'}(x) < -K_0 + 2V(g) + \ell M_g + 3$. Provided that K_0 was chosen large enough we actually have $\Delta_{(j+1)'}(x) < 0$.

If $j' \leq N_F + \ell$ the desired inequality is immediate from the choice of K_0. The inequality then follows for all j by induction.

By combining Claim 1 and Claim 2 we have that if $B_j(x) = B^-(x_j, x_{j+1})$ then $\Delta_{(j+1)'}(x) \leq K_0 + V(g) + 1 - D$.

In a similar way we can prove the corresponding lower bounds for $|\Delta_{j'}(x)|$, and hence we get:

- $|\Delta_{j'}(x)| \leq K_0 + V(g) + 1$ for all $x \in S_K$ and all $j \geq 0$;

- If a block was inserted after x_j then $|\Delta_{j'}(x)| \leq K_0 + V(g) + 1 - D$.

Finally, suppose we have any $x \in S_K$ and any $n \geq 0$. If $n \leq N_F + \ell$ then $|\Delta_n(x)| \leq K_0$ by the choice of K_0. Otherwise, we let j be the largest integer such that $j' \leq n$. Then we must have $j' \geq N_F$ and $n - j' \leq \ell$. Hence

$$g^n(x) = g^{j'}(x) + g^{n-j'}(\sigma^{j'}x) \leq g^{j'}(x) + \ell M_g;$$

and

$$\left|F\left(r^n(x)\right) - F\left(r^{j'}(x)\right)\right| \leq 1.$$

So

$$|\Delta_n(x)| \leq |\Delta_{j'}(x)| + \ell M_g + 1 \leq K_0 + V(g) + 1 + \ell M_g + 1.$$

This completes the proof that $g^n(\xi(x)) = F\left(r^n(\xi(x))\right) + O(1)$.

It remains to show that $\dim_H S'_K = \dim_H S_K$. First of all from Proposition 2.7 we know that ξ is Lipschitz continuous, and so $\dim_H S'_K \leq \dim_H S_K$. However we are more interested in the opposite inequality.

Let $\eta > 0$ be arbitrary. Then we can find $N_\eta \geq N_F$ such that whenever $t \geq N_\eta r_{min}$ we have

$$|F(t+\tau) - F(t)| < \eta \quad \forall 0 \leq \tau \leq r_{max}.$$

Claim 3: If $B_j(x)$ is non-empty (i.e. a block of length ℓ was inserted after x_j) for some j with $j' \geq N_\eta$, then B_{j+k} is empty for all $1 \leq k < \eta^{-1}$.

Proof of Claim 3: We use induction on k. Given $1 \leq k < \eta^{-1}$, assume that $B_{j+i}(x)$ is empty for all i such that $1 \leq i < k$. That is, the point $\xi(x)$ contains the string of symbols

$$x_{j+1}\, x_{j+2}\, x_{j+3}\, \ldots\, x_{j+k}$$

uninterrupted by blocks. We see that

$$x^{(j)} = x^{(j+1)} = x^{(j+2)} = \cdots = x^{(j+k-1)},$$

and also that $(j+k)' = (j+1)' + k - 1$.

Now, we showed previously that

$$\left|\Delta_{(j+1)'}(x)\right| \leq K_0 + V(g) + 1 - D.$$

Combining this with the inequality (2.2) we get

$$\left|g^{(j+1)'}(x^{(j)}) - F\left(r^{(j+1)'}(x^{(j)})\right)\right| \leq K_0 + 2V(g) + 2 - D. \tag{2.4}$$

Next we look at

$$g^{(j+k)'+1}(x^{(j+k-1)}) = g^{(j+1)'}(x^{(j)}) + g^k(\sigma^{(j+1)'}x^{(j)})$$

(using the fact that $x^{(j+k-1)} = x^{(j)}$). But $\sigma^{(j+1)'}x^{(j)} = \sigma^{j+1}x$ and so

$$\left|g^k(\sigma^{(j+1)'}x^{(j)})\right| = \left|g^k(\sigma^{j+1}x)\right| \leq 2K.$$

(This is where we use the fact that $x \in S_K$.) And so

$$\left|g^{(j+k)'+1}(x^{(j+k-1)}) - g^{(j+1)'}(x^{(j)})\right| \leq 2K. \tag{2.5}$$

And for the r term we have

$$\left|r^{(j+k)'+1}(x^{(j+k-1)}) - r^{(j+1)'}(x^{(j)})\right| \leq kr_{max}$$

(again using $x^{(j+k-1)} = x^{(j)}$) and so

$$\left|r^{(j+k)'+1}(x^{(j+k-1)}) - r^{(j+1)'}(x^{(j)})\right| \leq k\eta. \tag{2.6}$$

Combining (2.4), (2.5) and (2.6) gives

$$\left|g^{(j+k)'+1}(x^{(j+k-1)}) - F\left(r^{(j+k)'+1}(x^{(j+k-1)})\right)\right| \leq K_0 + 2V(g) + 2 + 2K + k\eta - D.$$

And so from (2.1), for any $y \in C_{(j+k)'+1}(x; S_K)$ we have

$$\left|g^{(j+k)'+1}(y^{(j+k-1)}) - F\left(r^{(j+k)'+1}(y^{(j+k-1)})\right)\right| \leq K_0 + 3V(g) + 3 + 2K + k\eta - D.$$

But $D \geq 3V(g) + 4 + 2K$, so

$$\left|g^{(j+k)'+1}(y^{(j+k-1)}) - F\left(r^{(j+k)'+1}(y^{(j+k-1)})\right)\right| \leq K_0 + k\eta - 1.$$

Thus if $k < \eta^{-1}$ this is less than K_0 for all $y \in C_{(j+k)'+1}(x; S_K)$, and so no block is inserted behind x_{j+k}.

This completes the inductive proof of Claim 3.

Suppose we let $P = \lceil \eta^{-1} \rceil$. Then for any $x \in S_K$ and any i with $i' \geq N_\eta$ we have

$$(i + P)' - i' \leq P + \ell.$$

And so whenever $j > i$ for $i' \geq N_\eta$, we have

$$\begin{aligned} j' - i' &\leq \left\lceil \frac{j-i}{P} \right\rceil (P + \ell) \\ &< \left(\frac{j-i}{P} + 1 \right) (P + \ell) \\ &\leq (j - i + P) + (j - i + P)\ell/P \\ &< j(1 + \ell/P) + P + \ell. \end{aligned}$$

If we take i to be the smallest integer such that $i' \geq N_\eta$, then $i' \leq N_\eta + \ell$ and so

$$j' < j(1 + \ell/P) + P + N_\eta + 2\ell,$$

i.e.

$$j' < C_\eta + j(1 + \eta\ell), \tag{2.7}$$

for a constant C_η which is independent of x. Note that we have proved this inequality under the assumption that $j > i$, but by changing the constant C_η if necessary we can ensure that it also holds whenever $j' \leq N_\eta + \ell$, and hence for all j.

Suppose we have $x, y \in S_K$ with $d(x, y) = \theta^j$ (i.e. $x_i = y_i$ for $i < j$, but $x_j \neq y_j$). Then, since ξ is defined on cylinders, we know that x_j and y_j are shifted to the same position j' in $\xi(x)$ and $\xi(y)$ respectively. So $d(\xi(x), \xi(y)) = \theta^{j'}$, and from the inequality (2.7) we have

$$\begin{aligned} d(\xi(x), \xi(y)) &> \theta^{C + j(1+\eta\ell)} \\ &> \theta^C . d(x, y)^{1+\eta\ell}. \end{aligned}$$

And hence

$$\dim_H S_K' \geq \frac{\dim_H S_K}{1 + \eta\ell}.$$

But η was arbitrary, so this shows

$$\dim_H S_K' \geq \dim_H S_K.$$

We proved the opposite inequality earlier, so we now finally have $\dim_H S_K' = \dim_H S_K$. This completes the proof of Theorem 2.8. □

Now we repeat the arguments which we used to deduce Theorem 2.4 from Theorem 2.3. We have

$$\begin{aligned}
\dim_H &\left\{x \in X_A^+ : g^n(x) = F(r^n(x)) + O(1) \quad \text{as} \quad n \to \infty\right\} \\
&\geq \sup\left\{\dim_H \mu : \mu \text{ an equilibrium state with } \int g\, d\mu = 0\right\} \\
&\geq \dim_H \mathrm{Ave}(g, 0),
\end{aligned}$$

by Theorem 2.2, provided that at least one such μ exists. To prove the opposite inequality, suppose $y \in X_A^+$ is such that $g^n(y) = F(r^n(y)) + O(1)$. Given $\epsilon > 0$ we can choose $t_0 > 0$ such that $\sup_{\tau\in[0,1]} |F(t+\tau) - F(t)| < \epsilon$ for all $t \geq t_0$. And so for all $n > t_0/r_{min}$ we have

$$\begin{aligned}
|F(r^n(y))| &\leq |F(t_0)| + \lceil r^n(y) - t_0 \rceil \epsilon \\
&\leq |F(t_0)| + (nr_{max} - t_0 + 1)\epsilon,
\end{aligned}$$

and so

$$\limsup_{n\to\infty} \frac{1}{n} |g^n(y)| \leq r_{max}\epsilon.$$

Since ϵ was arbitrary this shows $\lim_{n\to\infty} \frac{1}{n} g^n(y) = 0$, i.e. $y \in \mathrm{Ave}_{X_A^+}(g, 0)$.

Thus we have shown:

Theorem 2.9 *Suppose g, F, r are as in Theorem 2.8. Then if $0 \in (\underline{\alpha}, \overline{\alpha})$,*

$$\dim_H \left\{x \in X_A^+ : g^n(x) = F(r^n(x)) + O(1) \quad \textit{as} \quad n \to \infty\right\} = \dim_H \mathrm{Ave}_{X_A^+}(g, 0).$$

2.1.4 Two-sided subshifts and suspended flows

We want to define the sets in Theorems 2.3 and 2.8 more generally. Suppose we have a transformation $T : X \to X$ for some compact metric space X, with Hölder continuous functions $g : X \to \mathbb{R}$ and $r : X \to \mathbb{R}^+$, and a continuous function $F : \mathbb{R}^+ \to \mathbb{R}$. Then we shall define

$$\begin{aligned}
\mathrm{Bdd}_X(g) &= \{x \in X : g^n(x) \text{ is bounded}\}, \\
L_X(g, F, r) &= \{x \in X : g^n(x) = F(r^n(x)) + O(1)\}.
\end{aligned}$$

In the case of a one-sided subshift we have shown in the previous sections that, under certain assumptions, the dimensions of these sets are all equal, and equal to the dimension of $\mathrm{Ave}_{X_A^+}(g, 0)$.

Now consider a two-sided subshift $\sigma : X_A \longrightarrow X_A$. We have a projection $\pi_+ : X_A \longrightarrow X_A^+$ as defined in section 1.4 by $(\pi_+ x)_i = x_i$. For a symbol $s \in \{1, 2, \ldots, k\}$, look at the set

$$R_s := \{x \in X_A : x_0 = s\}.$$

As a metric space, we can think of R_s as being a product of a cylinder in X_A^+ with a cylinder in $X_{A^T}^+$, where A^T is the transpose of the matrix A. We will write these cylinders as $[s]^A$ and $[s]^{A^T}$ to avoid confusion. That is, we have a map

$$P_s : R_s \longrightarrow [s]^A \times [s]^{A^T}$$

given by

$$P_s(x) = (\pi_+(x), \pi_-(x)) = ((x_0, x_1, x_2, \ldots), (x_0, x_{-1}, x_{-2}, \ldots)).$$

Provided that the constant θ used to define the metric is the same for each space, this map P_s is a bi-Lipschitz homeomorphism.

If S is any subset of X_A^+, then

$$P_s(\pi_+^{-1}(S) \cap R_s) = (S \cap [s]^A) \times [s]^{A^T}.$$

But it is well-known (see [Pes], for example) that

$$\dim_H [s]^{A^T} = \overline{\dim}_B [s]^{A^T} = \dim_H X_{A^T}^+ = \overline{\dim}_B X_{A^T}^+,$$

and so by statement (e) from section 1.5,

$$\begin{aligned} \dim_H \left((S \cap [s]^A) \times [s]^{A^T} \right) &= \dim_H (S \cap [s]^A) + \dim_H [s]^{A^T} \\ &= \dim_H (S \cap [s]^A) + \dim_H X_{A^T}^+. \end{aligned}$$

Hence

$$\dim_H \left(\pi_+^{-1}(S) \cap R_s \right) = \dim_H (S \cap [s]^A) + \dim_H X_{A^T}^+.$$

And by maximizing over all possible s we get

$$\dim_H \left(\pi_+^{-1}(S)\right) = \dim_H S + \dim_H X_{A^T}^+.$$

This can be used to transfer our results about one-sided shifts to the two-sided case. In fact each of the sets we are interested in is of the form $\pi_+^{-1}(S)$ for some $S \subseteq X_A^+$, though for the following theorem we only need to use an easy special case of this statement.

Theorem 2.10 *Suppose there exists some equilibrium state μ on X_A such that $\int_{X_A} g\,d\mu = 0$. Then*

$$\dim_H \mathrm{Bdd}_{X_A}(g) = \dim_H \mathrm{Ave}_{X_A}(g, 0).$$

Furthermore if g is not cohomologous to a constant then

$$\dim_H L_{X_A}(g, F, r) = \dim_H \mathrm{Ave}_{X_A}(g, 0)$$

whenever the function F satisfies $\sup_{\tau\in[0,1]} |F(t+\tau) - F(t)| \to 0$ as $t \to \infty$.

Proof: If g is cohomologous to a constant then the first statement is trivial (as explained for one-sided subshifts). So now let us assume that g is not cohomologous to a constant. It is sufficient to prove the second statement, since the first is then the special case $F = 0$.

By Theorem 1.2 we can find Hölder continuous functions $g^{(u)}, r^{(u)} : X_A \to \mathbb{R}$ which are cohomologous to g, r respectively, such that whenever $x, y \in X_A$ with $x_i = y_i$ for all $i \geq 0$ we have $g^{(u)}(x) = g^{(u)}(y)$ and $r^{(u)}(x) = r^{(u)}(y)$. Furthermore we can require the function $r^{(u)}$ to be strictly positive. (Suppose $r^{(u)}$ is not strictly positive. $|r^n(x) - (r^{(u)})^n(x)|$ is bounded, and $r^n(x) \geq nr_{min}$, so there exists n such that $(r^{(u)})^n(x) > 0$ for all x. We can therefore replace $r^{(u)}$ by the cohomologous function $\frac{1}{n}(r^{(u)})^n$.) These functions can also be regarded as functions on X_A^+, which we will write as $g^+ : X_A^+ \to \mathbb{R}$, $r^+ : X_A \to \mathbb{R}^+$.

As in section 1.4, we look at the pushforward of μ to X_A^+ by the map π_+; this is an equilibrium state on X_A^+. And we have

$$\int_{X_A^+} g^+ \, d\mu^+ = \int_{X_A} g^{(u)} \, d\mu = \int_{X_A} g \, d\mu = 0.$$

So we can apply Theorem 2.8 to get

$$\dim_H L_{X_A^+}(g^+, F, r^+) = \dim_H \mathrm{Ave}_{X_A^+}(g^+, 0).$$

But by the definition of g^+, r^+, we have for any $x \in X_A$ and $n \geq 0$,

$$(g^+)^n(\pi_+ x) = (g^{(u)})^n(x), \quad (r^+)^n(\pi_+ x) = (r^{(u)})^n(x).$$

And so we have $L_{X_A}(g^{(u)}, F, r^{(u)}) = \pi_+^{-1}\left(L_{X_A^+}(g^+, F, r^+)\right)$ and $\mathrm{Ave}_{X_A}(g^{(u)}, 0) = \pi_+^{-1}\left(\mathrm{Ave}_{X_A^+}(g^+, 0)\right)$. Hence

$$\dim_H L_{X_A}(g^{(u)}, F, r^{(u)}) = \dim_H \mathrm{Ave}_{X_A}(g^{(u)}, 0).$$

Finally, because $g^{(u)}$, $r^{(u)}$ are cohomologous to g and r respectively, $|(g^{(u)})^n(x) - g^n(x)|$ and $|(r^{(u)})^n(x) - r^n(x)|$ are (uniformly) bounded; the latter implies that $\left|F\left((r^{(u)})^n(x)\right) - F\left(r^n(x)\right)\right| < 1$ for sufficiently large n. So $L_{X_A}(g^{(u)}, F, r^{(u)}) = L_{X_A}(g, F, r)$ and $\mathrm{Ave}_{X_A}(g^{(u)}, 0) = \mathrm{Ave}_{X_A}(g, 0)$. Hence

$$\dim_H L_{X_A}(g, F, r) = \dim_H \mathrm{Ave}_{X_A}(g, 0).$$

□

For a flow $\phi_t : X \to X$ we look at

$$\mathrm{Ave}_X(g, \alpha) := \left\{ x \in X : \frac{1}{t} \int_0^t g\left(\phi_\tau x\right) \, d\tau \to \alpha \ \text{ as } t \to \infty. \right\};$$

$$\mathrm{Bdd}_X(g) := \left\{ x \in X : \int_0^t g\left(\phi_\tau x\right) \, d\tau \text{ is bounded} \right\};$$

$$L_X(g, F) := \left\{ x \in X : \int_0^t g\left(\phi_\tau x\right) \, d\tau = F(t) + O(1) \right\}.$$

Notice that these sets are all ϕ-invariant. In the case of a suspended flow $\sigma_t^r : X_A^r \to X_A^r$, the metric on X_A^r is defined in such a way that, whenever S is a (Borel) ϕ-invariant set, we have

$$\dim_H S = \dim_H \{x \in X_A : \rho(x, 0) \in S\} + 1. \tag{2.8}$$

(Details are in [BS1].)

Now suppose μ is an equilibrium state on X_A^r; according to Proposition 1.7 this is of the form $(\nu \times l)/(\int r\, d\nu)$, where ν is an equilibrium state on X_A. And if $\int_{X_A^r} g\, d\mu = 0$ it follows that $\int_{X_A} \mathcal{I}g\, d\nu = 0$. Furthermore if g is not cohomologous to a constant then $\mathcal{I}g$ is not cohomologous to a constant. So we can apply Theorem 2.10 to show that

$$\dim_H \mathrm{Bdd}_{X_A}(\mathcal{I}g) = \dim_H L_{X_A}(\mathcal{I}g, F, r) = \dim_H \mathrm{Ave}_{X_A}(\mathcal{I}g, 0) \tag{2.9}$$

(whenever F satisfies the usual condition), where r is the roof function for the suspended flow. We note that

- $x \in \mathrm{Ave}_{X_A}(\mathcal{I}g, 0)$ if and only if $\rho(x, 0) \in \mathrm{Ave}_{X_A^r}(g, 0)$;
- $x \in \mathrm{Bdd}_{X_A}(\mathcal{I}g)$ if and only if $\rho(x, 0) \in \mathrm{Bdd}_{X_A^r}(g)$;
- $x \in L_{X_A}(\mathcal{I}g, F, r)$ if and only if $\rho(x, 0) \in L_{X_A^r}(g, F)$;

So by combining (2.9) with (2.8) we have:

Theorem 2.11 *Let $g : X_A^r \to \mathbb{R}$ be Hölder continuous, and suppose there exists some equilibrium state μ on X_A^r such that $\int_{X_A^r} g\, d\mu = 0$. Then*

$$\dim_H \mathrm{Bdd}_{X_A^r}(g) = \dim_H \mathrm{Ave}_{X_A^r}(g, 0).$$

Furthermore if g is not cohomologous to a constant then

$$\dim_H L_{X_A^r}(g, F) = \dim_H \mathrm{Ave}_{X_A^r}(g, 0)$$

for any continuous function F which satisfies $\sup_{\tau \in [0,1]} |F(t+\tau) - F(t)| \to 0$ as $t \to \infty$.

2.2 Moran covers

We will make heavy use of the techniques for dealing with conformal systems which are explained in Pesin's book [Pes] (for maps) and [PS] (for flows). This section explains what we will need.

We first look at the case of conformal repellers for expanding maps. If the coding map is $\chi : X_A^+ \to J$, then for any cylinder $C = [s_0\, s_1\, \ldots\, s_{n-1}]$ in X_A^+ we have the set

$$\chi(C) = \{\chi(x) : x \in C\} = R_{s_0} \cap f^{-1}(R_{s_1}) \cap \cdots \cap f^{-(n-1)}\left(R_{s_{n-1}}\right).$$

Now suppose we define $v : J \to \mathbb{R}$ by $v(z) = \log a(z) > 0$. We have:

Proposition 2.12 ([Pes]) *Let C be a cylinder in X_A^+ of length n. Then the set $\chi(C)$ is contained in a ball in J of radius $\overline{r}(C)$ and contains a ball in J of radius $\underline{r}(C)$, such that for any $z \in \chi(C)$ we have*

$$c_1 \exp\left(-v^n(z)\right) \le \underline{r}(C) \le \overline{r}(C) \le c_2 \exp\left(-v^n(z)\right),$$

for positive constants c_1, c_2.

In light of this proposition, we can construct for any small $r > 0$ a cover of X_A^+ by cylinders, such that (roughly speaking) the corresponding sets in J all have diameter close to r. We call this a *Moran cover.* The construction is as follows:

For each $x \in X_A^+$ we let $n(x)$ be the largest non-negative integer such that

$$\exp\left(-v^{n(x)}(\chi(x))\right) > r.$$

Since $v \ge v_{min} > 0$ this is well-defined, and bounded by $-\log r / v_{min}$. We can then define $C(x)$ to be the (unique) cylinder of length $n(x)$ such that $x \in C(x)$. Clearly the cylinders $\left\{C(x) : x \in X_A^+\right\}$ cover X_A^+. Furthermore, because the lengths of the cylinders are bounded, the cover is actually made up of finitely many sets, say $\{C_1, \ldots, C_m\}$.

Now, these sets may not be disjoint. However, because they are cylinders in X_A^+, we have that if $C_i \cap C_j \ne \varnothing$ then either $C_i \subset C_j$ or $C_j \subset C_i$. We shall throw out all those cylinders C_i which are contained in some other cylinder C_j. If we do this, then the remaining cylinders, which we relabel as $\{C_i : 1 \le i \le m'\}$, are a *disjoint* cover of X_A^+. This will be our Moran cover of X_A^+. The corresponding sets $\{\chi(C_i) : 1 \le i \le m'\}$ can also be called a Moran cover, in the sense that they cover J, and we denote this cover of J by $\mathfrak{U}_r$. Note that the sets $\chi(C_i)$ may overlap on their boundaries, but they have disjoint interiors.

Each cylinder C_i contains at least one point x_i such that $n(x_i)$ is the length of the cylinder C_i. Applying Proposition 2.12 to this point gives

$$c_1 \exp\left(-v^{n(x_i)}(\chi(x_i))\right) \le \underline{r}(C_i) \le \overline{r}(C_i) \le c_2 \exp\left(-v^{n(x_i)}(\chi(x_i))\right),$$

and so from the definition of $n(x)$,

$$c_1 r \le \underline{r}(C_i) \le \overline{r}(C_i) \le c_2 e^{v_{max}} r.$$

The number r is called the *size* of the Moran cover.

The main result concerning Moran covers is the following:

Proposition 2.13 ([Pes]) *There exists a constant M_{Moran}, independent of r, such that if we have a Moran cover $\mathfrak{U}_r$ of J and take any ball of radius r in J, the number of elements of the cover which intersect this ball is bounded by M_{Moran}.*

The constant M_{Moran} is called the *Moran multiplicity factor*. Essentially this proposition means that if a subset of J is covered by balls (as in one definition of the Hausdorff dimension), we can replace this cover by a new cover whose elements are cylinders.

Similar methods can be applied to conformal hyperbolic diffeomorphisms and flows, once we find the proper analogue of the sets $\chi(C)$.

Suppose we have a conformal hyperbolic diffeomorphism $f : \Lambda \to \Lambda$. Take any rectangle R^* of the Markov partition, and let $z^* = \chi(x^*)$ be a point in the interior of R^*. We look at the set $W^u_{R^*}(z^*)$, as defined in section 1.2. This set is modeled by the cylinder $[x_0^*] \subset X_A^+$: that is, we can define a function $\chi^{(u)}_{z^*} : [x_0^*] \to W^u_{R^*}(z^*)$

by

$$\chi^{(u)}_{z^*}(y) = \chi\left(\ldots\, x^*_{-2}\, x^*_{-1}\, x^*_0\, y_1\, y_2\, \ldots\right),$$

and this function is surjective.

Similarly, suppose we have a conformal hyperbolic flow $\phi_t : \Lambda \to \Lambda$, with $z^* \in \operatorname{int} T^*$ for some Markov section T^*. Then the set $W^u_{T^*}(z^*)$ is modeled by $[x^*_0] \subset X^+_A$, using a function which we will also write as $\chi^{(u)}_{z^*}$, where $\chi^{(u)}_{z^*} : [x^*_0] \to W^u_{T^*}(z^*)$ is surjective and defined by

$$\chi^{(u)}_{z^*}(y) = \rho\left(\left(\ldots\, x^*_{-2}\, x^*_{-1}\, x^*_0\, y_1\, y_2\, \ldots\right), 0\right).$$

Now if C is a cylinder in X^+_A with $C \subseteq [x^*_0]$, we can look at the sets $\chi^{(u)}_{z^*}(C)$:

Proposition 2.14 ([Pes], [PS]) *(a) Let R^* be an element of the Markov partition for a conformal hyperbolic diffeomorphism, and take $z^* = \chi(x^*) \in \operatorname{int} R^*$. Then for any cylinder $C \subseteq [x^*_0] \subset X^+_A$ of length n, we have that the set $\chi^{(u)}_{z^*}(C)$ is contained in a ball in $W^u_{R^*}(z^*)$ of radius $\overline{r}^{(u)}_{z^*}(C)$, and contains a ball in $W^u_{R^*}(z^*)$ of radius $\underline{r}^{(u)}_{z^*}(C)$. For any $z \in \chi^{(u)}_{z^*}(C)$ these radii satisfy*

$$c_1 \exp\left(-\upsilon^n(z)\right) \le \underline{r}^{(u)}_{z^*}(C) \le \overline{r}^{(u)}_{z^*}(C) \le c_2 \exp\left(-\upsilon^n(z)\right).$$

Here c_1, c_2 are positive constants and $\upsilon(z) := \log a^{(u)}(z) > 0$.

(b) Let T^* be a Markov section for a conformal hyperbolic flow, and take $z^* = \rho(x^*, 0) \in \operatorname{int} T^*$. Then for any cylinder $C \subseteq [x^*_0] \subset X^+_A$ of length n, we have that the set $\chi^{(u)}_{z^*}(C)$ is contained in a ball in $W^u_{T^*}(z^*)$ of radius $\overline{r}^{(u)}_{z^*}(C)$, and contains a ball in $W^u_{T^*}(z^*)$ of radius $\underline{r}^{(u)}_{z^*}(C)$. For any $z \in \chi^{(u)}_{z^*}(C)$, we can write $z = \rho(x, 0)$ where $x_i = x^*_i$ for all $i \le 0$, and then we have:

$$c_1 \exp\left(-\int_0^{r^n(x)} \upsilon^{(u)}(\phi_\tau z)\, d\tau\right) \le \underline{r}^{(u)}_{z^*}(C) \le \overline{r}^{(u)}_{z^*}(C)$$
$$\le c_2 \exp\left(-\int_0^{r^n(x)} \upsilon^{(u)}(\phi_\tau z)\, d\tau\right).$$

Here c_1, c_2 are positive constants and $\upsilon^{(u)}$ was defined earlier with $\upsilon^{(u)}(z) > 0$; r is the roof function for the suspended flow.

We can construct Moran covers $\mathfrak{U}_r$ of $W^u_{R^*}(z^*)$ and $W^u_{T^*}(z^*)$. For each point $x^+ \in [x_0^*] \subset X_A^+$ we define an integer $n(x^+)$, and we look at the cylinder $C(x^+) \in \mathrm{Cyl}(n(x^+))$ which contains the point x^+. Let $z = \chi^{(u)}_{z^*}(x^+)$. Then $n(x^+)$ is defined as follows:

- For a diffeomorphism, we take $n = n(x^+)$ to be the largest integer such that

$$\exp\left(-v^n(z)\right) > r.$$

- For a flow, if x is the point in X_A such that $x_i = x_i^*$ for $i \leq 0$ and $x_i = x_i^+$ for $i \geq 0$, then we take $n = n(x^+)$ to be the largest integer such that

$$\exp\left(-\int_0^{r^n(x)} v^{(u)}(\phi_\tau z)\, d\tau\right) > r.$$

We assume r is small enough that $n(x^+) \geq 1$, and so $C(x^+) \subseteq [x_0^*]$.

The rest of the construction is as for expanding maps. $\{C(x^+) : x^+ \in [x_0^*]\}$ is a finite collection of cylinders which cover $[x_0^*]$, and by throwing out those cylinders which are strictly contained in some other cylinder in the collection, we get a *disjoint* cover of $[x_0^*]$. The corresponding cylinders $\chi^{(u)}_{z^*}(C_i)$ are a cover of $W^u_{R^*}(z^*)$ (or $W^u_{T^*}(z^*)$ for a flow) which is our Moran cover $\mathfrak{U}_r$, and sets in this cover have disjoint interiors. We also have the important properties of Moran covers:

- For any $\chi^{(u)}_{z^*}(C_i) \in \mathfrak{U}_r$ we have

$$c_1 r \leq \underline{r}^{(u)}_{z^*}(C_i) \leq \overline{r}^{(u)}_{z^*}(C_i) \leq c_{\mathrm{Moran}} r.$$

- Given a ball of radius r in $W^u_{R^*}(z^*)$ (or in $W^u_{T^*}(z^*)$ for a flow), the number of elements of $\mathfrak{U}_r$ which intersect this ball is bounded by a constant M_{Moran}. This constant is independent of r.

