THE 1p-SHELL HYPERNUCLEI

AND

THE Λ-N INTERACTION

BY

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To Betty

With many thanks

RESEARCH EXPERIENCE

Since obtaining the Degree of Bachelor of Science in Physics in the University of Manchester in June 1960, I have been engaged on research in the Department of Theoretical Physics. The results of this research are contained in this thesis. From August 1962 I have held the position of Assistant Lecturer in the above department.

I declare that the work reported in this thesis has not been previously submitted in support of any degree at this or any other institution.

ABSTRACT

The p-shell hypernuclei have been studied in detail using two and three-body central Yukawa Λ -N interactions. Intermediate coupling and a range of density distributions are considered for the core nuclei. The Λ wave function and volume integrals are obtained by numerical solution of the appropriate two-body (Λ -core nucleus) eigenvalue problems: the results are conveniently expressed in terms of the numerical values of the relevant Slater integrals and Λ kinetic energies. The simple assumption of a spin-dependent and charge-independent two body interaction is found to be adequate to account for all the known B_Λ . Values of the spin-averaged volume integral of the two-body interaction, obtained from $_\Lambda He^5$, $_\Lambda Be^9$ and $_\Lambda C^{13}$, agree well in the absence of a three-body force; quite small upper limits can be placed on the permissable strength of the latter.

Nothing can be deduced about the range of the two-body forces or about their interactions in relative p-states. Implications of these results for the well depth in nuclear matter are discussed. The spin-dependent interaction energy is found to be completely masked by small uncertainties in the core sizes and, to a lesser extent, by uncertainties in the re-arrangement energies;

little can thus be deduced about the spin dependence. Plausible assumptions about the core sizes and energies can be made such that the spin dependence is consistent with the values obtained for the s-shell hypernuclei; conversely assuming such values for the spin dependence makes the Λ into a quite sensitive probe into small size differences. In particular, a detailed study of the mass 7 hypernuclei places special emphasis on the Λ in this role. It is found that a structure consisting of an α -particle plus two nucleons is very strongly indicated for both the T = 0 (Li^6) and the T = 1 (He^6, Be^6) are nuclei. These conclusions are further supported by a three-body (α - Λ -d) calculation of $_{\Lambda}Li^7$; a volume integral U_2, little different from the corresponding one from $_{\Lambda}H^3$, readily explains the value of B_{Λ} .

Finally, an equivalent two-body method for calculating the bound states of a three-body system is made more general by including states of arbitrary angular momentum and is then applied to $_{\Lambda}He^6$; no definite conclusion about the stability of this hypernucleus is obtained as its energy is found to lie close to that of the configuration $_{\Lambda}He^5$ + n.

CONTENTS

CHAPTER 1.

Introduction.

Phenomenological analyses of hypernuclear systems as yet form the main source of quantitative information about the interactions of the Λ^{o} hyperon with nucleons. In particular, assessments of the strength of these interactions depend essentially on the calculation of the energy, B_{Λ} , necessary to remove the Λ particle from the hypernucleus ground state, leaving the nucleons in their lowest energy configuration. Thus the measured value of this energy is used to determine the parameters of some assumed Λ-N force. This technique has mostly been applied to discuss the binding of the Λ particle to the s-shell nuclei, where the interactions take place predominantly in s states.

The work undertaken in this thesis is largely to apply the above approach to the observed p-shell hypernuclei and to discuss the implications so obtained with particular regard to the Λ -N force. To contrast with this it is interesting to remark that if one had sufficient confidence in the Λ -N interaction, e.g. assuming the relevant scattering data had been amassed and suitably analysed, then the hypernucleus could possibly be used to discuss the physics of the nuclei involved. It is thus possible to imagine the Λ particle acting as the source of a known perturbation, enabling statements to be made concerning the structure of the nucleus and also about its compressibility properties. Consideration is also given to this aspect of the hypernuclear problem, particularly in the case of the A = 7 hypernuclei, $_{\Lambda}Li^{7}$, $_{\Lambda}He^{7}$ and $_{\Lambda}Be^{7}$.

It should be noted that utilistion of the energy B_{Λ} in calculations of the above type is not the only way in which information about the Λ -N force has been obtained. Thus, assuming a spin-dependent two-body interaction, the observed branching ratios for the different decay modes of $_{\Lambda}H^{4}$ and the angular distributrions involved, very strongly suggest that the singlet interaction is more attractive than the triplet. This is also borne out for the p-shell hypernuclei by a similar analysis of the decay of $_{\Lambda}Li^{8}$ and is thus assumed to be true generally. However it is much less certain that the triplet interaction itself is attractive, as is indicated by the results for the s-shell hypernuclei, and one of the principal intentions of the work is to determine to what extent this situation is supported by the p-shell hypernuclei. The presence of a three-body Λ -N interaction could also modify this conclusion and a further aim of the work is to consider the effect of such forces in the p-shell and to estimate their magnitude. The relevant values of the Λ binding energy used are shown in table 1.[34)]

Hypernucleus	B_Λ (MeV)	Hypernucleus	B_Λ (MeV)
${}_\Lambda He^5$	3.1 ± 0.05	${}_\Lambda Be^8$	6.35 ± 0.3
${}_\Lambda He^7$	3.96 ± 0.7	${}_\Lambda Be^9$	6.5 ± 0.2
${}_\Lambda Be^7$	4.9 ± 0.5	${}_\Lambda Li^9$	8.0 ± 0.3
${}_\Lambda Li^7$	5.5 ± 0.2	${}_\Lambda B^{12}$	10.5 ± 0.2
${}_\Lambda Li^8$	6.65 ± 0.15	${}_\Lambda C^{13}$	10.9 ± 0.4

Table 1. Values of B_Λ in MeV.

1.1 The Λ -N Interactions.

For the discussion of the p-shell hypernuclei phenomonological two and three-body interactions are considered. Only charge independent, central, forces are used which are characterised by their 'volume' integrals and Yukawa ranges. Thus, following Dalitz and Downs,[1)] a soft, spin-dependent, two-body Λ -N interaction

$$V_{\Lambda N}(r_{\Lambda N}) = -\left(P_t U_t + P_s U_s\right) v(r_{\Lambda N}) \tag{1}$$

is used, where P_t and P_s are respectively the triplet and singlet spin projection operators for the Λ -N system and U_t and U_s are the corresponding volume integrals. For $v(r_{\Lambda N})$ a Yukawa shape,

$$v(r_{\Lambda N}) = \frac{\mu^2}{4\pi} \frac{\exp[-\mu r_{\Lambda N}]}{r_{\Lambda N}}, \tag{2}$$

is taken which is normalised to unity, μ^{-1} being the appropriate Yukawa range. In the main ordinary forces are considered and the two ranges, $\mu^{-1}_{2\pi}$ = 0.7 fm. and μ^{-1}_{K} = 0.4 fm. correspond respectively to the two pion and K meson exchange processes. This latter mechanism in fact gives rise to an exchange force and appropriate modifications are considered where necessary.

A central three-body force of the form

$$V_{\Lambda N_1 N_2}(r_{\Lambda N_1}, r_{\Lambda N_2}) = W(\vec{\sigma}_1 . \vec{\sigma}_2)(\vec{\tau}_1 . \vec{\tau}_2)\, \sigma(r_{\Lambda N_1})\, \sigma(r_{\Lambda N_2}) \tag{3}$$

which possesses an exchange character suggested by meson theoretical calculations[2], is considered. Here $\vec{\sigma}$ and $\vec{\tau}$ are the nucleon spin and isobaric spin operators respectively and W is the total "volume" integral of the three-body interaction. Then, with $W > 0$, the force is attractive for the interaction of a Λ with a nucleon pair which is in an even relative angular momentum state. The shape functions $\sigma(r_{\Lambda N})$ are again taken to be normalised Yukawa functions of range ν^{-1} fm. The values of ν used, namely ν = 1.0, 1.4 and 2.0 fm, where chosen so as to span a variety of ranges as well as shapes. The intermediate range, $\nu_\pi^{-1} = \hbar/m_\pi c$, corresponds to a force of Yukawa form with a Yukawa range appropriate to the exchange of a single pion with each of two nucleons, while the longer range is more nearly equivalent to an exponential shape with an <u>exponential</u> range of $\hbar/m_\pi c$ The shorter range is included to cover possibly important, modifications due to higher order processes. It should be noted that, although meson theory suggests that non-central contributions to the three-body force are very probably dominant, these are not considered explicitly.

1.2 The s-Shell Hypernuclei

The results of Bodmer and Sampanthar[3)#] for a soft spin-dependent two-body interaction of Yukawa form are summarised below in table 2, the errors quoted resulting from uncertainties in the Λ separation energy and the α-particle core size. Comparing these two hypernuclei* then leads to an estimate for the spin dependence, $\Delta = u_s - u_t$, of approximately 170 MeV fm^3 for $\mu_{2\pi}$ and 30 MeV fm^3 for μ_K or alternatively $u_s \sim$ 390 MeV fm^3 220 MeV fm^3 and $u_t \sim$ 220 MeV fm^3 and 190 MeV fm^3 for $\mu_{2\pi}$ and μ_K respectively. Qualitatively one can see the attractive triplet interaction results from the low statistical weight of the singlet spin state in the interaction of the Λ with the spin and isobaric spin saturated α-particle. It should be noted that for hypernuclei as light as these there exists an appreciable dependence of the volume integrals on the range of the Λ-N force.

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* The hypernucleus $_\Lambda H^4$ and its minor partner $_\Lambda He^4$ are not considered due to the much greater uncertainties in the size of their nucleon cores.

\# See also the work of Dalitz and Downs[1)], who initially performed these calculations for Gaussian interactions.

Hypernucleus	Total Volume Integral	$\mu_{2\pi}$ (MeV fm^3)	μ_K (MeV fm^3)
${}_\Lambda H^3$ #	$U_2 = \frac{1}{2}(3U_s + U_t)$	685 $\pm$ 15	422 $\pm$ 10
${}_\Lambda He^5$ x	$U_4 = 3U_t + U_s$	1038 $\pm$ 50	786 $\pm$ 40

Table 2. Two-Body Volume Integrals for the s-Shell Hypernuclei calculated with Yukawa forces.

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obtained with B_Λ $({}_\Lambda H^3) = 0.31 \pm 0.15$ MeV [34)]

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x The values quoted here refer to an r.m.s. radius $R_\alpha = 1.44 \pm 0.07$ fm for the matter distribution of the particle. The more recent measurements of Burleston and Kendall[5)], together with a value of 0.85 $\pm$ 0.05 fm for the r.m.s. radius of the proton, indicated by recent experiments, give $R_\alpha = 1.45 \pm 0.065$ fm and thus effectively the same value of U_4 with slightly reduced errors.

The dependence of these volume integrals on the precise shape of the two-body interaction can be illustrated by comparing these figures with those obtained by Dalitz and Downs[1] using a Gaussian shape with the same intrinsic range as the Yukawa's above. Thus Dalitz and Downs give $\bar{U}_4$ = 925 MeV fm^3 and 705 MeV fm^3 for $\mu_{2\pi}$ and μ_K respectively, which indicates a 10% dependence on the shape for the s-shell hypernuclei.* The dependence of the results on both the range and the shape are here, presumably, shown at their maximum. As one considers larger and larger hypernuclei it is to be expected that these differences become increasingly smaller as the Λ wave-function is compressed into the region where the nucleon density is constant. The fact that the K meson mechanism gives rise to an exchange force is unimportant in the s-shell as both the Λ and the single nucleon functions are 1s states and possess a large overlap.

The presence of a hard core in the two-body interaction has been considered by Downs, Smith and Truong[6] and Muller[7] for

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* Dalitz and Downs also considered the distortion of the α particle by the Λ and concluded that this resulted in corrections of approximately 3% to the volume integrals.

the case of the hypertriton. Downs et. al. using an hard core radius of 0.4 fm followed by a well with an exponential shape obtained a mean scattering length $\bar{a}$ = - 2.0 fm which is comparable with the value $\bar{a}$ = - 1.5 fm obtained for the soft force of Dalitz and Downs, the two interactions possessing the same intrinsic range. These figures again illustrate the insensitivity of the parameters of the low energy interaction for forces with the same intrinsic range. The results also agree well with those of Müller who considered an exponential well of longer range. Calculations of the other s-shell hypernuclei with hard core interactions have been performed by Dietrich, Folk and Mang[8] and Gutch[9] using square well shapes, again with the same intrinsic range as the soft forces. The scattering lengths obtained are in reasonable agreement with the other estimates quoted above.

Bodmer and Sampanthar also considered three-body forces of the type defined by eqn. 3 for the hypernuclei $_{\Lambda}H^3$ and $_{\Lambda}He^5$. Thus for each of these hypernuclei a relationship between the two-body volume integral $U_{\Lambda N}$ and the three-body strength W is obtained. However, as only two values of B_{Λ} are relevant no conclusions can be made about the possible magnitude of the three-body force. To achieve this one must consider other hypernuclear systems in addition to the s-shell ones. It is interesting to note that in the case of $_\Lambda He^5$ the relationship obtained is linear; a fact which arises because the Λ wavefunction is determined only by the total potential, which is itself insensitive to the ratio of two to three-body forces present.

1.3 The Λ Particle in Nuclear Matter.

The well depth felt by the Λ particle in nuclear matter has been estimated by Bodmer and Sampanthar[3)] using perturbation theory with the soft two and three-body forces previously discussed, the nucleons being considered as a Fermi gas. For the direct spin-dependent two-body interaction, eqn.1, the first order well depth is then given by

$$D_{20} = \left(\Psi_{q,A} \,,\ \sum_{i=1}^{A} V_{\Lambda N}\, \Psi_{q,A} \right)$$

$$= 1/4\ \bar{U}\rho \qquad (4)$$

where $\Psi_{q,A}$ is an unperturbed wavefunction constructed from the product of a Λ particle in a plane wave state of momentum $\hbar\vec{q}$ and a determinental nuclear wavefunction, corresponding to the ground state of A nucleons being represented by all the states with momentum less than the Fermi momentum k_F being occupied. The quantity ρ is then the nucleon density and the first order well depth is independent of the Λ momentum. For exchange forces the corresponding quantity D_{2E} now depends on q but approaches D_{20} as the range of the forces decreases. For the small values of the ratio $k_F/\mu = 1.35/2.5$ (corresponding to $\rho \sim 0.170$ fm^{-1} and $\mu_K^{-1} = 0.4$ fm) the q^2 dependent term is small (about 10% of the direct term of this range) and consideration of D_{2E} ($q = 0$) then gives an upper limit to the appropriate well depth. The well depths resulting from the three-body interactions are obtained in an exactly analogous manner. Second order contributions, the effects of nuclear pair correlations and velocity dependence are all found to be small, of order 10% corrections to the first order well depths. The results for the total well depth, $D = D_{20 \text{ or } E} + D_3$ is found to depend in a complicated manner on the particular two and three-body ranges involved, the relevant volume integrals being fixed by the results from the s-shell hypernuclei. However it is found that the value $D \sim 35$ MeV (for further discussion of this see section 3.2) is reasonably consistent with small three-body forces. For example with $\mu_{2\pi}$, [illegible] = 1025 MeV fm^3 and Δ = 200 MeV fm^3 the ratio of the three-body potential energy to the total potential energy is approximately 0.1 for all values of considered.

The Λ particle in nuclear matter has also been considered by Downs et al[10)] assuming the presence of an hard core in the Λ -N interaction. These calculations again indicate that a value of D in the range 30 - 40 MeV may well be expected although for these hard core interactions the contributions from states with $\ell \geqslant 1$ are relatively more important. In summary it may be stated that neither the results for forces with or without hard cores are necessarily inconsistent with the experimental estimates of the well depth D although as yet no significant information about the Λ -N interaction has been obtained from this type of calculation.

CHAPTER 2

THE p -SHELL HYPERNUCLEI

2.1 The Schrödinger Equation for the Λ Particle

The simplest method of analysing the systematics of the Λ separation energies in the p-shell is to consider that the excess of B_Λ over B_Λ (${}_\Lambda He^5$) arises entirely from the interaction of the Λ with the nucleons from the p-shell, as suggested by Lawson and Rotenberg[11] and Iwao[11]. Thus the Λ wavefunction is assumed to be the same for all the p-shell hypernuclei and a constant energy, C = 3.1 MeV, then results from the difference between the interaction energy of the Λ particle with the s-shell nucleons and the Λ kinetic energy. Hence, with two-body forces, this approximation leads to the expression

$$B_\Lambda = C + \tfrac{1}{4}\langle \bar{U} \rangle_p + K \langle \Delta \rangle_p \tag{5}$$

where N_p is the number of p-shell nucleons, $\tfrac{1}{4}\langle \bar{U} \rangle_p$ and $\langle \Delta \rangle_p$ are the expectation values of the spin-averaged interaction and spin dependence respectively and K is a number depending on the core nucleus structure and the spin of the hypernucleus state considered (see section 2.2 for the definition of K). If one considers the hypernuclei with spinless cores, namely $_\Lambda He^5$, $_\Lambda Be^7$ (or $_\Lambda He^7$), $_\Lambda Be^9$, $_\Lambda C^{13}$, which all have $\kappa = 0$, then this expression is linear in N_p, both actually and in theory. The slope of this line gives $\frac{1}{4}\langle \tau_\Lambda\cdot\tau \rangle_p = 0.9 \pm 0.05$ MeV. If, now, one considers the differences in B_Λ for the pairs of neighbouring hypernuclei in table 3 one obtains an estimate of $\langle \Delta \rangle_p$ in the p-shell. In table 3 the values of κ are those quoted by Dalitz[12)] obtained by using the intermediate coupled nucleon wavefunctions of Soper.

Hypernuclei	κ	$\langle \Delta \rangle_p$ (MeV)
$_\Lambda Be^7$ - $_\Lambda Li^7$	0.98	0.6 ± 0.5
$_\Lambda Be^8$ - $_\Lambda Be^9$	0.59	1.5 ± 0.5
$_\Lambda Be^9$ - $_\Lambda Li^9$	0.7	2.1 ± 0.4
$_\Lambda B^{12}$ - $_\Lambda C^{13}$	0.28	2.7 ± 1.3

Table 3. Expectation Values of Spin Dependence from the p-shell Hypernuclei

The estimate of $\langle \Delta \rangle_p$ so obtained in this manner is seen to increase significantly as N_p increases throughout the p-shell. In fact $\langle\Delta\rangle_p$ is a factor of three times $\frac{1}{4}\langle\sigma\rangle_p$ in the region of $_\Lambda C^{13}$, a situation which markedly contrasts with the results for the s-shell hypernuclei where $\Delta \lesssim {}^1/_{12}\,\sigma$). To understand this increase seems to require either one of two possibilities; first, the model assumed is not sufficiently flexible to give a reasonable account of the p-shell hypernuclei or, second, the increase reflects some feature of the Λ-N interaction not yet taken into consideration, e.g. possible spin-orbit or tensor forces. It is thought that the first possibility is the more likely and consequently the p-shell hypernuclei are reanalysed, relaxing the condition that the energy C is constant and also introducing the nuclear size explicitly.

Considerations are essentially based on a two-body model for the p-shell hypernuclei consisting of a Λ and the core nucleus. Thus the Λ, in a 1s state, moves in the potential well, $V_\Lambda(r)$, generated through its two and three-body interactions with the nucleons of the core and for the radial wave function, $\varphi_\Lambda(r)$, normalized to unity, the Schrödinger equation is

$$(T_\Lambda + V_\Lambda(r) + B_\Lambda)\,\varphi_\Lambda(r) = 0 \quad , \qquad (6)$$

where T_Λ is the kinetic energy operator for the Λ, with appropriate reduced mass and where B_Λ is the relevant separation energy with respect to the core.

For these hypernuclei, namely $_\Lambda He^5$, the mirror pair $_\Lambda He^7$ and $_\Lambda Be^7$, $_\Lambda Be^9$ and $_\Lambda C^{13}$, for which the total angular momentum, J_N, of the core nucleus is zero and with two-body forces, the potential $V_\Lambda(r)$ is obtained by folding the Λ -N interaction into the density distribution of the core. Then for the hypernucleus of mass number A (including the Λ) the eigenvalue problem, eqn. 6, may be solved for the total volume integral $\mho_{A-1}$, corresponding to the appropriate value of B_Λ In general, however, $J_N \neq 0$ and spin dependent effects will be present, in which case it is necessary to have a more detailed description of the core nucleus. To give this a shell model type wavefunction is assumed for the core and the spin dependent effects then arise from the coupling of the Λ with the p-shell nucleons only; the s-shell nucleons being treated as a spin and isobaric spin saturated system. The exchange character ~~also~~ assumed for the three-body force also requires such a more detailed description.

The decomposition into s and p-nucleons corresponds to single nucleon mass density distributions ρ_s and ρ_p , normalised to unity, which are mostly obtained from the corresponding radial harmonic oscillator functions

$$u_s(r) = \left[\frac{4}{\pi^{1/2} a_s^3}\right]^{1/2} \exp\left[-r^2/2a_s^2\right] \qquad (7)$$

$$u_p(r) = \left[\frac{8}{3\pi^{1/2}a_p^3}\right]^{1/2} \frac{r}{a_p} \exp\left[-r^2/2a_p^2\right] \tag{8}$$

where a_s and a_p are the oscillator size parameters for the s and p nucleons respectively. For all but the A = 7 hypernuclei these are taken to be equal, i.e. the s and p nucleons are assumed to move in the same harmonic oscillator well. It should be emphasised ~~that~~ that the density distributions used here refer to the centre of mass of the core nucleus and that for the undistorted core the values of a_s and a_p are taken to be those which give the same r.m.s. radii for, respectively, the s and p-nucleon mass density distributions as those obtained from analysis of the electron scattering experiments. Thus the oscillator parameters here differ slightly from the conventional ones which are obtained when centre of mass corrections are consistently incorporated in the harmonic oscillator wavefunctions. The mass distributions with respect to the centre of mass of the core are here always referred to directly as this is appropriate to the two-body model. For the case of small core distortion by the Λ, when the relevant density is that corresponding to an isolated nucleus, the value of B_Λ in eqn. 6 may be identified with the experimental Λ separation energy. If, on the other hand, distortion is important, the isolated core sizes are no longer appropriate and also the value of B_Λ used in eqn. 6 must now be increased by an amount corresponding to the rearrangment energy of the core nucleus as is discussed in more detail in the following sections. Whether or not core distortion is important depends on the particular core nucleus involved and will be discussed at the appropriate place for individual cases.

2.2 The Λ Potential due to Two-Body Forces

With only two-body forces the decomposition into s and p-shells gives

$$V_{\Lambda}(r) = V_{2s}(r) + V_{2p}(r) \tag{9}$$

where

$$V_{2s}(r) = \bar{U} \int d\vec{r}_1 \, \rho_s(r_1) \, v(r_{1\Lambda}) \tag{10}$$

and

$$V_{2p}(r) = \bar{U}_p \int d\vec{r}_1 \, \rho_p(r_1) \, v(r_{1\Lambda}) \tag{11}$$

In eqn. 10 the volume integral $\bar{U} = 3U_t + U_s$ is just four times the spin-averaged volume integral of the Λ-N interaction; while in eqn. 11 the volume integral $\bar{U}_p$ is defined by

$$\bar{U}_p = \langle J | \sum_i^{N_p} (P_t U_t + P_s U_s) | J \rangle , \tag{12}$$

the sum being taken over the N_p p-shell nucleons and $|J\rangle$ is the ground state wavefunction for the hypernucleus. Eqn. 12 may be rewritten as

$$U_p = 1/4\, U N_p + K \Delta \tag{13}$$

where $\Delta = U_s - U_t$ is the spin dependence of the Λ -N interaction and where

$$K = -\langle J | \sum_{i=1}^{N_p} \vec{\sigma}_i \cdot \vec{\sigma}_\Lambda | J \rangle \tag{14}$$

In general different states of the parent nucleus are involved and with $|\alpha; J_N, 1/2; J\rangle$ denoting the hypernuclear state obtained by coupling a core state of angular momentum J_N to the Λ spin to give J, one has

$$|J\rangle = \sum_{\alpha, J_N} a_{\alpha, J_N} |\alpha; J_N, 1/2; J\rangle . \tag{15}$$

The angular momentum of the core nucleus can only have the values $J_N = J \pm \frac{1}{2}$, α labelling the core states having the same value of

J_N. Only the states of each J_N lying lowest in energy are expected to be important and accordingly the label α will now be dropped. The coefficients a_{J_N} are then obtained by diagonalizing the appropriate, now two-dimensional, energy matrix with elements $E_{J_N}\delta_{J_N J_N'} + K_{J_N J_N'}\,\Delta\, F^{(o)}_{2p}$, where $F^{(o)}_{2p}$ is the relevant Slater integral and implicitly involves the wavefunction (see section 2.4) and where, corresponding to the shell model description, only core states arising from different p-nucleon configurations are considered. The energies E_{J_N} are just those of the parent nuclear state and the $K_{J_N J_N'}$ are given by

$$K_{J_N J_N'} = \langle J_N, 1/2, J \,|\, \sum^{N_p} \vec{\sigma}_i \cdot \vec{\sigma}_\Lambda \,|\, J_N', 1/2, J \rangle . \qquad (16)$$

For the case of jj coupling this matrix element is diagonal in J_N 11) and one has

$$K = -\frac{1}{6}\left[J(J+1) - J_N(J_N+1) - 3/4 \right] \qquad (17)$$

while for LS coupling

$$K_{J_N J_N'} = -\frac{1}{2}\left[(2J_N+1)(2J_N'+1)\right]^{1/2} (-)^{2S_N+J_N-J_N'} \times$$

$$\sum_{S=|S_N-1/2|}^{S_N+1/2} (2S+1)\, W(S_N S J_N J; {}^1\!/_2 L)\, W(S_N S J_N' J; {}^1\!/_2 L)\left[S(S+1)-S_N(S_N+1)-{}^3\!/_4\right] \tag{18}$$

where W(a b c d; e f) is a Racah coefficient. For intermediate coupling, which is in general appropriate for the p-shell nuclei, an expansion of the core wavefunctions in terms of either jj or LS coupled wavefunctions, then gives the required matrix elements. However, if the energy difference between the parent states of different J_N is large, then only the one corresponding to the ground state of the nuclear core is expected to be important and K will then be obtained without any diagonalization. For $_\Lambda Li^8$ and $_\Lambda Be^8$ in particular it is necessary to include both the $J_N = {}^3\!/_2^-$ ground state and the $J_N = \frac{1}{2}^-$ excited state at 0.48 MeV, as remarked by Dalitz[13], both of which contribute comparably to the J = 1 hypernuclear ground state. Using Soper's wavefunctions essentially the same results for K have been obtained as those quoted by Dalitz[12]. (See also appendix 1).

Expanding the shape functions $v(r_{\Lambda n})$ into a sum of Legendre polynomials one has

$$v(r_{\Lambda n}) = \sum_{k=0}^{\infty} v^{(k)}(r, r_n)\, P_k(\cos\omega_{\Lambda n}) \tag{19}$$

where $\omega_{\Lambda n}$ is the angle between $\vec{r}$ and $\vec{r}_n$. Averaging over the direction of $\vec{r}$, appropriate to a Λ in an s state, eqns. 10 and 11 then respectively become

$$V_{2s}(r) = U\, v_{2s}^{(0)}(r) \tag{20}$$

and

$$V_{2p}(r) = (1/4\, U N_p + \eta\Delta)\, v_{2p}^{(0)}(r) \tag{21}$$

where the potential shape function $v_{2\ell}^{(k)}(r)$ is given by

$$v_{2\ell}^{(k)}(r) = \int_0^{\infty} r_1^2\, u_\ell^2(r_1)\, v^{(k)}(r, r_1)\, dr_1 \tag{22}$$

Here ℓ refers to the angular momentum of the nucleon orbital and k to the appropriate Legendre polynomial. For Yukawa interactions these integrals can be expressed in terms of the tabulated function $Hh_n(x)$[14], the relevant expression being given in appendix 2. In practice the functions $v_{2p}^{(k)}(r)$ were obtained by numerical integration.

As remarked in section 2.1, for hypernuclei with $J_N = 0$ and with only two-body forces the above explicit separation into s and p-shell nucleons is not necessary as only the total density distribution and the spin-averaged volume integral enter. For these hypernuclei one then has

$$V_\Lambda(r) = \bar{U}_{A-1} \int d\vec{r}_1 \; \rho(r_1) \; v(r_1, \Lambda) \tag{23}$$

where $U_{A-1} = \frac{1}{4}(A-1)\,\bar{U}$ and the integral is just given by $\frac{1}{A-1}\left[4 \, v_{2s}^{(0)}(r) + N_p \, v_{2p}^{(0)}(r) \right]$.

2.3 The Λ Potential due to Three-Body Forces

The spin and isobaric spin exchange character assumed for the three-body interaction, eqn. 3, also requires that the s and p-nucleons are considered separately and the shell model description, as previously discussed, is used. The assumption that the 1s nucleons comprise a spin and isobaric spin saturated system has the consequence that the net contribution to the three-body potential, $V_3(r)$, arising from the coupling of the Λ with a p-nucleon and any of the s-nucleons is zero. Thus only the contributions $V_{3s}(r)$ and $V_{3p}(r)$ coming from the interactions of the Λ with nucleon pairs in the s-shell and the p-shell respectively need be considered. The former potential is given by

$$V_{3s}(r) = W \langle \chi | \sum_{i<j}^{4} (\vec{\sigma}_i . \vec{\sigma}_j)(\vec{\tau}_i . \vec{\tau}_j) | \chi \rangle \, v_{3s}^{(0)}(r)$$

$$= -18 W \, v_{3s}^{(0)}(r) \tag{24}$$

where $|\chi\rangle$ is the totally antisymmetric spin and isobaric spin wavefunction for the closed 1s nucleon shell and the shape function is just

$$v_{3s}^{(0)}(r) = \int d\vec{r}_1 d\vec{r}_2 \; \rho_s(r_1)\,\rho_s(r_2)\, v(r_{1\Lambda})\, v(r_{2\Lambda})$$

$$= \left[v_{2s}^{(0)}(r) \right]^2 \tag{25}$$

$v_{2s}^{(0)}(r)$ being calculated with the range parameter γ^{-1} appropriate to the three-body interaction.

The three-body interaction of the Λ with a single pair of p-nucleons, coupled to orbital angular momentum L, spin S and isobaric spin T, gives the potential

$$V_{3p}(r) = W \langle ST | (\vec{\sigma}_1 . \vec{\sigma}_2)(\vec{\tau}_1 . \vec{\tau}_2) | ST \rangle \, v_{3p}(r) \tag{26}$$

where

$$v_{3p}(r) = \int d\vec{r}_1 d\vec{r}_2 \; \left| \Psi_L(\vec{r}_1, \vec{r}_2) \right|^2 v(r_{1\Lambda})\, v(r_{2\Lambda}) \tag{27}$$

and $\Psi_L(\vec{r}_1,\vec{r}_2)$ is obtained by coupling the single p-nucleon states, $u_p(r)Y_1^m(\Omega)$, to angular momentum L. Only by ignoring the spatial correlations between the two nucleons is it possible to replace $|\Psi_L|^2$ by a product of single particle densities $\rho_p(r_1)\,\rho_p(r_2)$, as may be done for the s-nucleons. Expanding $v(r_{1\Lambda})$ and $v(r_{2\Lambda})$ in terms of Legendre polynomials and using the spherical harmonics addition theorem enables the integration over the nucleon angles to be performed. Thus one obtains

$$v_{3p}(r) = 4\pi \sum_{k,k'=0,2} \frac{C(1k1000)\,C(1k'1000)}{\left[(2k+1)(2k'+1)\right]^{1/2}}\; v_{2p}^{(k)}(r)\; v_{2p}^{(k')}(r)$$

$$\left[\sum_{m_k,m_{k'}} \Theta_{k,k'}(m_k,m_{k'},L)\; Y_k^{m_k}(\Omega_\Lambda)\, Y_{k'}^{m_{k'}}(\Omega_\Lambda)\right] \tag{28}$$

where Ω_Λ denotes the polar angles of the Λ, $C(a\,b\,c,\alpha\,\beta\,\delta)$ is a Clebsch-Gordan coefficient and $\Theta_{k,k'}$ is a number constructed from a sum of products of four such coefficients. Averaging over Ω_Λ, appropriate to a 1s state Λ, and performing the sum over m_{12} and m_{12}' the square bracket becomes $\frac{3}{4\pi} W(1k L 1; 11)$ where $W(abcd; ef)$ is a Racah coefficient.

Then

$$v_{3p}(r) = v_{3p}^{(0)}(r) + \frac{6}{25} W(12L1;11) v_{3p}^{(2)}(r) \tag{29}$$

which is now only a function of r and where

$$v_{3p}^{(k)}(r) = \left[v_{2p}^{(k)}(r) \right]^2 \tag{30}$$

If the spatial correlations between the two nucleons had been neglected then only the first term of eqn. 29 would have been obtained, the second term arises entirely from the angular correlations. For the three possible values of L in the p-shell, viz. L = 0, 1 and 2, the coefficients of $v_{3p}^{(2)}(r)$ in eqn. 29 are 2/25, -1/25 and 1/125 respectively. However, the exchange factor of the three-body force is related to the orbital wave function through the exclusion principle and one finds that the contribution of the correlation term is always attractive for $W > 0$. Thus, for the various two nucleon states, one obtains

$$V_{3p}(r) = -W(3v_{3p}^{(0)} + 6/25\, v_{3p}^{(2)}) \qquad \text{for } ^{13}S \text{ and } ^{31}S \text{ states} \tag{31a}$$

$$= -W(3v_{3p}^{(0)} + 3/175\, v_{3p}^{(2)}) \qquad \text{for } ^{13}D \text{ and } ^{31}D \text{ states} \tag{31b}$$

$$= W(9v_{3p}^{(0)} - 9/25\, v_{3p}^{(2)}) \qquad \text{for } ^{11}P \text{ states} \tag{31c}$$

$$= W(v_{3p}^{(0)} - 1/25\, v_{3p}^{(2)}) \qquad \text{for } ^{33}P \text{ states} \tag{31d}$$

The attractive potential, for $W > 0$, due to the angular correlations is normally smaller than the leading contribution involving $v_{3p}^{(0)}$. The opposite signs of the latter term, for S and D states on the one hand and for P states on the other, should be noted, this being a direct consequence of the exclusion principle. Thus the potential for a P state is predominantly repulsive, while for S and D states it is attractive.

If there are more than two p-nucleons one can obtain $V_{3p}(r)$ in terms of $v_{3p}^{(0)}(r)$ and $v_{3p}^{(2)}(r)$ by expanding the core wavefunctions in terms of two particle LS coupled states using the fractional parentage coefficients of Elliott, Hope and Jahn[15]. Thus, quite generally, for a hypernucleus of mass number A one can write

$$V_{3_p}(r) = -W\left(\alpha_{N_p} v_{3_p}^{(0)}(r) + \beta_{N_p} v_{3_p}^{(2)}(r)\right). \tag{32}$$

For an LS coupled nuclear state one has

$$\alpha_{N_p} = \frac{1}{2} N_p(N_p-1) \sum_{\substack{[\ell']L'S'T' \\ LST}} \left[[\ell]L_N S_N T_N \} [\ell']L'S'T', LST\right]^2 \alpha_2(S,T) \tag{33}$$

and

$$\beta_{N_p} = \frac{1}{2} N_p(N_p-1) \sum \left[[\ell]L_N S_N T_N \} [\ell']L'S'T', LST\right]^2 \beta_2(L,S,T) \tag{34}$$

with the values of $\alpha_2(S,T)$ and $\beta_2(L,S,T)$ given by eqn. 31 and with $[[\ell]L_N S_N T_N \} [\ell']L'S'T', LST]$ denoting the relevant f.p.c.. For an intermediate coupled state the generalisation is obvious.

The simple dependence of the three-body potential on the number of nucleon pairs, which has been included in the definition of α_{N_p} and β_{N_p}, can be strongly modified by the structure of the nuclear state principally due to the operation of the exclusion principle. This is indicated in table 4, where the coefficients are presented for the various hypernuclei and coupling schemes, the values for intermediate coupling have been obtained using Soper's wavefunctions for the nucleons. The results do not, in fact, depend sensitively on the coupling scheme adopted. It should be noted that if for the p-nucleons one were to assume that only S and D two particle antisymmetric states occur, i.e. the fractional parentage coefficients relating to two particle P states were zero, then the value $\alpha_{N_p} = 3 \times \frac{1}{2}N_p(N_p - 1)$ would be obtained. This is not in general true because of the antisymmetry of the many particle wavefunction. This requirement, together with $\beta_{N_p} = 0$, obtained by neglecting the angular correlations between the p-nucleons, gives the same dependence on the number of nucleon pairs as obtained in the s-shell, and essentially represents an upper limit for α_{N_p}. Thus considering as an example $_{\Lambda}C^{13}$ in LS coupling one obtains $\alpha_8 = 36$ and $(^3/_2)N_p(N_p - 1) = 84$. This latter value is relevant to a core state having permutational symmetry [8] which, of course, is not possible due to the exclusion principle. The maximum possible symmetry is [44] and this corresponds to a 3.6% and a 32% admixture of ^{11}P and ^{33}P two particle states

Hypernucleus	$\frac{3}{2}N_p(N_p-1)$	coupling scheme	α_{N_p}	β_{N_p}	Hypernucleus	$\frac{3}{2}N_p(N_p-1)$	coupling scheme	α_{N_p}	β_{N_p}
A=7	3	LS	3	0.24	$_\Lambda Li^8$ #	9	ic	8.72	0.42
		ic	28.9	2.15			LS	9	0.43
$_\Lambda C^{13}$	84	LS	36	2.30	$_\Lambda Be^9$	18	ic	17.55	0.87
		jj	20	2.08			LS	18	0.89
$_\Lambda B^{12}$	63	ic	21.8	1.42	$_\Lambda Li^9$	18	ic	9.84	0.53
		LS	27	1.73			LS	10	0.54

Table 4. The Coefficients α_{N_p} and β_{N_p} for the p-shell Hypernuclei

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\# The mixing of the nucleon states with angular momentum $J_N = \frac{1}{2}$ and $^3/2$, which is important for the calculation of K is here unimportant as the three-body interaction is effectively a scalar in the space of the nucleon variables for an s state Λ .

respectively. For $_{\Lambda}Be^9$, on the other hand, the value of α_4 in LS coupling is just the maximum possible since the symmetry [4] is now allowed, (and corresponds, in this sense, to a p-shell α-particle). The much smaller value of α_4 for $_{\Lambda}Li^9$, which has the same number of nucleons as $_{\Lambda}Be^9$, should also be noted and is due to the lower spatial symmetry ([31] in LS coupling) of Li^8 compared with that of Be^8. As is to be expected the inhibiting effect of the exclusion principle in reducing α_{N_p} below $^3/2\ N_p(N_p - 1)$ becomes progressively greater for larger A and this is quite similar to the situation found for nuclear matter [3)], the exchange character of the three-body force very much reducing its effect.