2.3 Results for conformal expanding maps

Let J be a conformal repeller for a $C^{1+\alpha}$ map $f : M \longrightarrow M$. Then, as explained in section 1.2, this is modeled by a subshift of finite type X_A^+ via the coding map $\chi : X_A^+ \longrightarrow J$. If $g : J \longrightarrow \mathbb{R}$ is Hölder continuous then the pullback to X_A^+ is a Hölder continuous function which we will call $\tilde{g}$.

Now, as in section 2.1 we look at

$$\left\{ \int_J g \, d\mu : \mu \in \mathcal{M}(J) \right\},$$

and if g is not cohomologous to a constant then this set is a closed interval $[\underline{\alpha}, \overline{\alpha}]$.

Theorem 2.2 carries across to this case:

Theorem 2.15 ([**BS2**])

1. *If* $\alpha \notin [\underline{\alpha}, \overline{\alpha}]$ *then* $\mathrm{Ave}_J(g, \alpha) = \varnothing$.

2. *If* $\alpha \in (\underline{\alpha}, \overline{\alpha})$ *then* $\mathrm{Ave}_J(g, \alpha) \neq \varnothing$, *and*

$$\dim_H \mathrm{Ave}_J(g, \alpha) = \sup \left\{ \dim_H \mu : \mu \in \mathcal{M}(J) \text{ and } \int g \, d\mu = \alpha \right\}.$$

Furthermore the supremum is attained for a measure μ *which is an equilibrium state for some Hölder continuous function.*

As before, we have that $\alpha \in (\underline{\alpha}, \overline{\alpha})$ if and only if there exists an equilibrium state ν such that $\int g d\nu = \alpha$.

We aim to prove versions of Theorems 2.3 and 2.8 for conformal repellers.

2.3.1 Block-adding processes for expanding maps

Our method of proof will be essentially the same as that for subshifts. We take a set $S_J \subset J$ on which we have good control over ergodic sums, and produce from this a new set S'_J which consists of points whose ergodic sums have the behaviour we are looking for. We then compare the dimension of S'_J to the dimension of S_J.

The set S'_J will be produced by means of a block-adding process on X_A^+. That is, we take a set $S_X \in X_A^+$ such that for each $z \in S_J$ there is at least one $x \in S_X$ with $\chi(x) = z$. Then we look for a block-adding process $\xi : S_X \to S'_X$. Finally the image of S'_X under χ will be S'_J.

$$\begin{array}{ccc} S_J & & S'_J \\ \chi\uparrow & & \uparrow\chi \\ S_X & \xrightarrow{\xi} & S'_X \end{array}$$

Note that if x and y are points in S_X with $\chi(x) = \chi(y)$ then it is not necessarily the case that $\chi(\xi(x)) = \chi(\xi(y))$. Thus there is not necessarily a well-defined 'block-adding function' which maps S_J to S'_J. Furthermore, even if such a function did exist (we could require that for each $z \in S_J$ there was only one $x \in S_X$ such that $\chi(x) = z$) it would not necessarily be injective.

However we can still get information about the dimension of S'_J by looking at cylinders in X_A^+, and their corresponding sets in J, rather than individual points. We need some estimates for the diameters of these sets. As explained in section 2.2, if C is a cylinder in X_A^+ then the set $\chi(C)$ contains a ball of radius $\underline{r}(C)$ and is contained in a ball of radius $\overline{r}(C)$. We have:

Lemma 2.16 *(i) There exist positive constants γ_1, γ_2, c_1, c_2 such that if C is a cylinder of length n in X_A^+ then*

$$c_1 \exp(-\gamma_1 n) \leq \underline{r}(C) \leq \overline{r}(C) \leq c_2 \exp(-\gamma_2 n).$$

(ii) Let $C = [x_0, \ldots, x_{m-1}]$ be a cylinder in X_A^+ and let C' be a cylinder produced by inserting a single block of length ℓ inside C, i.e.

$$C' = [x_0, \ldots x_{i-1}, b_0, \ldots, b_{\ell-1}, x_i \ldots x_{m-1}].$$

Then we have

$$\overline{r}(C') \geq \underline{r}(C') \geq \exp(-\gamma_3 \ell)\, \overline{r}(C),$$

for a constant γ_3.

Proof: Recall that $v : J \to \mathbb{R}$ is defined by $v(z) = \log a(z)$. Since a is Hölder continuous and $a(z) > 1$, we have that v is Hölder continuous and $v(z) > 0$. Let $\omega : X_A^+ \to \mathbb{R}^+$ be the pullback of v to X_A^+. Then ω is Hölder continuous, and we can write

$$0 < \omega_{min} \leq \omega(x) \leq \omega_{max}.$$

From Proposition 2.12 there are constants c_1, c_2 such that for any cylinder C in X_A^+ and any point $x \in C$,

$$c_1 \exp\left(-\omega^n(x)\right) \leq \underline{r}(C) \leq \overline{r}(C) \leq c_2 \exp\left(-\omega^n(x)\right), \tag{2.10}$$

where n is the length of the cylinder C. Part (i) (which is well-known) follows immediately from this with $\gamma_1 = \omega_{\max}$ and $\gamma_2 = \omega_{min}$.

Now for part (ii): Given the cylinders C, C', choose any points $x \in C$ and $y \in C'$. Applying (2.10) to y gives

$$\begin{aligned}
\underline{r}(C') &\geq c_1 \exp\left(-\omega^{m+\ell}(y)\right) \\
&\geq c_1 \exp\left(-\omega^i(y) - \omega^\ell(\sigma^i(y)) - \omega^{m-i}(\sigma^{i+\ell}(y))\right) \\
&\geq c_1 \exp\left(-\gamma_1 \ell\right) \exp\left(-\omega^i(y) - \omega^{m-i}(\sigma^{i+\ell}(y))\right).
\end{aligned}$$

Now x and y belong to the same cylinder of length i, so

$$\left|\omega^i(x) - \omega^i(y)\right| \leq V(\omega),$$

and similarly $\sigma^i(x)$ and $\sigma^{i+\ell}(y)$ belong to the same cylinder of length $m-i$, and so

$$\left|\omega^{m-i}(\sigma^i(x)) - \omega^{m-i}(\sigma^{i+\ell}(y))\right| \leq V(\omega).$$

Hence

$$\begin{aligned}
\underline{r}(C') &\geq c_1 \exp\left(-\gamma_1 \ell - 2V(\omega)\right) \exp\left(-\omega^i(x) - \omega^{m-i}(\sigma^i(x))\right) \\
&\geq c_1 \exp\left(-\gamma_1 \ell - 2V(\omega)\right) \exp\left(-\omega^m(x)\right).
\end{aligned}$$

And so by using (2.10) for the point x we get

$$\underline{r}(C') \geq \frac{c_1 \exp\left(-\gamma_1 \ell - 2V(\omega)\right)}{c_2} \overline{r}(C).$$

To put this in the form that we want we can take $\gamma_3 = \gamma_1 + 2V(\omega) + \log(c_1/c_2)$.

□

Now suppose Q is a cylinder with $Q \cap S'_X \neq \varnothing$. If ξ is defined on cylinders then (by Proposition 2.7) there is a well-defined way to 'remove blocks' from Q to produce a cylinder $\widehat{Q}$ which intersects S_X. We want to compare the diameters of the corresponding sets $\chi(Q)$ and $\chi(\widehat{Q})$.

Lemma 2.17 *Let Q be a cylinder of length n which intersects S'_X, and let $\widehat{n}$ be the length of the cylinder $\widehat{Q}$ obtained by removing the blocks from Q. Write $p = (n - \widehat{n})/\widehat{n}$. Then, provided that $p < 1$, we have*

$$\overline{r}(Q) \geq \underline{r}(Q) \geq c\left(\overline{r}(\widehat{Q})\right)^{1+\gamma p},$$

where c, γ are constants.

Proof: $n - \widehat{n}$ is the total length of all the blocks added to the cylinder $\widehat{Q}$ to produce Q. So by repeatedly applying Lemma 2.16 we have

$$\underline{r}(Q) \geq \exp\left(-\gamma_3(n - \widehat{n})\right) \overline{r}(\widehat{Q}).$$

But also from the inequality (2.10) we have

$$\overline{r}(\widehat{Q}) \leq c_2 \exp\left(-\omega_{min}\widehat{n}\right),$$

i.e.

$$\left(\overline{r}(\widehat{Q})\right)^{\gamma_3 p/\omega_{min}} \leq c_2^{\gamma_3 p/\omega_{min}} \exp\left(-\gamma_3(n - \widehat{n})\right).$$

Putting these inequalities together gives

$$\underline{r}(Q) \geq c_2^{-\gamma_3 p/\omega_{min}} \left(\overline{r}(\widehat{Q})\right)^{\gamma_3 p/\omega_{min}} \overline{r}(\widehat{Q}),$$

and so if $p < 1$ we can write

$$\underline{r}(Q) \geq c\left(\overline{r}(\widehat{Q})\right)^{1+\gamma p}$$

where $c = c_2^{-\gamma_3/\omega_{min}}$ and $\gamma = \gamma_3/\omega_{min}$. □

2.3.2 Points with bounded sums

For our Hölder continuous function $g : J \to \mathbb{R}$ we look at

$$\mathrm{Bdd}_J(g) := \{x \in J : g^n(x) \ \text{ is bounded}\}.$$

We will use block-adding processes to prove the following:

Theorem 2.18 *Let $g : J \to \mathbb{R}$ be Hölder continuous, and let ν be an equilibrium state for some Hölder continuous function. Suppose that $\int_J g\,d\nu = 0$. Then*

$$\dim_H \mathrm{Bdd}_J(g) \geq \dim_H \nu.$$

Like for subshifts of finite type, by combining this with Theorem 2.15 we have that

$$\dim_H \mathrm{Bdd}_J(g) = \dim_H \mathrm{Ave}_J(g, 0),$$

whenever there exists an equilibrium state ν such that $\int_J g\,d\nu = 0$ (or equivalently, whenever $0 \in (\underline{\alpha}, \overline{\alpha})$).

Proof of Theorem 2.18: Assume that g is not cohomologous to a constant (otherwise the result is trivial). We define $\tilde{g}$ to be the pullback of g to X_A^+ by the coding map; then $\tilde{g}$ must also be Hölder continuous and not cohomologous to a constant. Also, if ψ is the (Hölder continuous) potential for the equilibrium state ν, then we let μ be the measure on X_A^+ which is the equilibrium state for the pullback of ψ. Then we know (see section 1.4) that $\nu(S) = \mu(\chi^{-1}(S))$ for any set $S \subseteq J$. In particular we have $\int_{X_A^+} \tilde{g}\,d\mu = 0$.

We will think of the symbolic representation of the point $x \in X_A^+$ as being split up into sections of length n, i.e.

$$\boxed{x_0\, x_1\, \ldots\, x_{n-1}}\boxed{x_n\, x_{n+1}\, \ldots\, x_{2n-1}}\cdots\boxed{x_{in}\, x_{in+1}\, \ldots\, x_{(i+1)n-1}}\cdots.$$

In the block-adding process to be defined later, blocks will be inserted only at the ends of sections, i.e. behind x_{jn-1}.

First of all we define

$$\epsilon(n) = \mu\left(\left\{x \in X_A^+ : |\tilde{g}^n(x)| > n^{\frac{3}{4}} - V(\tilde{g})\right\}\right) + n^{-1}.$$

By the Central Limit Theorem we know that $\epsilon(n) \to 0$ as $n \to \infty$.

Next, define the collection of cylinders $\mathcal{C}(n) \subseteq \mathrm{Cyl}(n)$ by

$$\mathcal{C}(n) = \left\{C \in \mathrm{Cyl}(n) : |\tilde{g}^n(y)| > n^{\frac{3}{4}} \text{ for some } y \in C\right\}.$$

Then if $x \in \bigcup \mathcal{C}(n)$ we must have $|\tilde{g}^n(x)| > n^{\frac{3}{4}} - V(\tilde{g})$, so

$$\mu\left(\bigcup \mathcal{C}(n)\right) \leq \epsilon(n) - n^{-1} < \epsilon(n).$$

For each $i \geq 0$, define the set $E_i(n) \subseteq X_A^+$ by

$$E_i(n) = \bigcup_{C \in \mathcal{C}(n)} \sigma^{-in}(C).$$

Thus the section $\boxed{x_{in}\, x_{in+1}\, \ldots\, x_{(i+1)n-1}}$ determines whether x is an element of $E_i(n)$, and if $x \in X_A^+ \setminus E_i(n)$ we know $|\tilde{g}^n(\sigma^{in}x)| \leq n^{\frac{3}{4}}$.

We look at the set of 'good' points which do not belong to 'too many' of the sets $E_i(n)$, i.e.

$$G(n) := \left\{x \in X_A^+ : \limsup_{t \to \infty} \frac{\#\{0 \leq i < t : x \in E_i(n)\}}{t} < \epsilon(n)\right\}.$$

As explained in section 1.4, the measure μ is ergodic for the map $\sigma^n : X_A^+ \to X_A^+$. By applying the ergodic theorem to the indicator function of $\bigcup \mathcal{C}(n)$ we see that $\mu(G(n)) = 1$. So $\nu(\chi(G(n))) = 1$ and hence $\dim_H \chi(G(n)) \geq \dim_H \nu$.

For each n we will construct a block-adding process on $G(n)$. In the notation of section 2.3.1 we are taking $S_X = G(n)$ and $S_J = \chi(G(n))$.

We start by defining the blocks to be used in the construction. These will not depend on n. As before, for any symbols s, t we have a block $W(s,t)$ of length N such that $s\,W(s,t)\,t$ is an admissible sequence of symbols. Now if y is any point in X_A^+ and k a positive integer we can look at the periodic point u defined as

$$y_0\, y_1\, \ldots\, y_{k-1}\, W(y_{k-1}, y_0)\, y_0\, y_1\, \ldots\, y_{k-1}\, W(y_{k-1}, y_0)\, \ldots.$$

Then

$$\tilde{g}^k(y) - V(\tilde{g}) - NM_{\tilde{g}} \leq \tilde{g}^{k+N}(u) \leq \tilde{g}^k(y) + V(\tilde{g}) + NM_{\tilde{g}}.$$

By the Central Limit Theorem we can find y and k such that $\tilde{g}^k(y) \geq 1 + V(\tilde{g}) + NM_{\tilde{g}}$. So in this way we can construct a periodic point u^+, with period which divides some number p^+, such that

$$\tilde{g}^{p^+}\left(u^+\right) \geq 1.$$

Now for any symbols s, t and positive integer m we can define a block $B^+(s,t,m)$ by the sequence of symbols

$$W(s, u_0^+)\, u_0^+\, u_1^+\, \ldots\, u_{mp^+-1}^+\, W(u_{p^+-1}^+, t).$$

The block $B^+(s,t,m)$ has length $mp^+ + 2N$, and if $y \in [B^+(s,t,m)]$ then

$$m.\tilde{g}^{p^+}(u^+) - V(\tilde{g}) - 2NM_{\tilde{g}} \leq \tilde{g}^{mp^+ + 2N}(y) \leq m.\tilde{g}^{p^+}(u^+) + V(\tilde{g}) + 2NM_{\tilde{g}}. \quad (2.11)$$

Similarly we can define blocks $B^-(s,t,m)$ by

$$W(s, u_0^-)\, u_0^-\, u_1^-\, \ldots\, u_{mp^--1}^-\, W(u_{p^--1}^-, t),$$

where u^- is a periodic point with a period which divides p^-, and $\tilde{g}^{p^-}(u^-) \leq -1$; the block $B^-(s,t,m)$ has length $mp^- + 2N$, and if $y \in [B^-(s,t,m)]$ then

$$m.\tilde{g}^{p^-}(u^-) - V(\tilde{g}) - 2NM_{\tilde{g}} \leq \tilde{g}^{mp^- + 2N}(y) \leq m.\tilde{g}^{p^-}(u^-) + V(\tilde{g}) + 2NM_{\tilde{g}}. \quad (2.12)$$

As in Theorem 2.8 the block-adding process will be defined by an inductive construction. For each point $x \in G(n)$ we will define a sequence of points $\left(x^{(j)}\right)_{j \geq -1}$, starting with $x^{(-1)} = x$. For $j \geq 0$, the point $x^{(j)}$ will be the same as $x^{(j-1)}$ except that there may be a block inserted at the end of the section

$$\boxed{x_{jn}\, x_{jn+1}\, \ldots\, x_{jn+n-1}}\,.$$

(More precisely, we insert the block behind $x_{(jn)'+n-1}^{(j-1)}$, where $(jn)'$ is the position that the symbol x_{jn} has been shifted to in $x^{(j-1)}$.) Also as in Theorem 2.8 we promise that each stage of the construction will be defined on cylinders, and this ensures that the resulting block-adding process $\xi(x) = \lim_{j\to\infty} x^{(j)}$ is defined on cylinders.

So, let us take $j \geq 0$, and assume that we have already defined $y^{(j-1)}$ for all $y \in G(n)$, in such a way that $y \mapsto y^{(j-1)}$ is defined on cylinders. Given $x \in G(n)$ we look at the set

$$C_{jn+n}(x; G(n)) := \{y \in G(n) : x_i = y_i \ \forall i < jn+n\}.$$

Suppose the symbol x_{jn} is shifted to position $(jn)'$ in $x^{(j-1)}$; then for all $y \in C_{jn+n}(x; G(n))$ we know that y_{jn} is shifted to the same position $(jn)'$ in y, and furthermore $y_i^{(j-1)} = x_i^{(j-1)}$ for all $i < (jn)' + n$. So, if we define

$$\alpha = \inf_{y \in C_{jn+n}(x;G(n))} \tilde{g}^{(jn)'+n}(y^{(j-1)}), \qquad \beta = \sup_{y \in C_{jn+n}(x;G(n))} \tilde{g}^{(jn)'+n}(y^{(j-1)}),$$

then $\beta - \alpha \leq V(\tilde{g})$.

We proceed to define $x^{(j)}$ as follows:

(i) If $\alpha > 0$ then let m be the smallest positive integer such that $m\left|\tilde{g}^{p^-}(u^-)\right| \geq \alpha$. Then let $x^{(j)}$ be the result of taking $x^{(j-1)}$ and inserting the block $B^-(x_{jn+n-1}, x_{nj+n}, m)$ behind $x^{(j-1)}_{(jn)'+n-1}$.

(ii) If $\beta < 0$ then let m be the smallest positive integer such that $m.\tilde{g}^{p^+}(u^+) \geq -\beta$. Then let $x^{(j)}$ be the result of taking $x^{(j-1)}$ and inserting the block $B^+(x_{jn+n-1}, x_{nj+n}, m)$ behind $x^{(j-1)}_{(jn)'+n-1}$.

(iii) If $\alpha \leq 0$ but $\beta \geq 0$, then take $x^{(j)} = x^{(j-1)}$.

Clearly if we replace x with a different point in $C_{jn+n}(x; G(n))$ the values of α and β are unchanged. So the function $x \mapsto x^{(j)}$ is defined on cylinders, as required.

This completes the inductive definition of the sequence $\left(x^{(j)}\right)_{j\geq -1}$. As in Theorem 2.8, we see that for any i the sequence $x_i^{(j)}$ is eventually constant, and we take

$$\xi(x) = \lim_{j\to\infty} x^{(j)}.$$

We now prove that the ergodic sums $\tilde{g}^{i}\left(\xi(x)\right)$ are bounded. First we show that there is a constant K_1 such that for all $x \in G(n)$ and all $j \geq 0$,

$$\left|\tilde{g}^{((j+1)n)'}\left(x^{(j)}\right)\right| \leq K_1,$$

where $((j+1)n)'$ is the position that the symbol $x_{(j+1)n}$ is shifted to in $x^{(j)}$. Note that

$$\tilde{g}^{((j+1)n)'}\left(x^{(j)}\right) = \tilde{g}^{(jn)'+n}\left(x^{(j)}\right) + \tilde{g}^{\ell}\left(\sigma^{(jn)'+n}x^{(j)}\right),$$

where ℓ is the length of the block added behind $x^{(j-1)}_{(jn)'+n-1}$, and $\sigma^{(jn)'+n}x^{(j)}$ is a point whose first ℓ symbols coincide with those of that block. The inequalities (2.11) or (2.12) can therefore give bounds for that second term.

We look back at the three possibilities for how $x^{(j)}$ was defined:

(i) In this case we have

$$(m-1)\left|\tilde{g}^{p^-}(u^-)\right| \leq \alpha \leq \tilde{g}^{(jn)'+n}(x^{(j-1)}) \leq \alpha + V(\tilde{g}) \leq m\left|\tilde{g}^{p^-}(u^-)\right| + V(\tilde{g}),$$

and so

$$(m-1)\left|\tilde{g}^{p^-}(u^-)\right| - V(\tilde{g}) \leq \tilde{g}^{(jn)'+n}(x^{(j)}) \leq m\left|\tilde{g}^{p^-}(u^-)\right| + 2V(\tilde{g}).$$

Adding this to (2.12) gives

$$-\left|\tilde{g}^{p^-}(u^-)\right| - 2V(\tilde{g}) - 2NM_{\tilde{g}} \leq \tilde{g}^{((j+1)n)'}(x^{(j)}) \leq 3V(\tilde{g}) + 2NM_{\tilde{g}}.$$

(ii) Similarly in this case we get

$$-3V(\tilde{g}) - 2NM_{\tilde{g}} \leq \tilde{g}^{((j+1)n)'}(x^{(j)}) \leq \tilde{g}^{p^+}(u^+) + 2V(\tilde{g}) + 2NM_{\tilde{g}}.$$

(iii) In this case we have

$$-V(\tilde{g}) \leq \alpha \leq \tilde{g}^{(jn)'+n}(x^{(j-1)}) \leq \beta \leq V(\tilde{g}).$$

No block was inserted behind $x^{(j-1)}_{(jn)'+n-1}$ so we simply have $(jn)'+n = ((j+1)n)'$ and

$$-2V(\tilde{g}) \leq \tilde{g}^{((j+1)n)'}(x^{(j)}) \leq 2V(\tilde{g}).$$

Thus we always have

$$\left|\tilde{g}^{((j+1)n)'}(x^{(j)})\right| \leq 3V(\tilde{g}) + 2NM_{\tilde{g}} + \left|\tilde{g}^{p^-}(u^-)\right| + \tilde{g}^{p^+}(u^+) = K_1,$$

and the constant K_1 is independent of n.

Next we show that the lengths of the inserted blocks are bounded. Again we look at the block which was inserted behind $x^{(j-1)}_{(jn)'+n-1}$, i.e. the block B_{jn+n-1} in our block-adding process ξ. If this block was defined by (i) then it is $B^-(s,t,m)$ where

$$(m-1)\left|\tilde{g}^{p^-}(u^-)\right| \leq \alpha \leq \tilde{g}^{(jn)'+n}(x^{(j-1)}).$$

Whereas if the block was defined by (ii) then we have

$$(m-1)\tilde{g}^{p^+}(u^+) \leq -\beta \leq \left|\tilde{g}^{(jn)'+n}(x^{(j-1)})\right|.$$

Thus in either case we have

$$m \leq c_1\left|\tilde{g}^{(jn)'+n}(x^{(j-1)})\right| + c_2,$$

where

$$c_1 = \left(\min\left\{\left|\tilde{g}^{p^-}(u^-)\right|, \tilde{g}^{p^+}(u^+)\right\}\right)^{-1} \leq 1, \quad c_2 = c_1 \max\left\{\left|\tilde{g}^{p^-}(u^-)\right|, \tilde{g}^{p^+}(u^+)\right\}.$$

The length ℓ_{jn+n-1} of the block is either $mp^+ + 2N$ or $mp^- + 2N$, so we can find constants c_3 and c_4 (independent of n) such that

$$\ell_{jn+n-1} \leq c_3\left|\tilde{g}^{(jn)'+n}(x^{(j-1)})\right| + c_4. \tag{2.13}$$

But we know that $\left|\tilde{g}^{(jn)'}(x^{(j-1)})\right| \leq K_1$, and so $\left|\tilde{g}^{(jn)'+n}(x^{(j-1)})\right| \leq K_1 + nM_{\tilde{g}}$. Hence we have $\ell_{jn+n-1} \leq \ell_{max}(n)$ for all j, where $\ell_{max}(n) = O(n)$. We can write

$$\ell_{max}(n) \leq c_5 n. \tag{2.14}$$

Now the construction of ξ ensures that for any $x \in G(n)$ and any $j \geq 0$ we have $[\xi(x)]_i = x_i^{(j-1)}$ for all $i < (jn)'$. (In fact for all $i < (jn)' + n$.) Hence

$$\left|\tilde{g}^{(jn)'}(\xi(x))\right| \leq K_1 + V(\tilde{g}).$$

And $((j+1)n)' - (jn)' \le n + \ell_{max}(n)$, so for any $k \ge 1$ we have

$$\left|\tilde{g}^k\left(\xi(x)\right)\right| \le K_1 + V(\tilde{g}) + (n + \ell_{max}(n))\, M_{\tilde{g}}.$$

So the set $G'(n) := \xi(G(n))$ consists of points with bounded partial sums; and so

$$\chi(G'(n)) \subseteq \mathrm{Bdd}_J(g).$$

In order to compare the dimensions of $\chi(G'(n))$ and $\chi(G(n))$ we need to look at the restriction of ξ to the subset $S_X(n, t_0) \subseteq G(n)$ defined by

$$S_X(n, t_0) = \left\{ x \in X_A^+ : \frac{\#\{0 \le i < t : x \in E_i(n)\}}{t} < \epsilon(n) \text{ for all } t \ge t_0 \right\}.$$

Let us write $S'_X(n, t_0) = \xi\left(S_X(n, t_0)\right)$, $S_J(n, t_0) = \chi\left(S_X(n, t_0)\right)$ and $S'_J(n, t_0) = \chi\left(S'_X(n, t_0)\right)$. Note that the function $\xi : S_X(n, t_0) \to S'_X(n, t_0)$ is still a valid block-adding process that is defined on cylinders.

We have $\bigcup_{t_0 \ge 1} S_X(n, t_0) = G(n)$ and so $\dim_H S_J(n, t_0) \to \dim_H \chi\left(G(n)\right)$ as $t_0 \to \infty$.

Write $D = \dim_H S'_J(n, t_0)$ and let $\eta > 0$ be arbitrary. Then, by the definition of Hausdorff dimension, for all sufficiently small ρ we can find a cover of $S'_J(n, t_0)$ by a finite or countable collection of balls B_i $(i \ge 1)$ with radii $r_i < \rho$ such that

$$\sum_i r_i^{D+\eta} \le 1.$$

For each i we can construct a Moran cover $\mathfrak{U}_{r_i}$ of J with size r_i. Each set in this cover is of the form $\chi(Q)$ where Q is a cylinder in X_A^+. Let $\chi(Q_i^k)$ $(1 \le k \le m(i))$ be the sets of this cover for which $Q_i^k \cap \chi^{-1}(B_i) \cap S'_X(n, t_0) \ne \varnothing$.

By the basic properties of Moran covers we know that $m(i) \le M_{\mathrm{Moran}}$ and $\overline{r}(Q_i^k) \le c_{\mathrm{Moran}}.r_i$, for constants M_{Moran}, c_{Moran} which are independent of i. Thus

$$\sum_{i,k} \left(\overline{r}(Q_i^k)\right)^{D+\eta} \le M_{\mathrm{Moran}}.(c_{\mathrm{Moran}})^{D+\eta} =: K(\eta). \tag{2.15}$$

Suppose $y \in S'_X(n, t_0)$. Then $\chi(y) \in B_i$ for some i. The sets $\{Q : \chi(Q) \in \mathfrak{U}_{r_i}\}$ are a disjoint cover of X_A^+, so there is a (unique) Q with $y \in Q$. And since

$y \in \chi^{-1}(B_i) \cap S'_X(n, t_0)$ we must have $Q = Q_i^k$ for some k. So the sets Q_i^k cover $S'_X(n, t_0)$.

Now for each cylinder Q_i^k, we can 'remove blocks' from Q_i^k to produce the cylinder $\widehat{Q}_i^k$ as in Proposition 2.7. If $\xi(x) \in Q_i^k$ then $x \in \widehat{Q}_i^k$, so the sets $\widehat{Q}_i^k$ cover $S_X(n, t_0)$, and it follows that the sets $\chi(\widehat{Q}_i^k)$ are a cover of $S_J(n, t_0)$. We now want to use Lemma 2.16 to compare the diameters of $\chi(\widehat{Q}_i^k)$ and $\chi(Q_i^k)$.