2.4 Energy Expectation Values for the Λ Particle

It is convenient to discuss the results in terms of the expectation values of the Λ potential and kinetic energies using the Λ wavefunctions obtained from eqn. 6. Thus, with brackets denoting expectation values with respect to this wavefunction,

$$B_\Lambda = \langle V\rangle_{2s} + \langle V\rangle_{2p} + \langle V\rangle_{3s} + \langle V\rangle_{3p} - \langle T_\Lambda\rangle$$

$$= \bar{u} F_{2s}^{(0)} + \left(\tfrac{1}{4}\bar{u} N_p + k(A)\right) F_{2p}^{(0)} + 18 w F_{3s}^{(0)} + \bar{w}\left(\alpha_{N_p} F_{3p}^{(0)} + \beta_{N_p} F_{3p}^{(2)}\right) - \langle T_\Lambda\rangle \tag{35}$$

where in $\langle V\rangle_{n\ell}$ the suffix n = 2 or 3 denotes the type of force, ℓ = s or p relates to the nuclear shell and the Slater integral $F_{n\ell}^{(k)}$ is defined by

$$F_{n\ell}^{(k)} = \int_0^\infty \rho_\Lambda^2(r)\, v_{n\ell}^{(k)}(r)\, dr. \tag{36}$$

The two-body integrals $F_{2s}^{(0)}$ and $F_{2p}^{(0)}$ obtained from ${}_\Lambda C^{13}$ and ${}_\Lambda Be^9$ and the A = 7 hypernuclei are shown in figs. 1 and 2 as functions of a_s and a_p respectively and for both ranges $\mu_{2\pi}^{-1}$

<u>**Captions to figures 1 and 2**</u>

Fig. 1. The Slater Integral $F_{2s}^{(o)}$

The upper and lower dashed curves indicate the values for μ_K and $\mu_{2\pi}$ respectively for ${}_\Lambda He^5$ with B_Λ = 3.1 MeV, ~~the solid curves referring to a_s = 1.175 fm and the dotted ones to a_s = 1.254 fm~~. The values for the p-shell hypernuclei are given by the curves (a) to (h). For (a) to (d) a common oscillator well was used. ~~refers to this parameter.~~ For (e) to (h) the values of $F_{2s}^{(o)}$ are shown as a function of a_s with fixed a_p = 2.0 fm, $F_{2s}^{(o)}$ being almost independent of a_p as is indicated by the error bars on curve (f) at a_s = 1.2 fm and 1.5 fm which refer to a variation ~~fm~~ of ± 0.4 fm in a_p. The labelling of the curves is as follows :

(a) and (b) ${}_\Lambda C^{13}$, B_Λ = 10.9 ± 0.5 MeV for μ_K and $\mu_{2\pi}$ respectively.

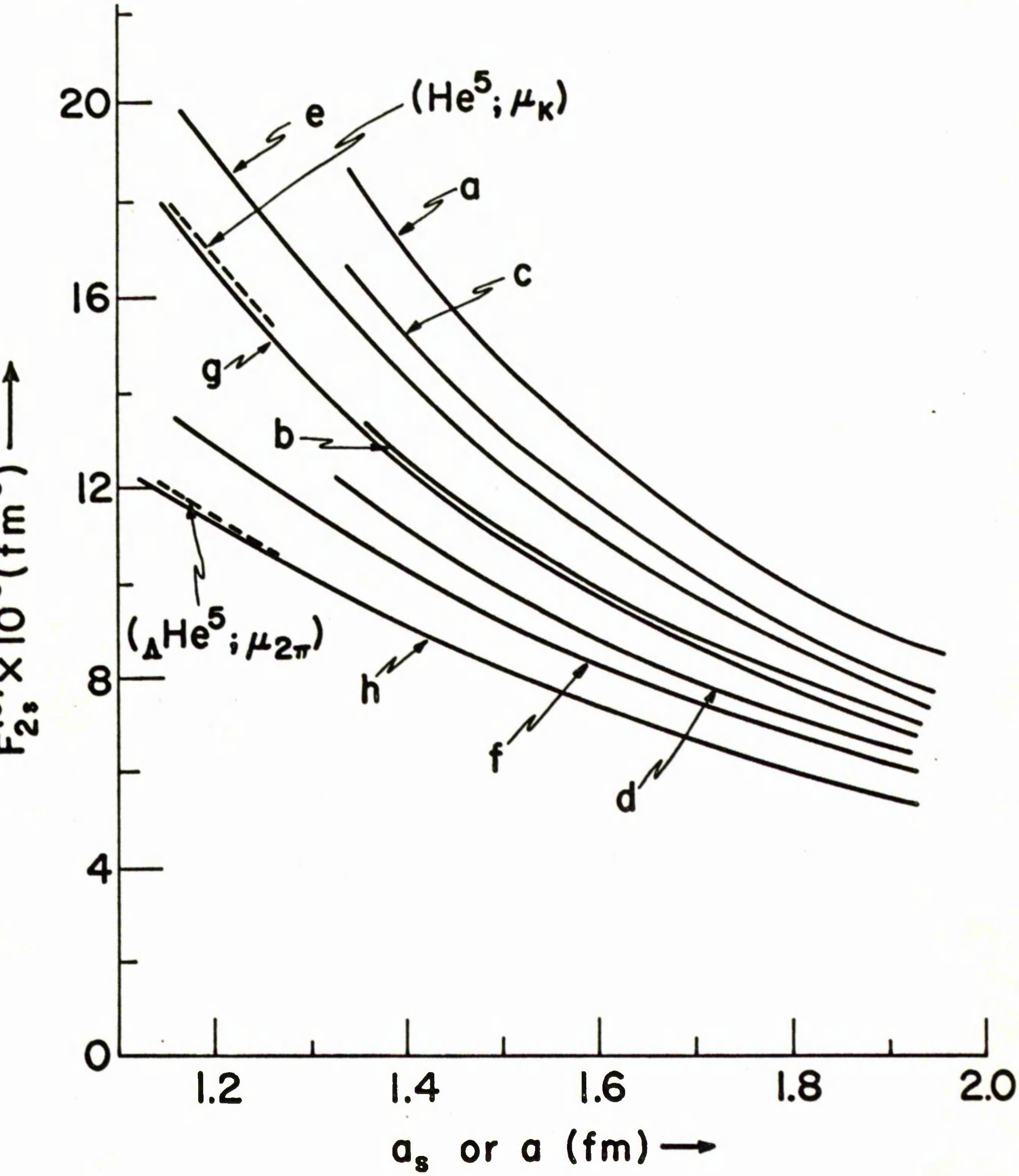
(c) and (d) ${}_\Lambda Be^9$, B_Λ = 6.5 ± 0.5 MeV for μ_K and $\mu_{2\pi}$ respectively.

(e) and (f) A = 7, T = 1, B_Λ = 5.0 MeV for μ_K and $\mu_{2\pi}$ respectively.

(g) and (h) A = 7, T = 1, B_Λ = 3.5 MeV for μ_K and $\mu_{2\pi}$ respectively.

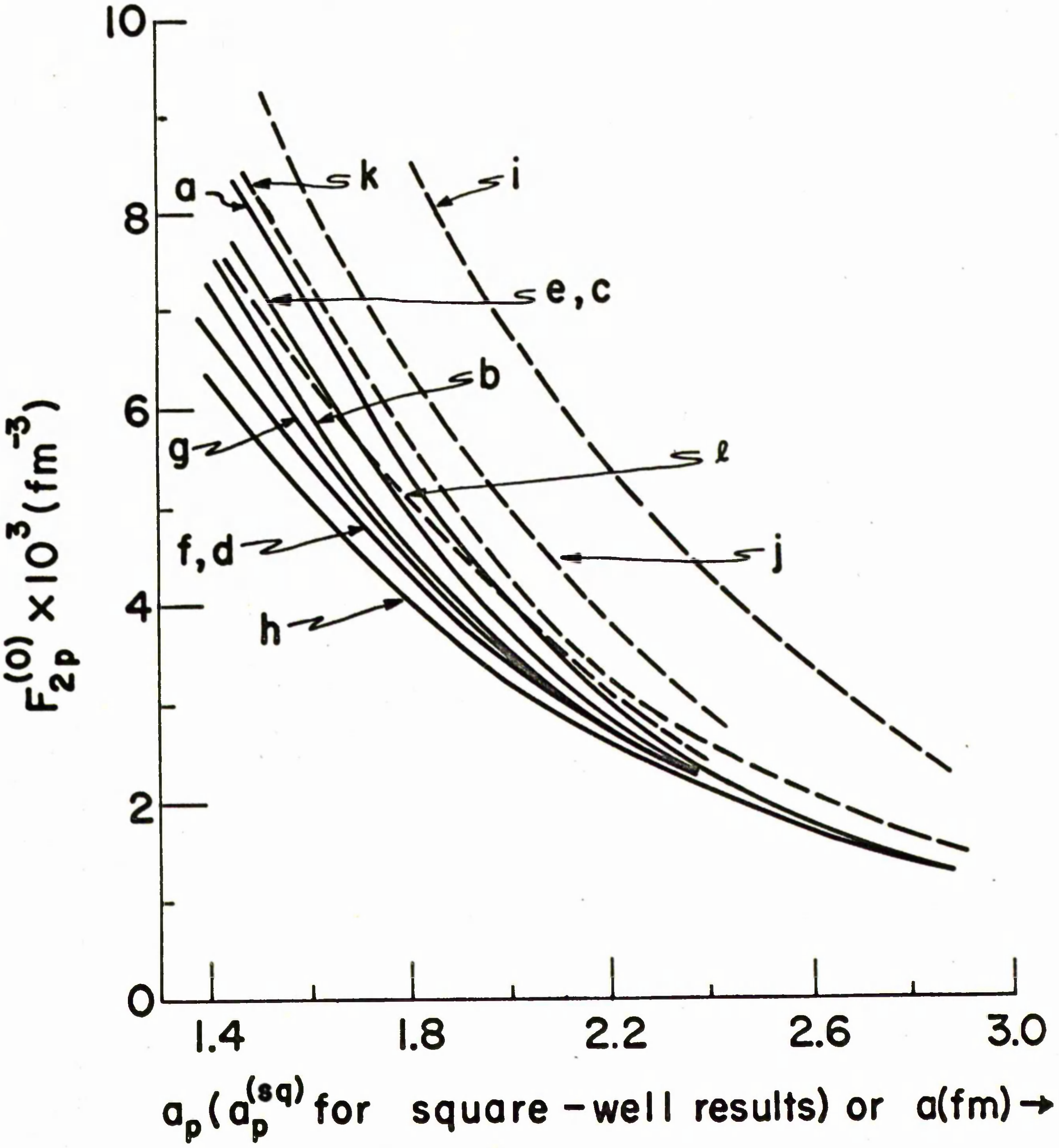
Fig. 2. The Slater Integral $F_{2p}^{(o)}$

The labelling is as for Fig. 1. The curves (a) to (d) are again for a common oscillator size parameter a, whereas (e) to (h) now show $F_{2p}^{(o)}$ as a function of a_p with fixed a_s = 1.2 fm. Curves (i) to (l) are for square-well p-nucleon wave functions for A = 7, T = 1, and for $\mu \, 2\pi$ and a_s = 1.2 fm; curves (i), (j) and (k) are for B_Λ = 5 MeV and B_p = 1, 3 and 5.66 MeV respectively, and curve (l) is for B_Λ = 3.5 MeV and B_p = 5.66 MeV.



20
16
12
8
4
0
1.2
1.4
1.6
1.8
2.0
$F_{2s}^{(0)} \times 10^3 (fm^{-3}) \longrightarrow$
a_s or a (fm) ⟶
$(He^5; \mu_K)$
$(_\Lambda He^5; \mu_{2\pi})$
e
a
c
g
b
h
f
d

Fig. 1.



10
8
6
4
2
0
$F_{2p}^{(0)} \times 10^3 (fm^{-3})$
a
k
i
e, c
b
g
ℓ
j
f, d
h
1.4
1.8
2.2
2.6
3.0
a_p ($a_p^{(sq)}$ for square-well results) or a(fm) →

Fig. 2.

and μ_K^{-1} . The monotonic decrease in the values of these integrals as the nucleon size parameter is increased is primarily a reflection of the fact that the forces considered are of much shorter range than the nuclear extensions involved and that for a δ -function interaction the potential shape functions $v_{2s}^{(0)}$ and $v_{2p}^{(0)}$ are just proportional to the appropriate single nucleon densities. Thus an increase in the size parameter for either s or p-nucleon densities results in a decrease in the magnitude of $v_{2s}^{(0)}$ or $v_{2p}^{(0)}$ respectively in the region where the Λ wavefunction is large. For the nucleon sizes of interest, for example $a_s = a_p \gtrsim 1.6$ fm, there is still a significant range dependence which leads to smaller volume integrals for μ_K than for $\mu_{2\pi}$, this difference decreasing as the size increases.

Apart from the core size and range of the force the integrals are also determined in part by the Λ wave function and thus in general depend on the particular hypernucleus considered and the value of B_Λ used. This results in the relatively small differences in the values obtained from the various hypernuclei and, as expected, these differences again diminish as the nucleon density is made more extended. It is a general feature that $F_{2s}^{(0)}$ is more sensitive to variations in $\rho_\Lambda(r)$, for given a and μ , than is $F_{2p}^{(0)}$ and hence the greater relative differences between the various hypernuclei for $F_{2s}^{(0)}$. This essentially arises from the behaviour of the potential functions $v_{2s}^{(0)}$ and $v_{2p}^{(0)}$ where the Λ wavefunction is large: $v_{2p}^{(0)}$ is here slowly varying while $v_{2s}^{(0)}$ decreases relatively rapidly as the distance from the origin increases. Variations in $V_\Lambda(r)$ and B_Λ which lead to a greater r.m.s. radius for the Λ then decreases the magnitude of the integrals, the effect being more pronounced for $F_{2s}^{(0)}$. Thus for both $_\Lambda C^{13}$ and $_\Lambda Be^9$ an increase in B_Λ of 1 MeV from the values specified gives approximately 4.5% and 3% increases in $F_{2s}^{(0)}$ and $F_{2p}^{(0)}$ respectively. Using these values the integrals for hypernuclei of intermediate mass numbers can be obtained by interpolation and a reasonable procedure is to consider the hypernuclei $_\Lambda Li^8$ and $_\Lambda Li^9$ as $_\Lambda Be^9$ systems and $_\Lambda B^{12}$ as $_\Lambda C^{13}$, suitably adjusting the B_Λ values. The variation of these two-body Slater integrals as the relative proportion of three to two-body force was increased was found to be insignificant as the overall potential $V_\Lambda(r)$ and the Λ wavefunction, $\phi_\Lambda(r)$, changed very little.

For the A = 7 hypernuclei the Λ wavefunction obtained is not expected to be appreciably different for the T = 0 and T = 1 systems, assuming the same core sizes and range, because the value of B_Λ considered is similar for both. However, with densities appropriate to the electron scattering data ($a_s \approx$ 1.65 fm, $a_p \approx$ 2.0 fm) and with B_Λ = 5.5 MeV, $F_{2s}^{(0)}$ for the T = 1 hyper nucleus was found to be approximately 10% greater than that from $_\Lambda Li^7$, while for $a_s \approx 1.2$ fm and the same values of a_p and B the integrals agree closely with each other and with the value obtained from $_\Lambda He^5$ (also shown in fig. 1). This difference originates from the fact that eqn. 6, for the T = 1 hypernucleus, was solved for the eigenvalue U while for $_\Lambda Li^7$, U = U4 was treated as a given parameter and the equation was solved for Δ. Then, for large values of a_s, φ_Λ (r) obtained from $_\Lambda Li^7$ is more extended relative to that from the T = 1 hypernuclei due to the necessarily exaggerated attraction of the Λ to the two p-shell nucleons in $_\Lambda Li^7$ (see further discussion in chapter 5) and a correspondingly reduced value of $F_{2s}^{(o)}$ is obtained. If for the T = 0 hypernucleus one had solved eqn. 6 with U = U (T = 1), where U(T = 1) is that value of U obtained from the T = 1 hypernucleus with the same core sizes and range, then the integrals would be very similar and accordingly only the results for the T = 1 hypernucleus are given.

The dependence of the two-body Slater integrals on B_Λ is indicated by the two curves given for B_Λ = 5.0 MeV and 3.5 MeV respectively. The difference between the integrals for these values of B_Λ is seen to decrease as the relevant nucleon size parameter is increased, as expected. Because the Λ wave-function is more strongly determined by a_s than by a_p (there are only two p-nucleons which are in general spatially remote) the value of $F_{2s}^{(o)}$ is essentially independent of a_p. Thus for a_s = 1.50 fm, B_Λ = 5.0 MeV and $\mu_{2\pi}$ the variation in $F_{2s}^{(o)}$ as a_p increases from 1.8 fm to 2.4 fm is $<$ 1%. However, the value of $F_{2p}^{(o)}$ for given a_p, B_Λ and μ was found to be more dependent than this on a_s, particularly for small values of a_p; for example with a_p = 1.8 fm, B_Λ = 5.0 MeV and $\mu_{2\pi}$ one obtains a 6% decrease in $F_{2p}^{(o)}$ as a_s is increased from 1.20 fm to 1.80 fm. For a_p = 2.4 fm the variation was negligible.

The three-body Slater integrals $F_{3s}^{(o)}$, $F_{3p}^{(o)}$ and $F_{3p}^{(2)}$ obtained from ${}_\Lambda C^{13}$ are shown in fig. 3 for the ranges ν^{-1} = 1.0 fm and 2.0 fm and for the two-body range $\mu_{2\pi}^{-1}$; the range ν^{-1} = 1.4 fm gives values intermediate to these. The Λ wavefunction relevant was obtained from eqn. 6 considering two-body forces alone as the slight dependence on the relative strengths of the two and three-body forces is again insignificant, particularly in the light of the discussion of section 3.4. Values appropriate to the two-body range μ_K^{-1} wan be obtained from those given by noting that the small relative compression of the Λ wavefunction for this shorter range leads to increases of 4.5%, 3% and 1% for $F_{3s}^{(o)}$ $F_{3p}^{(o)}$ and $F_{3p}^{(2)}$ respectively for all three-body ranges. The corresponding integrals from ${}_\Lambda Be^9$ with the same size and ranges, are approximately 10% smaller than those for ${}_\Lambda C^{13}$ for all two and three body ranges.

For completeness the values for $\langle T_\Lambda \rangle$ obtained from eqn. 6 for the various hypernuclei are shown in fig. 4 and, taken together with the various Slater integrals, enable the value of B_Λ to be estimated from eqn. 35 for all reasonable sizes and volume integrals. The dependence of $\langle T_\Lambda \rangle$ on B_Λ for the A = 7 hypernuclei is again indicated by the two curves given for B_Λ = 5.0 MeV and 3.5 MeV respectively. For these hypernuclei $\langle T_\Lambda \rangle$ is shown as a function of a_s for a_p = 2.0 fm, the actual value of a_p being unimportant as $\langle T_\Lambda \rangle$ does not depend sensitively on this. Thus for a_s = 1.5 fm, B_Λ = 5.0 MeV and $\mu_{2\pi}$ the total variation in $\langle T_\Lambda \rangle$ as a_p increases from 1.8 fm to 2.4 fm is approximately 1.5%. For $_\Lambda C^{13}$ and $_\Lambda Be^9$ an increase of 1 MeV in the value of B_Λ from that specified results in increases in $\langle T_\Lambda \rangle$ of approximately 0.4 MeV and 0.6 MeV respectively.