Say that the lengths of the cylinders $\widehat{Q}_i^k$ and Q_i^k are $\widehat{m}_i^k$ and m_i^k respectively. Then the number of blocks removed from Q_i^k to produce $\widehat{Q}_i^k$ is at most $\widehat{m}_i^k/n$, and each block has length at most $\ell_{max}(n)$. So

$$m_i^k - \widehat{m}_i^k \leq \frac{\widehat{m}_i^k \ell_{max}(n)}{n},$$

and so

$$\widehat{m}_i^k \geq \left(\frac{n}{\ell_{max}(n) + n}\right) m_i^k. \tag{2.16}$$

Now $\overline{r}(Q_i^k) \leq c_{\mathrm{Moran}} r_i$, so $\sup_{i,k} \overline{r}(Q_i^k) \to 0$ as $\rho \to 0$. By the first part of Lemma 2.16 this is equivalent to saying that $\inf_{i,k} m_i^k \to \infty$ as $\rho \to 0$. The inequality (2.16) therefore tells us that $\inf_{i,k} \widehat{m}_i^k \to \infty$ as $\rho \to 0$. From this we can deduce two things. Firstly, by using Lemma 2.16 again we see that $\sup_{i,k} \operatorname{diam} \chi(\widehat{Q}_i^k) \to 0$ as $\rho \to 0$. And secondly, by taking ρ sufficiently small we can ensure that $\widehat{m}_i^k \geq n t_0$ for all i, k. This last observation allows us to make use of the 'good' behaviour of sums for points in $S_X(n, t_0)$.

Consider a point x such that $\xi(x) \in Q_i^k \cap S'_X(n, t_0)$. Then $x \in \widehat{Q}_i^k \cap S_X(n, t_0)$. We are interested in the blocks $B_{jn+n-1}(x)$ for $0 \leq j < t$, where t is the largest integer such that $nt \leq \widehat{m}_i^k$. These are the blocks that are removed from the cylinder Q_i^k to produce $\widehat{Q}_i^k$ (except that if $\widehat{m}_i^k$ is exactly equal to nt, it is possible that only a truncated version of B_{tn+n-1} appears in Q_i^k). For each j we want to find a bound for the length ℓ_{jn+n-1} of the block. This bound will depend on whether x is an element of the set $E_j(n)$. If ρ has been taken to be sufficiently small that $m_i^k \geq nt_0$, then from the definition of $S_X(n, t_0)$ we have

$$\#\{0 \leq j < t : x \in E_j(n)\} < t\epsilon(n).$$

Now,

- If $x \in E_j$ then we just have the bound from (2.14),

$$\ell_{jn+n-1} \leq \ell_{max}(n) \leq c_5 n.$$

  However there are at most $t\epsilon(n)$ values of j for which $x \in E_j$. Thus the total length of all these blocks is at most $c_5 nt\epsilon(n)$.

- If $x \notin E_j$ then we look at $x^{(j-1)}$. We have the bound

$$\left|\tilde{g}^{(jn)'}\left(x^{(j-1)}\right)\right| \leq K_1.$$

  The section $\boxed{x_{jn}\, x_{jn+1}\, \cdots\, x_{jn+n-1}}$ is shifted to positions $(jn)'$ to $(jn)'+n-1$ in $x^{(j-1)}$, and because $x \notin E_j$ we have $|\tilde{g}^n(\sigma^{jn}x)| \leq n^{\frac{3}{4}}$. So

$$\left|\tilde{g}^n\left(\sigma^{(jn)'}x^{(j-1)}\right)\right| \leq n^{\frac{3}{4}} + V(\tilde{g}).$$

  Hence

$$\left|\tilde{g}^{(jn)'+n}\left(x^{(j-1)}\right)\right| \leq n^{\frac{3}{4}} + V(\tilde{g}) + K_1.$$

  So from (2.13) we have $\ell_{jn+n-1} \leq c_3\left(n^{\frac{3}{4}} + V(\tilde{g}) + K_1\right) + c_4$, which we can rewrite as

$$\ell_{jn+n-1} \leq c_3 n^{\frac{3}{4}} + c_6,$$

  where c_3, c_6 are constants independent of n. The total length of these blocks is therefore at most $t\left(c_3 n^{\frac{3}{4}} + c_6\right)$.

Adding these together we see that

$$m_i^k - \widehat{m}_i^k \leq c_5 nt\epsilon(n) + t\left(c_3 n^{\frac{3}{4}} + c_6\right) \leq \widehat{m}_i^k\left(c_5\epsilon(n) + c_3 n^{-\frac{1}{4}} + c_6 n^{-1}\right).$$

So we can write

$$\frac{m_i^k - \widehat{m}_i^k}{\widehat{m}_i^k} \leq q(n),$$

where $q(n) := c_5\epsilon(n) + c_3 n^{-\frac{1}{4}} + c_6 n^{-1}$. We see that $q(n) \to 0$ as $n \to \infty$.

We can now apply Lemma 2.17. Provided that n is sufficiently large that $q(n) < 1$, we get

$$\overline{r}\big(Q_i^k\big) \geq c\left(\overline{r}\big(\widehat{Q}_i^k\big)\right)^{1+\gamma q(n)}.$$

Combining this with (2.15) gives (for all sufficiently small ρ)

$$\sum_{i,k}\left(\overline{r}\big(\widehat{Q}_i^k\big)\right)^{(1+\gamma q(n))(D+\eta)} \leq c^{-(D+\eta)}K(\eta),$$

and so

$$\sum_{i,k}\left(\operatorname{diam}\chi\big(\widehat{Q}_i^k\big)\right)^{(1+\gamma q(n))(D+\eta)} \leq 2^{(1+\gamma q(n))(D+\eta)}c^{-(D+\eta)}K(\eta).$$

We showed earlier that the sets $\chi\big(\widehat{Q}_i^k\big)$ cover $S_J(n,t_0)$, and $\sup_{i,k}\operatorname{diam}\chi\big(\widehat{Q}_i^k\big) \to 0$ as $\rho \to 0$. So this shows

$$\dim_H S_J(n,t_0) \leq (1+\gamma q(n))\left(\dim_H S'_J(n,t_0) + \eta\right).$$

η was arbitrary, and so

$$\dim_H S_J(n,t_0) \leq (1+\gamma q(n))\dim_H S'_J(n,t_0).$$

But $S'_J(n,t_0) \subseteq \chi(G'(n)) \subseteq \operatorname{Bdd}_J(g)$ for all t_0, and so taking $t_0 \to \infty$ we have

$$\begin{aligned}\dim_H \operatorname{Bdd}_J(g) &\geq (1+\gamma q(n))^{-1}\dim_H \chi(G(n)) \\ &\geq (1+\gamma q(n))^{-1}\dim_H \nu.\end{aligned} \tag{2.17}$$

Finally, taking $n \to \infty$ gives

$$\dim_H \operatorname{Bdd}_J(g) \geq \dim_H \nu.$$

□

As explained earlier, by using a result such as Theorem 2.15 we can show that this is the 'best possible' result, in the sense that there is some measure ν for which we have equality. However even without any information about measures, we can get an upper bound for $\dim_H \mathrm{Bdd}_J(g)$ in terms of the 'good' sets $G(n)$. We go back to the inequality (2.17). Given any integer $m \geq 1$, for any sufficiently large $k \geq m$ we have

$$\begin{aligned}\dim_H \mathrm{Bdd}_J(g) &\geq (1+\gamma q(k))^{-1} \dim_H \chi(G(k)) \\ &\geq (1+\gamma q(k))^{-1} \dim_H \left(\bigcap_{n\geq m} \chi(G(n)) \right).\end{aligned}$$

Taking $k \to \infty$ we get

$$\dim_H \mathrm{Bdd}_J(g) \geq \dim_H \left(\bigcap_{n\geq m} \chi(G(n)) \right).$$

Now if we define

$$G = \bigcup_{m\geq 1} \bigcap_{n\geq m} G(n)$$

then $\dim_H \chi(G) = \sup_{m\geq 1} \dim_H \left(\bigcap_{n\geq m} \chi(G(n))\right)$ and so

$$\dim_H \mathrm{Bdd}_J(g) \geq \dim_H \chi(G).$$

But suppose $x \in \chi^{-1}(\mathrm{Bdd}_J(g))$; say $|\tilde{g}^j(x)| \leq K$ for all j. Then $|\tilde{g}^n(\sigma^j x)| \leq 2K$ for all n, j, and so $x \notin E_i(n)$ whenever n is sufficiently large that $2K + V(\tilde{g}) \leq n^{\frac{3}{4}}$. So $x \in G$. Thus $\dim_H \mathrm{Bdd}_J(g) \subseteq \dim_H \chi(G)$, and so we must have

$$\dim_H \mathrm{Bdd}_J(g) = \dim_H \chi(G).$$

This equality was already clear from the application of Theorem 2.15, but we now have a direct proof; this idea will be useful later in situations where Theorem 2.15 is not available without some modification. Also, notice that the fact that $\mu(G(n)) = 1$ immediately implies $\mu(G) = 1$, and this can be used as an alternative way to complete the proof that $\dim_H \mathrm{Bdd}_J(g) \geq \dim_H \nu$.

2.3.3 Multi-dimensional results

In [FS], Fan and Schmeling prove multi-dimensional versions of their Theorems 2.3 and 2.5. For a vector-valued Hölder continuous function $\mathbf{g} : X_A^+ \to \mathbb{R}^d$, they show

$$\dim_H \left\{ x \in X_A^+ : \mathbf{g}^n(x) \text{ is bounded} \right\} \geq \dim_H \mu,$$

for any equilibrium state μ such that $\int \mathbf{g}\, d\mu = \mathbf{0}$. As in the one-dimensional case, if there is at least one such μ, then there is guaranteed to be an equilibrium state for which equality holds.

This last statement follows from a generalisation of Theorem 2.2: say we define

$$\mathcal{D}(\mathbf{g}) = \left\{ \int_J \mathbf{g}\, d\mu : \mu \in \mathcal{M}(J) \right\}.$$

This is a compact, convex subset of $\mathbb{R}^d$. We say that $\mathbb{R}$-valued functions $g_1, \ldots, g_n$ are *cohomologously independent* if there is no non-trivial linear combination which is cohomologous to a a constant.

Theorem 2.19 ([BSS]) *Suppose* $\mathbf{g} : J \to \mathbb{R}^d$ *is Hölder continuous. Then:*

1. *If the components of* $\mathbf{g}$ *are cohomologously independent then the set* $\mathcal{D}(\mathbf{g})$ *is the closure of its interior.*
2. *If* $\boldsymbol{\alpha} \notin \mathcal{D}(\mathbf{g})$ *then* $\mathrm{Ave}_J(\mathbf{g}, \boldsymbol{\alpha}) = \varnothing$.
3. *If* $\boldsymbol{\alpha} \in \operatorname{int} \mathcal{D}(\mathbf{g})$ *then* $\mathrm{Ave}_J(\mathbf{g}, \boldsymbol{\alpha}) \neq \varnothing$, *and*

$$\dim_H \mathrm{Ave}_J(\mathbf{g}, \boldsymbol{\alpha}) = \sup \left\{ \dim_H \mu : \mu \in \mathcal{M}(J) \textit{ and } \int \mathbf{g}\, d\mu = \boldsymbol{\alpha} \right\}.$$

Furthermore the supremum is attained for a measure μ *which is an equilibrium state for some Hölder continuous function.*

If the components of $\mathbf{g}$ are cohomologously independent then $\int \mathbf{g}\, d\mu \in \operatorname{int} \mathcal{D}(\mathbf{g})$ whenever μ is an equilibrium state. (If $\int \mathbf{g}\, d\mu$ was on the boundary of $\mathcal{D}(\mathbf{g})$, then because $\mathcal{D}(\mathbf{g})$ is convex we could find a non-zero $\mathbf{v} \in \mathbb{R}^d$ such that $\left\langle \int \mathbf{g}\, d\nu, \mathbf{v} \right\rangle \geq \left\langle \int \mathbf{g}\, d\mu, \mathbf{v} \right\rangle$ for any other $\nu \in \mathcal{M}(J)$. This contradicts the Central Limit Theorem for the function $\langle \mathbf{g}, \mathbf{v} \rangle$.)

Note that if the components of $\mathbf{g}$ are not cohomologously independent then $\mathcal{D}(\mathbf{g})$ is contained in some proper subspace of $\mathbb{R}^d$.

We will need a multi-dimensional version of the Central Limit Theorem. For the moment we work with a subshift of finite type. The following is proved in [FS]:

Theorem 2.20 ([FS]) *Let $\sigma : X_A^+ \to X_A^+$ be a subshift of finite type, and $\mathbf{g} : X_A^+ \to \mathbb{R}^d$ a Hölder continuous function on X_A^+ whose components are cohomologously independent. Suppose that μ is an equilibrium state for some ($\mathbb{R}$-valued) Hölder continuous function ψ on X_A^+, with $\int \mathbf{g}\, d\mu = \mathbf{0}$. Then $\mathbf{g}^n/\sqrt{n}$ tends in distribution to a normal distribution (which is fully-supported on $\mathbb{R}^d$).*

In what follows we use the usual Euclidean norm on $\mathbb{R}^d$. We define $M_{\mathbf{g}} = \sup_{x \in X_A^+} \|\mathbf{g}(x)\|$ and we have the constant $V(\mathbf{g})$ such that $\|\mathbf{g}^n(x) - \mathbf{g}^n(y)\| \leq V(\mathbf{g})$ whenever $x_i = y_i$ for all $i < n$.

Lemma 2.21 *Suppose $\mathbf{g} : X_A^+ \to \mathbb{R}^d$ is Hölder continuous, and the components of $\mathbf{g}$ are cohomologously independent. Also suppose that there exists an equilibrium state μ such that $\int \mathbf{g}\, d\mu = \mathbf{0}$. Then there are constants $T > 0$ and $c > 0$ such that, for any $\mathbf{v} \in \mathbb{R}^d$, there exists a point $x \in X_A^+$ and integer $n \leq c\|\mathbf{v}\|$ such that*

$$\|\mathbf{v} + \mathbf{g}^n(x)\| \leq T.$$

Proof: Let $D_0 = 10\,(V(\mathbf{g}) + M_{\mathbf{g}} + 1)$. We can cover the set $\{\mathbf{v} \in \mathbb{R}^d : \|\mathbf{v}\| \leq D_0\}$ by finitely many balls of radius 1: say these balls have centres $\mathbf{w}^{(i)}$ $(1 \leq i \leq P)$. Now by the Central Limit Theorem, there is a number n_0 such that

$$\mu\left(x \in X_A^+ : \frac{\mathbf{g}^{n_0}(x)}{\sqrt{n_0}} \in B(\mathbf{w}^{(i)}, 1)\right) > 0 \;\forall i.$$

And so for each $i \leq P$ we can find a point $z^{(i)}$ with $\mathbf{g}^{n_0}(z^{(i)}) \in B(\sqrt{n_0}\mathbf{w}^{(i)}, \sqrt{n_0})$. And we can define a periodic point $y^{(i)}$ by the sequence of symbols

$$z_0^{(i)}\, z_1^{(i)}\, \ldots\, z_{n_0-1}^{(i)}\, W(z_{n_0-1}^{(i)}, z_0^{(i)})\, z_0^{(i)}\, z_1^{(i)}\, \ldots\, z_{n_0-1}^{(i)}\, W(z_{n_0-1}^{(i)}, z_0^{(i)})\, \ldots.$$

Let $n_1 = n_0 + N$ and $T_0 = 2(\sqrt{n_0} + V(\mathrm{g}) + NM_{\mathrm{g}})$. Then the period of $y^{(i)}$ divides n_1, and we have

$$\mathrm{g}^{n_1}(y^{(i)}) \in B(\sqrt{n_0}\mathbf{w}^{(i)}, \tfrac{1}{2}T_0).$$

So for any $\mathbf{v} \in \mathbb{R}^d$ with $\|\mathbf{v}\| \leq \sqrt{n_0}D_0$, we can find $y^{(i)}$ such that

$$\left\|\mathbf{v} + \mathrm{g}^{n_1}(y^{(i)})\right\| \leq T_0.$$

Claim: Given $\mathbf{v} \in \mathbb{R}^d$, $x \in X_A^+$ and integers $m \geq 0$, $k \geq 1$ such that

$$\|\mathbf{v} + \mathrm{g}^m(x)\| \leq 2^k\sqrt{n_0}D_0,$$

then there exists $x' \in X_A^+$ such that

$$\|\mathbf{v} + \mathrm{g}^{m+2^k n_1+N}(x')\| \leq 2^{k-1}\sqrt{n_0}D_0.$$

Proof of Claim: We can find $y^{(i)}$ as defined above with

$$\left\|\frac{1}{2^k}\left(\mathbf{v} + \mathrm{g}^m(x)\right) + \mathrm{g}^{n_1}(y^{(i)})\right\| \leq T_0.$$

Using the periodicity of $y^{(i)}$ this implies

$$\left\|\mathbf{v} + \mathrm{g}^m(x) + \mathrm{g}^{2^k n_1}(y^{(i)})\right\| \leq 2^k T_0.$$

Now let x' be the point defined by the sequence of symbols

$$x_0\, x_1\, \ldots\, x_{m-1}\, W(x_{m-1}, y_0^{(i)})\, y_0^{(i)}\, y_1^{(i)}\, y_2^{(i)}\, \ldots.$$

Then $\|\mathrm{g}^m(x') - \mathrm{g}^m(x)\| \leq V(\mathrm{g})$, $\left\|\mathrm{g}^N(\sigma^m x')\right\| \leq NM_{\mathrm{g}}$ and $\mathrm{g}^{2^k n_1}(\sigma^{m+N}x') = \mathrm{g}^{2^k n_1}(y^{(i)})$, so

$$\begin{aligned}
\left\|\mathbf{v} + \mathrm{g}^{m+N+2^k n_1}(y')\right\| &\leq 2^k T_0 + V(\mathrm{g}) + NM_{\mathrm{g}} \\
&\leq 2^{k-1}\left(4\sqrt{n_0} + 5V(\mathrm{g}) + 5NM_{\mathrm{g}}\right) \\
&\leq 2^{k-1}\sqrt{n_0}D_0.
\end{aligned}$$

This completes the proof of the Claim.

Now suppose $\mathbf{v} \in \mathbb{R}^d$ with $2^{k-1}\sqrt{n_0}D_0 \leq \|\mathbf{v}\| \leq 2^k\sqrt{n_0}D_0$ for some $k \geq 1$. By applying the above Claim repeatedly we find a point $x \in X_A^+$ and integer n such that

$$\|\mathbf{v} + \mathbf{g}^n(x)\| \leq \sqrt{n_0}D_0,$$

where

$$n = \sum_{i=1}^{k} \left(2^i n_1 + N\right) \leq 2^{k+1} n_1 + Nk \leq 2^{k-1}\left(4n_1 + N\right) \leq \left(4n_1 + N\right)\|\mathbf{v}\|.$$

Thus the statement of the lemma holds by taking $T = \sqrt{n_0}D_0$ and $c = 4n_1 + N$. (We have proved this in the case $\|\mathbf{v}\| \geq \sqrt{n_0}D_0$, but for $\|\mathbf{v}\| \leq \sqrt{n_0}D_0$ we may simply take $n = 0$.) $\square$

Theorem 2.22 *Let $\mathbf{g} : J \to \mathbb{R}^d$ be Hölder continuous and let ν be an equilibrium state with respect to some Hölder continuous $\mathbb{R}$-valued function on J. Suppose that $\int_J \mathbf{g}\, d\nu = \mathbf{0}$. Then*

$$\dim_H \mathrm{Bdd}_J(\mathbf{g}) \geq \dim_H \nu.$$

Proof: First suppose that the components of $\mathbf{g}$ are not cohomologously independent. Say without loss of generality that g_1 is cohomologous to the function $\sum_{i=2}^d \lambda_i g_i + c$. Since $\int \mathbf{g}\, d\nu = 0$ we see that $c = 0$. So by the basic properties of the cohomology condition we have that $g_1^n(x) - \sum_{i=2}^d \lambda_i g_i^n(x)$ is bounded. So if we define the function $\mathbf{g}' : J \to \mathbb{R}^{d-1}$ by $\mathbf{g}' = (g_2, g_3 \ldots, g_d)$, then

$$\mathrm{Bdd}_J(\mathbf{g}') = \mathrm{Bdd}_J(\mathbf{g}),$$

and so it is sufficient to prove the theorem for $\mathbf{g}'$. We can repeat this if necessary, throwing out components of $\mathbf{g}$ until the remaining components are cohomologously independent.

Thus we may assume that the components of $\mathbf{g}$ are cohomologously independent.

We now follow the proof of Theorem 2.18. Let $\tilde{\mathbf{g}}$ be the pullback of $\mathbf{g}$ to X_A^+, and let μ be the equilibrium state on X_A^+ which corresponds to the equilibrium state ν on J. Since we assume that the components of $\mathbf{g}$ are cohomologously independent, the same must be true of $\tilde{\mathbf{g}}$. Also we have $\int_{X_A^+} \tilde{\mathbf{g}}\, d\mu = 0$.

We define

$$\epsilon(n) = \mu\left(\left\{x \in X_A^+ : \|\tilde{\mathbf{g}}^n(x)\| > n^{\frac{3}{4}} - V(\tilde{\mathbf{g}})\right\}\right) + n^{-1};$$
$$\mathcal{C}(n) = \left\{C \in \mathrm{Cyl}(n) : \|\tilde{\mathbf{g}}^n(y)\| > n^{\frac{3}{4}} \text{ for some } y \in C\right\},$$

and the corresponding sets $E_i(n), G(n) \subseteq X_A^+$ as in Theorem 2.18.

We need a different method of finding the blocks to be used in the construction. In fact we shall look at *all* blocks of the form

$$W(s, u_0^{(i)})\, u_0^{(i)}\, u_1^{(i)} \ldots u_{m^{(i)}-1}^{(i)}\, W(u_{m^{(i)}-1}^{(i)}, t) \quad =: B(s,t,i),$$

where $u_0^{(i)}\, u_1^{(i)} \ldots u_{m^{(i)}-1}^{(i)}$ is any admissible sequence of symbols. Note that there are only countably many such blocks, so they can be indexed by $i \in \mathbb{N}$. We choose the labelling in such a way that if $i < j$ then $m^{(i)} \leq m^{(j)}$, i.e. the lengths of the blocks are non-decreasing. If $C^{(i)}$ is the cylinder $[u_0^{(i)}\, u_1^{(i)} \ldots u_{m^{(i)}-1}^{(i)}] \subseteq X_A^+$, we define

$$I^{(i)} = \frac{\int_{C^{(i)}} \mathbf{g}^{m^{(i)}}\, d\mu}{\mu\left(C^{(i)}\right)}.$$

So for any $y \in C^{(i)}$ we have

$$\left\|\mathbf{g}^{m^{(i)}}(y) - I^{(i)}\right\| \leq V(\tilde{\mathbf{g}}). \tag{2.18}$$

It follows from Lemma 2.21 that for any $\mathbf{v} \in \mathbb{R}^d$ we can find $i \in \mathbb{N}$ with $m^{(i)} \leq c\|\mathbf{v}\|$ such that

$$\left\|\mathbf{v} + I^{(i)}\right\| \leq T + V(\tilde{\mathbf{g}}).$$

Our block-adding construction $\xi : G(n) \to G'(n)$ will be the same as in Theorem 2.18 except for the definition of which block to insert behind x_{jn+n-1}. Recall that we consider the set

$$C_{jn+n}(x; G(n)) := \{y \in G(n) : x_i = y_i \ \forall i < jn+n\},$$

and that $y^{(j-1)}$ is defined in such a way that if $y \in C_{jn+n}(x; G(n))$ then $y_i^{(j-1)} = x_i^{(j-1)}$ for all $i < (jn)' + n$. We can then find the smallest $i \in \mathbb{N}$ such that

$$\left\|\tilde{\mathbf{g}}^{(jn)'+n}(y^{(j-1)}) + I^{(i)}\right\| \leq T + V(\mathbf{g}) \text{ for some } y \in C_{jn+n}(x; G(n)).$$

Because the $m^{(i)}$ were chosen to be non-decreasing this ensures that

$$m^{(i)} \leq c\left\|\tilde{\mathbf{g}}^{(jn)'+n}(y^{(j-1)})\right\| \leq c\left(\left\|\tilde{\mathbf{g}}^{(jn)'+n}(x^{(j-1)})\right\| + V(\tilde{\mathbf{g}})\right). \tag{2.19}$$

The block that we insert behind x_{jn+n-1} will be $B(x_{jn+n-1}, x_{jn+n}, i)$ for this value of i. As in the proof of Theorem 2.18, our definition ensures that $x \mapsto x^{(j)}$ is defined on cylinders.

Now we look at

$$\begin{aligned}\tilde{\mathbf{g}}^{(jn+n)'}(x^{(j)}) = \tilde{\mathbf{g}}^{(jn)'+n}(x^{(j)}) + \tilde{\mathbf{g}}^{N}(\sigma^{(jn)'+n}x^{(j)}) \\ + \tilde{\mathbf{g}}^{m^{(i)}}(\sigma^{(jn)'+n+N}x^{(j)}) + \tilde{\mathbf{g}}^{N}(\sigma^{jn+n+N+m^{(i)}}x^{(j)}).\end{aligned}$$

We have

$$\left\|\tilde{\mathbf{g}}^{(jn)'+n}(y^{(j-1)}) + I^{(i)}\right\| \leq T + V(\tilde{\mathbf{g}})$$

for some $y \in C_{jn+n}(x; G(n))$, from the definition of i. Also

$$\left\|\tilde{\mathbf{g}}^{(jn)'+n}(x^{(j)}) - \tilde{\mathbf{g}}^{(jn)'+n}(y^{(j-1)})\right\| \leq V(\tilde{\mathbf{g}})$$

because $x_k^{(j)} = x_k^{(j-1)} = y_k^{(j-1)}$ for all $k < (jn)' + n$; and

$$\left\|\tilde{\mathbf{g}}^{m^{(i)}}(\sigma^{(jn)'+n+N}x^{(j)}) - I^{(i)}\right\| \leq V(\tilde{\mathbf{g}})$$

from the inequality (2.18). Putting these all together gives

$$\left\|\tilde{\mathbf{g}}^{(jn+n)'}(x^{(j)})\right\| \leq T + 3V(\mathbf{g}) + 2NM_{\tilde{\mathbf{g}}} =: K_1.$$

Next we look at ℓ_{jn+n+1} (the length of the block inserted behind x_{jn+n+1}). We have $\ell_{jn+n+1} = 2N + m^{(i)}$, where $m^{(i)} \leq c\left(\left\|\tilde{\mathbf{g}}^{(jn)'+n}(x^{(j-1)})\right\| + V(\tilde{\mathbf{g}})\right)$ from (2.19). Hence we have constants c_3, c_4 such that

$$\ell_{jn+n+1} \leq c_3\left\|\tilde{\mathbf{g}}^{(jn)'+n}(x^{(j-1)})\right\| + c_4. \tag{2.20}$$

(Compare this with (2.13).) As in Theorem 2.18 this implies that the lengths of the blocks are bounded by some number $\ell_{max}(n)$, and

$$\ell_{max}(n) \leq c_5 n. \tag{2.21}$$

And this in turn implies that for any $k \geq 1$,

$$\left\|\tilde{\mathbf{g}}^k(\xi(x))\right\| \leq K_1 + V(\tilde{\mathbf{g}}) + (n + \ell_{max})M_{\tilde{\mathbf{g}}},$$

proving that

$$\chi(G'(n)) \subseteq \mathrm{Bdd}_J(\mathbf{g}).$$

The rest of the proof goes through exactly as in Theorem 2.18, making use of the inequalities (2.20) and (2.21). □

2.3.4 Points with sums which grow at a specified rate

Finally for conformal repellers we wish to prove an analogue of Theorem 2.8. Again this will be done by considering a block-adding process on the related subshift of finite type. We could simply choose to use the block-adding process ξ that was defined in the proof of Theorem 2.8; however, we will give a generalisation to the multi-dimensional case. The following lemma will provide the blocks to be used in the construction:

Lemma 2.23 *Suppose $\mathbf{g} : X_A^+ \to \mathbb{R}^d$ is Hölder continuous, and the components of $\mathbf{g}$ are cohomologously independent. Also suppose that there exists an equilibrium state μ such that $\int \mathbf{g}\, d\mu = \mathbf{0}$. Then, given $H > 0$, we can find an $m \geq 1$ and finitely many points $u^{(i)} \in X_A^+$ $(1 \leq i \leq P)$ such that for any $\mathbf{v} \in \mathbb{R}^d$ with $\|\mathbf{v}\|$ sufficiently large, we have*

$$\|\mathbf{v} + \mathbf{g}^m(u^{(i)})\| \leq \|\mathbf{v}\| - H$$

for some i.

Proof: We start in a similar way to the proof of Lemma 2.21. We can cover the set $\{\mathbf{v} \in \mathbb{R}^d : \|\mathbf{v}\| \leq H+2\}$ by finitely many balls of radius 1: say these balls have centres $\mathbf{w}^{(i)}$ ($1 \leq i \leq P$). By the Central Limit Theorem, we can choose $m \geq 1$ such that

$$\mu\left(x \in X_A^+ : \frac{\mathbf{g}^m(x)}{\sqrt{m}} \in B(\mathbf{w}^{(i)}, 1)\right) > 0 \ \forall i.$$

And so for each $i \leq P$ we can find a point $u^{(i)}$ with $\mathbf{g}^m(u^{(i)}) \in B(\sqrt{m}\mathbf{w}^{(i)}, \sqrt{m})$. We will show that these points have the desired property.