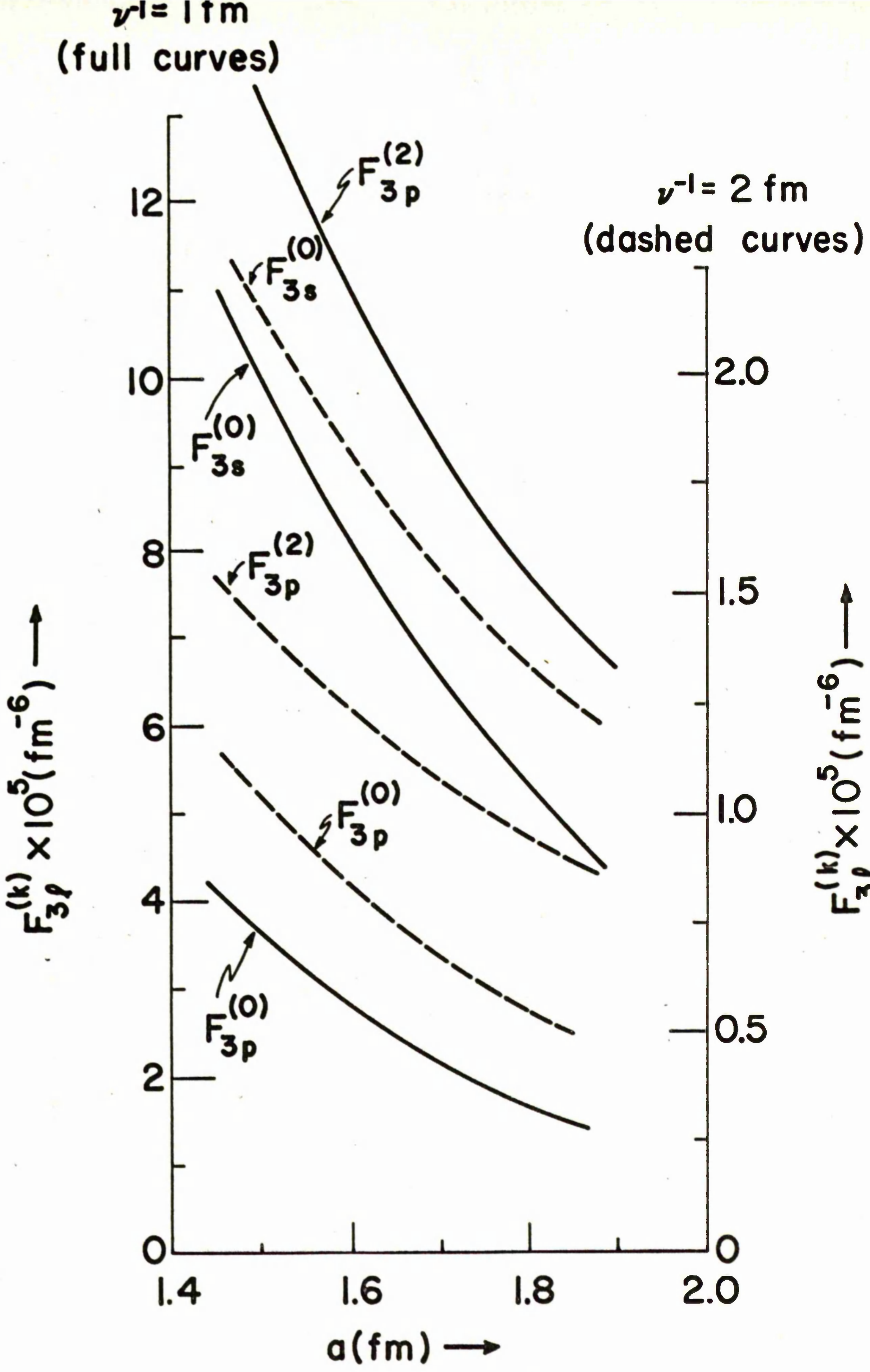
Captions to figures 3 and 4

Fig. 3 Three-Body Slater Integrals for $_{\Lambda}C^{13}$ with Two-Body forces of range $\mu_{2\pi}^{-1}$ fm

The three-body interaction Slater integrals are shown for $_{\Lambda}C^{13}$ with B_{Λ} = 10.9 MeV and for two-body forces of range $\mu_{2\pi}^{-1}$. The solid curves are for ν^{-1} = 1.0 fm (left hand scale) and the dashed curves for ν^{-1} = 2.0 fm (right hand scale).

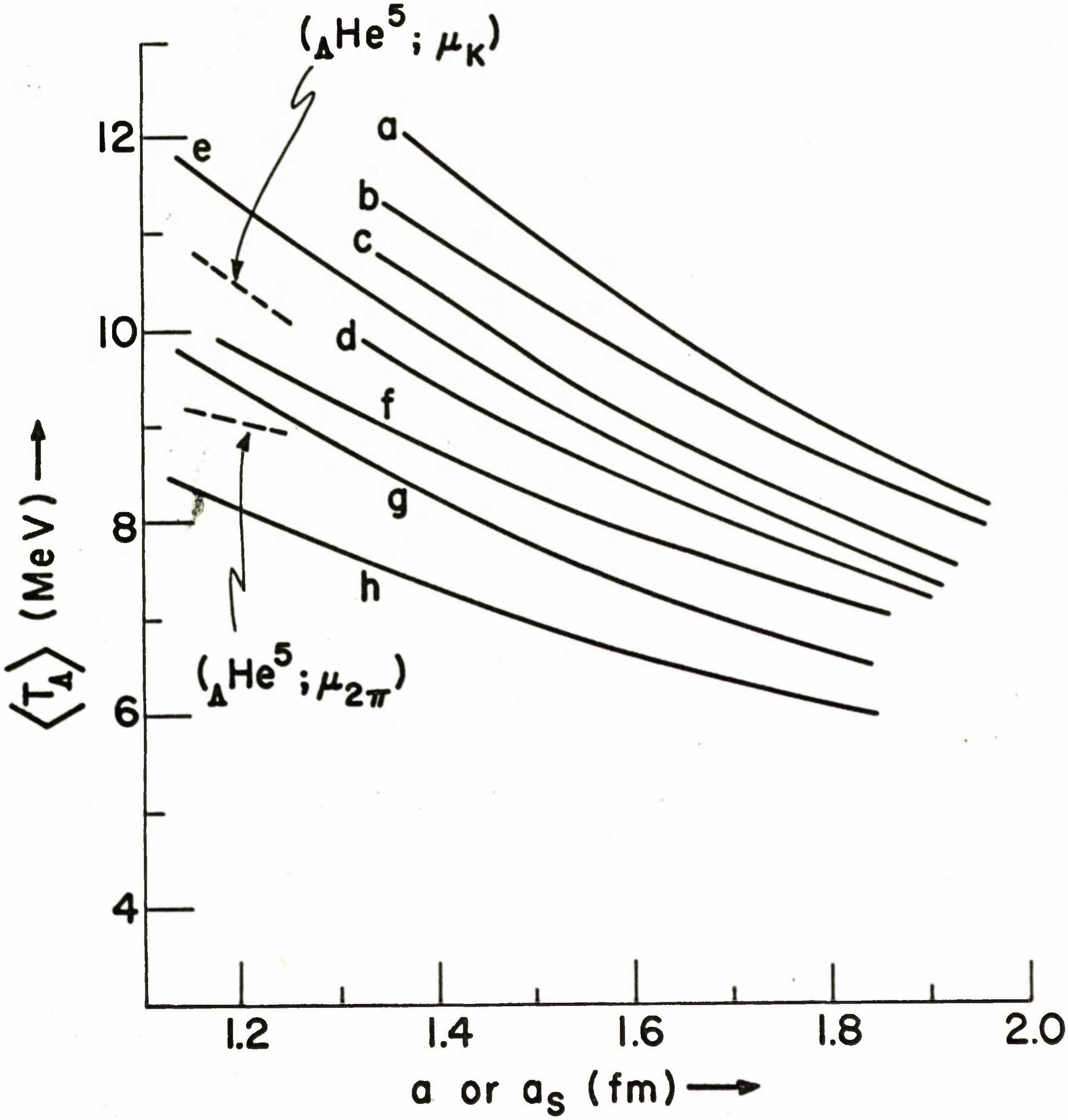
Fig. 4 The Λ kinetic energy

The caption is the same as for Fig. 1 but for the computed values of the Λ kinetic energy.



ν⁻¹ = 1 fm
(full curves)
ν⁻¹ = 2 fm
(dashed curves)
$F_{3p}^{(2)}$
$F_{3s}^{(0)}$
$F_{3s}^{(0)}$
$F_{3p}^{(2)}$
$F_{3p}^{(0)}$
$F_{3p}^{(0)}$
$F_{3\ell}^{(k)} \times 10^5 (\mathrm{fm}^{-6})$
$F_{3\ell}^{(k)} \times 10^5 (\mathrm{fm}^{-6})$
0
2
4
6
8
10
12
0
0.5
1.0
1.5
2.0
1.4
1.6
1.8
2.0
a(fm)

Fig. 3.



($_\Lambda He^5$; μ_K)
($_\Lambda He^5$; $\mu_{2\pi}$)
a
b
c
d
e
f
g
h
12
10
8
6
4
1.2
1.4
1.6
1.8
2.0
$\langle T_\Lambda \rangle$ (MeV)
a or a_s (fm)

Fig. 4.

CHAPTER 3

The Hypernuclei $_\Lambda C^{13}$ and $_\Lambda Be^9$

3.1 Two Body Λ -N Forces for $_\Lambda C^{13}$

The relevant hypernuclei for which B_Λ is known and whose cores have zero spin ($J_N = 0$) are the mirror pair, $_\Lambda He^7$ and $_\Lambda Be^7$, $_\Lambda Be^9$ and $_\Lambda C^{13}$ of which only the last two will be considered here, the A = 7 hypernuclei being discussed separately in chapter 5 because of their special features. With only two-body forces the potential well for these and also for $_\Lambda He^5$ is given by eqn. 23. The volume integrals $1/2U_8$ and $1/3\, U_{12}$ obtained for $_\Lambda Be^9$ and $_\Lambda C^{13}$ respectively should then agree with each other and with the value of U_4 obtained from $_\Lambda He^5$ if two body forces alone are to give a satisfactory description of these hypernuclei. In this section we shall mainly consider $_\Lambda C^{13}$ since the C^{12} core is stable and its density distribution is known from electron scattering experiments, neither of which is true for the Be^8 core of $_\Lambda Be^9$. Also the effects of the compression of the C^{12} core by the Λ is expected to be quite small as such effects are, in general, proportional to $1/(A - 1)$ since there are $(A - 1)$ nucleons and only one Λ . Thus for $_\Lambda C^{13}$ estimates on the lines of ref.[3] indicate a decrease of $\lesssim$ 2% for U_{12} and $\lesssim$ 3% for the core radius using reasonable values of the compressibility coefficient

K($\gtrsim$ 100 MeV).# Such changes are small compared with the errors due to uncertainties in B_Λ and in the undistorted core density distribution and justify the use of the latter for the evaluation of $V_\Lambda(r)$.

For ${}_\Lambda C^{13}$ we have considered both an oscillator and a Fermi nucleon density distribution, some results for the former having already been given by Dalitz [16]. With the same oscillator size parameter a for both the s and p-nucleons the total oscillator density distribution is given by

$$\rho_{H.O.}(r) = \frac{(A-1)}{3\pi^{3/2}a^3}\left[1 + \frac{(Z-2)}{3}\frac{r^2}{a^2}\right]\exp\left[-r^2/a^2\right], \qquad (37)$$

Thus one obtains for the increase of B_Λ arising from compression of the core the expression $\delta B_\Lambda = a^2\left(\frac{\partial B_\Lambda}{\partial a}\right)^2_U / K(A-1)$ $= -\left(\frac{\partial B_\Lambda}{\partial U}\right)_a \delta U$ where δU is the change in U(= 1/3 U_{12}) needed to compensate the change δB_Λ and where a is the oscillator size parameter appropriate to the undistorted core. The corresponding change in the core size is given by $\delta a = a^2\left(\frac{\partial B_\Lambda}{\partial a}\right)_U / K(A-1)$. For $A=13$, $U = 1000$ MeV fm^3 and with a = 1.65 fm and the calculated values $\left(\frac{\partial B_\Lambda}{\partial a}\right)_U = -2.0$ MeV fm^{-1} and $\left(\frac{\partial B_\Lambda}{\partial U}\right)_a = 0.02$ fm^{-3} one obtains $\frac{\delta U}{U} = -\frac{2.8}{K}$ and $\frac{\delta a}{a} = -\frac{2.75}{K}$.

the electron scattering data for $C^{12}(Z = 6)$ being well fitted with $a = 1.64 \pm 0.05$ fm [17]. The Fermi distribution is given by

$$\rho_F = \rho_0 \left[1 + \exp\left[\frac{r-c}{d}\right]\right]^{-1} \quad ; \quad \rho_0 = \frac{3(A-1)}{4\pi c^3}\left[1 + \frac{\pi^2 d^2}{c^2}\right]^{-1} \tag{38}$$

To investigate the effect of the shape of ρ in U_{12} we have considered in particular 'equivalent' Fermi distributions having the same half-density radius c and the same 90% - 10% surface thickness s (= 4.40d) as the oscillator distributions used. Thus, for example, with a = 1.64 fm the equivalent values for ρ_F are c = 2.30 fm and s = 1.82 fm. The parameters c and s are essentially the only ones determined by electron scattering experiments and one might expect that they also effectively determine U_{12} uniquely, independent of the precise shape of ρ. In fact direct fits of a Fermi distribution to the electron scattering results give values for c and s which are in reasonable agreement with those obtained by fitting an oscillator distribution [17].

The results for $1/3\ U_{12}$, to be compared with U_4 are shown in table 5 for $\rho_{H.O.}$ and for the most recent value B_Λ = 10.9 MeV [16], the errors indicated being due to those in B_Λ only. Our results for ordinary forces, are roughly in agreement with those given by Dalitz.# The values of U_{12} for 'equivalent' Fermi distributions are

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\# Thus from ref [16] one obtains $1/3\ U_{12}$ = 950 ± 150 MeV fm^3 and 740 ± 130 MeV fm^3 for $\mu_{2\pi}$ and μ_K respectively. These calculations relate to Gaussian interactions with the same intrinsic range as our Yukawa's.

consistently about 4% smaller for both $\mu_{2\pi}$ and μ_K than for the corresponding oscillator distributions for all values of the size parameter a considered. Thus uncertainties due to the shape of ρ are rather smaller than those due to errors in the size and considerably smaller than differences in U_4 obtained with Gaussian and Yukawa interactions of the same intrinsic range.# For the dependence on c and s one obtains $\frac{1}{3}\left(\frac{\partial U_{12}}{\partial c}\right)_s$ = 430 MeV fm^2 and 1/3 $\left(\frac{\partial U_{12}}{\partial s}\right)_c$ = 255 MeV fm^2 for $\mu_{2\pi}$. Thus for the values c = 2.24 fm and s = 2.2 fm obtained by Meyer-Berkhout et al.[17] from fitting a Fermi charge distribution directly to the electron scattering data, one obtains 1/3 U_{12} = 1060 MeV fm^3. Folding out the proton charge distribution would reduce this and in fact Meyer-Berkhout et al. claim a somewhat more satisfactory fit with an oscillator than with a Fermi distribution. On the basis of these results one reasonably obtains 1/3 U_{12} = 990 $\pm$ 60 MeV fm^3 and 815 $\pm$ 50 MeV fm^3 for ordinary Yukawa interactions of ranges $\mu_{2\pi}^{-1}$ and μ_K^{-1} respectively, most of the error now being due to size uncertainties.

The corresponding difference for 1/3 U_{12} is expected to be rather less than for U_4.

Table 5 Results for Two Body Forces for ${}_{\Lambda}C^{13}$ with B_{Λ} = 10.9 ± 0.5 MeV

The indicated errors are due to those in B_{Λ} only.

a (fm)	$\frac{1}{3}U_{12}$ (MeV fm^3)	C (MeV)	$\frac{1}{12}U_{12}F_{2p}^{(o)}$ (MeV)	$\frac{1}{3}U_{12}$ (ordinary) (MeV fm^3)	$\frac{1}{3}U_{12}$ (exchange) (MeV fm^3)	C (MeV)	$\frac{1}{12}U_{12}F_{2p}^{(o)}$ (MeV)
1.55	903 ±15	−0.54 ±0.08	1.43 ±0.03	750 ±20	790 ±20	−0.36 ±0.15	1.41 ±0.05
1.65	998 ±20	−0.13 ±0.08	1.38 ±0.04	824 ±20	860 ±20	−0.02 ±0.16	1.37 ±0.04
1.75	1093 ±25	0.31 ±0.16	1.33 ±0.04	915 ±20	960 ±20	0.40 ±0.18	1.32 ±0.04

For the K meson range the exchange nature of the force should also be taken into account. This was achieved by using the expression of Dalitz and Downs [1] for the ratio $\eta = \langle V \rangle^{(exch)} / \langle V \rangle^{(ord)}$, of the potential energies for ordinary and exchange Gaussian Λ-N interactions and using a Gaussian Λ wavefunction. Since Gaussian and Yukawa interactions with the same intrinsic range are very nearly equivalent in binding the Λ, the corresponding ratios may be expected to be in excellent agreement with each other. For the size parameter of the Λ wavefunction we use a_Λ = 1.80 fm and 1.75 fm for the ranges $\mu_{2\pi}^{-1}$ and μ_K^{-1} respectively, these values giving the same r.m.s. radii as obtained from the eigensolutions of eqn. 6 with a = 1.65 fm; the results in fact do not depend sensitively on the prewise value of a_Λ . For $\eta_\ell = \langle v \rangle_\ell^{(exch)} / \langle v \rangle_\ell^{(ord)}$ where ℓ = s or p for the interactions with s or p-nucleons respectively, one finds $\eta_s(\mu_{2\pi})$ = 0.99 and $\eta_s(\mu_K)$ = 1.00 and thus effectively no difference between ordinary and exchange forces, while $\eta_p(\mu_{2\pi})$ = 0.75 and $\eta_p(\mu_K)$ = 0.91. The resulting values for the total ratio η are $\eta(\mu_{2\pi})$ = 0.86 and $\eta(\mu_K)$ = 0.95 and thus the values of 1/3 U_{12} obtained with exchange forces will be correspondingly larger than for ordinary forces. The results for the physically interesting range μ_K^{-1} are given in table 5 and one now has 1/3 $U_{12}^{(exch)}$ ~~(exch.)~~ (μ_K) = 830 $\pm$ 50 MeV fm^3, the error including size uncertainties. For comparison the corresponding values for nuclear matter (ρ = 0.168 fm^3) are $\eta(\mu_{2\pi})$ = 0.7 and $\eta(\mu_K)$ = 0.86; exchange effects thus being considerably greater here than in $_\Lambda C^{13}$. The ratio, P, of the total interaction energy in relative p-states to the total interaction energy is given, to a very good approximation, by $P = \frac{1}{2}(\langle v \rangle^{(ord)} - \langle v \rangle^{(exch)}) / \langle v \rangle^{(ord)} = \frac{1}{2}(1-\eta)$ since interactions in relative Λ-N states with $\ell >$ 1 are expected to be negligible. Thus for $_\Lambda C^{13}$ one obtains $P_{12}(\mu_{2\pi})$ =

7% and P_{12} (μ_K) = 2.5% for a = 1.65 fm and, as expected, P_{12} is smaller for the shorter range. For nuclear matter the corresponding values are $P_\infty(\mu_{2\pi})$ = 15% and $P_\infty(\mu_K)$ = 7%. Interactions in relative Λ -N angular momentum states with $l \geqslant 1$ are thus considerably less effective in ${}_\Lambda C^{13}$ than in nuclear matter. This presumably is mainly a reflection of the smaller average density of C^{12} associated with its large surface and in this respect conditions in the two are still substantially different. If, as an extreme case of an ordinary force of range $\mu_{2\pi}^{-1}$ for which interactions in relative p-states are weakened compared with the s states, we consider a force with only relative s state interactions, then one obtains 1/3 U_{12} = 1070 MeV for a = 1.65 fm and B_Λ = 10.9 MeV, corresponding to a 10% increase as compared with the value for a static force.

It is seen that within the uncertainties of 1/3 U_{12} and U_4 these are consistent with each other for both ranges $\mu_{2\pi}^{-1}$ and μ_K^{-1} and to draw any conclusions about the range of the interaction from hypernuclei as light as the p-shell ones requires, above all, a very accurate knowledge of the relevant core sizes. The small values of the quantity C = 1/3 U_{12} $F_{2s}^{(o)} - \langle T_\Lambda \rangle$ should be noted, implying a near cancellation between the Λ kinetic energy and the potential energy due to interactions with the s-nucleons, although slightly larger values for C are obtained for ${}_\Lambda Be^9$ as shown in table 7. For ${}_\Lambda He^5$ (i.e. no p-nucleons) C has, of course, just the value of B_Λ = 3.1 ± 0.05 MeV and the conclusions of chapter 5 indicate a value similar to this also for the A = 7 hypernuclei. The much smaller values of C for the heavier p-shell hypernuclei are mainly due to the larger s-shell sizes for these, the Slater integral $F_{2s}^{(o)}$ decreasing as the size increases. The assumption that C is constant and equal to 3.1 MeV throughout the p-shell made in other investigations [11, 12] of these hypernuclei implies that the expectation value of the spin averaged interaction is also constant and given by $1/4\ UF_{2p}^{(o)} \simeq 1.0$ MeV. This is therefore an underestimate for the heavier p-shell nuclei. As seen from tables 5 and 7 the values of $1/4\ UF_{2p}^{(o)}$ are approximately 1.4 MeV for both $_{\Lambda}Be^{9}$ and $_{\Lambda}C^{13}$ and do not depend much on the size a.

Results for the potential well $V_{\Lambda}(r)$ are shown in table 6, the values for the two ranges $\mu_{2\pi}^{-1}$ and μ_{K}^{-1} and also for $\rho_{H.O.}$, and the 'equivalent' ρ_F agreeing closely. For $\rho_{H.O.}$ the maximum density is $\rho_{H.O.}(r = 0.8\ fm) = 0.169$ fm. However $V_{\Lambda}(r)$ does not show a corresponding dip at r = 0. This is because the finite range of the Λ-N interaction smooths out the effects of density variations over distances of the order of the range μ^{-1} (see also the discussion in the appendix 2).

Table 6 Potentials in ${}_\Lambda C^{13}$ for a = 1.64 fm and B_Λ = 10.9 MeV

For $\rho_{H.O.}$ the volume integrals obtained from eqn. 6 are $1/3\ U_{12}$ = 988 and 830 MeV fm^3 for $\mu_{2\pi}$ and μ_K respectively: those for ρ_F are $1/3\ U_{12}$ = 950 and 800 MeV fm^3

	Harmonic Oscillator Density Distribution					Fermi Density Distribution				
		$\mu_{2\pi}^{-1} = 0.7$ fm		$\mu_K^{-1} = 0.4$ fm			$\mu_{2\pi}^{-1} = 0.7$ fm		$\mu_K^{-1} = 0.4$ fm	
r (fm)	$\rho_{H.O.}$ (fm^{-3})	$\frac{1}{12}\bar{U}_{12}\rho_{H.O.}$ (MeV)	$V_\Lambda(r)$ (MeV)	$\frac{1}{12}\bar{U}_{12}\rho_{H.O.}$ (MeV)	$V_\Lambda(r)$ (MeV)	ρ_F (fm^{-3})	$\frac{1}{12}\bar{U}_{12}\rho_F$ (MeV)	$V_\Lambda(r)$ (MeV)	$\frac{1}{12}\bar{U}_{12}\rho_F$ (MeV)	$V_\Lambda(r)$ (MeV)
0	0.163	40.2	33.5	33.8	33.2	0.178	42.25	33.7	35.6	33.5
0.4	0.167	41.2	33.1	34.6	33.0	0.177	42.0	33.2	35.4	33.2
0.8	0.169	41.7	31.8	35.0	32.2	0.173	41.0	31.8	34.6	32.1
1.2	0.163	40.2	29.1	33.8	29.9	0.167	39.6	29.2	33.4	29.9
2.0	0.109	26.9	20.0	22.6	20.1	0.120	28.5	20.3	24.0	20.6

3.2 Well Depth for Nuclear Matter and B_Λ for Heavy Hypernuclei

Also shown in table 6 are the relevant values of $D = \frac{1}{2}U\rho$ this being the first order well depth for nuclear matter of density ρ for ordinary Λ -N forces. Second order contributions to the well depth are small and probably less than about 10% [3]. Again due to the effects of the finite range and finite core size the values of $V_\Lambda(r)$ for the longer range $\mu_{2\pi}^{-1}$ are substantially smaller than the corresponding values of D, while for μ_K the differences are much less (for zero range forces one has $V_\Lambda = D$). Thus, especially for $\mu_{2\pi}$, surface effects in ${}_\Lambda C^{13}$ are sufficiently important for the central value of the potential, $V_\Lambda(0) = 33.5 \pm 2$ MeV, for either range, not to be a reliable indication of the corresponding well depth for nuclear matter, which for $\mu_{2\pi}$ is D = 42 MeV for U = 1000 MeV fm^3 and $\rho = 0.168\ fm^{-3}$ in spite of the fact that the central density of C^{12} is quite close to that of nuclear matter. For exchange forces of range μ_K^{-1} one finds $D^{(exch)} = \gamma\ D^{(ord)} = 29.5$ MeV for U = 800 MeV fm^3 and $\gamma = 0.88$ instead of $D^{(ord)} = 33.6$ MeV for ordinary forces.

Experimental results [19] for heavy nuclei with $60 \leq A - 1 \leq 100$ (corresponding to heavy emulsion nuclei) indicate a value of B_Λ in the region of 25 MeV and not much in excess of 30 MeV. Even for these nuclei B_Λ is still substantially less than D both because the Λ kinetic energy is still appreciable (thus for A = 60 one has $\langle T_\Lambda \rangle$ = 6.5 and 7.5 MeV for B_Λ = 25 and 35 MeV respectively) and because an appreciable fraction of the nucleus is still contained in the nuclear surface. Thus using a Fermi density distribution, eqn. 38, with the values ρ_0 = 0.168 fm^{-3} and s = 2.49 fm, appropriate to electron scattering experiments for heavier nuclei, one obtains for $B_\Lambda^{(A)}$ (D) the values, in MeV: $B_\Lambda^{(120)}$ (40) = 32.3, $B_\Lambda^{(120)}$ (30) = 22.9, $B_\Lambda^{(60)}$ (40) = 28.2, $B_\Lambda^{(60)}$ (30) = 19.5 for $\mu_{2\pi}$ and $B_\Lambda^{(120)}$ (40) = 34.3, $B_\Lambda^{(120)}$ (30) = 24.5, $B_\Lambda^{(60)}$ (40) = 30.2, $B_\Lambda^{(60)}$ (30) = 21 for μ_K ; the dependence of B_Λ on D being linear for all cases. Even for A = 200 there is still a difference of about 5 MeV between D and B_Λ , although for these large values of A the effects of the finite range is very slight. The predicted values for a mean value of A = 80, to be compared with the experimental ones, are then B_Λ = 30 $\pm$ 1 MeV for $\mu_{2\pi}$ with U = U_4 = 1025 $\pm$ 30 MeV fm^3 and B_Λ = 20.5 $\pm$ 1 MeV for exchange forces of range μ_K^{-1} with U = U_4 = 785 $\pm$ 25 MeV fm^3. Neither of these values seems inconsistent with the experimental results and it seems doubtful whether significance should be attached, at present, to the slightly large values for $\mu_{2\pi}$, especially if uncertainties in U of about 10%, or possibly even more, are accepted.† Such uncertainties may arise not only from errors in the value of the phenomenological volume integrals U_4 and U_{12} but also from possible uncertainties coming from a lack of knowledge of the shape and velocity dependence of the Λ -N interaction as well as from possible differences in second order effects of the force in heavy and light hypernuclei. The value $D \simeq 40$ MeV for $\mu_{2\pi}$ obtained here with Yukawa interactions agrees well with that of Ram and Downs [10] obtained by using a hard core (of radius 0.4 fm) together with an attractive well, which, in the asymptotic region, possesses an exponential behaviour appropriate to the two pion exchange mechanics, although for such a force the relative p-state contributions are greatly enhanced relative to those for 'soft' forces such as we use.

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† The predicted value would in fact be reduced to $\simeq$ 28 MeV if there was an experimental bias towards smaller values of A in the range $60 \leq A - 1 \leq 100$.

3.3 The Hypernucleus $_{\Lambda}Be^9$

Results for $1/2\ U_8$ obtained using appropriate oscillator density distributions, eqn. 37 with Z = 4, are shown in table 7 for B_{Λ} = 6.5 ± 0.25 MeV. Since the isolated Be^8 core nucleus is unstable, its size and energy when the Λ is present are uncertain. As discussed in more detail in sections 5.2 and 5.3 in connection with the A = 7 hypernuclei, the rearrangement energy of the core, i.e. the difference between the total core energy with the Λ present and that of the isolated core, must be positive. Thus the value of $1/2\ U_8$ obtained from the two-body model calculations for a given B_{Λ} and some assumed core size will be a lower limit to the value which would be obtained if the re-arrangement energy, appropriate to the actual core size, was included. Bearing this in mind, inspection of table 7 then indicates that a size 1.5 fm $\lesssim$ a $\lesssim$ 1.6 fm is consistent with the results for U_4 and $1/3\ U_{12}$ as well as with a reasonable re-arrangement energy of $\lesssim$ 1 MeV.

Table 7 Results for Two-Body Forces for $_{\Lambda}Be^{9}$ with B_{Λ} = 6.5 ± 0.25 MeV

The indicated errors are due to those in B_{Λ} only.

a (fm)	$\frac{1}{2}U_8$ (MeV fm^3)	C (MeV)	$\frac{1}{8}U_8F_{2p}^{(o)}$ (MeV)	$\frac{1}{2}U_8$ (MeV fm^3)	C (MeV)	$\frac{1}{8}U_8F_{2p}^{(o)}$ (MeV)
1.50	960 ±15	0.62 ±0.09	1.48 ±0.03	785 ±15	0.68 ±0.11	1.45 ±0.03
1.60	1044 ±18	0.81 ±0.09	1.43 ±0.02	865 ±15	0.92 ±0.10	1.40 ±0.03
1.70	1126 ±20	1.01 ±0.19	1.38 ±0.04	950 ±15	1.24 ±0.13	1.34 ±0.03

To obtain more definite conclusions a more dynamical approach is needed which includes the degrees of freedom of the Be^{8} core.

Such an approach is a three-body model consisting of two α particles and the Λ which has been considered by Wilhelmsson and Zielinski [20] and more especially by Suh [20]. A more detailed investigation has been made by Bodmer and Ali [32] (to be published) using more realistic α-α interactions as well as better trial wave functions than used by Suh. With α-α interactions which give phase shifts in reasonable agreement with the experimental values one then obtains $1/2\ U_s = 1038 \pm 20$ MeV fm^3 for $\mu_{2\pi}$ and with $B_\Lambda = 6.5 \pm 0.15$ MeV which is in very good agreement with the corresponding value obtained from ${}_\Lambda He^5$ and ${}_\Lambda C^{13}$. The size of the Be^8 core obtained from the three-body calculation may be expressed in terms of an equivalent oscillator size parameter which is more or less directly comparable with our size parameter a. For this equivalent size parameter one obtains 1.65 ± 0.03 fm, while the rearrangement energy turns out to be somewhat less than one MeV. The results of table 7 are thus quite consistent with those of a three-body model calculation and can therefore be more significantly interpreted in terms of the latter.

3.4 Two and Three-Body Λ -N Interactions

For investigating the strength of the three-body force mainly $_\Lambda He^5$ and $_\Lambda C^{13}$ will be considered, in view of the inherent uncertainty attached to a two-body model analysis of $_\Lambda Be^9$. Although there is a good agreement between $_\Lambda He^5$ and $_\Lambda C^{13}$ with only two-body forces one cannot necessarily conclude from this that three-body forces are negligible. Thus with the inclusion of three-body forces one will obtain from each hypernucleus a relation between U and W and if this should be similar for both hypernuclei then consistency could be obtained for a large range of three-body strengths including zero. On the other hand if the two relations are sufficiently different then combining them will give significant limits on the strength of the three-body force.

Results for $_\Lambda C^{13}$ including three-body forces have been obtained using the procedure discussed in chapter 2. As for the case of $_\Lambda He^5$, previously reported [3] and for the same reasons as discussed there, the relation between U and W for $_\Lambda C^{13}$ was also found to be linear within the computational accuracy. Thus one has

$$U = U_4 - Z_4 W \qquad \text{for } {}_\Lambda He^5 \tag{39}$$

$$U = 1/3\, U_{12} - Z_{12} W \quad \text{for } {}_\Lambda C^{13} \tag{40}$$

where the values obtained for the coefficient Z_{A-1} are given in table 8 and depend on the ranges μ^{-1} and ν^{-1} ; U_4 and U_{12} being the volume integrals obtained with two-body forces only. From eqn. 35 one has the following expressions for Z_{A-1} in terms of the relevant Slater integrals (whose values depend of course on the particular hypernucleus considered).

$$Z_4 = \frac{18 F_{3s}^{(0)}}{F_{2s}^{(0)}} \tag{41}$$

$$Z_{12} = \frac{18 F_{3s}^{(0)} + \alpha_8 F_{3p}^{(0)} + \beta_8 F_{3p}^{(2)}}{F_{2s}^{(0)} + 2 F_{2p}^{(0)}} \tag{42}$$

It is seen that Z_4 is approximately twice as large as Z_{12}, corresponding to the effect of the three-body force being greater in $_\Lambda He^5$ than in $_\Lambda C^{13}$. This can be understood as due to the large s-nucleon density of He^4 on the one hand, the three and two-body contributions being approximately proportional to ρ^2 and ρ respectively, while on the other hand the effect of the rather large number of p-nucleon pairs in $_\Lambda C^{13}$ is very much reduced by the exclusion principle, as discussed in section 2.3. For $_\Lambda Be^9$, in intermediate coupling, one has for comparison $Z_8 = 0.125$ fm^{-3} and 0.105 fm^{-3} with $\nu^{-1} = 1.0$ fm and $Z_8 = 0.030$ fm^{-3} and 0.025 fm^{-3}

with $\nu^{-1} = 2.0$ fm for $\mu_{2\pi}$ and μ_{K} respectively and for B_{Λ} = 6.5 MeV and a = 1.60 fm. These are quite close to the corresponding values of Z_{12}, the three-body force contributions due to both s and p-nucleons now being quite similar for $_{\Lambda}C^{13}$ and $_{\Lambda}Be^{9}$. In fact, for intermediate coupling the ratio $Z_{12}/Z_8 \lesssim 1$ for all reasonable sizes for C^{13} and this is smaller than the ratio of the number of p-nucleons for these nuclei. The similar values of Z_8 and Z_{12} imply that, even if the uncertainties due to the core structure in $_{\Lambda}Be^{9}$ could be avoided (as, for example, is achieved to some extent by an α-α-Λ model), no useful limits could be expected to be placed on the strength of the three-body forces from a comparison of $_{\Lambda}Be^{9}$ with $_{\Lambda}C^{13}$, while a comparison with $_{\Lambda}He^{5}$ would not be expected to give anything substantially different from comparing $_{\Lambda}He^{5}$ with $_{\Lambda}C^{13}$.

Combining the results for $_{\Lambda}He^{5}$ and $_{\Lambda}C^{13}$ then gives the values of U shown in table 8. The results for $\nu^{-1} = 1.4$ fm have not been given but are intermediate to those for $\nu^{-1} = 1.0$ fm and 2.0 fm. Also shown are the corresponding ratios of the three-body to the total potential energy which are given by $\rho_4 = (U_4 - U)/U_4$ and $\rho_{12} = (\frac{1}{3}U_{12} - U)/\frac{1}{3}U_{12}$ for $_{\Lambda}He^{5}$ and $_{\Lambda}C^{13}$ respectively as well as the corresponding ratio for nuclear matter, viz. $\rho_{\infty} = (D_2 - D)/D_2$ which has been obtained using the results of ref[3)] for the total and two-body well depths D and D_2 respectively.

Table 8

Results for two-body and three-body forces. The errors shown for U are due to the errors in $B_{\Lambda}({}_{\Lambda\Lambda}C^{13}) = 10.9 \pm 0.5$ MeV and in the values of B_{Λ} and the core size of ${}_{\Lambda}He^{5}$. The errors shown for ξ_{A-1} effectively arise only from the errors in U. The results are for intermediate coupling unless otherwise stated.