Take $\mathbf{v} \in \mathbb{R}^d$ with $\|\mathbf{v}\| \geq \sqrt{m}(H+2)$, and let $\mathbf{v}' = \sqrt{m}(H+2)\frac{\mathbf{v}}{\|\mathbf{v}\|}$. Then we can find i such that $-\mathbf{v}' \in B(\sqrt{m}\mathbf{w}^{(i)}, \sqrt{m})$, and so

$$\left\|\mathbf{v}' + \mathbf{g}^m(u^{(i)})\right\| \leq 2\sqrt{m}.$$

Hence

$$\left\|\mathbf{v} + \mathbf{g}^m(u^{(i)})\right\| \leq 2\sqrt{m} + \|\mathbf{v} - \mathbf{v}'\|.$$

But $\|\mathbf{v} - \mathbf{v}'\| = \left\|\left(1 - \frac{\sqrt{m}(H+2)}{\|\mathbf{v}\|}\right)\mathbf{v}\right\| = \|\mathbf{v}\| - \sqrt{m}(H+2)$, so we have

$$\begin{aligned}\left\|\mathbf{v} + \mathbf{g}^m(u^{(i)})\right\| &\leq \|\mathbf{v}\| - H\sqrt{m} \\ &\leq \|\mathbf{v}\| - H.\end{aligned}$$

□

Following the notation of section 2.1, if $\mathbf{g} : J \to \mathbb{R}^d$ and $r : J \to \mathbb{R}^+$ are Hölder continuous, and $\mathbf{F} : \mathbb{R}^+ \to \mathbb{R}^d$ is continuous, then we define

$$L_J(\mathbf{g}, \mathbf{F}, r) = \{x \in J : \mathbf{g}^n(x) = \mathbf{F}(r^n(x)) + O(1)\}.$$

Theorem 2.24 *Let $\mathbf{g}$, $\mathbf{F}$, r be as above, and let ν be an equilibrium state for some Hölder continuous function on J, with $\int \mathbf{g}\, d\nu = 0$. If the components of $\mathbf{g}$ are cohomologously independent and $\sup_{\tau\in[0,1]} \|\mathbf{F}(t+\tau) - \mathbf{F}(t)\| \to 0$ as $t \to \infty$, then*

$$\dim_H L_J(\mathbf{g}, \mathbf{F}, r) \geq \dim_H \nu.$$

Proof: Again we use the coding map $\chi : X_A^+ \to J$ to produce the pullback function $\tilde{\mathbf{g}} : X_A^+ \to \mathbb{R}^d$ (whose components are cohomologously independent) and the equilibrium state μ on X_A^+ for which $\int_{X_A^+} \tilde{\mathbf{g}}\, d\mu = 0$. We also define $\tilde{r} : X_A^+ \to \mathbb{R}^+$ to be the pullback of r.

We look at the set

$$S_K := \left\{ x \in X_A^+ : \|\tilde{\mathbf{g}}^n(x)\| < K \;\forall n \right\}.$$

As in the proof of Theorem 2.8, we will construct a block-adding process $\xi : S_K \to S'_K$.

We apply Lemma 2.23 to the function $\tilde{\mathbf{g}}$, taking $H = 20(V(\tilde{\mathbf{g}}) + NM_{\tilde{\mathbf{g}}} + K + 1)$. This gives us points $u^{(i)} \in X_A^+$ $(1 \le i \le P)$ and an integer $m \ge 1$ such that whenever $\|\mathbf{v}\|$ is sufficiently large there exists i such that $\left\|\mathbf{v} + \tilde{\mathbf{g}}^m(u^{(i)})\right\| \le \|\mathbf{v}\| - H$. We can then define the blocks $B(s,t,i)$ $(1 \le i \le P)$ by

$$B(s,t,i) = W(s, u_0^{(i)})\, u_0^{(i)}\, u_1^{(i)}\, \ldots\, u_{m-1}^{(i)}\, W(u_{m-1}^{(i)}, t).$$

These blocks all have the same length $\ell = m + 2N$.

As in the proof of Theorem 2.8, we construct for each $x \in X_A^+$ a sequence of points $\left(x^{(j)}\right)_{j \ge -1}$, starting with $x^{(-1)} = x$, in such a way that each function $x \mapsto x^{(j)}$ is a block-adding process that is defined on cylinders.

We find $N_{\mathbf{F}}$ such that whenever $t \ge N_{\mathbf{F}} \tilde{r}_{min}$,

$$\|\mathbf{F}(t+\tau) - \mathbf{F}(t)\| < 1 \quad \forall 0 \le \tau \le \max\left\{V(\tilde{r}), (\ell+1)\tilde{r}_{max}\right\}.$$

The constant K_0 is then chosen to be sufficiently large that for all $n \le N_{\mathbf{F}} + \ell$ and all $z \in X_A^+$ we have

$$\|\tilde{\mathbf{g}}^n(z) - \mathbf{F}\left(\tilde{r}^n(z)\right)\| < K_0.$$

As before, given $j \ge 0$ and $x \in S_K$ we define j' to be the position that the symbol x_j is shifted to in $x_j^{(j-1)}$. If $j' \ge N_{\mathbf{F}}$ then for $y, z \in C_{j'+1}(x; S_K)$ we have the inequality

$$\left\|\left(\tilde{\mathbf{g}}^{j'+1}(y^{(j-1)}) - \mathbf{F}\left(\tilde{r}^{j'+1}(y^{(j-1)})\right)\right) - \left(\tilde{\mathbf{g}}^{j'+1}(z^{(j-1)}) - \mathbf{F}\left(\tilde{r}^{j'+1}(z^{(j-1)})\right)\right)\right\| \le V(\tilde{\mathbf{g}}) + 1 \quad (2.22)$$

(derived in the same way as inequality (2.1) from Theorem 2.8). The point $x^{(j)}$ is defined in terms of $\left\{y^{(j-1)} : y \in C_{j+1}(x; S_K)\right\}$ as follows:

- If $j' < N_{\mathbf{F}}$, or if $\left\|\tilde{\mathbf{g}}^{j'+1}(y^{(j-1)}) - \mathbf{F}\left(\tilde{r}^{(j'+1)}(y^{(j-1)})\right)\right\| \le K_0$ for all $y \in C_{j+1}(x; S_K)$, then we take $x^{(j)} = x^{(j-1)}$.
- If $j' \ge N_{\mathbf{F}}$ and $\left\|\tilde{\mathbf{g}}^{j'+1}(y^{(j-1)}) - \mathbf{F}\left(\tilde{r}^{(j'+1)}(y^{(j-1)})\right)\right\| > K_0$ for some $y \in C_{j+1}(x; S_K)$, we find the smallest i such that there exists $z \in C_{j+1}(x; S_K)$ with

  $$\left\|\tilde{\mathbf{g}}^{j'+1}(z^{(j-1)}) - \mathbf{F}\left(\tilde{r}^{(j'+1)}(z^{(j-1)})\right) + \tilde{\mathbf{g}}^{m}(u^{(i)})\right\| \le \left\|\tilde{\mathbf{g}}^{j'+1}(z^{(j-1)}) - \mathbf{F}\left(\tilde{r}^{(j'+1)}(z^{(j-1)})\right)\right\| - H.$$

  (Lemma 2.23 tells us that at least one such i exists, so long as K_0 was chosen sufficiently large.) We then take $B_j(x) = B(x_j, x_{j+1}, i)$; that is, $x^{(j)}$ is the same as $x^{(j-1)}$ but with this block $B(x_j, x_{j+1}, i)$ inserted behind $x_{j'}^{(j-1)}$.

If $x \mapsto x^{(j-1)}$ was defined on cylinders, then this definition ensures that $x \mapsto x^{(j)}$ also is. As usual, we take $\xi(x) = \lim_{j\to\infty} x^{(j)}$, and it follows that this block-adding process ξ is also defined on cylinders.

Continuing to follow the proof of Theorem 2.8, we define for $x \in S_K$ and $n \ge 0$,

$$\mathbf{\Delta}_n(x) = \tilde{\mathbf{g}}^n(\xi(x)) - \mathbf{F}\left(\tilde{r}^n(\xi(x))\right).$$

And for $j' \ge N_{\mathbf{F}}$ we have the inequalities

$$\left\|\mathbf{\Delta}_{j'}(x) - \left(\tilde{\mathbf{g}}^{j'}(x^{(j-1)}) - \mathbf{F}\left(\tilde{r}^{j'}(x^{(j-1)})\right)\right)\right\| \le V(\tilde{\mathbf{g}}) + 1; \quad (2.23)$$

$$\left\|\mathbf{\Delta}_{j'+1}(x) - \left(\tilde{\mathbf{g}}^{j'+1}(x^{(j-1)}) - \mathbf{F}\left(\tilde{r}^{j'+1}(x^{(j-1)})\right)\right)\right\| \le V(\tilde{\mathbf{g}}) + 1. \quad (2.24)$$

We now need a revised version of Claim 1 from the proof of Theorem 2.8:

Claim 1′: If $j' \geq N_{\mathbf{F}}$ and $B_j(x) = B(x_j, x_{j+1}, i)$ (for any $1 \leq i \leq P$), then

$$\left\|\mathbf{\Delta}_{(j+1)'}(x)\right\| \leq \left\|\mathbf{\Delta}_{j'}(x)\right\| - D,$$

where $D \geq 3V(\tilde{\mathbf{g}}) + 4 + 2K$.

Proof of Claim 1′: From the definition of the block-adding process we know there is some $z \in C_{j+1}(x; S_K)$ such that

$$\begin{aligned}\left\|\tilde{\mathbf{g}}^{j'+1}(z^{(j-1)}) - \mathbf{F}\left(\tilde{r}^{j'+1}(z^{(j-1)})\right) + \tilde{\mathbf{g}}^m(u^{(i)})\right\| \\ \leq \left\|\tilde{\mathbf{g}}^{j'+1}(z^{(j-1)}) - \mathbf{F}\left(\tilde{r}^{j'+1}(z^{(j-1)})\right)\right\| - H.\end{aligned}$$

Combining this with (2.22) gives

$$\begin{aligned}\left\|\tilde{\mathbf{g}}^{j'+1}(x^{(j-1)}) - \mathbf{F}\left(\tilde{r}^{j'+1}(x^{(j-1)})\right) + \tilde{\mathbf{g}}^m(u^{(i)})\right\| \\ \leq \left\|\tilde{\mathbf{g}}^{j'+1}(x^{(j-1)}) - \mathbf{F}\left(\tilde{r}^{j'+1}(x^{(j-1)})\right)\right\| + 2V(\tilde{\mathbf{g}}) + 2 - H. \quad (2.25)\end{aligned}$$

Now

$$\tilde{\mathbf{g}}^{(j+1)'}(x^{(j)}) = \tilde{\mathbf{g}}^{j'+1}(x^{(j)}) + \tilde{\mathbf{g}}^N(\sigma^{j'+1}x^{(j)}) + \tilde{\mathbf{g}}^m(\sigma^{j'+1+N}x^{(j)}) + \tilde{\mathbf{g}}^N(\sigma^{j'+1+N+m}x^{(j)}),$$

and $\sigma^{j'+1+N}x^{(j)} \in [u_0^{(i)}\, u_1^{(i)}\, \ldots\, u_{m-1}^{(i)}]$, so

$$\left\|\tilde{\mathbf{g}}^m(\sigma^{j'+1+N}x^{(j)}) - \tilde{\mathbf{g}}^m(u^{(i)})\right\| \leq V(\tilde{\mathbf{g}}).$$

Hence

$$\left\|\tilde{\mathbf{g}}^{(j+1)'}(x^{(j)}) - \left(\tilde{\mathbf{g}}^{j'+1}(x^{(j)}) + \tilde{\mathbf{g}}^m(u^{(i)})\right)\right\| \leq 2NM_{\tilde{\mathbf{g}}} + V(\tilde{\mathbf{g}}).$$

And because $x_i^{(j)} = x_i^{(j-1)}$ for all $i < j' + 1$, this implies

$$\left\|\tilde{\mathbf{g}}^{(j+1)'}(x^{(j)}) - \left(\tilde{\mathbf{g}}^{j'+1}(x^{(j-1)}) + \tilde{\mathbf{g}}^m(u^{(i)})\right)\right\| \leq 2NM_{\tilde{\mathbf{g}}} + 2V(\tilde{\mathbf{g}}). \quad (2.26)$$

Furthermore we have $|\tilde{r}^{(j+1)'}(x^{(j)}) - \tilde{r}^{j'+1}(x^{(j)})| \leq (\ell + 1)\tilde{r}_{max}$ and $|\tilde{r}^{j'+1}(x^{(j)}) - \tilde{r}^{j'+1}(x^{(j-1)})| \leq V(\tilde{r})$, so by the assumption that $j' \geq N_{\mathbf{F}}$ we have

$$\left\|\mathbf{F}\left(\tilde{r}^{(j+1)'}(x^{(j)})\right) - \mathbf{F}\left(\tilde{r}^{j'+1}(x^{(j-1)})\right)\right\| \leq 2. \quad (2.27)$$

Now by applying (2.26) and (2.27) to the left hand side of (2.25) we get

$$\left\|\tilde{\mathbf{g}}^{(j+1)'}(x^{(j)}) - \mathbf{F}\left(\tilde{r}^{(j+1)'}(x^{(j)})\right)\right\|$$
$$\leq \left\|\tilde{\mathbf{g}}^{j'+1}(x^{(j-1)}) - \mathbf{F}\left(\tilde{r}^{j'+1}(x^{(j-1)})\right)\right\| + 4V(\tilde{\mathbf{g}}) + 2NM_{\tilde{\mathbf{g}}} + 4 - H. \quad (2.28)$$

And so from inequalities (2.23) and (2.24) we have

$$\left\|\mathbf{\Delta}_{(j+1)'}(x)\right\| \leq \|\mathbf{\Delta}_{j'+1}(x)\| + 6V(\tilde{\mathbf{g}}) + 2NM_{\tilde{\mathbf{g}}} + 6 - H.$$

But $\|\mathbf{\Delta}_{j'+1}(x) - \mathbf{\Delta}_{j'}(x)\| \leq M_{\tilde{\mathbf{g}}} + 1$ and so

$$\left\|\mathbf{\Delta}_{(j+1)'}(x)\right\| \leq \|\mathbf{\Delta}_{j'}(x)\| + 6V(\tilde{\mathbf{g}}) + (2N+1)M_{\tilde{\mathbf{g}}} + 7 - H.$$

We can then take $D = H - 6V(\tilde{\mathbf{g}}) - (2N+1)M_{\tilde{\mathbf{g}}} - 7$, and since we chose $H = 20(V(\tilde{\mathbf{g}}) + NM_{\tilde{\mathbf{g}}} + K + 1)$ we certainly have $D \geq 3V(\tilde{\mathbf{g}}) + 4 + 2K$. This completes the proof of Claim 1$'$.

From this point onwards the calculations in the multi-dimensional case are the same as for the block-adding process for one dimension used in Theorem 2.8. In particular, following the proof of Theorem 2.8 we have

- $\|\mathbf{\Delta}_{j'}(x)\| \leq K_0 + V(\tilde{\mathbf{g}}) + 1$ for all $x \in S_K$ and all $j \geq 0$;
- If a block was inserted after x_j then $\|\mathbf{\Delta}_{j'}(x)\| \leq K_0 + V(\tilde{\mathbf{g}}) + 1 - D$.
- For any $n \geq 0$ we have $\|\mathbf{\Delta}_n(x)\| \leq K_0 + V(\tilde{\mathbf{g}}) + 1 + \ell M_{\tilde{\mathbf{g}}} + 1$.

And so $\tilde{\mathbf{g}}^n(\xi(x)) = \mathbf{F}\left(\tilde{r}^n(\xi(x))\right) + O(1)$. Transferring this back to J we have that $\chi(S'_K) \subseteq L_J(\mathbf{g}, \mathbf{F}, r)$.

Now, by using the arguments from Theorem 2.8, for any $\eta > 0$ we can find a constant C_η such that for each $x \in S_K$,

$$j' < C_\eta + j(1 + \eta\ell). \quad (2.29)$$

(As always, j' is the position that the symbol x_j is shifted to in $\xi(x)$. Thus it depends on x, but it is constant on cylinders of length $j+1$.) We want this in a slightly different form: it follows from (2.29) that there exists a constant N'_η such that whenever $j \geq N'_\eta$ we have

$$j' + \ell + 1 \leq j(1 + 2\eta\ell). \tag{2.30}$$

We wish to prove that $\dim_H \chi(S'_K) = \dim_H \chi(S_K)$, and this part of the argument is similar to the use of Moran covers in Theorem 2.18. We write $D = \dim_H S'_K$ and let $\eta > 0$ be arbitrary. Then for all sufficiently small ρ we can find a cover of S'_K by a finite or countable collection of balls B_i with radii $r_i < \rho$ such that

$$\sum_i r_i^{D+\eta} \leq 1.$$

For each i we construct a Moran cover $\mathfrak{U}_{r_i}$ of J with size r_i. We let $\chi(Q_i^k)$ $(1 \leq k \leq m(i))$ be the sets of this cover for which $Q_i^k \cap \chi^{-1}(B_i) \cap S'_K \neq \varnothing$. This implies that $\bigcup_{i,k} Q_i^k \supseteq S'_K$.

We have $m(i) \leq M_{\text{Moran}}$ and $\overline{r}(Q_i^k) \leq c_{\text{Moran}} \cdot r_i$, and so

$$\sum_{i,k} \left(\overline{r}(Q_i^k)\right)^{D+\eta} \leq M_{\text{Moran}} \left(c_{\text{Moran}}\right)^{D+\eta} =: K(\eta). \tag{2.31}$$

For each cylinder Q_i^k, we can 'remove blocks' to produce the cylinder $\widehat{Q}_i^k$. If $\xi(x) \in Q_i^k$ then $x \in \widehat{Q}_i^k$, so the sets $\widehat{Q}_i^k$ cover S_K, and hence the sets $\chi(\widehat{Q}_i^k)$ cover $\chi(S_K)$.

Say that the lengths of the cylinders $\widehat{Q}_i^k$ and Q_i^k are $\widehat{m}_i^k$ and m_i^k respectively. We know that $\overline{r}(Q_i^k) \to 0$ as $\rho \to 0$, and by the first part of Lemma 2.16 this implies $\inf_{i,k} m_i^k \to \infty$ as $\rho \to 0$. But $m_i^k \leq (\ell+1)\widehat{m}_i^k$, so this implies $\inf_{i,k} \widehat{m}_i^k \to \infty$ as $\rho \to 0$. In particular, if ρ is sufficiently small then $\widehat{m}_i^k - 1 \geq N'_\eta$ for all i, k. Also by applying Lemma 2.16 again we have that $\sup_{i,k} \operatorname{diam} \chi(\widehat{Q}_i^k) \to 0$ as $\rho \to 0$.

Since $Q_i^k \cap S'_K \neq \varnothing$ we can find a point $x \in S_K$ with $\xi(x) \in Q_i^k$. We now apply (2.30) for $j = \widehat{m}_i^k - 1$. From Proposition 2.7 we have

$$j' \leq m_k^i - 1 < (j+1)',$$

and so

$$m_k^i \leq (j+1)' \leq j' + 1 + \ell.$$

If ρ is sufficiently small that $\inf_{i,k} \widehat{m}_i^k - 1 \geq N'_\eta$ then (2.30) applies and so

$$m_k^i \leq j(1 + 2\eta\ell) < \widehat{m}_k^i(1 + 2\eta\ell).$$

Hence

$$\frac{m_i^k - \widehat{m}_k^i}{\widehat{m}_k^i} < 2\eta\ell.$$

So from Lemma 2.17,

$$\overline{r}(Q_i^k) \geq \underline{r}(Q_i^k) \geq c\left(\overline{r}(\widehat{Q}_i^k)\right)^{1+2\gamma\eta\ell}.$$

Combining this with (2.31) gives (for sufficiently small ρ)

$$\sum_{i,k} \left(\overline{r}(\widehat{Q}_i^k)\right)^{(1+2\gamma\eta\ell)(D+\eta)} \leq c^{-(D+\eta)} K(\eta),$$

and so

$$\sum_{i,k} \left(\operatorname{diam} \widehat{Q}_i^k\right)^{(1+2\gamma\eta\ell)(D+\eta)} \leq 2^{(1+2\gamma\eta\ell)(D+\eta)} c^{-(D+\eta)} K(\eta).$$

We have shown that the sets $\chi(\widehat{Q}_i^k)$ cover $\chi(S_K)$ and that $\sup_{i,k} \operatorname{diam} \chi(\widehat{Q}_i^k) \to 0$ as $\rho \to 0$. Hence

$$\dim_H \chi(S_K) \leq (1 + 2\gamma\eta\ell)(\dim_H \chi(S'_K) + \eta).$$

Taking $\eta \to 0$ gives

$$\dim_H \chi(S_K) \leq \dim_H \chi(S'_K).$$

But $\chi(S'_K) \subseteq L_J(\mathbf{g}, \mathbf{F}, r)$ so

$$\dim_H L_J(\mathbf{g}, \mathbf{F}, r) \geq \dim_H \chi(S_K).$$

And $\bigcup_{K\in\mathbb{N}} \chi(S_K) = \mathrm{Bdd}_J(\mathbf{g})$, so

$$\begin{aligned} \dim_H L_J(\mathbf{g}, \mathbf{F}, r) &\geq \dim_H \mathrm{Bdd}_J(\mathbf{g}) \\ &\geq \dim_H \nu \qquad \text{(from Theorem 2.22).} \end{aligned}$$

□

As usual, by observing that $L_J(\mathbf{g}, \mathbf{F}, r) \subseteq \mathrm{Ave}_J(\mathbf{g}, \mathbf{0})$ we can restate this result in the following way:

Theorem 2.25 *Let* $\mathbf{g}$, $\mathbf{F}$, r, ν *be as in Theorem 2.24. Then*

$$\dim_H \mathrm{Bdd}_J(\mathbf{g}) = \dim_H L_J(\mathbf{g}, \mathbf{F}, r) = \dim_H \mathrm{Ave}_J(\mathbf{g}, \mathbf{0}).$$

2.4 Hyperbolic diffeomorphisms and flows

2.4.1 Product structure

When dealing with one-sided subshifts and conformal repellers, we compared the Hausdorff dimensions of sets to the Hausdorff dimensions of equilibrium states. However, for hyperbolic diffeomorphisms and flows this is no longer fruitful. For example, suppose we have a function $g : \Lambda \to \mathbb{R}$ and an equilibrium state ν such that $\int_\Lambda g\, d\nu = \alpha$. Then we can still say that

$$\dim_H \mathrm{Ave}_\Lambda(g, \alpha) \geq \dim_H \nu.$$

(This is immediate from the definition of Hausdorff dimension of a measure and the ergodic theorem.) But it is no longer necessarily the case that this is a 'best possible' bound: there may not be a measure ν for which we have equality. For one-sided subshifts and conformal repellers such a measure ν was guaranteed by Theorems 2.2 and 2.15 respectively, but for the hyperbolic case we do not have a result that can be applied in that way. Thus while we *could* still show that $\dim_H \mathrm{Bdd}_\Lambda(g) \geq \dim_H \nu$, this is not really the appropriate bound to consider, and in particular this bound is not good enough to be able to deduce that $\dim_H \mathrm{Bdd}_\Lambda(g) = \dim_H \mathrm{Ave}_\Lambda(g, \alpha)$.

Roughly speaking, the 'problem' is that looking at $\dim_H \nu$ does not take account of the splitting into stable and unstable directions. Say we consider a hyperbolic diffeomorphism $f : \Lambda \to \Lambda$, and an equilibrium state ν on Λ. Then ν has a product structure, as described in [Pes]: that is, if we consider a rectangle R^* and a point $z^* \in \operatorname{int} R^*$, then we can find measures ν^+ on $W^u_{R^*}(z^*)$ and ν^- on $W^s_{R^*}(z^*)$ such that ν is equivalent to the product $\nu^+ \times \nu^-$ on R^*. (If μ is the equilibrium state on X_A which corresponds to ν on Λ, then ν^+ is obtained from the measure μ^+ defined in section 1.4; for the stable directions, the measures μ^- and ν^- are defined similarly.) Rather than comparing the dimension of a set to $\dim_H \nu$, we should be comparing it to a sum of the dimensions of two measures, one of the form ν^+ and one of the form ν^-. And, crucially, these measures might be derived from two different equilibrium states, ν_1 and ν_2.

An illustration of this is that, in general, there do not exist measures of maximal dimension:

Theorem 2.26 (Manning, McCluskey [MM]) *There exist Axiom A diffeomorphisms f, with corresponding basic sets $\Omega(f)$, for which there is a strict inequality*

$$\dim_H \Omega(f) > \sup \{\dim_H \mu : \mu(\Omega(f)) = 1,\ \ \mu \text{ ergodic}\}.$$

Essentially the issue is that we do have a 'variational principle' for $\dim_H \Omega(f)$, but only by maximizing for stable and unstable directions separately and taking a sum. Thus the inequality above is strict unless the two suprema are attained by the same ν (and generically this is not the case).

However, the sets that we are interested in all 'depend only on the future':

Proposition 2.27 ([Pes])

(a) Let $f : \Lambda \to \Lambda$ be a hyperbolic diffeomorphism and let $g : \Lambda \to \mathbb{R}$ be Hölder continuous. Then there is a constant $K(g)$ such that whenever $z, z' \in \Lambda$

with $z' \in W^s_\epsilon(z)$ we have

$$|g^n(z') - g^n(z)| \leq K(g) \ \forall n \geq 0.$$

(b) *Let $\phi_t : \Lambda \longrightarrow \Lambda$ be a hyperbolic flow and let $g : \Lambda \longrightarrow \mathbb{R}$ be Hölder continuous. Then there is a constant $K(g)$ such that whenever $z, z' \in \Lambda$ with $z' \in W^s_\epsilon(z)$ we have*

$$\left| \int_0^t g(\phi_\tau z')\, d\tau - \int_0^t g(\phi_\tau z)\, d\tau \right| \leq K(g) \ \forall t \geq 0.$$

(The proof is essentially the same as the derivation of the constant $V(g)$ in section 2.1.2.) This means that $z \in \mathrm{Bdd}_\Lambda(g)$ if and only if $z' \in \mathrm{Bdd}_\Lambda(g)$; and similarly for the sets $\mathrm{Ave}_\Lambda(g, \alpha)$ and $L_\Lambda(g, F, r)$ (or $L_\Lambda(g, F)$ for a flow), provided that F satisfies the usual condition.

So, when we consider the splitting into stable and unstable directions, it is only the unstable direction which distinguishes the sets. More precisely, suppose we look at one of the rectangles R^* in the Markov partition for a hyperbolic diffeomorphism. We know that this has a product structure $R^* \longrightarrow W^u_{R^*}(z^*) \times W^s_{R^*}(z^*)$. Then if S is a set which depends only on the future, the intersection $S \cap R^*$ is represented in the product structure by $(S \cap W^u_{R^*}(z^*)) \times W^s_{R^*}(z^*)$. Thus, in order to compare the dimensions of such sets, we are mainly interested in the intersections $S \cap W^u_{R^*}(z^*)$.

In order to make use of the symbolic model, we consider a slightly weaker condition than 'depending only on the future'.

Suppose we have a hyperbolic diffeomorphism $f : \Lambda \longrightarrow \Lambda$, which is coded by the subshift of finite type $\sigma : X_A \longrightarrow X_A$. For a set $S_X \subseteq X_A$, we say that S_X 'depends only on future co-ordinates' if, whenever $x, y \in X_A$ with $x_i = y_i$ for all $i \geq 0$, we have $x \in S_X$ if and only if $y \in S_X$. We will then say that the set $S \subseteq \Lambda$ 'depends only on the future *in the coding*' if $S = \chi(S_X)$ for a set S_X which has this property. Then for a rectangle $R_j = \{\chi(x) : x \in X_A \text{ with } x_0 = j\}$, we can define

$$S_{R_j} = \{\chi(x) : x \in S_X \text{ with } x_0 = j\}.$$

Thus $S = \bigcup_j S_{R_j}$. (This definition depends on the choice of S_X; when we write S_{R_j} we are implicitly assuming we know which set S_X we are working with.) Note that if S is a set which satisfies the stronger condition of depending only on the future in Λ (as for the sets $\mathrm{Ave}_\Lambda(g,\alpha)$, $\mathrm{Bdd}_\Lambda(g)$ and $L_\Lambda(g,F,r)$) then we can simply take $S_X = \chi^{-1}(S)$, which implies $S_{R_j} = S \cap R_j$. However this does not hold if S is produced from a more general set S_X as there may be complications at the boundaries of rectangles. Note also that if S_X is σ-invariant then $\chi(S_X)$ is f-invariant.

This condition is still strong enough to ensure that S looks like a product when restricted to a rectangle. That is, suppose we pick a rectangle R^* and a point $z^* \in \operatorname{int} R^*$ as above. Then if S depends only on the future in the coding, the set S_{R^*} is represented in the product structure by $(S_{R^*} \cap W^u_{R^*}(z^*)) \times W^s_{R^*}(z^*)$.