$\mu_{2\pi}$; $U_4 = 1038 \pm 30$ MeV fm^3												
		$\nu^{-1} = 1$ fm, $Z_4 = 0.228$ fm^{-3}					$\nu^{-1} = 2$ fm, $Z_4 = 0.062$ fm^{-3}					
a (fm)	$\frac{1}{3}U_{12}$ (MeV fm^3)	Z_{12} (fm^{-3})	U (MeV fm^3)	$\xi_{12} \times 100$	$\xi_4 \times 100$	$\xi_\infty \times 100$	Z_{12} (fm^{-3})	U (MeV fm^3)	$\xi_{12} \times 100$	$\xi_4 \times 100$	$\xi_\infty \times 100$
1.55	903 ± 15	0.122	748 ± 38	17.2 ∓ 4.3	28.0 ∓ 3.7	21.8 ∓ 4.2	0.029	780 ± 40	13.4 ∓ 4.5	24.8 ∓ 3.8	25.5 ∓ 5.5
1.65	998 ± 20	0.110	958 ± 44	4.0 ∓ 4.4	7.7 ∓ 4.3	6.75 ∓ 4.8	0.027	965 ± 50	3.3 ∓ 5.0	7.0 ∓ 4.8	5.7 ∓ 4.5
1.65 (LS)	998 ± 20	0.119	950 ± 50	4.8 ∓ 5.0	8.5 ∓ 4.8	6.8 ∓ 4.0	0.029	958 ± 55	3.9 ∓ 5.5	7.6 ∓ 5.3	6.5 ∓ 5.3
1.65 (jj)	998 ± 20	0.099	964 ± 38	3.4 ∓ 3.8	7.1 ∓ 3.7	5.0 ∓ 3.6	0.023	972 ± 42	2.4 ∓ 4.2	6.2 ∓ 4.1	5.0 ∓ 3.8
1.75	1093 ± 25	0.099	1128 ± 53	-3.2 ∓ 4.9	-8.7 ∓ 5.1	-5.7 ∓ 2.1	0.025	1130 ± 42	-3.6 ∓ 3.9	-9.0 ∓ 4.1	-7.5 ∓ 2.5
μ_K; $U_4 = 786 \pm 25$ MeV fm^3											
		$\nu^{-1} = 1.0$ fm, $Z_4 = 0.172$ fm^{-3}					$\nu^{-1} = 2.0$ fm, $Z_4 = 0.047$ fm^{-3}				
a (fm)	$\frac{1}{3}U_{12}$ (MeV fm^3)	Z_{12} (fm^{-3})	U (MeV fm^3)	$\xi_{12} \times 100$	$\xi_4 \times 100$	$\xi_\infty \times 100$	Z_{12} (fm^{-3})	U (MeV fm^3)	$\xi_{12} \times 100$	$\xi_4 \times 100$	$\xi_\infty \times 100$
1.55	750 ± 20	0.102	692 ± 46	7.1 ∓ 6.1	11.8 ∓ 5.9	8.6 ∓ 5.8	0.024	710 ± 50	5.3 ∓ 6.6	9.7 ∓ 6.4	10.2 ∓ 8.6
1.65	824 ± 20	0.091	862 ± 55	-4.6 ∓ 6.7	-9.9 ∓ 7.0	-9.1 ∓ 3.9	0.022	855 ± 42	-3.8 ∓ 5.1	-9.0 ∓ 5.4	-11.2 ∓ 4.8
1.65 (LS)	824 ± 20	0.098	872 ± 52	-5.8 ∓ 6.5	-11.2 ∓ 6.7	-9.5 ∓ 5.3	0.024	862 ± 35	-4.5 ∓ 4.3	-9.8 ∓ 4.5	-12.0 ∓ 7.3
1.65 (jj)	824 ± 20	0.082	856 ± 60	-3.9 ∓ 7.3	-9.1 ∓ 7.6	-7.5 ∓ 6.5	0.019	848 ± 55	-2.8 ∓ 6.7	-8.0 ∓ 7.0	-9.4 ∓ 7.4
1.75	915 ± 20	0.084	1033 ± 50	-12.9 ∓ 5.5	-31.6 ∓ 6.4	-20.0 ∓ 2.8	0.021	1015 ± 42	-10.9 ∓ 4.6	-29.4 ∓ 5.4	-26.0 ∓ 3.5

For a given size of C^{12} the values ξ_4, ξ_{12} and ξ_∞ are comparable, the values of ξ_4 being on the whole somewhat larger than ξ_{12} and ξ_∞ for the reasons already discussed. The precise numerical values are seen to depend in a rather involved way on the particular ranges and densities considered. For a given two-body range and the size of C^{12} the proportion of three-body force does not depend much on the three-body range, while, for given μ and ν, the values of ξ decrease as the size increases coresponding to attractive three-body forces for small values of a, i.e. somewhat less than electron scattering sizes, and repulsive forces for larger values of a. In particular it is striking that for the actual size expected for C^{12} core, i.e. a = 1.64 $\pm$ 0.05 fm, the corresponding magnitude obtained for the three-body force must be quite small, viz. $|\xi_{12}| = 0 \pm 15\%$ for both two-body ranges. The most probable values correspond to weak, attractive, three-body forces for $\mu_{2\pi}$ and weak, but repulsive, forces for μ_K.

Hypernuclei with Non-Zero Spin Cores.

Apart from ${}_\Lambda Li^7$, which is discussed in Chapter 5, the hypernuclei whose cores have non-zero spin, $J_N \neq 0$, and for which B_Λ is reasonably well known are the mirror pair ${}_\Lambda Li^8$ and ${}_\Lambda Be^8$, ${}_\Lambda Li^9$ and ${}_\Lambda B^{12}$. If these hypernuclei were considered individually and an absolute calculation of the total B_Λ attempted, as has been done so far, then the spin-dependent contribution to B_Λ, which is roughly of magnitude $1/A$ relative to the dominant contribution depending only on U (and W), would be masked by even fairly small uncertainties in U. This is largely avoided by considering the differences, δB_Λ, in B_Λ for the following pairs: ${}_\Lambda Be^9$ and ${}_\Lambda Li^8$, ${}_\Lambda Be^9$ and ${}_\Lambda Li^9$, ${}_\Lambda C^{13}$ and ${}_\Lambda B^{12}$.

From eq. 35 one then has

$$\delta B_\Lambda = \frac{1}{4} U(W) F_{2p}^{(0)} - K \Delta F_{2p}^{(0)} + \left(\frac{\partial B_\Lambda}{\partial a}\right)\delta a + \left[\delta\alpha F_{3p}^{(0)} + \delta\beta F_{3p}^{(2)}\right] W \qquad (43)$$

where δB_Λ may strictly only be identified with the experimental value if the rearrangement energies are the same for both members of a pair. For the pair ${}_\Lambda Be^9 - {}_\Lambda Li^9$ the first term is not present since the mass numbers in this case are the same. In eq. 43 the term depending on U(W) is now comparable with the other terms since it arises from the interaction of the Λ with only one nucleon.

In fact the values $U(W = 0) = 1000$ and 760 MeV fm^3 for $\mu_{2\kappa}$ and μ_K respectively are taken, consistent with the results for the $J_N = 0$ hypernuclei, the precise value used now being unimportant. The third term is due to differences in the size between members of a pair. To a good approximation the quantity denoted by $(\partial B_\Lambda/\partial a)_\sigma$ is given by the rate of increase of B_Λ with respect to the oscillator size parameter, with U constant, for the $J_N = 0$ partner, i.e. for ${}_\Lambda Be^9$ and ${}_\Lambda C^{13}$ assuming only two body forces. It may be obtained directly from the results of tables 5 and 7 or from the results for the relevant Slater integrals given in section 2.4. In fact, as is the case for the $A = 7$ hypernuclei, $(\partial B_\Lambda/\partial a)_\sigma$ is given to a fair approximation by just thechange of interaction energy of the Λ with the p - nucleons, since the remaining part of the energy, i.e. $C = UF_2^{(o)} - \langle T_\Lambda \rangle$ is small and does not change very much with a. One has always $(\partial B_\Lambda/\partial a)_\sigma < 0$ corresponding to $F_{2p}^{(o)}$ increasing as the size a decreases. The last term in eq.43 is the difference in three body potential energy, the relevant values of $\delta\alpha$ and $\delta\beta$ being obtained from table 4. Instead of W it is of more immediate physical significance, especially with reference to the results of section 3.4, to use ρ_{12}, i.e. the ratio of the three-body interaction to the total potential energy in ${}_\Lambda C^{13}$. Correspondingly one then has $W = U(W{=}0)\ \rho_{12}/Z_{12}$ in eq. 43. Alternatively, of course, ρ_4 or ρ_∞ could have been used.

The results obtained from eq. 43 using the appropriate values for the various pairs, are shown in table 9 for both $\mu_{2\pi}$ and μ_K and for the three-body ranges $\nu^{-1} = 1.0$ and 2.0 fm, the results for $\nu^{-1} =$ 1.4 fm being intermediate to these. The changes in Δ with respect to the size a and ρ_{12} are insensitive to the precise value of a $(J_N = 0)$ used, while the value of Δ for $\delta a = 0$, $\rho_{12} = 0$ depends somewhat more on the size a, although not at all critically. The values of K used are those given by Dalitz (see also appendix 1).

Considering first the results for only two-body forces, i.e. for $\rho_{12} = 0$, one notes that the values of Δ for $\delta a = 0$, i.e. for the same core sizes, are very considerably larger than the values obtained from the s -shell hypernuclei, using the experimental values of δB_Λ. This seems true even when the substantial uncertainties due to erros in δB_Λ are taken into account. This conclusion is in agreement with the results given by Dalitz[12]. However, the values of Δ are seen to depend very sensitively on the size differences δa. This is because the size dependence of the major part of B_Λ is involved, with the result that even for quite small $|\delta a|$ the corresponding change in energy, $(\partial B_\Lambda / \partial a)_{\sigma} \delta a$ can easily be compared with or is larger in magnitude than the energy differences due to the effect of essentially a single nucleon given by the first two

Table 9

The spin dependence Δ and related quantities for the p-shell hypernuclei with $A > 7$.

Hypernuclei	$\kappa(J_N \neq 0)$	$\delta\alpha$	$\delta\beta$	$\delta B_\Lambda = B_\Lambda(J_N = 0) - B_\Lambda(J_N \neq 0)$ (MeV)	$a\ (J_N = 0)$ (fm)	$(\partial B_\Lambda / \partial a)_U$ (MeV fm^{-1})	Δ for $\delta a = 0$, $\xi_{12} = 0$ (MeV fm^3) $\mu_{2\pi}$	μ_K	$\delta\Delta$ for $\delta a = \pm 0.1$ fm and $\xi_{12} = 0$ (MeV fm^3) $\mu_{2\pi}$	μ_K	$\delta\Delta$ for $\xi_{12} = \mp 0.1$, $\delta a = 0$ (MeV fm^3) $\nu^{-1} = 1.0$ fm $\mu_{2\pi}$	$\nu^{-1} = 1.0$ fm μ_K	$\nu^{-1} = 2.0$ fm $\mu_{2\pi}$	$\nu^{-1} = 2.0$ fm μ_K
${}_\Lambda Be^9 - {}_\Lambda Li^8$	0.6	8.83	0.44	-0.15 ± 0.3	1.5	-13	465 ∓ 80	350 ∓ 65	∓ 345	∓ 285	± 30	± 27	± 27	±19
					1.6		470 ∓ 80	355 ∓ 75	∓ 395	∓ 330	± 30	± 25	± 30	±20
				0.85 ± 0.3*	1.5		200 ∓ 80	130 ∓ 65	∓ 345	∓ 285	± 30	± 27	± 27	±19
					1.6		170 ± 90	100 ∓ 75	∓ 395	∓ 330	± 30	± 25	± 30	±20
${}_\Lambda Be^9 - {}_\Lambda Li^9$	0.7	7.71	0.34	-1.5 ± 0.3	1.5	-13	340 ∓ 70	280 ∓ 60	∓ 345	∓285	± 55	± 45	± 60	± 50
					1.6		390 ∓ 80	327 ∓ 65	∓ 395	∓ 330	± 50	± 40	± 50	± 40
				-0.5 ± 0.3*	1.5		110 ∓ 70	90 ∓ 60	∓ 345	∓ 285	± 55	± 45	± 60	± 50
					1.6		130 ∓ 80	110 ∓ 65	∓ 395	∓ 330	± 50	± 40	± 50	± 40
${}_\Lambda C^{13} - {}_\Lambda B^{12}$	0.3	7.1	0.73	0.4 ± 0.4	1.6	-19.5	610 ∓ 225	440 ∓ 200	± 1100	∓ 930	± 55	± 49	± 50	± 36

* Including 1 MeV rearrangement energy for ${}_\Lambda Be^9$.

terms of eqn. 43. A reflection of this is the increase of $(\partial B_\Lambda/\partial a)_{\sigma}$ with A.

For the pair ${}_\Lambda C^{13}$ - ${}_\Lambda B^{12}$ one probably has positive δa = 0.1 ± 0.1 fm, corresponding to the values $a(C^{12})$ = 1.64 ± 0.05 fm and $a(B^{11})$ = 1.55 ± 0.1 fm obtained from analysis of electron scattering results [17]. For this case the rearrangement energies are expected to be very small and also very similar for both hypernuclei and the resulting uncertainties to be much less than those due to the error in δB_Λ. It is then immediately clear from table 9 that for very reasonable positive values of δa the value of Δ is reduced from that for $\delta a = 0$ to values quite consistent with those obtained from the s - shell hypernuclei. Thus for $\mu_{2\pi}$ one gets $\Delta = 60 \pm 225$ Mev fm^3 and -500 ± 225 MeV fm^3 for δa = 0.05 fm and 0.1 fm respectively, the errors being due to those in δB_Λ. Then to obtain any significant information about the spin-dependence it is necessary not only to have quite accurate experimental values for δB_Λ but also very accurate knowledge of the core sizes. In fact one can reasonably reverse the procedure and supposing that Δ has a value of the order obtained from the light ($A \leq 5$) hypernuclei, i.e. assuming that the Λ-N interaction is approximately known, one can consider the as a fairly sensitive probe into size differences. In this way one obtains $\delta a = a(C^{12}) - a(B^{11}) = 0.04 \pm 0.02$ fm, independently of the range of the Λ-N interaction.

The situation is quite similar for ~~the other two~~ the other two pairs although for these there is a further complication due to the rearrangement energy which is almost certainly substantial for $_\Lambda Be^9$. Thus an α-α-Λ model of $_\Lambda Be^9$ gives about 1 MeV for this,[321] the corresponding oscillator size parameter being about 1.6 fm. It seems not unreasonable to expect the rearrangement energies for $_\Lambda Li^8$ and $_\Lambda Li^9$ to be quite considerably less since for these the distorting effect of the Λ is expected to affect all the nucleons of the core more or less equally and thus be a fairly small 1/A effect; the procedure used for $_\Lambda C^{13}$ together with a reasonable compressibility coefficient giving $\lesssim$ 0.2 MeV. This is in contrast to the situation for $_\Lambda Be^9$ where, within the framework of an α-α-Λ model,[321] the distorting effect of the ~~α~~ Λ on each of the tightly bound α-particles will be very small but where there will be a quite large effect on the relative motion of the two α-particles as these are not even bound in Be^8. Assuming, then, zero rearrangement energies for $_\Lambda Li^8$ and $_\Lambda Li^9$ and one MeV for $_\Lambda Be^9$ one gets the larger 'effective' values of $\delta B'_\Lambda = 0.85 \pm 0.3$ and -0.5 ± 0.3 for $_\Lambda Be^9$ - $_\Lambda Li^8$ and $_\Lambda Be^9$ - $_\Lambda Li^9$ respectively. The actual values are likely to be somewhat, but not much, less than these. The effect of the rearrangement energy is then to reduce Δ to the values shown in the table. These are consistent with the values obtained from the s-shell hypernuclei even when the core sizes are taken to be the same. In fact, for values of $\delta B_{\Lambda}'$ of the order of those just considered, this then implies that the core sizes (in the hypernuclei) are very nearly the same. It is clear that because of uncertainties in both the rearrangement energies as well as in the core sizes very little can be said about Δ except that there is certainly no obvious inconsistency with the values obtained from the light hypernuclei. Because of the increasing relative importance with A of differential size effects the corresponding uncertainties will also become progressively larger, other things being equal. It seems that one is only likely to obtain a reasonably reliable value for the spin-dependence from the p - shell hypernuclei if the excitation energy of the spin flip state corresponding to the ground state of some hypernucleus can be determined reasonably accurately since the core sizes can be expected to be almost the same for both states.

The effect on Δ of three-body forces, shown for $\delta a = 0$, is seen to be small, although not entirely negligible, for strengths consistent with those obtained in section 3.4 and will thus be completely masked by even very small size differences of the order of 0.01 fm. It should be noted that the difference in the three-body potential energy, and correspondingly the effect on Δ is seen to be about the same in magnitude for the equal mass number pair, $_{\Lambda}Be^{9}$ - $_{\Lambda}Li^{9}$, as for the other pairs whose partners differ by one nucleon. It is interesting to observe that for ~~the shorter~~ all the three ranges considered ~~range γ^{-1} = 1.0 fm~~ the difference in the three-body potential energy has the same sign and comparable magnitude for all three pairs. ~~For this three-body range~~ Consistency with the value of Δ obtained from s - shell hypernuclei could then be approximately obtained for equal sizes of the partners of each pair with a value of ξ_{12} of roughly about - 0.5 corresponding to an extremely strong repulsive three-body force which is, of course, quite unacceptable in view of the previous results. ~~For the longer range the three-body energy for the pair ${}_{\Lambda}Be^{9}$ - ${}_{\Lambda}Li^{9}$ is seen to have the opposite sign to that for the other two pairs.~~

Other contributions to B_{Λ} for $J_N \neq 0$ hypernuclei, and thus to the differences δB_{Λ} for the pairsconsidered, may arise from non-central Λ -N forces. Thus, assuming equal sizes, Lawson [21] has obtained good agreement for all the B_{Λ} of the p-shell hypernuclei with a two-body tensor Λ - N force in addition to a spin-dependent central force. Three-body non-central forces may perhaps also make an appreciable contribution for $J_N \neq 0$ in view of the fact that these are indicated to be quite strong by meson - theoretical calculations [2].

It is interesting to note that the results for $\delta a = 0$ and $\xi_{12} = 0$ can also be obtained by calculating the relevant Slater integral, $F_{2p}^{(o)}$, assuming Gaussian Λ - N interactions of the same intrinsic range as the Yukawas and by using a Λ wavefunction of Gaussian form with the same size parameter as the nucleon distribution. Thus, with $a_s = a_p = a_\Lambda = 1.6$ fm, one finds for Δ in MeV. fm^3, the values, $\Delta = 450 \mp 72$, 310 ∓ 62, 636 ∓ 194 for $\mu_{2\pi}$ and $\Delta = 350 \mp 66$, 290 ∓ 58, 452 ∓ 178 for μ_K, relevant to the pairs of hypernuclei ${}_\Lambda Be^9 - {}_\Lambda Li^8$, ${}_\Lambda Be^9 - {}_\Lambda Li^9$ and ${}_\Lambda C^{13} - {}_\Lambda B^{12}$. It appears that for a calculation of Δ in this manner the approximation that the Λ has the same density distribution as an s-shell nucleon is then a reasonable one to make. For the pair ${}_\Lambda Be^9 - {}_\Lambda Li^9$ the above procedure slightly underestimates Δ compared with the values in table 9, and this is because the wave-function has here a smaller extension than that obtained from eqn.6 and correspondingly $F_{2p}^{(o)}$ is larger. For the other two pairs of hypernuclei this effect is masked by the presence of the U dependent term (for $\delta B_\Lambda = 0$ one just has $\Delta = U/4\eta$).

Chapter 5.

The $A = 7$ Hypernuclei

5.1 Introduction

These hypernuclei are discussed separately because firstly, the spin-dependence of the Λ - N interaction is expected to have a relatively greater effect for the $T = 0$ hypernucleus $_{\Lambda}Li^7$ ($J_N = 1$) than for the other p-shell hypernuclei with $J_N \neq 0$. The second, and more important reason, is that both the $T = 0$ and $T = 1$ nuclei, Li^6 and He^6, Be^6 are exceptional in that the two p-nucleons are quite weakly bound, with separation energies of about 5 MeV for Li^6 and 2 MeV and 0.6 MeV for He^6 and Be^6 respectively, while the s-nucleons on the other hand are very tightly bound, probably corresponding to an α-particle like core*. This situation is also reflected by the nucleon density distribution. Thus the electron scattering data for Li^6 indicate oscillator size parameters $a_s = 1.6$ fm and $a_p = 2.1$ fm for the s and p-nucleon distributions respectively, corresponding to a considerably greater extension of the latter. These values of the size parameter are those appropriate to the present usage, i.e. without centre of mass or any other corrections. They are interpreted as giving the nucleon distribution with respect to the centre of mass, having been obtained from the corresponding r.m.s. radii of s and p-nucleon distributions[22]. For He^6 the p nucleons are probably even more

* Both these features are very clearly evident in the quasi-free scattering of protons by Li^6

extended corresponding to a value $a_p \approx 2.5$ fm obtained from the Coulomb energy difference between He^6 and the lowest T = 1 state of Li^6 [23]. The much larger value of a_s than obtained for the α-particle (a_s = 1.175 fm) can very plausibly be explained on the basis of an α-particle plus two nucleon model as due to the spreading out of the α-particle density by its recoil motion, relative to the centre of mass of the whole nucleus, due to the action of the two p-nucleons[22, 23, 24]. Because of the special structural features of the A = 6 nuclei neither a two body treatment of the A = 7 hypernuclei, which assumes the Λ to interact with the core as a composite whole, nor the assumption of small distortion of the core by the Λ may be adequate.

The results of such a two-body analysis discussed below show that the former, even if not the latter, is certainly the case and that rather than obtaining information about the Λ -N interaction from the A = 7 hypernuclei the Λ may be considered as a reasonably sensitive probe into the structure of the A = 6 nuclei.

In view of the results of section 3.4 we shall for our two-body model analysis consider only two-body Λ -N forces. The numerical results were obtained by solving the appropriate Schrödinger equation, eqn.6, the Λ potential, V_Λ (r), having been obtained assuming L-S coupling for the nucleons.

5.2 The T = 1 Hypernuclei. $_\Lambda He^7$ and $_\Lambda Be^7$

For these the value B_Λ = 5.0 ± 0.5 MeV (> 4.5 MeV) obtained for $_\Lambda Be^7$ [24] * is used, the lower value of B_Λ = 3.96 ± 0.7 MeV obtained for the mirror nucleus being interpreted as corresponding to an excited isomeric state[25]. An alternative explanation [26] interprets the difference between the two values as due to the differing structures of the core nuclei Be^6 and He^6. Further below reasons are given why this explanation does not seem plausible.

For the T = 1 hypernuclei only U is relevant since the core nuclei have zero spin; thus $U_6(T = 1) = 3/2\ U$. The fact that both $_\Lambda He^5$ and $_\Lambda C^{13}$, as well as $_\Lambda Be^9$, give essentially the same value for U, in spite of their very different density distributions, seems strong justification for assuming this sort of value also for the A = 7 hypernuclei. This, as will be seen, is then tantamount to considering the Λ as a probe whose nuclear interaction is known. The results obtained for the Λ energies with $V_\Lambda(r)$ generated from nucleon oscillator density distributions and assuming U = 1040 ± 40 MeV fm^3, and 780 ± 30 MeV fm^3 for $\mu_{2\pi}$ and μ_K respectively are shown in fig. 5 as a function of a_s and a_p. These energies, denoted by B_Λ' cannot in general be identified with the actual Λ separation energy B_Λ but are <u>larger</u> than this by the rearrangement energy of the core nucleus appropriate to its

* The existence and therefore stability of $_\Lambda Be^7$ against heavy particle breakup into $_\Lambda He^5$ + p + p gives a lower limit of 4.5 MeV for B_Λ .

configuration in the hypernucleus and the energy of the isolated core nucleus in its ground state, where its total energy is a minimum. Thus $B_\Lambda' = B_\Lambda + \varepsilon_\Lambda \geqslant B_\Lambda$ the equality sign applying only when the core configuration is the isolated core with the size parameters $a_s \approx 1.7$ fm & $a_p \approx 2.5$ fm for He^6. For such values the calculated B_Λ' which may now be identified with B_Λ is only about 1.5 MeV (and probably even less for ${}_\Lambda Be^7$ appropriate to a somewhat larger value of a_p for Be^6 than for He^6) and this is much less than the experimental value. This shows quite clearly that it is unjustified to consider the Λ as interacting with the undistorted core as a composite. Further, remembering $B_\Lambda' \geqslant B \geqslant$ 4.5 MeV, it is clear from fig.5 that unless a_p is to be unreasonably small, e.g. $a_p < a_s$, one must conclude that the effectively sees an s - shell distribution which is much closer to the α - particle size than to that of the isolated nucleus. In fact the presence of the Λ is not expected to reduce a_p very much from its value for the isolated core nucleus since the p - nucleon seperation energy will only be increased by approximately one MeV, from about 2 to 3 MeV for ${}_\Lambda He^7$ and 0.6 to 1.6 MeV for ${}_\Lambda Be^7$, due to the additional binding caused by the Λ. For such an increase, calculations, discussed below, for p - nucleons moving in a square well (which allows the introduction of the nucleon binding energy in contrast to the case of an oscillator well) indicate a decrease

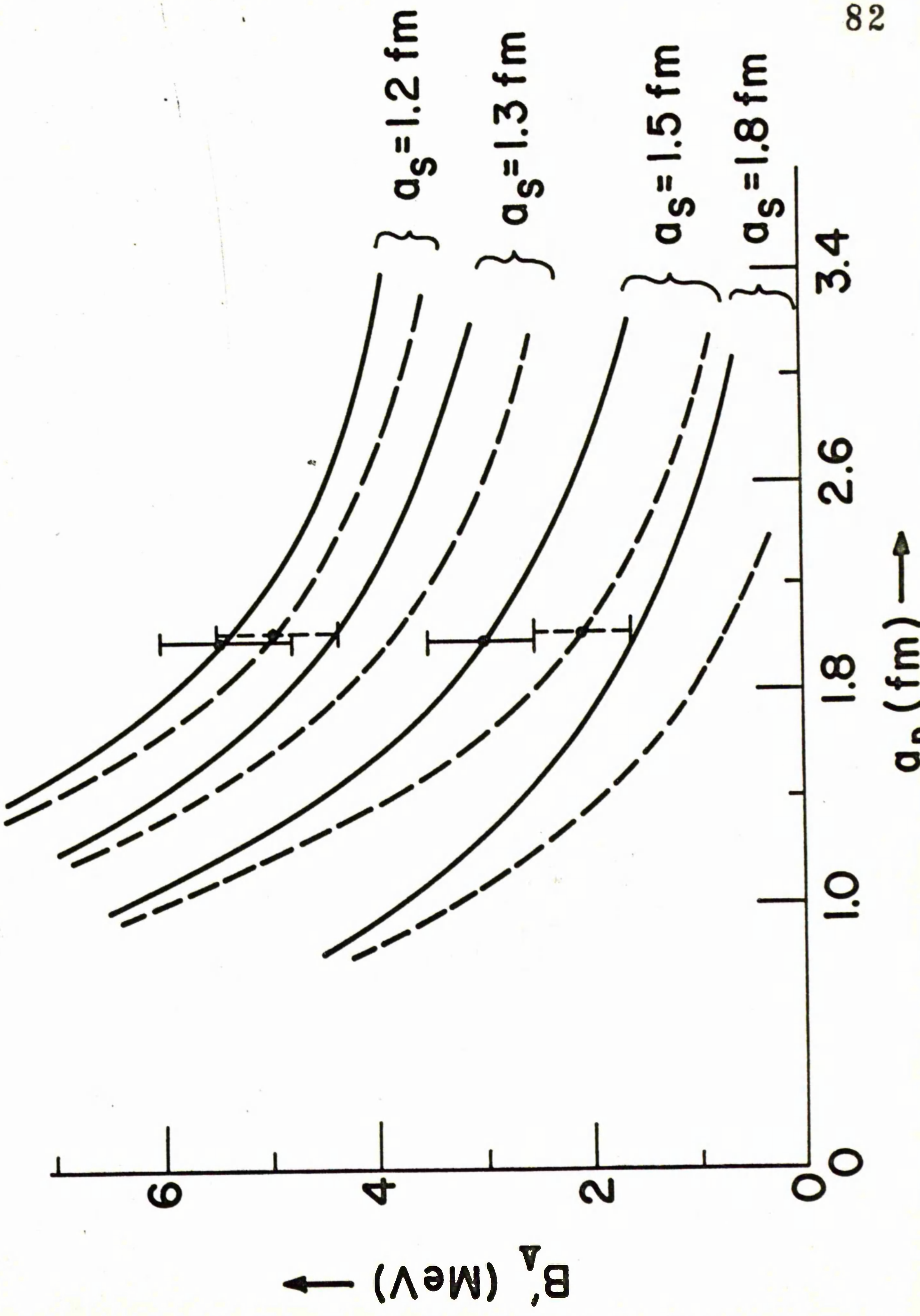
δa_p of magnitude $|\delta a_p| \lesssim$ 0.2 fm and 0.3 fm for $_\Lambda He^7$ and $_\Lambda Be^7$ respectively. Such a decrease is also indicated by an estimate of the effect of the Λ in compressing the p-nucleon distribution of He^6 using a compressibility coefficient for the p-nucleons of $K_p \approx$ 60 Mev/ nucleon appropriate to $a_p \approx$ 2.5 fm[23]. Assuming then, fairly generally, that with the Λ present a_p is not less than about 2 fm one can conclude that the Λ must see an effective s - shell distribution which can at most be very little larger than an α-particle. One may further conclude, since always $B_\Lambda' \gtrsim B_\Lambda$ that the rearrangement energy must be rather small. Thus, from inspection of fig. 5, even a reduction of a_p to about 2 fm is seen to imply $\varepsilon_\Lambda \lesssim$ 1 Mev (assuming U is not greater than about 1080 MeV ^{3} fm and $a_s \approx$ 1.2 fm and $B_\Lambda \gtrsim$ 4.5 MeV). For a reduction by 0.2 fm to $a_p \approx$ 2.3 fm for $_\Lambda He^7$ one gets $\varepsilon_\Lambda \lesssim$ 0.5 MeV.

These results can be readily interpreted if one accepts an α-particle plus two nucleon structure for the core nuclei. It is then quite reasonable to expect the Λ to become strongly correlated with the α -particle, the interaction with this dominating that with the p-nucleons of which there are only two and these, moreover, having a very extended distribution. The Λ will then effectively see an s-nucleon extension comparable with that of a free α -particle. The picture thus revealed in

Figure 5 **The Λ Separation Energy B_Λ' for A = 7, T = 1 Hypernuclei**

The Λ separation energy B_Λ' obtained without taking account of any re-arrangement energy, shown as a function of a_s and a_p for the A = 7, T = 1 hypernuclei: the solid curves are for $\mu_{2\pi}$ with U = 1090 $\pm$ 50 MeV fm^3 and the dashed curves for μ_K with U = 780 $\pm$ 40 MeV fm^3.

Fig 5



a_s=1.2 fm
a_s=1.3 fm
a_s=1.5 fm
a_s=1.8 fm
0
1.0
1.8
2.6
3.4
a_p (fm) ⟶
0
2
4
6
B'_Λ (MeV) ⟶

which the Λ preferentially attaches itself to the α-particle instead of to the core nucleus as a composite probably corresponds quite closely to a three-body model, briefly discussed by Dalitz[6], whose constituents are ${}_{\Lambda}He^{5}$, and the two p - nucleons. The preferred correlation of the Λ with the α-particle is quite consistent with only a relatively slight distortion of the core nucleus and thus a small value of ε_{Λ}. That this is quite conceivable is most readily visualized for the corresponding situation for ${}_{\Lambda}Li^{7}$, discussed below, if for the Li^{6} core one assumes - purely for the sake of illustration - an α - d model. Within a shell model context it is in fact only necessary that the effective total potential seen by each of the two p-nucleons is not too different from that without the Λ - which seems very likely. These conclusions, obtained by requiring U to be consistent with the values obtained from ${}_{\Lambda}He^{5}$ and ${}_{\Lambda}C^{13}$ (and also ${}_{\Lambda}Be^{9}$) and thus basically considering the Λ as a nuclear probe, may in fact be regarded as fairly strong and direct confirmation of an α-particle plus two nucleon structure for the A = 6, T = 1 nuclei.

It has been argued[26] that the small experimental value of B_{Λ} of about 3.96 MeV for ${}_{\Lambda}He^{7}$ may be reflection of the more open structure of Be^{6} as compared with He^{6}. Thus, if the size of the He^6 and Be^6 cores in the hypernucleus are similar and more nearly equal to that of the, more extended, isolated Be^6 than the He^6 nucleus and the latter, therefore, expands when the Λ is added while the former remains about the same, then the rearrangement energy for ${}_{\Lambda}He^7$ will be larger than for ${}_{\Lambda}Be^7$ and its B_{Λ} will be correspondingly smaller in agreement with experiment. Although, because of nuclear saturation the core nucleus will in general expand when a further <u>nucleon</u> is added to it the reverse is true when a Λ is added since B_{Λ} always increases as the core size is decreased and the hypernuclear configuration of minimum total energy will therefore be obtained by a contraction of the core from its isolated size. In fact the Λ will compress the more extended Be^6 nucleons more than He^6 (thus the increase due to the Λ of about 1 MeV in the nucleon seperation energy is relatively greater for ${}_{\Lambda}Be^7$ than ${}_{\Lambda}He^7$) and if anything the rearrangement energy will be greater for ${}_{\Lambda}Be^7$ and its B_{Λ} therefore correspondingly less than for ${}_{\Lambda}He^7$. In fact the rearrangement energies are not likely to be very different. Thus for the Coulomb energy difference between He^6 and Be^6 one obtains, with LS coupled wavefunctions and using the p - shell size parameter of He^6 [23], thd value $2 \times 0.81 + 0.41 = 2.03$ MeV instead of the experimental difference of 2.47 ± 0.2 MeV, where 0.81 MeV is the Coulomb energy due to the interaction of one p - proton with the s - protons and where 0.41 MeV is the calculated Coulomb energy between the two p - protons.* Thus nearly all of the energy of Be^6 is accounted for by using the He^6 configuration and the difference in rearrangement energies must therefore be quite small. It thus seems very unlikely that the difference between B_Λ for ${}_\Lambda Be^7$ and ${}_\Lambda He^7$ is due to any rearrangement effects but that this is much more likely to be due to an isomeric state of ${}_\Lambda He^7$. It may be remarked, as is clear from fig. 5 that even the small value B_Λ = 3.9 MeV would still imply a quite strong $\alpha - \Lambda$ correlation in ${}_\Lambda He^7$.

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* The calculated difference, appropriate to the He^6 size, is in fact <u>less</u> than the experimental difference. Unless one is prepared to accept absurdly small values of a_p, this seems to imply either a quite strong charge dependence of the nuclear forces or else, and perhaps more plausibly, a considerably stronger angular correlation between the two p - protons than is implied by the shell model. The latter explanation is the analogue for the T = 1 nuclei of a possible tendency towards an α -deuteron clustering in Li^6.

5.3 Calculations for the T = 1 Hypernuclei using Square Well p - nucleon Wave Functions.

Since the p - nucleons in Li^6 and He^6 are rather loosely bound the incorrect asymptotic behaviour of oscillator functions may lead to some error, although this is expected to be reduced because of the peaking of the p- nucleon wavefunctions at a finite radius. Thus calculatios, have been made for $\mu_{2\pi}$, in which $v_{2p}^{(0)}(r)$ is generated from more realistic wavefunctions; $v_{2s}^{(0)}(r)$ being obtained from a Gaussian s-nucleon density as before. The p-nucleonsare considered as moving in a square well of radius R and depth V_0 where the corresponding binding energy B_p may be approximately identified with the p-nucleon seperation energy, although because of rearrangement affects the actual seperation energy will be somewhat less. For comparison with the oscillator results it is convenient to use the 'equivalent' harmonic oscillator size parameter.

$$a_p^{(eq)} = \frac{2}{3} \langle r^2 \rangle_p^{1/2} \tag{44}$$

where $\langle r^2 \rangle_p^{1/2}$ is the r.m.s. radius obtained with a square well wavefunction and considered, consistent with the previous prescription, to refer to the p-nucleon density distribution with respect tp the centre of mass of the core.

Instead of showing for the T = 1 hypernuclei the results for B_Λ corresponding to the assumed value of U as in fig.5, the results obtained for U for the assumed experimental values of $B_\Lambda = 5.0 \pm 0.5$ MeV are given in Fig.6. For any set of values of a_s and $a_p^{(sq)}$ assumed to give the core size in the hypernucleus the values of U will always be a lower limit to the value which would be obtained if the rearrangement energy was taken into account.* For B_p between 1.5 MeV and 3 MeV, corresponding to the seperation energy expected for a p-nucleon in the T = 1 hypernuclei, one sees that the square well results are quite similar to the oscillator ones, also shown in Fig.6. The mostly somewhat smaller values for the former are a reflection of the correspondingly larger values of $F_{2p}^{(o)}$ for a square well, as illustrated in Fig.2, the crossing of the two curves for large a_p being due to the difference in $\langle T_\Lambda \rangle$ for the two cases. All the previous conclusions, in particular these

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* The value which would be determined if the rearrangement energy was included would necessarily be an upper limit if the assumed core size is the one actually realized in the hypernucleus, but not otherwise.

concerning the strong α-Λ correlations, are thus unchanged unless, again, quite unreasonably small values of $a_p^{(sq)}$ are admitted.

In this connection the value of $\langle r^2\rangle_p^{1/2}$ required to give the experimental difference of 0.81 MeV between He^6 and the corresponding T = 1 state of Li^6 has been calculated. If this difference is interpreted as being entirely due to the Coulomb interaction of a single p - proton with the s - shell protons which, for simplicity, are taken to act as a point change, i.e. assuming a_s = o, one then obtains $\langle r^2\rangle_p^{1/2}$ = 4.35 fm, very nearly independently of the precise value of R_p. The comparable value for the harmonic oscillator wavefunction is 4.19 fm.* [23] Assuming the ratio of the two values i.e. 1.04, to remain about the same also for an extended s- shell proton distribution then gives $\langle r^2\rangle_p^{1/2}$ = 4.05fm. for a_s = 1.8fm. corresponding to $\langle r^2\rangle_p^{1/2}$ = 3.9fm. obtained for an oscillator distribution including centre of mass and exchange corrections as discussed in ref.[23]. Correspondingly one then has $a_p^{(sq)}$ = 2.6 fm instead of a_p = 2.5 fm. for the oscillator case.

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*The closeness of these two values must be regarded as a reflection of the peaking of the p-nucleon density at a finite radius in view of the fact that the Coulomb energy and the r.m.s. radius involve difference moments of the distributions. Because of this such close agreement could by no means be expected for two s-nucleon type distributions of different shapes.

Fig. 6.

The volume integral U_{Λ} obtained neglecting any rearrangement energy, is shown as a function of the size parameters for the A = 7, T = 1 hypernuclei with B_{Λ} = 5.0 ± 0.5 MeV and for $\mu_{2\pi}$. The dashed curves are for harmonic oscillator p-nucleon wave functions characterised by a_p. The full curves are for square well p-nucleon wave functions and correspond to the use of the equivalent size parameter $a_p^{(sq)}$. The curves (a) to (c) are for B_p = 1, 3 and 5.66 MeV respectively.