Similarly, suppose we have a hyperbolic flow $\phi_t : \Lambda \to \Lambda$, coded by a suspended flow $\sigma^r_t : X^r_A \to X^r_A$ over a subshift of finite type $\sigma : X_A \to X_A$. Here we will say that a ϕ-invariant set $S \subseteq \Lambda$ 'depends only on the future in the coding' if it is of the form

$$S = \bigcup_{t \in \mathbb{R}} \phi_t(\{\rho(x,0) : x \in S_X\}),$$

where $S_X \subseteq X_A$ is a σ-invariant set which depends only on future co-ordinates. (We need the sets to be invariant here because we have to make use of the projection along lines of the flow.) We recall from the construction of the symbolic dynamics that we can find $\tau > 0$ such that Λ is covered by the sets $\bigcup_{t \in (-\tau,\tau)} \phi_t(T_j)$, diffeomorphic to the product $T_j \times (-\tau,\tau)$. If we define

$$S_{T_j} = \bigcup_{t \in (-\tau,\tau)} \phi_t\big(\{\rho(x,0) : x \in S_X \text{ with } x_0 = j\}\big),$$

then $S = \bigcup_j S_{T_j}$. When looking at the rectangle T^* with product structure $T^* \to W^u_{T^*}(z^*) \times W^s_{T^*}(z^*)$, the set $S_{T^*} \cap T^*$ is represented in the product structure by $(S_{T^*} \cap W^u_{T^*}(z^*)) \times W^s_{T^*}(z^*)$.

If our diffeomorphism (or flow) is conformal, then we have a formula for the dimension of the intersection of Λ with a stable manifold, in terms of the function $a^{(s)}$ (or $\upsilon^{(s)}$ for a flow) defined in section 1.5:

Theorem 2.28 ([Pes],[PS]) *(a) Suppose $f : \Lambda \to \Lambda$ is a conformal hyperbolic diffeomorphism. Then for any rectangle R^* and point $z^* \in \operatorname{int} R^*$ we have*

$$\dim_H W^s_{R^*}(z^*) = \overline{\dim}_B W^s_{R^*}(z^*) = t^{(s)},$$

where $t^{(s)}$ is the unique number such that $P(t^{(s)} \log a^{(s)}) = 0$. (Here P is the pressure function on Λ.)

(b) Suppose $\phi_t : \Lambda \to \Lambda$ is a conformal hyperbolic flow. Then for any rectangle T^* and point $z^* \in \operatorname{int} T^*$ we have

$$\dim_H W^s_{T^*}(z^*) = \overline{\dim}_B W^s_{T^*}(z^*) = t^{(s)},$$

where $t^{(s)}$ is the unique number such that $P(t^{(s)}\upsilon^{(s)}) = 0$.

(The references [Pes] and [PS] also give similar results for the unstable manifolds, and these can be combined to find the dimension of the set Λ itself.)

Because the Hausdorff dimension and upper box dimension of $W^s_{R^*}(z^*)$ coincide, we can apply property (e) from section 1.5 to any product $V \times W^s_{R^*}(z^*)$ for $V \subseteq W^u_{R^*}(z^*)$. (And similarly for flows.) Furthermore, Proposition 1.8 tells us that, with the conformality assumption, the product structure on each rectangle is Lipschitz, and so preserves dimension. Combining all these observations we have the following:

Lemma 2.29 *(a) Suppose $f : \Lambda \to \Lambda$ is a conformal hyperbolic diffeomorphism, and suppose the set $S \subseteq \Lambda$ depends only on the future (in the coding). Then for any rectangle R^* and any $z^* \in \operatorname{int} R^*$ we have*

$$\dim_H S_{R^*} = \dim_H(S_{R^*} \cap W^u_{R^*}(z^*)) + t^{(s)}.$$

(b) *Suppose $\phi_t : \Lambda \to \Lambda$ is a conformal hyperbolic flow, and suppose the ϕ-invariant set $S \subseteq \Lambda$ depends only on the future (in the coding). Then for any rectangle T^* and any $z^* \in \operatorname{int} T^*$ we have*

$$\dim_H S_{T^*} = \dim_H(S_{T^*} \cap W^u_{T^*}(z^*)) + t^{(s)} + 1.$$

Applying this to the sets which depend only on the future in Λ (rather than just in the coding), we have:

Lemma 2.30 *(a) Suppose $f : \Lambda \to \Lambda$ is a conformal hyperbolic diffeomorphism. Then for any rectangle R^* and any $z^* \in \operatorname{int} R^*$ we have*

$$\dim_H(\mathrm{Ave}_\Lambda(\mathbf{g}, \boldsymbol{\alpha}) \cap R^*) = \dim_H(\mathrm{Ave}_\Lambda(\mathbf{g}, \boldsymbol{\alpha}) \cap W^u_{R^*}(z^*)) + t^{(s)};$$

$$\dim_H(\mathrm{Bdd}_\Lambda(\mathbf{g}) \cap R^*) = \dim_H(\mathrm{Bdd}_\Lambda(\mathbf{g}) \cap W^u_{R^*}(z^*)) + t^{(s)};$$

$$\dim_H(L_\Lambda(\mathbf{g}, \mathbf{F}, r) \cap R^*) = \dim_H(L_\Lambda(\mathbf{g}, \mathbf{F}, r) \cap W^u_{R^*}(z^*)) + t^{(s)}.$$

(b) *Suppose $\phi_t : \Lambda \to \Lambda$ is a conformal hyperbolic flow. Then for any rectangle T^* and any $z^* \in \operatorname{int} T^*$, if we write $T^*_\tau = \bigcup_{t \in (-\tau, \tau)} \phi_t(T^*)$ we have*

$$\dim_H(\mathrm{Ave}_\Lambda(\mathbf{g}, \boldsymbol{\alpha}) \cap T^*_\tau) = \dim_H(\mathrm{Ave}_\Lambda(\mathbf{g}, \boldsymbol{\alpha}) \cap W^u_{T^*}(z^*)) + t^{(s)} + 1;$$

$$\dim_H(\mathrm{Bdd}_\Lambda(\mathbf{g}) \cap T^*_\tau) = \dim_H(\mathrm{Bdd}_\Lambda(\mathbf{g}) \cap W^u_{T^*}(z^*)) + t^{(s)} + 1;$$

$$\dim_H(L_\Lambda(\mathbf{g}, \mathbf{F}) \cap T^*_\tau) = \dim_H(L_\Lambda(\mathbf{g}, \mathbf{F}) \cap W^u_{T^*}(z^*)) + t^{(s)} + 1.$$

(Here the functions $\mathbf{g} : \Lambda \to \mathbb{R}^d$ and $r : \Lambda \to \mathbb{R}^+$ are assumed to be Hölder continuous, and $\mathbf{F}$ satisfies the condition of Theorem 2.24.)

This is all analogous to the method for working with two-sided subshifts of finite type in section 2.1.4. There we looked at the projection to the one-sided subshift: this is much the same as looking at the intersection with an unstable manifold. However, for subshifts of finite type the projection was to the 'nice' space X_A^+ which we had already studied. In order for a similar method to work for hyperbolic diffeomorphisms (and flows), we need a way to get information about the intersections $S_{R^*} \cap W^u_{R^*}(z^*)$, and in particular we need an analogue of Theorem 2.2. The ideas that we need are described in the next section.

2.4.2 BS-dimension

We follow the descriptions in [Pes] and [BSS]. Let $T : X \to X$ be a continuous map of a compact metric space, and let $\mathcal{U}$ be a finite open cover of X. We consider strings of sets, which we write as $\mathbf{U} = (U_0, U_1, \ldots, U_{m(\mathbf{U})-1})$, where $U_i \in \mathcal{U}$ for each i. Let $\mathcal{S}_m(\mathcal{U})$ be the set of all such strings $\mathbf{U}$ for which $m(\mathbf{U}) = m$.

Each $\mathbf{U} \in \bigcup_{m\geq 0} \mathcal{S}_m(\mathcal{U})$ defines a set

$$X(\mathbf{U}) := \left\{x \in X : T^i x \in U_i \ \ \forall 0 \leq i < m(\mathbf{U})\right\}.$$

And then for any continuous real-valued function ψ on X we can define

$$\psi(\mathbf{U}) = \sup_{x\in X(\mathbf{U})} \psi^{m(\mathbf{U})}(x).$$

Now let $u : X \to \mathbb{R}^+$ be a strictly positive continuous function. For any set $Z \subseteq X$ and $\alpha \in \mathbb{R}$ we define

$$M(Z, \alpha, u, \mathcal{U}) =$$
$$\liminf_{n\to\infty} \left\{ \sum_{\mathbf{U}\in\Gamma} \exp(-\alpha u(\mathbf{U})) : \Gamma \subseteq \bigcup_{m\geq n} \mathcal{S}_m(\mathcal{U}) \text{ with } \bigcup_{\mathbf{U}\in\Gamma} X(\mathbf{U}) \supseteq Z \right\},$$

and we use this to define

$$\dim_{u,\mathcal{U}} Z = \inf\left\{\alpha : M(Z, \alpha, u, \mathcal{U}) = 0\right\}.$$

We can then consider what happens as we take $\operatorname{diam}\mathcal{U} \to 0$ (where $\operatorname{diam}\mathcal{U} := \max_{U\in\mathcal{U}} \operatorname{diam} U$). It turns out that the limit

$$\dim_u Z := \lim_{\operatorname{diam}\mathcal{U}\to 0} \dim_{u,\mathcal{U}} Z$$

always exists. (In fact thinking of this as a limit is slightly misleading: we expect $\dim_{u,\mathcal{U}}$ to be *independent* of $\mathcal{U}$, provided that $\operatorname{diam}\mathcal{U}$ is sufficiently small.) This quantity was first defined by Barreira and Schmeling in [BSc] and so is referred to in [Pes] as *BS-dimension.*

When $u = 1$, $\dim_u Z$ gives the topological entropy of T on Z. (Or, to be more precise, it coincides with the usual definition of topological entropy for compact invariant sets, and extends it to more general sets.)

BS-dimension shares some of the simple properties of Hausdorff dimension. In particular, if $Z_1 \subseteq Z_2 \subseteq X$ then $\dim_u Z_1 \leq \dim_u Z_2$, and if $Z = \bigcup_{i \geq 1} Z_i$ for sets $Z_i \subseteq X$ then

$$\dim_u Z = \sup_{i \geq 1} \dim_u Z_i. \tag{2.32}$$

We can also define BS-dimension for measures. For a Borel probability measure μ we let

$$\dim_{u,\mathcal{U}} \mu = \inf \left\{ \dim_{u,\mathcal{U}} Z : \mu(Z) = 1 \right\},$$

and then $\dim_u \mu$ is defined by

$$\dim_u \mu = \lim_{\operatorname{diam} \mathcal{U} \to 0} \dim_{u,\mathcal{U}} \mu.$$

Again this limit is guaranteed to exist. For us the important consequence of this definition is that if Z is a set with $\mu(Z) = 1$ then $\dim_u Z \geq \dim_u \mu$.

Proposition 2.31 ([**BSc**]) *If μ is ergodic then*

$$\dim_u \mu = \frac{h_\mu(T)}{\int_X u \, d\mu},$$

where $h_\mu(T)$ is the measure-theoretic entropy.

We can now state a generalisation of Theorems 2.2/2.15/2.19. Suppose that $\mathbf{g} : X \longrightarrow \mathbb{R}^d$ is continuous, and define as before

$$\mathcal{D}(\mathbf{g}) = \left\{ \int_X \mathbf{g} \, d\mu : \mu \in \mathcal{M}(X) \right\}.$$

Then we have:

Theorem 2.32 ([**BSS**]) *Suppose that the function $\mu \mapsto h_\mu(T)$ is upper semi-continuous, and the continuous function $\mathbf{g} : X \longrightarrow \mathbb{R}^d$ is such that $\lambda u + \sum_{i=1}^d \lambda_i g_i$ has a unique equilibrium state for any $\lambda, \lambda_1, \ldots, \lambda_d \in \mathbb{R}$. Then:*

1. *If* $\boldsymbol{\alpha} \notin \mathcal{D}(\mathbf{g})$ *then* $\mathrm{Ave}_X(\mathbf{g}, \boldsymbol{\alpha}) = \varnothing$.

2. *If* $\boldsymbol{\alpha} \in \operatorname{int} \mathcal{D}(\mathbf{g})$ *then* $\mathrm{Ave}_X(\mathbf{g}, \boldsymbol{\alpha}) \neq \varnothing$, *and*

$$\dim_u \mathrm{Ave}_X(\mathbf{g}, \boldsymbol{\alpha}) = \sup \left\{ \frac{h_\mu(T)}{\int_X u \, d\mu} : \mu \in \mathcal{M}(X) \text{ and } \int_X \mathbf{g} \, d\mu = \boldsymbol{\alpha} \right\}.$$

Furthermore the supremum is attained by some ergodic measure μ, which is an equilibrium state for the function

$$\langle \mathbf{q}(\boldsymbol{\alpha}), \mathbf{g} \rangle - (\dim_u \mathrm{Ave}_X(\mathbf{g}, \boldsymbol{\alpha}))\, u,$$

for some $\mathbf{q}(\boldsymbol{\alpha}) \in \mathbb{R}^d$.

The conditions of this theorem are satisfied for the maps that we are interested in (expanding maps and hyperbolic diffeomorphisms), when $\mathbf{g}$ and u are Hölder continuous. Also in that case we have that if the components of $\mathbf{g}$ are cohomologously independent then the set $\mathcal{D}(\mathbf{g})$ is the closure of its interior.

The link to Theorem 2.19 is given by the following:

Proposition 2.33 ([BSc]) *Suppose we take X to be a repeller J of a conformal $C^{1+\alpha}$ expanding map. Then if we set $u(x) = v(x) = \log a(x)$, we have $\dim_H Z = \dim_u Z$ for any $Z \subseteq J$, and $\dim_H \mu = \dim_u \mu$ for any Borel probability measure μ.*

We can use this to deduce Theorem 2.19 as a special case of Theorem 2.32.

There is also a concept of BS-dimension for *flows*, introduced in [BS1]. We restrict attention to conformal hyperbolic flows $\phi_t : \Lambda \to \Lambda$ since this is the only type of flow we will want to consider.

For $x \in \Lambda$, $t > 0$, $\epsilon > 0$ we define the set

$$B(x, t, \epsilon) = \{ y \in \Lambda : d(\phi_\tau y, \phi_\tau x) < \epsilon \text{ whenever } 0 \leq \tau \leq t \},$$

and then for a continuous function $\psi : \Lambda \to \mathbb{R}$ we can define

$$\psi(x, t, \epsilon) = \sup \left\{ \int_0^t \psi(\phi_\tau y) \, d\tau : y \in B(x, t, \epsilon) \right\}.$$

Now suppose $u : \Lambda \longrightarrow \mathbb{R}^+$ is a strictly positive continuous function. Then for any set $Z \subseteq \Lambda$ and $\alpha \in \mathbb{R}$ we define

$$M(Z, \alpha, u, \epsilon) = \lim_{T \to \infty} \inf_{\Gamma} \sum_{(x,t) \in \Gamma} \exp(-\alpha u(x, t, \epsilon)),$$

where the infimum is taken over all countable subsets Γ of $\Lambda \times [T, \infty)$ for which $\bigcup_{(x,t)\in\Gamma} B(x, t, \epsilon) \supseteq Z$. We then take

$$\dim_{u,\epsilon} Z = \inf \{\alpha : M(Z, \alpha, u, \epsilon) = 0\}$$

and set

$$\dim_u Z = \lim_{\epsilon \to 0} \dim_{u,\epsilon} Z.$$

For a Borel probability measure μ on Λ we define

$$\dim_{u,\epsilon} \mu = \inf \{\dim_{u,\epsilon} Z : \mu(Z) = 1\},$$

and then

$$\dim_u \mu = \lim_{\epsilon \to 0} \dim_{u,\epsilon} \mu.$$

Again these limits are guaranteed to exist, and we have:

Proposition 2.34 ([BD]) *If μ is ergodic then*

$$\dim_u \mu = \frac{h_\mu(\phi)}{\int_\Lambda u \, d\mu}.$$

And we have a version of Theorem 2.32:

Theorem 2.35 ([BD]) *Let $\phi_t : \Lambda \longrightarrow \Lambda$ be a hyperbolic flow. Suppose the functions $\mathbf{g} : \Lambda \longrightarrow \mathbb{R}^d$ and $u : \Lambda \longrightarrow \mathbb{R}^+$ are Hölder continuous. Then*

1. *If $\boldsymbol{\alpha} \notin \mathcal{D}(\mathbf{g})$ then $\mathrm{Ave}_\Lambda(\mathbf{g}, \boldsymbol{\alpha}) = \varnothing$.*

2. *If $\boldsymbol{\alpha} \in \operatorname{int} \mathcal{D}(\mathbf{g})$ then $\mathrm{Ave}_\Lambda(\mathbf{g}, \boldsymbol{\alpha}) \neq \varnothing$, and*

$$\dim_u \mathrm{Ave}_X(\mathbf{g}, \boldsymbol{\alpha}) = \sup \left\{ \frac{h_\mu(\phi)}{\int_\Lambda u \, d\mu} : \mu \in \mathcal{M}(\Lambda) \text{ and } \int_\Lambda \mathbf{g} \, d\mu = \boldsymbol{\alpha} \right\}.$$

Furthermore the supremum is attained by some ergodic measure μ, which is an equilibrium state for the function

$$\langle \mathbf{q}(\boldsymbol{\alpha}), \mathbf{g} \rangle - (\dim_u \operatorname{Ave}_X(\mathbf{g}, \boldsymbol{\alpha}))\, u,$$

for some $\mathbf{q}(\boldsymbol{\alpha}) \in \mathbb{R}^d$.

The reason that BS-dimension is useful to us is that, if we choose the right function u, we get information about the Hausdorff dimension. In the case of a conformal repeller for an expanding map, Proposition 2.33 gives that the Hausdorff dimension is actually equal to the BS-dimension for a particular u. For hyperbolic diffeomorphisms and flows, we instead get information about the dimensions of subsets of unstable (or stable) manifolds.

Lemma 2.36 *(a) Let $f : \Lambda \longrightarrow \Lambda$ be a conformal hyperbolic diffeomorphism, and define the function u by $u(z) = v^{(u)}(z) = \log a^{(u)}(z)$. Suppose S is a f-invariant subset of Λ which depends only on the future (in the coding). Then for a rectangle R^* and point $z^* \in \operatorname{int} R^*$ we have*

$$\dim_H \left(S_{R^*} \cap W^u_{R^*}(z^*) \right) = \dim_u S_{R^*}.$$

(b) Let $\phi_t : \Lambda \longrightarrow \Lambda$ be a conformal hyperbolic flow, and take u to be the function $v^{(u)}$. Suppose S is a ϕ_t-invariant subset of Λ which depends only on the future (in the coding). Then for a rectangle T^* and point $z^* \in \operatorname{int} T^*$ we have

$$\dim_H \left(S_{T^*} \cap W^u_{T^*}(z^*) \right) = \dim_u S_{T^*}.$$

Part (b) is implied in [BD] without the conditions on S. (Though clearly it cannot be true for every set S because it requires $\dim_H S_{T^*} \cap W^u_{T^*}(z^*)$ to be independent of z^*.) Along with Proposition 2.33, this result follows from the 'bounded distortion' property of conformal systems: e.g. for a conformal hyperbolic flow we have

$$c_1 \le \frac{(\operatorname{diam} B(x,t,\epsilon) \cap W^u_\epsilon(x))^\alpha}{\exp(-\alpha v^{(u)}(x,t,\epsilon))} \le c_2.$$

While I believe Lemma 2.36 is well known, I have been unable to find an explicit reference (except for the incomplete statement of (b) in [BD]), so present a sketch proof of part (a) below. (Part (b) can be proved by a similar method.) But for the moment we note that by combining Lemma 2.36 with Theorem 2.29 we have:

Lemma 2.37 *(a) Let $f : \Lambda \longrightarrow \Lambda$ be a conformal hyperbolic diffeomorphism, and define the function u by $u(z) = v^{(u)}(z) = \log a^{(u)}(z)$. Suppose S is a f-invariant subset of Λ which depends only on the future (in the coding). Then*

$$\dim_H S = \dim_u S + t^{(s)}.$$

(b) Let $\phi_t : \Lambda \longrightarrow \Lambda$ be a conformal hyperbolic flow, and take u to be the function $v^{(u)}$. Suppose S is a ϕ_t-invariant subset of Λ which depends only on the future (in the coding). Then

$$\dim_H S = \dim_u S + t^{(s)} + 1.$$

Proof: For part (a), consider the rectangles R_j of the Markov partition, and choose a point $z^{(j)} \in \operatorname{int} R_j$ for each j. We have

$$\begin{aligned}
\dim_H S &= \sup_j \dim_H S_{R_j} \\
&= \sup_j \left(\dim_H \left(S_{R_j} \cap W^u_{R_j}(z^{(j)})\right) + t^{(s)}\right) \qquad \text{(from Theorem 2.29)} \\
&= \sup_j \dim_u S_{R_j} + t^{(s)} \qquad \text{(from Lemma 2.36)} \\
&= \dim_u S + t^{(s)} \qquad \text{(from (2.32))}.
\end{aligned}$$

Part (b) is similar. □

Sketch proof of Lemma 2.36(a): We consider covers $\mathcal{U}$ whose elements are small *open rectangles* in Λ. That is, each $U \in \mathcal{U}$ is an open set in Λ such that whenever $x, y \in U$ we have $[x, y] \in U$; we also require that U is 'connected' in the sense that if $x, y \in U$ with $y \in W^u_\epsilon(x)$ (or $y \in W^s_\epsilon(x)$) then U contains the entire segment of $W^u_\epsilon(x) \cap \Lambda$ (respectively $W^s_\epsilon(x) \cap \Lambda$) between x and y.

If each U is an open rectangle it follows that every set $\Lambda(\mathbf{U})$ is also an open rectangle.

Now, for each $U \in \mathcal{U}$ we want to produce a slightly larger open rectangle U' by 'expanding U in the stable direction'. Suppose we are given $\delta \ll \operatorname{diam} \mathcal{U}$. The rectangle U is bounded by two local stable manifolds and two local unstable manifolds. By moving the unstable manifolds slightly, we can produce U' with $\operatorname{diam} U' \leq \operatorname{diam} U + \delta$ with the property that there exists $\delta' > 0$ such that

$$U' \supseteq \bigcup_{x \in U} \{y \in W^s_\epsilon(x) : d(x,y) < \delta'\}.$$

Since $\mathcal{U}$ is finite we may choose δ' independent of U. We now have a new cover $\mathcal{U}' = \{U' : U \in \mathcal{U}\}$, and if $\mathbf{U} = (U_0, U_1, \ldots, U_{m(\mathbf{U})-1})$ for $U_i \in \mathcal{U}$ we may define $\mathbf{U}' = (U_0', U_1', \ldots, U'_{m(\mathbf{U})-1})$, which has the property that

$$\Lambda(\mathbf{U}') \supseteq \bigcup_{x \in \Lambda(\mathbf{U})} \{y \in W^s_\epsilon(x) : d(x,y) < \delta'\}.$$

Now suppose we have a cover of $S_{R^*} \cap W^u_{R^*}(z^*)$, as in the definition of Hausdorff dimension. Then by an argument similar to that for Moran covers, we can replace each set of this cover by a bounded number of sets of the form $\Lambda(\mathbf{U})$, such that $\operatorname{diam} \Lambda(\mathbf{U})$ is no more than a constant multiple of the diameter of the original set. But if the sets $\Lambda(\mathbf{U}_j)$ cover $S_{R^*} \cap W^u_{R^*}(z^*)$, then the sets $\Lambda(\mathbf{U}_j')$ cover

$$S_{R^*} \cap \bigcup_{x \in W^u_{R^*}(z^*)} \{y \in W^s_{R^*}(z^*) : d(x,y) < \delta'\}.$$

The 'bounded distortion' property for a conformal hyperbolic diffeomorphism tells us

$$c_1 \left(\operatorname{diam} \Lambda(\mathbf{U}) \cap W^u_\epsilon(z^*)\right)^\alpha \leq \exp(-\alpha u(\mathbf{U})) \leq c_2 \left(\operatorname{diam} \Lambda(\mathbf{U}) \cap W^u_\epsilon(z^*)\right)^\alpha. \tag{2.33}$$

Comparing the definitions of Hausdorff dimension and BS-dimension then gives

$$\dim_{u,\mathcal{U}'} \left(S_{R^*} \cap \bigcup_{x \in W^u_{R^*}(z^*)} \{y \in W^s_{R^*}(z^*) : d(x,y) < \delta'\} \right) \leq \dim_H S_{R^*} \cap W^u_{R^*}(z^*).$$

But the right hand side is independent of the choice of $z^* \in \operatorname{int} R^*$, and the sets on the left hand side cover S_{R^*} when taken over a suitable finite number of choices of z^*. Then taking $\operatorname{diam} \mathcal{U} \longrightarrow 0$ (which implies $\operatorname{diam} \mathcal{U}' \longrightarrow 0$) we get

$$\dim_u S_{R^*} \leq \dim_H S_{R^*} \cap W^u_{R^*}(z^*).$$

The opposite inequality is easier since any cover of S_{R^*} is automatically a cover of $S_{R^*} \cap W^u_{R^*}(z^*)$, and we can apply (2.33) again.

2.4.3 Results for conformal hyperbolic diffeomorphisms

Our aim is to prove the following:

Theorem 2.38 *Let $f : \Lambda \longrightarrow \Lambda$ be a conformal hyperbolic diffeomorphism, and let $\mathbf{g} : \Lambda \longrightarrow \mathbb{R}^d$ be Hölder continuous. Suppose there exists an equilibrium state ν on Λ such that $\int_\Lambda \mathbf{g}\, d\nu = \mathbf{0}$. Then*

$$\dim_H \mathrm{Bdd}_\Lambda(\mathbf{g}) = \dim_H \mathrm{Ave}_\Lambda(\mathbf{g}, \mathbf{0}).$$

Furthermore if the components of $\mathbf{g}$ are cohomologously independent then

$$\dim_H L_\Lambda(\mathbf{g}, \mathbf{F}, r) = \dim_H \mathrm{Ave}_\Lambda(\mathbf{g}, \mathbf{0}),$$

for any strictly positive Hölder continuous function $r : \Lambda \longrightarrow \mathbb{R}^+$ and any continuous function $\mathbf{F} : \mathbb{R}^+ \longrightarrow \mathbb{R}^d$ with the property that $\sup_{\tau \in [0,1]} \|\mathbf{F}(t + \tau) - \mathbf{F}(t)\| \longrightarrow 0$ as $t \longrightarrow \infty$.

Although the first part of this theorem does not require the components of $\mathbf{g}$ to be cohomologously independent, we can always reduce to a case where the components are cohomologously independent by 'throwing out' components if necessary, as explained in the proof of Theorem 2.22. So in what follows we will always assume that the components of $\mathbf{g}$ are cohomologously independent.

We look at the subshift of finite type $\sigma : X_A \to X_A$ which is the model for f. We have the projection $\pi_+ : X_A \to X_A^+$, and if $z^* = \chi(x^*) \in \operatorname{int} R^*$ for a rectangle R^* of the Markov partition for f, the set $W^u_{R^*}(z^*)$ is coded by the map $\chi^{(u)}_{z^*} : [x_0^*] \to W^u_{R^*}(z^*)$ as defined in section 2.2. We look at the sets $\chi^{(u)}_{z^*}(C)$ where $C \subseteq [x_0^*]$ is a a cylinder in X_A^+. According to Proposition 2.14, each such set $\chi^{(u)}_{z^*}(C)$ is contained in a ball (in $W^u_{R^*}(z^*)$) of radius $\overline{r}^{(u)}_{z^*}(C)$ and contains a ball of radius $\underline{r}^{(u)}_{z^*}(C)$. By copying the proofs of Lemmas 2.16 and 2.17, we have analogous results for this situation:

Lemma 2.39 *(i) There exist positive constants γ_1, γ_2, c_1, c_2 such that if $C \subseteq [x_0^*]$ is a cylinder of length n in X_A^+ then*

$$c_1 \exp(-\gamma_1 n) \leq \underline{r}^{(u)}_{z^*}(C) \leq \overline{r}^{(u)}_{z^*}(C) \leq c_2 \exp(-\gamma_2 n).$$

(ii) Let $C = [x_0, \ldots, x_{m-1}]$ be a cylinder in X_A^+ with $x_0 = x_0^*$, and let C' be a cylinder produced by inserting a single block of length ℓ inside C, i.e.

$$C' = [x_0, \ldots x_{i-1}, b_0, \ldots, b_{\ell-1}, x_i \ldots x_{m-1}].$$

Then we have

$$\overline{r}^{(u)}_{z^*}(C') \geq \underline{r}^{(u)}_{z^*}(C') \geq \exp(-\gamma_3 \ell)\, \overline{r}^{(u)}_{z^*}(C),$$

for a constant γ_3.

Lemma 2.40 *Suppose we have a block-adding process $\xi : S \to S'$ (for some $S \subseteq X_A^+$) which is defined on cylinders. Let $Q \subseteq [x_0^*]$ be a cylinder of length n which intersects S', and let $\widehat{n}$ be the length of the cylinder $\widehat{Q}$ obtained by removing the blocks from Q. Write $p = (n - \widehat{n})/\widehat{n}$. Then, provided that $p < 1$, we have*

$$\overline{r}^{(u)}_{z^*}(Q) \geq \underline{r}^{(u)}_{z^*}(Q) \geq c\left(\overline{r}^{(u)}_{z^*}(\widehat{Q})\right)^{1+\gamma p},$$

where c, γ are constants.

Now suppose we are given the Hölder continuous function $\mathbf{g} : \Lambda \to \mathbb{R}^d$, and an equilibrium state ν on Λ such that $\int_\Lambda \mathbf{g}\, d\nu = \mathbf{0}$. We transfer these to X_A, producing a function $\hat{\mathbf{g}} : X_A \to \mathbb{R}^d$ and an equilibrium state μ on X_A such that $\int_{X_A} \hat{\mathbf{g}}\, d\mu = \mathbf{0}$. By applying Proposition 1.2 (to each of the components of $\hat{\mathbf{g}}$) we can find a Hölder continuous function $\hat{\mathbf{g}}^{(u)}$ which is cohomologous to $\hat{\mathbf{g}}$ and which depends only on future co-ordinates. This gives rise to a well-defined function $\hat{\mathbf{g}}^+ : X_A^+ \to \mathbb{R}^d$, by taking $\hat{\mathbf{g}}^+(\pi_+ x) = \hat{\mathbf{g}}^{(u)}(x)$, for which $\int_{X_A^+} \hat{\mathbf{g}}^+\, d\mu^+ = \mathbf{0}$.

If the components of $\mathbf{g}$ are cohomologously independent, this implies that the components of $\hat{\mathbf{g}}^+$ are cohomologously independent.

Suppose $z^* = \chi(x^*) \in \operatorname{int} R^*$ as above. Then if y is a point in $[x_0^*] \subseteq X_A^+$, we have (by definition) $\chi_{z^*}^{(u)}(y) = \chi(x)$ where $x \in X_A$ is the point defined by

$$x = (\dots\, x_{-2}^*\, x_{-1}^*\, x_0^*\, y_1\, y_2\, \dots).$$

We then have

$$\begin{aligned} y \in \mathrm{Bdd}_{X_A^+}(\hat{\mathbf{g}}^+) &\Leftrightarrow x \in \mathrm{Bdd}_{X_A}(\hat{\mathbf{g}}^{(u)}) \\ &\Leftrightarrow x \in \mathrm{Bdd}_{X_A}(\hat{\mathbf{g}}) \\ &\Leftrightarrow \chi_{z^*}^{(u)}(y) \in \mathrm{Bdd}_\Lambda(\mathbf{g}) \end{aligned} \tag{2.34}$$

And similarly

$$y \in \mathrm{Ave}_{X_A^+}(\hat{\mathbf{g}}^+, \mathbf{0}) \Leftrightarrow \chi_{z^*}^{(u)}(y) \in \mathrm{Ave}_\Lambda(\mathbf{g}, \mathbf{0}). \tag{2.35}$$

Also, given $r : \Lambda \to \mathbb{R}^+$ we can produce in the same way the function $\hat{r}^+ : X_A^+ \to \mathbb{R}^+$. (We explained in section 2.1.4 why we may take $\hat{r}^+ > 0$.) And then we have

$$y \in L_{X_A^+}(\hat{\mathbf{g}}^+, \mathbf{F}, \hat{r}^+) \Leftrightarrow \chi_{z^*}^{(u)}(y) \in L_\Lambda(\mathbf{g}, \mathbf{F}, r), \tag{2.36}$$

assuming $\mathbf{F}$ satisfies the usual condition.

We now look at the sets $G(n) \subseteq X_A^+$ as defined in the proof of Theorem 2.22. Recall that these are defined in terms of a Hölder continuous function

$\tilde{\mathbf{g}} : X_A^+ \to \mathbb{R}^d$ and an equilibrium state on X_A^+ which integrates $\tilde{\mathbf{g}}$ to zero. We will take $\tilde{\mathbf{g}} = \hat{\mathbf{g}}^+$, and the equilibrium state to be μ^+. That is, we define

$$\epsilon(n) = \mu^+ \left(\left\{ x \in X_A^+ : \|\tilde{\mathbf{g}}^n(x)\| > n^{\frac{3}{4}} - V(\tilde{\mathbf{g}}) \right\} \right) + n^{-1};$$
$$\mathcal{C}(n) = \left\{ C \in \mathrm{Cyl}(n) : \|\tilde{\mathbf{g}}^n(y)\| > n^{\frac{3}{4}} \text{ for some } y \in C \right\};$$

and then

$$G(n) = \left\{ x \in X_A^+ : \limsup_{t \to \infty} \frac{\#\{0 \le i < t : x \in E_i(n)\}}{t} < \epsilon(n) \right\},$$

where $E_i(n) = \bigcup_{C \in \mathcal{C}(n)} \sigma^{-in}(C)$. We know that $\mu^+(G(n)) = 1$.

Note that the set $G(n)$ is σ^n-invariant. We would prefer to work with σ-invariant sets, so we we will actually consider instead

$$\tilde{G}(n) := \bigcap_{0 \le i < n} \sigma^{-i}(G(n)).$$

Since $\mu^+(\sigma^{-i}(G(n))) = \mu^+(G(n)) = 1$ for all i, we have $\mu^+\left(\tilde{G}(n)\right) = 1$.

We can now define the set $\Gamma(n) \subseteq \Lambda$ by

$$\Gamma(n) = \chi\left(\pi_+^{-1}(\tilde{G}(n))\right).$$

Since $\mu^+\left(\tilde{G}(n)\right) = 1$ we have $\mu\left(\pi_+^{-1}\left(\tilde{G}(n)\right)\right) = 1$ and so $\nu(\Gamma(n)) = 1$. If we then put

$$\Gamma = \bigcup_{m \ge 1} \bigcap_{n \ge m} \Gamma(n)$$

we also have $\nu(\Gamma) = 1$.

In order to compare the dimensions of the sets $\mathrm{Ave}_\Lambda(\mathbf{g}, \mathbf{0})$, $\mathrm{Bdd}_\Lambda(\mathbf{g})$ and $L_\Lambda(\mathbf{g}, \mathbf{F}, r)$, we will relate all of these sets to the set Γ. We start with a simple inclusion:

Lemma 2.41 *If* $\mathbf{F}$ *satisfies the condition that* $\sup_{\tau \in [0,1]} \|\mathbf{F}(t+\tau) - \mathbf{F}(t)\| \to 0$ *as* $t \to \infty$, *then* $L_\Lambda(\mathbf{g}, \mathbf{F}, r) \subseteq \Gamma$.

Proof: Suppose $z = \chi(x) \in L_\Lambda(\mathbf{g}, \mathbf{F}, r)$, and let $y = \pi_+ x \in X_A^+$. Then we know $y \in L_{X_A^+}(\hat{\mathbf{g}}^+, \mathbf{F}, \hat{r}^+)$; so we can find $K > 0$ such that

$$\left\|(\hat{\mathbf{g}}^+)^m(y) - \mathbf{F}\left((\hat{r}^+)^m(y)\right)\right\| \leq K \quad \forall m \geq 0.$$

Choose n_0 sufficiently large that $n_0^{\frac{3}{4}} > 2K + V(\hat{\mathbf{g}}^+) + 1$. Given $n \geq n_0$, we can find $t_0(n)$ such that whenever $t \geq t_0(n)$ we have $\|\mathbf{F}(t+\tau) - \mathbf{F}(t)\| < 1$ for all $\tau \in [0, n\hat{r}^+_{max}]$. Then whenever m is sufficiently large that $m\hat{r}^+_{min} \geq t_0(n)$ we have

$$\begin{aligned}
\left\|(\hat{\mathbf{g}}^+)^n(\sigma^m y)\right\| &= \left\|(\hat{\mathbf{g}}^+)^{m+n}(y) - (\hat{\mathbf{g}}^+)^m(y)\right\| \\
&\leq \left\|\mathbf{F}\left((\hat{r}^+)^{m+n}(y)\right) - \mathbf{F}\left((\hat{r}^+)^m(y)\right)\right\| + 2K \\
&< 2K + 1 \\
&< n^{\frac{3}{4}} - V(\hat{\mathbf{g}}^+),
\end{aligned}$$

and so $\sigma^m y \notin E_0(n)$. So the points y, σy, $\ldots$, $\sigma^{n-1} y$ each belong to only finitely many of the sets $E_i(n)$, which implies that $y \in \bigcap_{n \geq n_0} \tilde{G}(n)$. Since $\Gamma(n) = \chi\left(\pi_+^{-1}(G(n))\right)$ it follows that $z \in \bigcap_{n \geq n_0} \Gamma(n) \subseteq \Gamma$. $\square$

Our next result is the one that gives us the important lower bound for $\dim_H \mathrm{Bdd}_\Lambda(\mathbf{g})$:

Lemma 2.42 $\dim_H \mathrm{Bdd}_\Lambda(\mathbf{g}) = \dim_H \Gamma$.

Proof: For each $n \geq 1$ we look at the block-adding process $\xi : G(n) \to G'(n)$ defined in the proof of Theorem 2.22 (taking $\tilde{\mathbf{g}} = \hat{\mathbf{g}}^+$ as above). Then $G'(n) \subseteq \mathrm{Bdd}_{X_A^+}(\hat{\mathbf{g}}^+)$. Also as in Theorem 2.22 we consider the sets

$$S_X(n, t_0) := \left\{x \in X_A^+ : \frac{\#\{0 \leq i < t : x \in E_i(n)\}}{t} < \epsilon(n) \text{ for all } t \geq t_0\right\}.$$

Now suppose $z^* = \chi(x^*) \in \operatorname{int} R^*$ where R^* is a rectangle of the Markov partition for f. We define

$$\tilde{S}_X(n, t_0, x_0^*) = S_X(n, t_0) \cap \tilde{G}(n) \cap [x_0^*],$$

and look at the restriction of ξ to $\tilde{S}_X(n, t_0, x_0^*)$. We write

$$\tilde{S}'_X(n, t_0, x_0^*) = \xi\left(\tilde{S}_X(n, t_0, x_0^*)\right).$$

For any $x \in G(n)$ we have $[\xi(x)]_0 = x_0$; so $\tilde{S}'_X(n, t_0, x_0^*) \subseteq [x_0^*]$. Thus the map $\chi_{z^*}^{(u)} : [x_0^*] \to W^u_{R^*}(z^*)$ is defined on the set $\tilde{S}'_X(n, t_0, x_0^*)$, as well as on $\tilde{S}_X(n, t_0, x_0^*)$. We will compare the dimensions of the sets $\chi_{z^*}^{(u)}\left(\tilde{S}_X(n, t_0, x_0^*)\right)$ and $\chi_{z^*}^{(u)}\left(\tilde{S}'_X(n, t_0, x_0^*)\right)$, using the same arguments as in Theorem 2.18.

We write $D = \dim_H \chi_{z^*}^{(u)}\left(\tilde{S}'_X(n, t_0, x_0^*)\right)$, and let $\eta > 0$ be arbitrary. For all sufficiently small ρ we can find a cover of $\chi_{z^*}^{(u)}\left(\tilde{S}'_X(n, t_0, x_0^*)\right)$ by a finite or countable collection of balls B_i in $W^u_{R^*}(z^*)$ with radii $r_i < \rho$ such that

$$\sum_i r_i^{D+\eta} \leq 1.$$

For each i we can construct a Moran cover $\mathfrak{U}_{r_i}$ of $W^u_{R^*}(z^*)$ with size r_i; this consists of sets of the form $\chi_{z^*}^{(u)}(Q)$ where Q is a cylinder in X_A^+ with $Q \subseteq [x_0^*]$. We let $\chi_{z^*}^{(u)}(Q_i^k)$ $(1 \leq k \leq m(i))$ be the sets of this cover for which $Q_i^k \cap (\chi_{z^*}^{(u)})^{-1}(B_i) \cap \tilde{S}'_X(n, t_0, x_0^*) \neq \varnothing$.

We have $m(i) \leq M_{\text{Moran}}$ and $\overline{r}_{z^*}^{(u)}(Q_i^k) \leq c_{\text{Moran}}.r_i$, so

$$\sum_{i,k} \left(\overline{r}_{z^*}^{(u)}(Q_i^k)\right)^{D+\eta} \leq M_{\text{Moran}}.(c_{\text{Moran}})^{D+\eta} =: K(\eta). \tag{2.37}$$

For each cylinder Q_i^k, we remove the blocks from Q_i^k to produce the cylinder $\widehat{Q}_i^k$. The sets $\widehat{Q}_i^k$ then cover $\tilde{S}_X(n, t_0, x_0^*)$, and so the images $\chi_{z^*}^{(u)}(\widehat{Q}_i^k)$ cover $\chi_{z^*}^{(u)}\left(\tilde{S}_X(n, t_0, x_0^*)\right)$.

If the lengths of the cylinders $\widehat{Q}_i^k$ and Q_i^k are $\widehat{m}_i^k$ and m_i^k respectively, then as in Theorem 2.18 or 2.22 we have

$$\widehat{m}_i^k \geq \left(\frac{n}{\ell_{max}(n) + n}\right) m_i^k. \tag{2.38}$$

Combining this with Lemma 2.39(i) now gives that $\sup_{i,k} \operatorname{diam} \chi_{z^*}^{(u)}(\widehat{Q}_i^k) \to 0$ as $\rho \to 0$, and that if ρ is sufficiently small then $\widehat{m}_i^k \geq nt_0$ for all i, k. We also showed in the proof of Theorem 2.18/2.22 that if $\widehat{m}_i^k \geq nt_0$ then

$$\frac{m_i^k - \widehat{m}_i^k}{\widehat{m}_i^k} \leq q(n),$$

where $q(n) := c_5\epsilon(n) + c_3 n^{-\frac{1}{4}} + c_6 n^{-1} \to 0$ as $n \to \infty$.

So, by Lemma 2.40, if n is sufficiently large that $q(n) < 1$ then

$$\overline{r}_{z^*}^{(u)}\left(Q_i^k\right) \geq c\left(\overline{r}_{z^*}^{(u)}\left(\widehat{Q}_i^k\right)\right)^{1+\gamma q(n)}.$$

Combining this with (2.15) gives (for all sufficiently small ρ)

$$\sum_{i,k}\left(\overline{r}_{z^*}^{(u)}\left(\widehat{Q}_i^k\right)\right)^{(1+\gamma q(n))(D+\eta)} \leq c^{-(D+\eta)}K(\eta),$$

and so

$$\sum_{i,k}\left(\operatorname{diam}\chi_{z^*}^{(u)}\left(\widehat{Q}_i^k\right)\right)^{(1+\gamma q(n))(D+\eta)} \leq 2^{(1+\gamma q(n))(D+\eta)}c^{-(D+\eta)}K(\eta).$$

But the sets $\chi_{z^*}^{(u)}\left(\widehat{Q}_i^k\right)$ cover $\chi_{z^*}^{(u)}\left(\tilde{S}_X(n,t_0,x_0^*)\right)$, and $\sup_{i,k}\operatorname{diam}\chi_{z^*}^{(u)}\left(\widehat{Q}_i^k\right) \to 0$ as $\rho \to 0$, so

$$\dim_H \chi_{z^*}^{(u)}\left(\tilde{S}_X(n,t_0,x_0^*)\right) \leq (1+\gamma q(n))\left(\dim_H \chi_{z^*}^{(u)}\left(\tilde{S}'_X(n,t_0,x_0^*)\right) + \eta\right).$$

As η was arbitrary,

$$\dim_H \chi_{z^*}^{(u)}\left(\tilde{S}_X(n,t_0,x_0^*)\right) \leq (1+\gamma q(n))\dim_H \chi_{z^*}^{(u)}\left(\tilde{S}'_X(n,t_0,x_0^*)\right).$$

Because $\bigcup_{t_0\geq 1} S_X(n,t_0) = G(n)$ we have $\bigcup_{t_0\geq 1}\tilde{S}_X(n,t_0,x_0^*) = \tilde{G}(n)\cap[x_0^*]$, and so $\dim_H \chi_{z^*}^{(u)}(\tilde{S}_X(n,t_0,x_0^*)) \to \dim_H \chi_{z^*}^{(u)}(\tilde{G}(n)\cap[x_0^*])$ as $t_0 \to \infty$. Also we have $\tilde{S}'_X(n,t_0,x_0^*) \subseteq G'(n)\cap[x_0^*] \subseteq \operatorname{Bdd}_{X_A^+}(\hat{\mathbf{g}}^+)\cap[x_0^*]$. So

$$\dim_H \chi_{z^*}^{(u)}\left(\operatorname{Bdd}_{X_A^+}(\hat{\mathbf{g}}^+)\cap[x_0^*]\right) \geq (1+\gamma q(n))^{-1}\dim_H \chi_{z^*}^{(u)}\left(\tilde{G}(n)\cap[x_0^*]\right). \quad (2.39)$$

Now, we have defined $\Gamma(n) = \chi\left(\pi_+^{-1}(\tilde{G}(n))\right)$; as such, $\Gamma(n)$ depends only on the future in the coding, in the sense defined in section 2.4.1; by comparing this definition with the definition of $\chi_{z^*}^{(u)}$ we see that

$$[\Gamma(n)]_{R^*}\cap W_{R^*}^u(z^*) = \chi_{z^*}^{(u)}\left(\tilde{G}(n)\cap[x_0^*]\right).$$

And so from Lemma 2.29 we have

$$\dim_H[\Gamma(n)]_{R^*} = \dim_H \chi_{z^*}^{(u)}\left(\tilde{G}(n)\cap[x_0^*]\right) + t^{(s)}.$$

Similarly, $\mathrm{Bdd}_\Lambda(\mathbf{g})$ depends only on the future (in the coding), and here (as explained in section 2.4.1) we have more simply $[\mathrm{Bdd}_\Lambda(\mathbf{g})]_{R^*} = \mathrm{Bdd}_\Lambda(\mathbf{g}) \cap R^*$. Furthermore we showed that $y \in \mathrm{Bdd}_{X_A^+}(\hat{\mathbf{g}}^+) \cap [x_0^*]$ if and only if $\chi_{z^*}^{(u)}(y) \in \mathrm{Bdd}_\Lambda(\mathbf{g})$, and so

$$[\mathrm{Bdd}_\Lambda(\mathbf{g})]_{R^*} \cap W_{R^*}^u(z^*) = \chi_{z^*}^{(u)}\left(\mathrm{Bdd}_{X_A^+}(\hat{\mathbf{g}}^+) \cap [x_0^*]\right).$$

Applying Lemma 2.29 to this we get

$$\dim_H [\mathrm{Bdd}_\Lambda(\mathbf{g})]_{R^*} = \dim_H \chi_{z^*}^{(u)}\left(\mathrm{Bdd}_{X_A^+}(\hat{\mathbf{g}}^+) \cap [x_0^*]\right) + t^{(s)}.$$

Substituting into (2.39) then gives

$$\left(\dim_H [\mathrm{Bdd}_\Lambda(\mathbf{g})]_{R^*} - t^{(s)}\right) \geq (1+\gamma q(n))^{-1}\left(\dim_H [\Gamma(n)]_{R^*} - t^{(s)}\right).$$

By maximizing over all the rectangles of the Markov partition we then get

$$\dim_H \mathrm{Bdd}_\Lambda(\mathbf{g}) - t^{(s)} \geq (1+\gamma q(n))^{-1}\left(\dim_H \Gamma(n) - t^{(s)}\right).$$

And now by taking $n \to \infty$ as in the remarks at the end of section 2.3.2 we have

$$\dim_H \mathrm{Bdd}_\Lambda(\mathbf{g}) - t^{(s)} \geq \dim_H \left(\bigcup_{m\geq 1}\bigcap_{n\geq m} \Gamma(n)\right) - t^{(s)},$$

i.e.

$$\dim_H \mathrm{Bdd}_\Lambda(\mathbf{g}) \geq \dim_H \Gamma.$$

The opposite inequality follows from the fact that $\mathrm{Bdd}_\Lambda(\mathbf{g}) \subseteq \Gamma$, which is a special case of Lemma 2.41. □

Now we have the analogue of Theorem 2.24:

Lemma 2.43 $\dim_H L_\Lambda(\mathbf{g}, \mathbf{F}, r) = \dim_H \Gamma$.

Proof: In light of Lemmas 2.41 and 2.42 it remains to prove $\dim_H L_\Lambda(\mathbf{g}, \mathbf{F}, r) \geq \dim_H \mathrm{Bdd}_\Lambda(\mathbf{g})$.

In the proof of Theorem 2.24 we defined a block-adding process $\xi : S_K \to S'_K$ in terms of functions $\tilde{\mathbf{g}} : X_A^+ \to \mathbb{R}^d$ and $\tilde{r} : X_A^+ \to \mathbb{R}^+$. We will make use of this block-adding process, taking $\tilde{\mathbf{g}} = \hat{\mathbf{g}}^+$ and $\tilde{r} = \hat{r}^+$. That is, the block-adding process is defined on the set

$$S_K := \left\{ x \in X_A^+ : \left\| (\hat{\mathbf{g}}^+)^n(x) \right\| < K \ \forall n \right\},$$

and we showed in the proof of Theorem 2.24 that $S'_K \subseteq L_{X_A^+}(\hat{\mathbf{g}}^+, \mathbf{F}, \hat{r}^+)$.

If $z^* = \chi(x^*) \in \operatorname{int} R^*$ for some rectangle R^*, we look at the restriction of χ to the set

$$S_K(x_0^*) := S_K \cap [x_0^*].$$

We define $S'_K(x_0^*)$ to be the image of $S_K(x_0^*)$ under ξ. Then $S'_K(x_0^*) \subseteq [x_0^*]$, so we may compare the dimensions of $\chi_{z^*}^{(u)}(S_K(x_0^*))$ and $\chi_{z^*}^{(u)}(S'_K(x_0^*))$.

Now, in Theorem 2.24, where we had a coding $\chi : X_A^+ \to J$, we showed that $\dim_H \chi(S'_K) \geq \dim_H \chi(S_K)$. By modifying this part of the proof appropriately we can show for our hyperbolic diffeomorphism that

$$\dim_H \chi_{z^*}^{(u)}(S'_K(x_0^*)) \geq \dim_H \chi_{z^*}^{(u)}(S_K(x_0^*)).$$

(We omit the details: the only changes are that we look at covers of $S'_K(x_0^*)$ rather than the whole of S'_K, and that we have the coding $\chi_{z^*}^{(u)} : X_A^+ \to W_{R^*}^u(z^*)$ rather than χ. We went through these modifications explicitly in the proof of Lemma 2.42, where we were copying the arguments of Theorem 2.22.)

Now $\chi_{z^*}^{(u)}(S'_K(x_0^*)) \subseteq \chi_{z^*}^{(u)}\left(L_{X_A^+}(\hat{\mathbf{g}}^+, \mathbf{F}, \hat{r}^+) \cap [x_0^*]\right) \subseteq L_\Lambda(\mathbf{g}, \mathbf{F}, r) \cap W_{R^*}^u(z^*)$ from (2.36), so

$$\dim_H \left(L_\Lambda(\mathbf{g}, \mathbf{F}, r) \cap W_{R^*}^u(z^*)\right) \geq \dim_H \chi_{z^*}^{(u)}(S_K(x_0^*)).$$

For the right hand side we have $\bigcup_{K \geq 1} S_K(x_0^*) = \operatorname{Bdd}_{X_A^+}(\hat{\mathbf{g}}^+) \cap [x_0^*]$ so

$$\dim_H \chi_{z^*}^{(u)}\left(\operatorname{Bdd}_{X_A^+}(\hat{\mathbf{g}}^+) \cap [x_0^*]\right) = \sup_{K \geq 1} \dim_H \chi_{z^*}^{(u)}(S_K(x_0^*));$$

and $\chi_{z^*}^{(u)}\left(\operatorname{Bdd}_{X_A^+}(\hat{\mathbf{g}}^+) \cap [x_0^*]\right) = \operatorname{Bdd}_\Lambda(\mathbf{g}) \cap W_{R^*}^u(z^*)$ from (2.34). Hence

$$\dim_H \left(L_\Lambda(\mathbf{g}, \mathbf{F}, r) \cap W_{R^*}^u(z^*)\right) \geq \dim_H \left(\operatorname{Bdd}_\Lambda(\mathbf{g}) \cap W_{R^*}^u(z^*)\right).$$

So from Lemma 2.30 we have

$$\dim_H \left(L_\Lambda(\mathbf{g}, \mathbf{F}, r) \cap R^*\right) \geq \dim_H \left(\mathrm{Bdd}_\Lambda(\mathbf{g}) \cap R^*\right),$$

and maximizing over all rectangles of the Markov partition gives

$$\dim_H L_\Lambda(\mathbf{g}, \mathbf{F}, r) \geq \dim_H \mathrm{Bdd}_\Lambda(\mathbf{g}).$$

□

We are now ready to complete the proof of our main result for hyperbolic diffeomorphisms.

Proof of Theorem 2.38: We have already shown that $\dim_H \mathrm{Bdd}_\Lambda(\mathbf{g}) = \dim_H L_\Lambda(\mathbf{g}, \mathbf{F}, r) = \dim_H \Gamma$, and we also know that $\mathrm{Bdd}_\Lambda(\mathbf{g}) \subseteq \mathrm{Ave}_\Lambda(\mathbf{g}, \mathbf{0})$. Thus it is sufficient to prove that $\dim_H \Gamma \geq \dim_H \mathrm{Ave}_\Lambda(\mathbf{g}, \mathbf{0})$. This is where we will make use of the BS-dimension. We set $u(z) = v^{(u)}(z)$, which we know is Hölder continuous.

Because $\int_\Lambda \mathbf{g}\, d\nu = \mathbf{0}$ and ν is an equilibrium state we know that $\mathbf{0} \in \operatorname{int} \mathcal{D}(\mathbf{g})$. So by Theorem 2.32,

$$\dim_u \mathrm{Ave}_\Lambda(\mathbf{g}, \mathbf{0}) = \sup \left\{ \frac{h_\mu(f)}{\int_\Lambda u\, d\mu} : \mu \in \mathcal{M}(\Lambda) \text{ and } \int_\Lambda \mathbf{g}\, d\mu = \mathbf{0} \right\},$$

and the supremum is attained for a measure which is an equilibrium state for a Hölder continuous function on Λ. We will now assume that ν is the measure for which the supremum is attained. (It is sufficient to prove this case, because although the set Γ is defined in terms of ν, one consequence of Lemma 2.42 is that $\dim_H \Gamma$ is independent of the ν that was given.) That is, we may assume that

$$\dim_u \mathrm{Ave}_\Lambda(\mathbf{g}, \mathbf{0}) = \frac{h_\nu(f)}{\int_\Lambda u\, d\nu}.$$

And by Proposition 2.31 this implies

$$\dim_u \mathrm{Ave}_\Lambda(\mathbf{g}, \mathbf{0}) = \dim_u \nu.$$

If μ is the measure on X_A which corresponds to ν, we have $\mu\left(\pi_+^{-1}(\tilde{G}(n))\right) = 1$ for all n, and so $\mu\left(\bigcap_{n\geq 1}\pi_+^{-1}(\tilde{G}(n))\right) = 1$. Hence $\nu\left(\chi\left(\bigcap_{n\geq 1}\pi_+^{-1}(\tilde{G}(n))\right)\right) = 1$, and so from the definition of BS-dimension for a measure we have

$$\dim_u \chi\left(\bigcap_{n\geq 1}\pi_+^{-1}(\tilde{G}(n))\right) \geq \dim_u \mathrm{Ave}_\Lambda(\mathbf{g},\mathbf{0}).$$

So from Lemma 2.37,

$$\dim_H \chi\left(\bigcap_{n\geq 1}\pi_+^{-1}(\tilde{G}(n))\right) \geq \dim_H \mathrm{Ave}_\Lambda(\mathbf{g},\mathbf{0}).$$

But $\chi\left(\bigcap_{n\geq 1}\pi_+^{-1}(\tilde{G}(n))\right) \subseteq \bigcap_{n\geq 1}\chi\left(\pi_+^{-1}(\tilde{G}(n))\right) \subseteq \Gamma$, and so

$$\dim_H \Gamma \geq \dim_H \mathrm{Ave}_\Lambda(\mathbf{g},\mathbf{0}),$$

which is what we wanted to prove. □

2.4.4 Results for conformal hyperbolic flows

We can adapt the methods for diffeomorphisms to work for flows. For the flow $\phi_t : \Lambda \longrightarrow \Lambda$ the sets we are interested in are

$$\begin{aligned}
\mathrm{Ave}_\Lambda(\mathbf{g},\boldsymbol{\alpha}) &:= \left\{x \in \Lambda : \frac{1}{t}\int_0^t \mathbf{g}\left(\phi_\tau x\right)\, d\tau \longrightarrow \boldsymbol{\alpha} \ \text{ as } t \longrightarrow \infty.\right\};\\
\mathrm{Bdd}_\Lambda(\mathbf{g}) &:= \left\{x \in \Lambda : \int_0^t \mathbf{g}\left(\phi_\tau x\right)\, d\tau \text{ is bounded}\right\};\\
L_\Lambda(\mathbf{g},\mathbf{F}) &:= \left\{x \in \Lambda : \int_0^t \mathbf{g}\left(\phi_\tau x\right)\, d\tau = \mathbf{F}(t) + O(1)\right\}.
\end{aligned}$$

Our main result for flows will be this:

Theorem 2.44 *Let $\phi_t : \Lambda \longrightarrow \Lambda$ be a conformal hyperbolic flow, and let $\mathbf{g} : \Lambda \longrightarrow \mathbb{R}^d$ be Hölder continuous. Suppose there exists an equilibrium state ν on Λ such that $\int_\Lambda \mathbf{g}\, d\nu = \mathbf{0}$. Then*

$$\dim_H \mathrm{Bdd}_\Lambda(\mathbf{g}) = \dim_H \mathrm{Ave}_\Lambda(\mathbf{g},\mathbf{0}).$$

Furthermore if the components of $\mathbf{g}$ *are cohomologously independent then*

$$\dim_H L_\Lambda(\mathbf{g}, \mathbf{F}) = \dim_H \mathrm{Ave}_\Lambda(\mathbf{g}, \mathbf{0}),$$

whenever the continuous function $\mathbf{F} : \mathbb{R}^+ \to \mathbb{R}^d$ *has the property that* $\sup_{\tau \in [0,1]} \|\mathbf{F}(t+\tau) - \mathbf{F}(t)\| \to 0$ *as* $t \to \infty$.

As for diffeomorphisms, by throwing out components if necessary we may assume throughout that the components of $\mathbf{g}$ are cohomologously independent.

The flow is modeled by a suspended flow $\sigma_t^r : X_A^r \to X_A^r$ over a subshift of finite type $\sigma : X_A \to X_A$. We look at a point $z^* = \rho(x^*, 0) \in \operatorname{int} T^*$, where T^* is one of the Markov sections used to construct the symbolic dynamics. Then the set $W_{T^*}^u(z^*)$ is thought of as being represented by the cylinder $[x_0^*]$ in X_A^+, via the map $\chi_{z^*}^{(u)} : [x_0^*] \to W_{T^*}^u(z^*)$. As for diffeomorphisms, if we have a cylinder $C \subseteq [x_0^*]$ in X_A^+ we can look at $\underline{r}_{z^*}^{(u)}(C)$ and $\overline{r}_{z^*}^{(u)}(C)$. Lemmas 2.39 and 2.40 carry over word for word to the flow case.

We let $\hat{\mathbf{g}} : X_A^r \to \mathbb{R}^d$ be the pullback of $\mathbf{g}$ via the coding map ρ. We can also transfer the equilibrium state ν to X_A^r, giving an equilibrium state which we will call $\hat{\nu}$, which satisfies $\int_{X_A^r} \hat{\mathbf{g}}\, d\hat{\nu} = \mathbf{0}$. Now as explained in section 1.4 we can consider the function $\mathcal{I}\hat{\mathbf{g}} : X_A \to \mathbb{R}^d$ defined by

$$\mathcal{I}\hat{\mathbf{g}}(x) = \int_0^{r(x)} \mathbf{g}(x, s)\, ds.$$

Furthermore from Proposition 1.7 the measure $\hat{\nu}$ is of the form $(\mu \times l)/(\int_{X_A} r\, d\mu)$, where μ is an equilibrium state on X_A and l is Lebesgue measure, and it follows that

$$\int_{X_A} \mathcal{I}\hat{\mathbf{g}}\, d\mu = \mathbf{0}.$$

We now look at the function $(\mathcal{I}\hat{\mathbf{g}})^{(u)}$ which depends only on future co-ordinates and which is cohomologous to $\mathcal{I}\hat{\mathbf{g}}$. As before, this defines a function $(\mathcal{I}\hat{\mathbf{g}})^+ : X_A^+ \to \mathbb{R}^d$ by taking $(\mathcal{I}\hat{\mathbf{g}})^+(\pi_+ x) = (\mathcal{I}\hat{\mathbf{g}})^{(u)}(x)$. We set $\tilde{\mathbf{g}} = (\mathcal{I}\hat{\mathbf{g}})^+$.

Now suppose $y \in [x_0^*]$; then $\chi_{z^*}^{(u)}(y) = \rho(x, 0)$ where x is given by

$$x = (\ldots\, x_{-2}^*\, x_{-1}^*\, x_0^*\, y_1\, y_2\, \ldots).$$

We have

$$\begin{aligned} y \in \mathrm{Bdd}_{X_A^+}(\tilde{\mathbf{g}}) &\Leftrightarrow x \in \mathrm{Bdd}_{X_A}\left((\mathcal{I}\hat{\mathbf{g}})^{(u)}\right) \\ &\Leftrightarrow x \in \mathrm{Bdd}_{X_A}(\mathcal{I}\hat{\mathbf{g}}) \\ &\Leftrightarrow (x, 0) \in \mathrm{Bdd}_{X_A^r}(\hat{\mathbf{g}}), \end{aligned}$$

and so

$$y \in \mathrm{Bdd}_{X_A^+}(\tilde{\mathbf{g}}) \Leftrightarrow \chi_{z^*}^{(u)}(y) \in \mathrm{Bdd}_\Lambda(\mathbf{g}). \tag{2.40}$$

We could prove a similar statement for $\mathrm{Ave}_\Lambda(\mathbf{g}, \mathbf{0})$ but this will not be necessary; however we do need to consider the sets $L_\Lambda(\mathbf{g}, \mathbf{F})$. We look at $r^{(u)} : X_A \to \mathbb{R}^+$, which is cohomologous to r (where r is the roof function for the suspended flow X_A^r), and use this to define $r^+ : X_A^+ \to \mathbb{R}^+$ as we have done previously. Then we have

$$\begin{aligned} y \in L_{X_A^+}(\tilde{\mathbf{g}}, \mathbf{F}, r^+) &\Leftrightarrow x \in L_{X_A}\left((\mathcal{I}\hat{\mathbf{g}})^{(u)}, \mathbf{F}, r^{(u)}\right) \\ &\Leftrightarrow x \in L_{X_A}(\mathcal{I}\hat{\mathbf{g}}, \mathbf{F}, r) \\ &\Leftrightarrow x \in L_{X_A^r}(\hat{\mathbf{g}}, \mathbf{F}) \end{aligned}$$

(because $\int_0^{r^n(x)} \hat{\mathbf{g}}(\sigma_\tau^r(x, 0))\, d\tau = (\mathcal{I}\hat{\mathbf{g}})^n(x)$ for all n). And so

$$y \in L_{X_A^+}(\tilde{\mathbf{g}}, \mathbf{F}, r^+) \Leftrightarrow \chi_{z^*}^{(u)}(y) \in L_\Lambda(\mathbf{g}, \mathbf{F}). \tag{2.41}$$

We now consider the σ-invariant sets $\tilde{G}(n)$ defined in section 2.4.3. We define sets $\Gamma(n)$ for the flow by

$$\Gamma(n) = \bigcup_{t \in \mathbb{R}} \phi_t\left(\left\{\rho(x, 0) : \pi_+ x \in \tilde{G}(n)\right\}\right).$$

Thus $\Gamma(n)$ depends only on the future in the coding, in the sense defined in section 2.4.1. And, as for diffeomorphisms, we define

$$\Gamma = \bigcup_{m \geq 1} \bigcap_{n \geq m} \Gamma(n).$$

Lemma 2.45 *If* $\mathbf{F}$ *satisfies the condition that* $\sup_{\tau\in[0,1]} \|\mathbf{F}(t+\tau)-\mathbf{F}(t)\| \to 0$ *as* $t\to\infty$, *then* $L_\Lambda(\mathbf{g},\mathbf{F}) \subseteq \Gamma$.

Proof: Because $L_\Lambda(\mathbf{g},\mathbf{F})$ and Γ are both ϕ-invariant it is sufficient to show that $L_\Lambda(\mathbf{g},\mathbf{F}) \cap T^* \subseteq \Gamma$ for each rectangle T^*. Suppose $z=\rho(x,0)\in L_\Lambda(\mathbf{g},\mathbf{F})\cap T^*$, and let $y=\pi_+ x$; then from (2.41) we have $y\in L_{X_A^+}(\tilde{\mathbf{g}},\mathbf{F},r^+)$. So from the proof of Lemma 2.41 we have $y\in\bigcap_{n\geq n_0}\tilde{G}(n)$ for some n_0. Hence

$$z\in\bigcap_{n\geq n_0}\left\{\rho(x,0):\pi_+x\in\tilde{G}(n)\right\}\subseteq\bigcap_{n\geq n_0}\Gamma(n)\subseteq\Gamma.$$

□

Lemma 2.46 $\dim_H \mathrm{Bdd}_\Lambda(\mathbf{g}) = \dim_H \Gamma$.

Proof: Let $z^*=\rho(x^*,0)\in\operatorname{int}T^*$ for a rectangle T^*. As in Lemma 2.42 we consider the block-adding process

$$\xi:\tilde{S}_X(n,t_0,x_0^*)\longrightarrow\tilde{S}'_X(n,t_0,x_0^*)$$

(defined in terms of the function $\tilde{\mathbf{g}}:X_A^+\to\mathbb{R}^d$), where

$$\tilde{S}_X(n,t_0,x_0^*)=S_X(n,t_0)\cap\tilde{G}(n)\cap[x_0^*].$$

We know that $\tilde{S}'_X(n,t_0,x_0^*)\subseteq G'(n)\subseteq\mathrm{Bdd}_{X_A^+}(\tilde{\mathbf{g}})$.

We look at the images of the sets $\tilde{S}_X(n,t_0,x_0^*)$ and $\tilde{S}'_X(n,t_0,x_0^*)$ under the map $\chi_{z^*}^{(u)}:[x_0^*]\to W^u_{T^*}(z^*)$. By repeating the calculations of Lemma 2.42 we find that

$$\dim_H\chi_{z^*}^{(u)}\left(\mathrm{Bdd}_{X_A^+}(\tilde{\mathbf{g}})\cap[x_0^*]\right)\geq(1+\gamma q(n))^{-1}\dim_H\chi_{z^*}^{(u)}\left(\tilde{G}(n)\cap[x_0^*]\right).\quad(2.42)$$

Now, by definition we have

$$[\Gamma(n)]_{T^*}=\bigcup_{t\in(-\tau,\tau)}\phi_t\left(\left\{\rho(x,0):\pi_+x\in\tilde{G}(n)\text{ and }x_0=x_0^*\right\}\right),$$

and so

$$[\Gamma(n)]_{T^*}\cap W^u_{T^*}(z^*)=\chi_{z^*}^{(u)}\left(\tilde{G}(n)\cap[x_0^*]\right).$$

So from Lemma 2.29,

$$\dim_H [\Gamma(n)]_{T^*} = \dim_H \chi_{z^*}^{(u)} \left(\tilde{G}(n) \cap [x_0^*]\right) + t^{(s)} + 1.$$

And similarly,

$$\dim_H [\mathrm{Bdd}_\Lambda(\mathbf{g})]_{T^*} = \dim_H \chi_{z^*}^{(u)} \left(\mathrm{Bdd}_{X_A^+}(\tilde{\mathbf{g}}) \cap [x_0^*]\right) + t^{(s)} + 1.$$

Substituting into (2.42) gives

$$\dim_H [\mathrm{Bdd}_\Lambda(\mathbf{g})]_{T^*} - t^{(s)} - 1 \geq (1 + \gamma q(n))^{-1} \left(\dim_H [\Gamma(n)]_{T^*} - t^{(s)} - 1\right),$$

and by maximizing over all rectangles T^* we get

$$\dim_H \mathrm{Bdd}_\Lambda(\mathbf{g}) - t^{(s)} - 1 \geq (1 + \gamma q(n))^{-1} \left(\dim_H \Gamma(n) - t^{(s)} - 1\right).$$

Taking $n \to \infty$ gives

$$\dim_H \mathrm{Bdd}_\Lambda(\mathbf{g}) - t^{(s)} - 1 \geq \dim_H \left(\bigcup_{m \geq 1} \bigcap_{n \geq m} \Gamma(n)\right) - t^{(s)} - 1,$$

i.e.

$$\dim_H \mathrm{Bdd}_\Lambda(\mathbf{g}) \geq \dim_H \Gamma.$$

The opposite inequality follows from Lemma 2.45. □

Lemma 2.47 $\dim_H L_\Lambda(\mathbf{g}, \mathbf{F}) = \dim_H \Gamma$.

Proof: We follow the proof of Lemma 2.43. We aim to show that $\dim_H L_\Lambda(\mathbf{g}, \mathbf{F}) \geq \dim_H \Gamma$.

Let $z^* = \rho(x^*, 0) \in \operatorname{int} T^*$ for a rectangle T^*. As in Lemma 2.43 we consider the block-adding process

$$\xi : S_K(x_0^*) \to S'_K(x_0^*),$$

where $S_K(x_0^*) = S_K \cap [x_0^*]$. This block-adding process ξ is defined in terms of functions $\tilde{\mathbf{g}} : X_A^+ \to \mathbb{R}^d$ and $\tilde{r} : X_A^+ \to \mathbb{R}^+$; we have already defined our function $\tilde{\mathbf{g}}$ in terms of $\mathbf{g}$, and we set $\tilde{r} = r^+$ as defined above. The construction of ξ then gives that $S'_K(x_0^*) \subseteq L_{X_A^+}(\tilde{\mathbf{g}}, \mathbf{F}, r^+)$.

We look at the images of the sets $S_K(x_0^*)$ and $S'_K(x_0^*)$ under the map $\chi_{z^*}^{(u)} : [x_0^*] \to W^u_{T^*}(z^*)$. As in Lemma 2.43 we can show

$$\dim_H \chi_{z^*}^{(u)}(S'_K(x_0^*)) \geq \dim_H \chi_{z^*}^{(u)}(S_K(x_0^*)).$$

Now $\chi_{z^*}^{(u)}(S'_K(x_0^*)) \subseteq \chi_{z^*}^{(u)}(L_{X_A^+}(\tilde{\mathbf{g}}, \mathbf{F}, r^+) \cap [x_0^*]) \subseteq L_\Lambda(\mathbf{g}, \mathbf{F}) \cap W^u_{T^*}(z^*)$ from (2.41), so

$$\dim_H (L_\Lambda(\mathbf{g}, \mathbf{F}) \cap W^u_{T^*}(z^*)) \geq \dim_H \chi_{z^*}^{(u)}(S_K(x_0^*)).$$

But $\bigcup_{K\geq 1} S_K(x_0^*) = \mathrm{Bdd}_{X_A^+}(\tilde{\mathbf{g}}) \cap [x_0^*]$ so

$$\dim_H \left(\chi_{z^*}^{(u)}(\mathrm{Bdd}_{X_A^+}(\tilde{\mathbf{g}}) \cap [x_0^*])\right) = \sup_{K\geq 1} \dim_H \chi_{z^*}^{(u)}(S_K(x_0^*));$$

and $\chi_{z^*}^{(u)}(\mathrm{Bdd}_{X_A^+}(\tilde{\mathbf{g}}) \cap [x_0^*]) = \mathrm{Bdd}_\Lambda(\mathbf{g}) \cap W^u_{T^*}(z^*)$ from (2.40). So

$$\dim_H (L_\Lambda(\mathbf{g}, \mathbf{F}) \cap W^u_{T^*}(z^*)) \geq \dim_H (\mathrm{Bdd}_\Lambda(\mathbf{g}) \cap W^u_{T^*}(z^*)).$$

So from Lemma 2.30,

$$\dim_H (L_\Lambda(\mathbf{g}, \mathbf{F}) \cap T^*_\tau) \geq \dim_H (\mathrm{Bdd}_\Lambda(\mathbf{g}) \cap T^*_\tau),$$

where $T^*_\tau = \bigcup_{t\in(-\tau,\tau)} \phi_t(T^*)$. If we allow T^* to vary, the sets T^*_τ cover Λ. So by maximizing over all rectangles T^* we get

$$\dim_H L_\Lambda(\mathbf{g}, \mathbf{F}) \geq \dim_H \mathrm{Bdd}_\Lambda(\mathbf{g}).$$

$\square$

As for diffeomorphisms, we can put all these results together to complete the proof of our main result.

Proof of Theorem 2.44: It remains to show that $\dim_H \Gamma \geq \dim_H \mathrm{Ave}_\Lambda(\mathbf{g}, \mathbf{0})$. Like for diffeomorphisms, we do this by making use of the BS-dimension. We take $u(z) = v^{(u)}(z)$.

Because $\int_\Lambda \mathbf{g}\, d\nu = \mathbf{0}$, we know that $\mathbf{0} \in \mathrm{int}\, \mathcal{D}(\mathbf{g})$. So by Theorem 2.35,

$$\dim_u \mathrm{Ave}_\Lambda(\mathbf{g}, \mathbf{0}) = \sup \left\{ \frac{h_\mu(\phi)}{\int_\Lambda u\, d\mu} : \mu \in \mathcal{M}(\Lambda) \text{ and } \int_\Lambda \mathbf{g}\, d\mu = \mathbf{0} \right\},$$

and the supremum is attained for an equilibrium state on Λ. As in the proof of Theorem 2.38 we may assume that ν is the measure for which the supremum is attained, i.e.

$$\dim_u \mathrm{Ave}_\Lambda(\mathbf{g}, \mathbf{0}) = \frac{h_\nu(\phi)}{\int_\Lambda u\, d\nu}.$$

And so by Theorem 2.34,

$$\dim_u \mathrm{Ave}_\Lambda(\mathbf{g}, \mathbf{0}) = \dim_u \nu.$$

Now, recall that the measure $\hat{\nu}$ on X_A^r is of the form $(\mu \times l)/(\int_{X_A} r\, d\mu)$. And we have $\mu\left(\pi_+^{-1}(\tilde{G}(n))\right) = 1$ for all n, which implies $\mu\left(\bigcap_{n\geq 1} \pi_+^{-1}(\tilde{G}(n))\right) = 1$. Thus

$$\hat{\nu}\left(\left\{(x,t) \in X_A^r : \pi_+ x \in \bigcap_{n\geq 1} \tilde{G}(n),\ 0 \leq t < r(x)\right\}\right) = 1.$$

But

$$\rho\left(\left\{(x,t) \in X_A^r : \pi_+ x \in \bigcap_{n\geq 1} \tilde{G}(n),\ 0 \leq t < r(x)\right\}\right)$$
$$\subseteq \bigcup_{t\in\mathbb{R}} \phi_t\left(\left\{\rho(x,0) : \pi_+ x \in \bigcap_{n\geq 1} \tilde{G}(n)\right\}\right),$$

so we have

$$\nu\left(\bigcup_{t\in\mathbb{R}} \phi_t\left(\left\{\rho(x,0) : \pi_+ x \in \bigcap_{n\geq 1} \tilde{G}(n)\right\}\right)\right) = 1.$$

And so from the definition of BS-dimension for ν,

$$\dim_u \bigcup_{t\in\mathbb{R}} \phi_t\left(\left\{\rho(x,0) : \pi_+ x \in \bigcap_{n\geq 1} \tilde{G}(n)\right\}\right) \geq \dim_u \nu = \dim_u \mathrm{Ave}_\Lambda(\mathbf{g}, \mathbf{0}).$$

Now applying Lemma 2.37 gives

$$\dim_H \bigcup_{t\in\mathbb{R}} \phi_t\left(\left\{\rho(x,0) : \pi_+ x \in \bigcap_{n\geq 1} \tilde{G}(n)\right\}\right) \geq \dim_H \mathrm{Ave}_\Lambda(\mathbf{g}, \mathbf{0}).$$

But

$$\bigcup_{t\in\mathbb{R}} \phi_t\left(\left\{\rho(x,0) : \pi_+ x \in \bigcap_{n\geq 1} \tilde{G}(n)\right\} \subseteq \bigcap_{n\geq 1}\left(\bigcup_{t\in\mathbb{R}} \phi_t\left(\left\{\rho(x,0) : \pi_+ x \in \tilde{G}(n)\right\}\right)\right),$$

and the right hand side here is just $\bigcap_{n\geq 1}\Gamma(n)$, so

$$\bigcup_{t\in\mathbb{R}}\phi_t\left(\left\{\rho(x,0):\pi_+x\in\bigcap_{n\geq 1}\tilde{G}(n)\right\}\subseteq\Gamma,\right.$$

and hence

$$\dim_H\Gamma\geq\dim_H\mathrm{Ave}_\Lambda(\mathbf{g},\mathbf{0}).$$

$\square$

Chapter 3

Directions in homology for periodic orbits

In this chapter we consider the periodic orbits of a transitive Anosov flow $\phi_t : M \to M$. As with any closed curve in M, if we are given a periodic orbit γ we can look at its homology class $[\gamma] \in H_1(M, \mathbb{Z})$. Our aim is to describe how the periodic orbits of ϕ are distributed amongst the homology classes.

In particular we want to be able to talk about the 'directions' of homology classes. For this to make sense we have to regard a homology class as being an element of $\mathbb{Z}^b$, in the way described in section 1.6. That is, we use the fact that $H_1(M, \mathbb{Z})$ is isomorphic to $\mathbb{Z}^b \oplus \mathrm{Tor}$, and ignore the torsion component. We allow ourselves to write $[\gamma]$ to mean the point in $\mathbb{Z}^b$ that represents (the torsion-free part of) the homology class of γ, as well as the homology class itself. Of course, to make this definition we have to fix a particular choice of the map $H_1(M, \mathbb{Z}) \to \mathbb{Z}^b$; to put it another way, we are choosing a basis for the torsion-free part of $H_1(M, \mathbb{Z})$ which will correspond to the standard basis of $\mathbb{Z}^b$.

We assume that the Betti number b is strictly positive, otherwise everything becomes trivial.

Once we have a point in $\mathbb{Z}^b$ (or $\mathbb{R}^b$) we can define its direction as being the projection onto the (Euclidean) unit sphere. The projection map $p_S : \mathbb{R}^b \setminus \{0\} \to$

S^{b-1} is defined by $p_S(\mathbf{v}) = \frac{\mathbf{v}}{\|\mathbf{v}\|_2}$, where $\|\cdot\|_2$ is the usual Euclidean norm; if $[\gamma] \neq 0$ then we can define

$$\theta(\gamma) = p_S([\gamma]).$$

(When $[\gamma] = 0$ we will leave $\theta(\gamma)$ undefined.)

Now for $T > 0$ we can define a measure ν_T on the unit sphere by

$$\nu_T = \frac{1}{\pi(T)} \sum_{l(\gamma) \leq T, [\gamma] \neq 0} \delta_{\theta(\gamma)},$$

where $\delta_{\theta(\gamma)}$ is the Dirac measure at $\theta(\gamma)$. This fails to be a probability measure because there may be some periodic orbits with $[\gamma] = 0$; but we do know that $\#\{\gamma : l(\gamma) \leq T, [\gamma] \neq 0\} \sim \pi(T)$ as $T \to \infty$, so the measure of the whole sphere tends to 1 as $T \to \infty$.

The problem is to determine whether ν_T has a (weak*) limit as $T \to \infty$, and, if so, to describe the limit ν_∞.

We will show that the limit always exists, and the nature of ν_∞ depends on the *asymptotic cycle* for the measure of maximal entropy.

Asymptotic cycles were introduced by Schwartzman in [Sch]. We look at the first cohomology group of M, $H^1(M, \mathbb{R})$, defined to be the set of smooth closed 1-forms on M, modulo the exact 1-forms. $H^1(M, \mathbb{R})$ is the dual space to $H_1(M, \mathbb{R})$, and so also has dimension b. Suppose μ is an invariant measure on M; then for any closed 1-form ω we can look at the integral $\int \omega(\mathcal{X})\, d\mu$, where $\mathcal{X}$ is the tangent vector field for the flow. If ω is an *exact* 1-form, so that $\omega = df$ for some function f, then $\omega(\mathcal{X})$ is simply the derivative of f with respect to the flow; since μ is invariant this implies $\int \omega(\mathcal{X})\, d\mu = 0$. So, more generally, we have $\int \omega_1(\mathcal{X})\, d\mu = \int \omega_2(\mathcal{X})\, d\mu$ whenever ω_1 and ω_2 belong to the same cohomology class. Hence there is a well-defined map $\Phi_\mu : H^1(M, \mathbb{R}) \to \mathbb{R}$ given by

$$\Phi_\mu([\omega]) = \int \omega(\mathcal{X})\, d\mu.$$

This Φ_μ is then called the μ-asymptotic cycle (or winding cycle). It can be regarded as being an element of $H_1(M, \mathbb{R})$.

It is shown in [Sha1] that the flow is homologically full (i.e. every homology class contains a periodic orbit) if and only if there is some fully supported invariant measure μ for which Φ_μ is identically zero. The constant h^* in Theorem 1.9 is then given by $\sup\{h_\mu(\phi) : \Phi_\mu = 0\}$. In particular, if $\Phi_{\mu_0} = 0$, where μ_0 is the measure of maximal entropy, then the asymptotic in Theorem 1.9 holds with $h^* = h$.

As mentioned above, this asymptotic cycle Φ_{μ_0} turns out to be particularly important for our problem. We write $\Phi_0 = \Phi_{\mu_0}$.

If ϕ_t is the geodesic flow on the unit tangent bundle of a Riemannian manifold, then we always have $\Phi_0 = 0$. But for a general Anosov flow it is possible to have $\Phi_0 \neq 0$.

3.1 Obtaining homology from integration

As described in the introduction, by ignoring torsion and choosing a basis for $H_1(M,\mathbb{Z})$, we are thinking of $[\gamma]$ as being a point in $\mathbb{Z}^b$. We would like to write $[\gamma]$ as an integral of an $\mathbb{R}^b$-valued function around γ. The following methods are described in [BaL] and [Sha2].

Suppose we consider closed 1-forms $\omega_1, \omega_2, \ldots, \omega_b$, whose cohomology classes form a basis for $H^1(M,\mathbb{R})$. Then for a periodic orbit γ we can look at the vector

$$\mathbf{v}_\gamma = \left(\int_\gamma \omega_1(\mathcal{X}), \int_\gamma \omega_2(\mathcal{X}), \ldots, \int_\gamma \omega_b(\mathcal{X})\right).$$

(Here we are writing $\int_\gamma \psi$ to mean $\int_{s=0}^{l(\gamma)} \psi(\gamma(s))\, ds$.) Taken over the basis we have chosen for $H_1(M,\mathbb{Z})$, the vectors $\mathbf{v}_{\gamma_i}$ must span $\mathbb{R}^b$. It follows (by a suitable change-of-basis transformation of the ω_i) that the 1-forms $\omega_1, \omega_2, \ldots, \omega_b$ can be chosen so that we actually have $\mathbf{v}_\gamma = [\gamma]$. That is,

$$[\gamma] = \int_\gamma \mathbf{F},$$

where $F_i = \omega_i(\mathcal{X})$.

Clearly this is closely related to the definition of asymptotic cycles. Indeed, for an invariant measure μ we have by definition

$$\left(\Phi_\mu([\omega_1]), \ldots, \Phi_\mu([\omega_b])\right) = \left(\int F_1 \, d\mu, \ldots, \int F_b \, d\mu\right),$$

and so if we are regarding Φ_μ as an element of $H_1(M, \mathbb{R})$, then its representation as a point in $\mathbb{R}^b$ is given by $\int \mathbf{F} \, d\mu$.

There is an alternative approach if we work with the suspended flow $\sigma_t^r : X_A^r \to X_A^r$ which is the model for the Anosov flow.

We look back at the definition of the coding $\rho : X_A^r \to M$ in terms of Markov sections T_j. Because M is compact, we can find $\epsilon_M > 0$ such that if we have two closed curves γ_1 and γ_2 for which $d(\gamma_1(t), \gamma_2(t)) < \epsilon_M$ for all t, then they must belong to the same homology class. We want to choose Markov sections which are small enough, and close enough, such that all the T_j have diameter much smaller than ϵ_M, and the distance between T_i and T_j for which $A_{ij} = 1$ is also much smaller than ϵ_M.

For each T_j $(1 \leq j \leq k)$ we pick a point $z_j \in T_j$. Also pick some base point $z \in M$. Then for any j we pick a curve c_j which joins z to z_j. Similarly, for any pair (i, j) such that $A_{ij} = 1$ we choose a curve c_{ij} which joins z_i to z_j. Because the Markov sections are small, and T_i and T_j are close if $A_{ij} = 1$, we can make sure that the curve c_{ij} is short (i.e. much smaller than ϵ_M). We can then define a closed curve γ_{ij} which consists of the curve c_i from z to z_i, followed by the curve c_{ij} from z_i to z_j, followed by the reverse of c_j, which takes us back to z. We can now define the function $\mathbf{g} : X_A \to H_1(M, \mathbb{Z})$ by taking $\mathbf{g}(x)$ to be the homology class of $\gamma_{x_0 x_1}$ (or its representation as a point in $\mathbb{Z}^b$). We see that if γ is a periodic orbit in M which corresponds to the periodic orbit $\{x, \sigma x, \ldots, \sigma^{n-1} x\}$ in X_A, then $\mathbf{g}^n(x) = [\gamma]$.

This function $\mathbf{g}$ is Hölder continuous – indeed it is locally constant, depending only on the first two co-ordinates. We can therefore construct a Hölder continuous function $\mathbf{f}$ on X_A^r such that $\mathcal{I}\mathbf{f} = \mathbf{g}$. Then if γ is a periodic orbit in M, and x is a point in X_A such that $\rho(x,0) \in \gamma$, we have

$$\int_0^{l(\gamma)} \mathrm{f}(\sigma_t^r(x,0))\,dt = [\gamma].$$

The drawback of this approach is that, as noted in section 1.3, there is not a one-to-one correspondence between periodic orbits of ϕ and those of the suspended flow.

3.2 The case $\Phi_0 \neq 0$

We make use of a result of Lalley which can be thought of as being a form of the weak law of large numbers. It applies to a general hyperbolic flow $\phi_t : \Lambda \to \Lambda$.

Theorem 3.1 (Lalley [Lal]) *Let $\phi_t : \Lambda \to \Lambda$ be a hyperbolic flow, and let $F : \Lambda \to \mathbb{R}$ be continuous. Then for any $\epsilon > 0$, we have*

$$\lim_{T\to\infty} \frac{1}{\pi(T)} \# \left\{ \gamma : l(\gamma) \leq T, \left| \frac{\int_\gamma F}{l(\gamma)} - \int F\, d\mu_0 \right| > \epsilon \right\} = 0.$$

We can also obtain a version of this theorem for a vector-valued function $\mathbf{F} : \Lambda \to \mathbb{R}^d$, by applying the theorem to each component of $\mathbf{F}$.

We apply this to the function $\mathbf{F} : M \to \mathbb{R}^b$ defined in the previous section, for which $[\gamma] = \int_\gamma \mathbf{F}$. The quantity $\int \mathbf{F}\, d\mu_0$ in Theorem 3.1 is then simply Φ_0, the asymptotic cycle for the measure of maximal entropy. Thus we have

$$\lim_{T\to\infty} \frac{1}{\pi(T)} \# \left\{ \gamma : l(\gamma) \leq T, \left\| \frac{[\gamma]}{l(\gamma)} - \Phi_0 \right\|_2 > \epsilon \right\} = 0. \tag{3.1}$$

(We write $\|.\|_2$ to mean the Euclidean norm on $\mathbb{R}^b$.)

Now suppose $\Phi_0 \neq 0$. Given an open set $D \subset S^{b-1}$, we look the sector $p_S^{-1}(D) \subset \mathbb{R}^b \setminus \{0\}$. Since this is itself an open set, it follows from (3.1) that if $\Phi_0 \in p_S^{-1}(D)$ (or equivalently if $\Phi_0/\|\Phi_0\|_2 \in D$) then

$$\lim_{T\to\infty} \frac{1}{\pi(T)} \# \left\{ \gamma : l(\gamma) \leq T, \frac{[\gamma]}{l(\gamma)} \in p_S^{-1}(D) \right\} = 1.$$

But $p_S([\gamma]/l(\gamma)) = p_S([\gamma]) = \theta(\gamma)$ and so

$$\lim_{T\to\infty} \frac{1}{\pi(T)} \# \{\gamma : l(\gamma) \leq T, \theta(\gamma) \in D\} = 1.$$

Thus we have shown that if D is any open neighbourhood of $\Phi_0/\|\Phi_0\|_2$ in S^{b-1} we have

$$\lim_{T\to\infty} \nu_T(D) = 1.$$

So we have proved the following:

Theorem 3.2 *If $\Phi_0 \neq 0$ then the measures ν_T have a weak* limit ν_∞ as $T \to \infty$, and ν_∞ is the Dirac measure at $\Phi_0/\|\Phi_0\|_2$.*

3.3 The case $\Phi_0 = 0$

3.3.1 A norm on homology

From now on we will assume that ϕ is $C^{1+\epsilon}$, which ensures that the functions $\omega(\mathcal{X})$ are Hölder continuous. This is merely a simplifying assumption – if ϕ is only C^1 then we can work instead with the suspended flow $\sigma_t^r : X_A^r \to X_A^r$, where by the construction in section 3.1 we know that homology is given by integrating the Hölder continuous function $\mathbf{f}$ around a periodic orbit. But by assuming that the flow is $C^{1+\epsilon}$ we can analyse the flow itself and not have to start by working with the model.

We define $\mathcal{B}_\phi = \{\Phi_\mu : \mu \in \mathcal{M}(M)\}$. This is a compact convex set in $\mathbb{R}^b$, and we always have $\Phi_0 \in \operatorname{int} \mathcal{B}_\phi$ (cf. Theorem 2.19 and the set $\mathcal{D}(\mathbf{g})$). We define a function $\mathfrak{h} : \operatorname{int} \mathcal{B}_\phi \to \mathbb{R}$ by

$$\mathfrak{h}(\rho) = \sup \{h_\mu(\phi) : \Phi_\mu = \rho\}.$$

Immediately from the definition we can see that $\mathfrak{h}(\Phi_0) = h$, and if $\rho \neq \Phi_0$ then $\mathfrak{h}(\rho) < h$. Furthermore by making use of the thermodynamic formalism it can be shown ([Sha1], [BaL]) that $\mathfrak{h}$ is a strictly concave, analytic function (with $\nabla\mathfrak{h}(\Phi_0) = 0$), and that $\mathcal{H} := -\nabla^2\mathfrak{h}(\Phi_0)$ is positive definite.

Now suppose $\Phi_0 = 0$; we define a norm $\|\cdot\|$ on $H_1(M,\mathbb{R})$ (or $\mathbb{R}^b$) by

$$\|\rho\|^2 = \langle \rho, \mathcal{H}\rho \rangle.$$

We then have

$$\mathfrak{h}(\rho) = h - \tfrac{1}{2}\|\rho\|^2 + O(\|\rho\|^3). \tag{3.2}$$

We will see that this norm is, in a sense, the 'correct' norm to use on homology in our problem.

We can also define a function $\mathfrak{p} : H^1(M,\mathbb{R}) \to \mathbb{R}$ by $\mathfrak{p}([\omega]) = P(\omega(\mathcal{X}))$, where P is the pressure as defined in section 1.4. Then $\nabla\mathfrak{p}$ maps $\mathbb{R}^b$ to $\operatorname{int}\mathcal{B}_\phi$ (regarding $H^1(M,\mathbb{R})$ and $H_1(M,\mathbb{R})$ as copies of $\mathbb{R}^b$ according to the bases we have chosen), and indeed $\nabla\mathfrak{p} : \mathbb{R}^b \to \operatorname{int}\mathcal{B}_\phi$ is a diffeomorphism ([BaL]). Given $\rho \in \operatorname{int}\mathcal{B}_\phi$, we write $\xi(\rho) = (\nabla\mathfrak{p})^{-1}(\rho)$.

In fact the functions $-\mathfrak{h}$ and $\mathfrak{p}$ are a Legendre conjugate pair: the map $-\nabla\mathfrak{h} : \operatorname{int}\mathcal{B}_\phi \to \mathbb{R}^b$ is the inverse of $\nabla\mathfrak{p}$, and we have

$$\mathfrak{h}(\rho) = \mathfrak{p}(\xi(\rho)) - \langle \xi(\rho), \rho \rangle.$$

3.3.2 An 'equidistribution' result

Given a set $A \subseteq \mathbb{Z}^b$, we define

$$\pi(T, A) = \#\{\gamma : l(\gamma) \leq T, [\gamma] \in A\}.$$

We also define

$$d_{\|\cdot\|}(A) = \lim_{r\to\infty} \frac{\#\{\alpha \in A : \|\alpha\| \leq r\}}{\#\{\alpha \in \mathbb{Z}^b : \|\alpha\| \leq r\}},$$

if this limit exists. This quantity $d_{\|\cdot\|}(A)$ is called the *density* of A with respect to the norm $\|\cdot\|$. We could define the density for any norm on $\mathbb{R}^b$, but the norm $\|\cdot\|$ from section 3.3.1 is of particular interest. We will prove the following result:

Theorem 3.3 *Suppose $\Phi_0 = 0$. Then if $A \subseteq \mathbb{Z}^b$ is a set for which the density $d_{\|\cdot\|}(A)$ exists, we have*

$$\lim_{T\to\infty} \frac{\pi(T, A)}{\pi(T)} = d_{\|\cdot\|}(A).$$

In the case of the geodesic flow on surfaces of constant curvature, a result of this type was originally suggested by Petridis and Risager ([PR]).

Suppose we take A to be a set of the form $p_S^{-1}(D) \cap \mathbb{Z}^b$, where $D \subset S^{b-1}$ is an open set. Then we have

$$\frac{\pi(T, A)}{\pi(T)} = \frac{1}{\pi(T)} \# \{\gamma : l(\gamma) \leq T, \theta(\gamma) \in D\} = \nu_T(D).$$

Furthermore, if U is any bounded open subset of $\mathbb{R}^b$, then by a standard integration result its volume $\mathrm{Vol}(U)$ is given by

$$\mathrm{Vol}(U) = \lim_{r\to\infty} \frac{1}{r^b} \# \left\{\alpha \in \mathbb{Z}^b : \alpha/r \in U\right\}.$$

By applying this to the sets $p_S^{-1}(D) \cap B_{\|\cdot\|}$ and $B_{\|\cdot\|}$, where $B_{\|\cdot\|}$ is the unit ball for the norm $\|\cdot\|$, we have

$$d_{\|\cdot\|}(A) = \frac{\mathrm{Vol}(p_S^{-1}(D) \cap B_{\|\cdot\|})}{\mathrm{Vol}(B_{\|\cdot\|})}.$$

And so as a consequence of Theorem 3.3 we have the following:

Theorem 3.4 *If $\Phi_0 = 0$, then the measures ν_T have a weak* limit ν_∞ which is fully-supported on S^{b-1}; indeed for any open set $D \subseteq S^{b-1}$ we have*

$$\nu_\infty(D) = \frac{\mathrm{Vol}(p_S^{-1}(D) \cap B_{\|\cdot\|})}{\mathrm{Vol}(B_{\|\cdot\|})}.$$

The main ingredient of the proof of Theorem 3.3 is an asymptotic estimate for $\pi(T, \alpha(T))$, in the case where $\alpha(T)$ depends linearly on T. In order to state this result precisely we need to consider the way that $H_1(M, \mathbb{Z})$ is embedded as a lattice inside $H_1(M, \mathbb{R})$. We choose a fundamental domain $\mathcal{F}$ for $H_1(M, \mathbb{Z})$. Then given $\rho \in H_1(M, \mathbb{R})$ we can define $\lfloor \rho \rfloor \in H_1(M, \mathbb{Z})$ as the unique element of $H_1(M, \mathbb{Z})$ for which $\rho - \lfloor \rho \rfloor \in \mathcal{F}$.

Theorem 3.5 ([BaL]) *Suppose* $0 \in \operatorname{int} \mathcal{B}_\phi$. *Then for* $\rho \in \operatorname{int} \mathcal{B}_\phi$,

$$\pi(T, \lfloor T\rho \rfloor) \sim C(\rho)\, e^{\langle \xi(\rho), T\rho - \lfloor T\rho \rfloor \rangle} \frac{e^{\mathfrak{h}(\rho)T}}{T^{1+b/2}} \quad \textit{as } T \to \infty,$$

uniformly for ρ *in any compact subset of* $\operatorname{int} \mathcal{B}_\phi$. *Here* $C(\rho)$ *is given by*

$$C(\rho) = \frac{\sqrt{|\det \nabla^2 \mathfrak{h}(\rho)|}}{(2\pi)^{b/2} \mathfrak{p}(\xi(\rho))}.$$

We also note that if $\Phi_0 = 0$ then the condition of Theorem 1.9 is satisfied and the flow is weak-mixing ([Sha1]); in particular we know that $\pi(T) \sim e^{hT}/hT$.

We will also need the following consequence of a 'central limit theorem' from [Lal]:

Lemma 3.6 *Suppose* $\Phi_0 = 0$. *Then given any* $\epsilon > 0$, *we can find* $\Delta > 0$ *such that*

$$\limsup_{T \to \infty} \frac{1}{\pi(T)} \# \left\{ \gamma : l(\gamma) \le T, \frac{\|[\gamma]\|}{\sqrt{T}} > \Delta \right\} < \epsilon.$$

(This is obtained from the results in [Lal] by considering the functions F_i from section 3.1.)

Proof of Theorem 3.3: From Theorem 3.5 we can find $\delta > 0$ such that

$$\lim_{T \to \infty} \sup_{\|\rho\| \le \delta} \left| \frac{T^{1+b/2} \pi(T, \lfloor T\rho \rfloor)}{C(\rho) e^{\mathfrak{h}(\rho)T} e^{\langle \xi(\rho), T\rho - \lfloor T\rho \rfloor \rangle}} - 1 \right| = 0.$$

By considering $\rho = \alpha/T$ where $\alpha \in H_1(M, \mathbb{Z})$ we have

$$\lim_{T \to \infty} \sup_{\|\alpha\| \le \delta T} \left| \frac{T^{1+b/2} \pi(T, \alpha)}{C(\alpha/T) e^{\mathfrak{h}(\alpha/T)T}} - 1 \right| = 0.$$

Now let Δ be a large positive constant. We have $\Delta\sqrt{T} \le \delta T$ for all sufficiently large T, and so

$$\lim_{T \to \infty} \sup_{\|\alpha\| \le \Delta\sqrt{T}} \left| \frac{T^{1+b/2} \pi(T, \alpha)}{C(\alpha/T) e^{\mathfrak{h}(\alpha/T)T}} - 1 \right| = 0.$$

Because $C(\rho)$ is a continuous function of ρ it follows that

$$\lim_{T \to \infty} \sup_{\|\alpha\| \le \Delta\sqrt{T}} \left| \frac{T^{1+b/2} \pi(T, \alpha)}{C(0) e^{\mathfrak{h}(\alpha/T)T}} - 1 \right| = 0, \tag{3.3}$$

and we know $C(0) = \sigma^{-b}(2\pi)^{-b/2}h^{-1}$, where $\sigma^b = |\det \nabla^2 \mathfrak{h}(0)|^{-1/2}$. Also from (3.2) we have (for small ρ) $\mathfrak{h}(\rho) = h - \frac{1}{2}\|\rho\|^2 + r(\rho)$, where $|r(\rho)| \leq c\|\rho\|^3$. So for $\|\alpha\| \leq \Delta\sqrt{T}$ we can write $\mathfrak{h}(\alpha/T)T = hT - \|\alpha\|^2/2T + r(\alpha/T)T$, with $|r(\alpha/T)T| \leq c\Delta^3 T^{-1/2}$. Substituting this into (3.3) and making use of the fact that $r(\alpha/T)T$ is bounded we get

$$\lim_{T\to\infty} \sup_{\|\alpha\|\leq\Delta\sqrt{T}} \left| \frac{hT^{1+b/2}\pi(T,\alpha)}{e^{hT}} - \frac{e^{-\|\alpha\|^2/2T}e^{r(\alpha/T)T}}{(2\pi)^{b/2}\sigma^b} \right| = 0.$$

Now we would like to sum over $\{\alpha \in A : \|\alpha\| \leq \Delta\sqrt{T}\}$. We have $\#\{\alpha : \|\alpha\| \leq \Delta\sqrt{T}\} = O(T^{b/2})$, and so

$$\lim_{T\to\infty} \frac{1}{T^{b/2}} \sum_{\alpha\in A:\|\alpha\|\leq\Delta\sqrt{T}} \left(\frac{hT^{1+b/2}\pi(T,\alpha)}{e^{hT}} - \frac{e^{-\|\alpha\|^2/2T}e^{r(\alpha/T)T}}{(2\pi)^{b/2}\sigma^b} \right) = 0,$$

i.e.

$$\lim_{T\to\infty} \sum_{\alpha\in A:\|\alpha\|\leq\Delta\sqrt{T}} \left(\frac{hT\pi(T,\alpha)}{e^{hT}} - \frac{e^{-\|\alpha\|^2/2T}e^{r(\alpha/T)T}}{(2\pi)^{b/2}\sigma^b T^{b/2}} \right) = 0. \tag{3.4}$$

In order to deal with the second term we need the following lemma, which we will prove later:

Lemma 3.7 *(i) Given any $\epsilon > 0$, there exists Δ such that*

$$\frac{1}{(2\pi)^{b/2}\sigma^b T^{b/2}} \sum_{\|\alpha\|\geq\Delta\sqrt{T}} e^{-\|\alpha\|^2/2T} < \epsilon \quad \forall T \geq 1.$$

(ii) If the set A has density $d_{\|\cdot\|}(A)$ with respect to the norm $\|\cdot\|$, then

$$\lim_{T\to\infty} \frac{1}{(2\pi)^{b/2}\sigma^b T^{b/2}} \sum_{\alpha\in A} e^{-\|\alpha\|^2/2T} = d_{\|\cdot\|}(A).$$

Now since $|r(\alpha/T)T| \leq c\Delta^3 T^{-1/2}$ we have

$$\liminf_{T\to\infty} \sum_{\alpha\in A:\|\alpha\|\leq\Delta\sqrt{T}} \frac{e^{-\|\alpha\|^2/2T}e^{r(\alpha/T)T}}{(2\pi)^{b/2}\sigma^b T^{b/2}}$$

$$\geq \left(\liminf_{T\to\infty} e^{-c\Delta^3 T^{-1/2}} \right) \left(\liminf_{T\to\infty} \sum_{\alpha\in A:\|\alpha\|\leq\Delta\sqrt{T}} \frac{e^{-\|\alpha\|^2/2T}}{(2\pi)^{b/2}\sigma^b T^{b/2}} \right).$$

So, given $\epsilon > 0$, we know from Lemma 3.7 that for all sufficiently large Δ we have

$$\liminf_{T\to\infty} \sum_{\alpha\in A:\|\alpha\|\leq\Delta\sqrt{T}} \frac{e^{-\|\alpha\|^2/2T}e^{r(\alpha/T)T}}{(2\pi)^{b/2}\sigma^b T^{b/2}} \geq d_{\|\cdot\|}(A) - \epsilon.$$

And so from (3.4),

$$\liminf_{T\to\infty} \sum_{\alpha\in A:\|\alpha\|\leq\Delta\sqrt{T}} \frac{hT\pi(T,\alpha)}{e^{hT}} \geq d_{\|\cdot\|}(A) - \epsilon.$$

Furthermore we know $\pi(T) \sim e^{hT}/hT$, and so

$$\liminf_{T\to\infty} \frac{\pi(T,A)}{\pi(T)} \geq \liminf_{T\to\infty} \frac{1}{\pi(T)} \sum_{\alpha\in A:\|\alpha\|\leq\Delta\sqrt{T}} \pi(T,\alpha) \geq d_{\|\cdot\|}(A) - \epsilon.$$

Since ϵ was arbitrary this shows

$$\liminf_{T\to\infty} \frac{\pi(T,A)}{\pi(T)} \geq d_{\|\cdot\|}(A).$$

Similarly, we have

$$\limsup_{T\to\infty} \sum_{\alpha\in A:\|\alpha\|\leq\Delta\sqrt{T}} \frac{e^{-\|\alpha\|^2/2T}e^{r(\alpha/T)T}}{(2\pi)^{b/2}\sigma^b T^{b/2}}$$
$$\leq \left(\limsup_{T\to\infty} e^{c\Delta^3 T^{-1/2}}\right)\left(\limsup_{T\to\infty} \sum_{\alpha\in A:\|\alpha\|\leq\Delta\sqrt{T}} \frac{e^{-\|\alpha\|^2/2T}}{(2\pi)^{b/2}\sigma^b T^{b/2}}\right),$$

and so from Lemma 3.7 (not needing part (i) here),

$$\limsup_{T\to\infty} \sum_{\alpha\in A:\|\alpha\|\leq\Delta\sqrt{T}} \frac{e^{-\|\alpha\|^2/2T}e^{r(\alpha/T)T}}{(2\pi)^{b/2}\sigma^b T^{b/2}} \leq d_{\|\cdot\|}(A),$$

from which we get

$$\limsup_{T\to\infty} \frac{1}{\pi(T)} \sum_{\alpha\in A:\|\alpha\|\leq\Delta\sqrt{T}} \pi(T,\alpha) \leq d_{\|\cdot\|}(A).$$

Now,

$$\frac{\pi(T,A)}{\pi(T)} = \frac{1}{\pi(T)} \sum_{\alpha\in A:\|\alpha\|\leq\Delta\sqrt{T}} \pi(T,\alpha) + \frac{1}{\pi(T)} \sum_{\alpha\in A:\|\alpha\|>\Delta\sqrt{T}} \pi(T,\alpha).$$

Given $\epsilon > 0$, from Lemma 3.6 we can find Δ sufficiently large that

$$\limsup_{T\to\infty} \frac{1}{\pi(T)} \sum_{\alpha\in A:\|\alpha\|>\Delta\sqrt{T}} \pi(T,\alpha) < \epsilon.$$

We then have

$$\limsup_{T\to\infty} \frac{\pi(T,A)}{\pi(T)} \leq d_{\|\cdot\|}(A) + \epsilon.$$

Since ϵ was arbitrary this implies

$$\limsup_{T\to\infty} \frac{\pi(T,A)}{\pi(T)} \leq d_{\|\cdot\|}(A).$$

□

It remains to prove Lemma 3.7. We will deduce this as a special case of the following result:

Lemma 3.8 *Let $\|\cdot\|$ be an arbitrary norm on $\mathbb{R}^k$ and let $f : \mathbb{R}^k \to \mathbb{R}$ be a continuous integrable function such that $f(t)$ depends only on $\|t\|$. Suppose also that $|f| \leq F$, where $F : \mathbb{R}^k \to \mathbb{R}^+$ is an integrable function depending only on $\|t\|$, such that $F(\|t\|)$ is decreasing in $\|t\|$. Then*

(i) Given any $\epsilon > 0$, there exists $R > 0$ such that

$$\sum_{\alpha\in\mathbb{Z}^k:\|\alpha\|>xR} x^{-k}\,|f(\alpha/x)| < \epsilon \quad \forall x \geq 1.$$

(ii) If $A \subseteq \mathbb{Z}^k$ has density $d_{\|\cdot\|}(A)$ with respect to the norm $\|\cdot\|$, then

$$\lim_{x\to\infty} \sum_{\alpha\in A} x^{-k} f(\alpha/x) = d_{\|\cdot\|}(A) \int_{\mathbb{R}^k} f(t)\,dt.$$

Remark: The condition that f is bounded by an integrable function decreasing in $\|t\|$ is stronger than necessary. However, some control is needed on the behaviour of f as $\|t\| \to \infty$, as there exist continuous integrable functions f for which the limit in (ii) does not exist.

Proof: For any $\alpha \in \mathbb{Z}^k \setminus \{0\}$, define

$$\alpha' = \alpha \left(\frac{\|\alpha\|_2 - \frac{1}{4}}{\|\alpha\|_2} \right),$$

where $\|\cdot\|_2$ is the usual Euclidean norm on $\mathbb{R}^k$. Because any two norms on $\mathbb{R}^k$ are equivalent, we can find constants c_1 and c_2 such that

$$c_1 \|t\|_2 \leq \|t\| \leq c_2 \|t\|_2$$

for any $t \in \mathbb{R}^k$. By applying this to $t = \alpha' - \alpha$ we see that for any $\alpha \neq 0$, $\|\alpha'\| \leq \|\alpha\| - c_1/4$.

Now let $r = c_1/4c_2$, and consider $B_2(\alpha', r)$, the open ball centre α' and radius r in the Euclidean norm. This ball has a volume v which is independent of α, and for different values of α these balls are disjoint. Note also that if $t \in B_2(\alpha', r)$ then $\|t - \alpha'\| < c_1/4$ and so $\|t\| < \|\alpha\|$.

Similarly, if $x \in [1, \infty)$ then for any $t \in B_2(\alpha'/x, r/x)$ we have $\|t\| < \|\alpha/x\|$, and so $F(t) \geq F(\alpha/x)$. Hence

$$\int_{B_2(\alpha'/x, r/x)} F(t)\, dt \geq v x^{-k} F(\alpha/x) \geq v x^{-k} |f(\alpha/x)|.$$

Summing over $\|\alpha\| \geq xR$ and using the fact that the balls $B_2(\alpha'/x, r/x)$ are disjoint, we get

$$\sum_{\alpha \in \mathbb{Z}^k : \|\alpha\| > xR} x^{-k} |f(\alpha/x)| \leq \frac{1}{v} \int_{\|t\| > R-1} F(t)\, dt.$$

But F is integrable, so this last integral can be made as small as desired by choosing R sufficiently large. This completes the proof of part (i).

Now for part (ii): because of (i), it is sufficient to prove

$$\lim_{x \to \infty} \sum_{\alpha \in A : \|\alpha\| \leq xR} f(\alpha/x) = d_{\|\cdot\|}(A) \int_{\|t\| \leq R} f(t)\, dt,$$

for large R.

We may at this point assume that f is strictly positive. (If not, we can write f in terms of its positive and negative parts, $f = f_+ - f_-$, and consider $f_+ + 1$ and $f_- + 1$.)

The case $A = \mathbb{Z}^k$ is straightforward from the Riemann definition of integration. We wish to extend this result to a general set A for which $d_{\|\cdot\|}(A)$ exists. For a large integer n, define the sets

$$S_1 := \left\{t \in \mathbb{R}^k : \|t\| \leq Rn^{-1}\right\};$$
$$S_m := \left\{t \in \mathbb{R}^k : R(m-1)n^{-1} < \|t\| \leq Rmn^{-1}\right\}, \quad 2 \leq m \leq n.$$

So $\{t : \|t\| \leq R\}$ is the disjoint union of these sets. We claim that for any m,

$$P_m(x, A) := \frac{\#\{\alpha \in A : x^{-1}\alpha \in S_m\}}{\#\{\alpha \in \mathbb{Z}^k : x^{-1}\alpha \in S_m\}} \to d_{\|\cdot\|}(A) \text{ as } x \to \infty.$$

For $m = 1$ this is immediate from the definition of the density. For $m > 1$, we see that $P_m(x, A)$ is equal to

$$\frac{\#\{\alpha \in A : x^{-1}\|\alpha\| \leq Rmn^{-1}\} - \#\{\alpha \in A : x^{-1}\|\alpha\| \leq R(m-1)n^{-1}\}}{\#\{\alpha \in \mathbb{Z}^k : x^{-1}\|\alpha\| \leq Rmn^{-1}\} - \#\{\alpha \in \mathbb{Z}^k : x^{-1}\|\alpha\| \leq R(m-1)n^{-1}\}}.$$

By the definition of $d_{\|\cdot\|}$ we have

$$\lim_{x\to\infty} \frac{\#\{\alpha \in A : x^{-1}\|\alpha\| \leq Rmn^{-1}\}}{\#\{\alpha \in \mathbb{Z}^k : x^{-1}\|\alpha\| \leq Rmn^{-1}\}} = d_{\|\cdot\|}(A).$$

Also

$$\lim_{x\to\infty} x^{-k}\#\left\{\alpha \in \mathbb{Z}^k : x^{-1}\|\alpha\| \leq Rmn^{-1}\right\} = \mathrm{Vol}\left\{t : \|t\| \leq Rmn^{-1}\right\},$$

which follows from the case $A = \mathbb{Z}^k$ by letting f approximate the indicator function of $\{t : \|t\| \leq Rmn^{-1}\}$. Hence, as $x \to \infty$, $P_m(x, A)$ converges to

$$d_{\|\cdot\|}(A)\left(\frac{\mathrm{Vol}\{t : \|t\| \leq Rmn^{-1}\}}{\mathrm{Vol}\{t : \|t\| \leq Rmn^{-1}\} - \mathrm{Vol}\{t : \|t\| \leq R(m-1)n^{-1}\}}\right)$$
$$- d_{\|\cdot\|}(A)\left(\frac{\mathrm{Vol}\{t : \|t\| \leq R(m-1)n^{-1}\}}{\mathrm{Vol}\{t : \|t\| \leq Rmn^{-1}\} - \mathrm{Vol}\{t : \|t\| \leq R(m-1)n^{-1}\}}\right),$$

which is equal to $d_{\|\cdot\|}(A)$ as claimed.

We now use the fact that $f(t)$ depends only on $\|t\|$. Together with the assumption that f is positive and (uniformly) continuous on $\|t\| \leq R$, this implies that for any $\epsilon > 0$, we can choose n sufficiently large that

$$\sup_{t \in S_m} f(t) < (1+\epsilon) \inf_{t \in S_m} f(t),$$

simultaneously for all $1 \leq m \leq n$. And for each such n, we know that if x is sufficiently large then for all $1 \leq m \leq n$,

$$d_{\|.\|}(A) - \epsilon < P_m(x, A) < d_{\|.\|}(A) + \epsilon.$$

We then have

$$\begin{aligned}
\sum_{\alpha \in A:\, x^{-1}\alpha \in S_m} x^{-k} f(\alpha/x) &\geq \sum_{\alpha \in A:\, x^{-1}\alpha \in S_m} x^{-k} \inf_{t \in S_m} f(t) \\
&\geq \left(d_{\|.\|}(A) - \epsilon\right) \sum_{\alpha \in \mathbb{Z}^k:\, x^{-1}\alpha \in S_m} x^{-k} \inf_{t \in S_m} f(t) \\
&\geq \frac{d_{\|.\|}(A) - \epsilon}{1+\epsilon} \sum_{\alpha \in \mathbb{Z}^k:\, x^{-1}\alpha \in S_m} x^{-k} f(\alpha/x);
\end{aligned}$$

so, by summing over m,

$$\sum_{\alpha \in A: \|\alpha\| \leq xR} x^{-k} f(\alpha/x) \geq \frac{d_{\|.\|}(A) - \epsilon}{1+\epsilon} \sum_{\alpha \in \mathbb{Z}^k: \|\alpha\| \leq xR} x^{-k} f(\alpha/x).$$

Taking $x \to \infty$ and using the result for the case $A = \mathbb{Z}^k$ gives

$$\liminf_{x \to \infty} \sum_{\alpha \in A: \|\alpha\| \leq xR} x^{-k} f(\alpha/x) \geq \frac{d_{\|.\|}(A) - \epsilon}{1+\epsilon} \int_{\|t\| \leq R} f(t) dt.$$

A similar argument shows

$$\limsup_{x \to \infty} \sum_{\alpha \in A: \|\alpha\| \leq xR} x^{-k} f(\alpha/x) \leq \left(d_{\|.\|}(A) + \epsilon\right)(1+\epsilon) \int_{\|t\| \leq R} f(t) dt.$$

And now taking $\epsilon \to 0$ completes the proof. $\square$

Proof of Lemma 3.7: We apply Lemma 3.8 with $f(t) = e^{-\|t\|^2/2}$ and $x = \sqrt{T}$, and note that

$$\int_{\mathbb{R}^b} e^{-\|t\|^2/2}\, dt = \frac{(2\pi)^{b/2}}{\sqrt{\det \mathcal{H}}} = (2\pi)^{b/2} \sigma^b.$$

$\square$

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