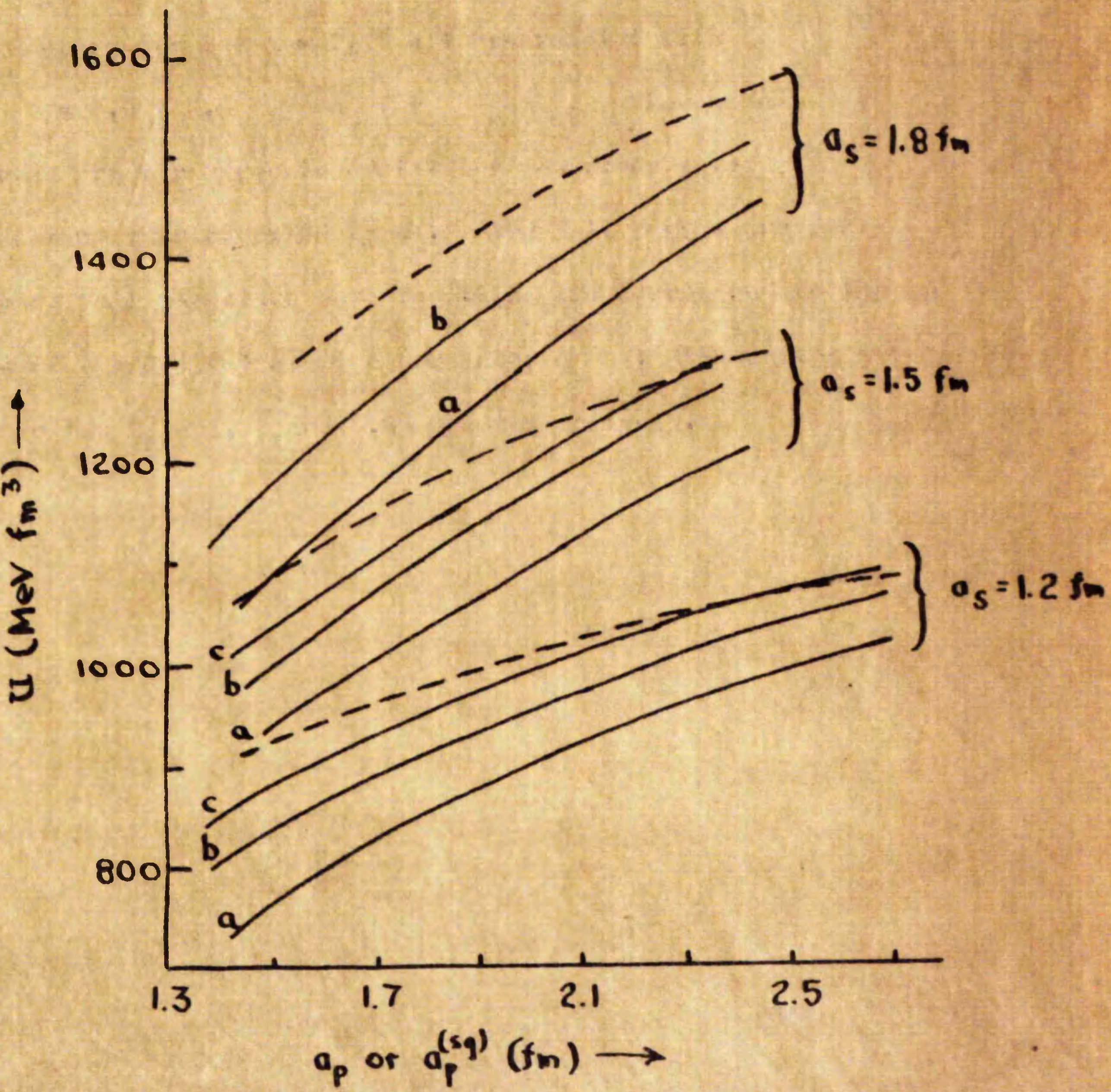
1600
1400
1200
1000
800
U (MeV fm^3) →
1.3
1.7
2.1
2.5
a_p or a_p^(sq) (fm) →
a_s = 1.8 fm
a_s = 1.5 fm
a_s = 1.2 fm
a
b
c

Fig. 6.

Fig. 6 shows that the dependence of U on $\langle r^2\rangle_p^{1/2}$ is in general considerably stronger than that on B_p unless the former is quite large orthe latter quite small*. This is mainly a reflection of the stronger dependence of the p-shell extension, and hence of $F_{2p}^{(o)}$, on the well radius R than on B_p, except for very small values of the latter. Thus for B_p = 3MeV one has $\frac{\partial\langle r^2\rangle^{1/2}}{\partial B_p} = -0.2$ fm MeV^{-1} and $\frac{\partial\langle r^2\rangle^{1/2}_p}{\partial R} = 0.7$ Further, for the sort of values of R and B_p of interest, the well depth V_o depends mainly on R. It is then clear from Fig.6 that for a definite, assumed, value of U this will determine $a_p^{(sq)}$, R and V_o approximately as a function of a_s, largely independently of the precise value of B_p. Table 10 shows the values obtained for U = 1025 $\pm$ 30 MeV fm^3 for B_Λ = 5 MeV and also for B_Λ = 5.5 MeV for only B_p = 5.66 MeV. The effect of the errors in U is in fact seen to be larger than that due to even quite substantial variations in B_p. It is important to note that any rearrangement energy will increase the effective value of B_Λ to be used above the experimental value of about 5 MeV and will lead to a corresponding decrease of $a_p^{(sq)}$ and R and an increase of V_o. The main conclusion to be drawn from table 10 is then that only for values of a_s close to 1.2 fm and for the associated large values

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* The increase of U with B_p for fixed $\langle r^2\rangle_p^{1/2}$ is due to the fact that R must be made larger as B_p is increased with the result that the bulk of the wavefunction which is inside the well becomes more extended resulting in a smaller value of $F_{2p}^{(o)}$.

Table 10 Square Well Results for A = 7, T = 1

Hypernucleus

with U = 1025 + 30 MeV fm^3 and $\mu_{\Lambda\pi}$

a_s (fm)	B (MeV)	Harmonic Oscillator a_p (fm)	B_p (MeV)	(sq) a_p (fm)	R (fm)	V_o (MeV)
1.2	5.0	2.1 ± 0.2	1.0	2.65 ± 0.2	3.1 ± 0.4	29 ∓ 9
			3.0	2.39 ± 0.2	3.48 ± 0.4	27 ∓ 8
			5.66	2.23 ± 0.2	3.6 ± 0.4	29 ∓ 7
	5.5	1.9 ± 0.2	5.66	1.99 ± 0.2	3.05 ± 0.4	37.5∓10
1.3	5.0	1.7 ± 0.15	1.0	2.29 ± 0.15	2.4 ± 0.3	47 ∓ 10
			3.0	2.00 ± 0.15	2.62 ± 0.3	43 ∓ 10
			5.66	1.89 ± 0.15	2.85 ± 0.3	40 ∓ 10
	5.5	1.55 ± 0.15	5.66	1.68 ± 0.15	2.4 ± 0.3	55.5∓11
1.5	5.0		1.0	1.85 ± 0.1	1.65 ± 0.15	97 ∓ 18
			3.0	1.60 ± 0.1	1.9 ± 0.15	84 ∓ 14
			5.66	1.49 ± 0.1	2.0 ± 0.15	73 ∓ 12
	5.5		5.66	1.33 ± 0.1	1.68 ± 0.15	94 ∓ 14

of $a_p^{(sq)}$ <u>and also</u> only for fairly small rearrangement energies ($\lesssim$ 0.5 MeV) are the nucleon well depths V_0 in reasonable agreement with the type of value to be expected. Thus Miss Jackson[22)] and Johansson and Sakomoto[27)] have fitted the Li^6 electron scattering form factor with a smoothly varying well for the p-nucleons having a central, maximum, value of about 40 MeV and half radius of approximately 2.5 fm. Allowing for the difference in shape between such a well and a square well, these values are in good agreement with the square well results with V_0 = 25 MeV, R = 3.5 fm. for a_s = 1.2 fm., whereas they would already be inconsistent with the square well results for a_s = 1.3 fm. These square well results thus further support and sharpen the conclusions already previously reached.

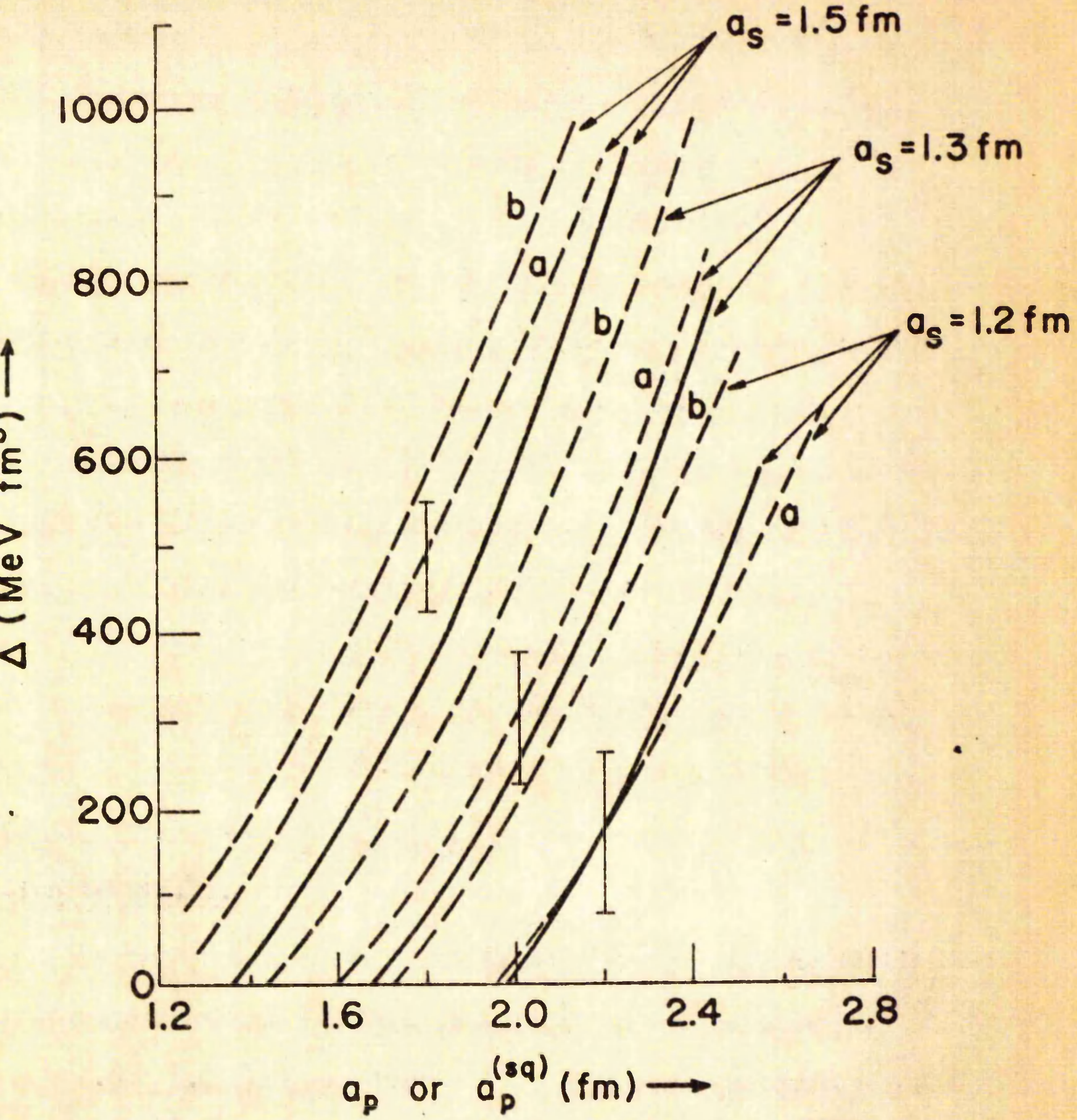
5.4 The T = 0 Hypernucleus $_{\Lambda}Li^{7}$

For $_{\Lambda}Li^{7}$, whose core nucleus Li^{6} has $J_N = 1$, both U and Δ are relevant. Fig.7 shows the values of Δ for $\mu_{2\pi}$ as a function of a_s and a_p or $a_p^{(sq)}$ for the experimental value $B_{\Lambda} = 5.5 \pm 0.25$ MeV [13] and for U = 1000 MeV fm^3 and 1040 MeV fm^3. The p - nucleon square well results, which are quite similar to the oscillator ones, have been obtained for $B_p = 5.66$ MeV in an exactly analogous manner as for the T = 1 hypernuclei. Again the values shown are lower limits to those which would be obtained if the appropriate rearrangement energy ε_{Λ} was taken into account. The rapid increase of Δ with a_p for given a_s is a reflection of the fact that the spin dependent part of the energy comes only from the p - nucleons ($F_{2p}^{(o)}$ decreases as a_p increases and a correspondingly larger value of Δ is thus required to give the same B_{Λ}).

The situation is quite analogous to that for the T = 1 hypernuclei and similar arguments apply. Thus for size parameters corresponding to the electron scattering results for Li^{6} ($a_s \approx$ 1.6 fm, $a_p \approx$ 2.1 fm), for which $\varepsilon_{\Lambda} = o$, Δ must have quite unreasonably large values of about 1000 MeV fm^3 as compared with the value obtained from analysis of the s - shell hypernuclei $_{\Lambda}H^{3}$ and $_{\Lambda}He^{5}$ [2,3] which give $\Delta \approx$ 130 MeV fm^3 for $\mu_{2\pi}$ ($\approx$ 30 MeV fm^3 for μ_K). To get such values of Δ with the value of a_p not much less than 2.1 fm, remembering that the results shown are lower limits, one is again forced to conclude that the Λ sees the s - nucleons very nearly as a free α -particle. The reduction in a_p due to the presence of the Λ is expected to be about 0.1 fm, rather less than for the T = 1 hypernuclei since the relative increase in B_p brought about by the Λ is now correspondingly less. It is interesting to note that <u>assuming</u> a_s = 1.2 fm. one must have $a_p \gtrsim$ 1.8 fm. as otherwise Δ becomes negative which is unacceptable if the singlet is stronger than the triplet Λ-N interaction. Also the rearrangement energy must again be correspondingly small: thus for a_p = 2 fm and Δ = 150 MeV fm^3 one has $\varepsilon_\Lambda \lesssim$ 0.3 MeV. Thus, as for the T = 1 hypernucleus, the Λ may essentially be considered as a probe into the Li^6 core, it being clear that very little can be deduced about the Λ -N interaction.

<u>Caption to Fig. 7.</u>

The spin dependence Δ obtained for $_\Lambda Li^7$, neglecting any rearrangement energy, is shown as a function of the size parameters for B_Λ = 5.5 ± 0.25 MeV and for $\mu_{2\pi}$. The dashed curves are for square well p-nucleon wavefunctions with B_p = 5.66 MeV and U = 1000 MeV fm^3. The full curves are for harmonic oscillator wavefunctions with U = 1040 and 1000 MeV fm^3 for (a) and (b) respectively.



$a_s = 1.5$ fm
$a_s = 1.3$ fm
$a_s = 1.2$ fm
b
a
b
a
b
a
1000
800
600
400
200
0
Δ (MeV fm^3) ⟶
1.2
1.6
2.0
2.4
2.8
a_p or $a_p^{(sq)}$ (fm) ⟶

Fig. 7.

Because the spin - dependent contribution to B_Λ is (roughly) due to only a single nucleon and thus of order of magnitude 1/A relative to the bulk of the energy which depends only on U, any uncertainties in U will be correspondingly magnified in the value obtained for Δ. Thus, e.g. for B_Λ = 5.5 MeV and a_s = 1.2 fm, a_p = 2.1 fm one gets Δ = 290 $\mp$ 180 MeV. fm^3 corresponding to U = 1040 $\pm$ 40 MeV fm^3 while errors of 0.25 MeV in B_Λ give errors of approximately $\pm$ 80 MeV fm^3 in Δ. Quite apart from errors in B_Λ one would have to know U quite accurately in order to obtain a reasonably good value for Δ even if one could make a much better - dynamical - calculation for the total B_Λ. However, most of this uncertainty can in principle be avoided bymaking a calculation of the Λ energy relative to that of ${}_\Lambda He^5$, i.e. for δB_Λ = $B_\Lambda({}_\Lambda Li^7) - B_\Lambda({}_\Lambda He^5)$ = 2.4 $\pm$ 0.25 MeV although at the cost of proportionally somewhat larger errors in δB_Λ than in $B_\Lambda({}_\Lambda Li^7)$. One has

$$\delta B_\Lambda' = \delta B_\Lambda + \varepsilon_\Lambda = (\tfrac{1}{2}\bar{\sigma} + \Delta) F_{2p}^{(0)}({}_\Lambda Li^7) + \delta C \qquad (45)$$

where $C = \langle V \rangle_S - \langle T_\Lambda \rangle$, with $\langle V \rangle_S = UF_{2S}^{(0)}$ (for ${}_\Lambda He^5$ the value of C is just the corresponding B_Λ)*, and where the rearrangement energy for ${}_\Lambda He^5$ is assumed to be negligible. The effect of errors in U is now quite acceptably small since they occur only through the U dependent p - nucleon contribution to this energy.

*i.e. corresponding to the relevant value of U.

Thus for $\mu_{2\pi}$ and $a_s(\,_{\Lambda}Li^7) = 1.2$ fm one has $\delta C = 0.5$ MeV and then for $a_p = 2.1$ fm and $\varepsilon_{\Lambda} = 0$, giving a lower limit, one gets $\Delta = 122$ MeV fm^3 with errors now only $\pm$ 20 MeV fm^3 due to $\pm$ 40 MeV fm^3 in U but of $\pm$ 80 MeV fm^3 due to 0.25 MeV in δB_{Λ}. However for the value of Δ size uncertainties (those in a_s enter through the value of δC) are now quite critical (thus for $a_s = 1.3$ fm, $a_p = 2.1$ fm, one gets $\delta C = -0.3$ MeV and $\Delta = 370$ MeV fm^3) and, as might be expected, one just gets back to the same situation as discussed for the total B_{Λ} for $_{\Lambda}Li^7$ except that the relevant errors due to U are now reduced, those due to δB_{Λ} being slightly increased - exactly the same result can be achieved in the discussion of the total B_{Λ} ~~there~~ with the value U = 1040 MeV fm^3 with errors of about $\pm$ 5 MeV fm^3 and with errors of about $\pm$ 0.25 MeV in B_{Λ}. The T = 1 hypernuclei can, of course, also be considered relative to $_{\Lambda}He^5$ with precisely the same conclusions as already reached.

5.5 Direct Comparison of the T = 0 and T = 1 Hypernuclei.

Especially for the spin-dependance Δ it is of interest to make a calculation of the Λ energy of the T = 0 hypernucleus relative to that of the T = 1 hypernuclei i.e. ~~from~~ for $\delta B_{\Lambda} = B_{\Lambda}(T = 0) - B_{\Lambda}(T = 1) = 0.5 \pm 0.5$ MeV. Although the uncertainties due to $\bar{\tau}$ are now largely avoided there will of course be large errors due to the very large relative errors in δB_{Λ}. In view of the previous results we shall assume that the effective s - shell density seen by the Λ is the same for T = 0 and T = 1 and also that both rearrangement energies are small and similar. In fact the rearrangement energy for T = 1 is expected to be somewhat larger than for T = 0 resulting in a decrease of the effective value of δB_{Λ} to be used, this decrease, however, is expected to be well within the experimental errors. One then has

$$\delta B_{\Lambda} = \Delta F_{2p}^{(0)}(T=0) + \frac{1}{2}\bar{\tau}\left[F_{2p}^{(0)}(T=0) - F_{2p}^{(0)}(T=1)\right] \qquad (46)$$

differences in $C = U F_{2s}^{(0)} - \langle T_{\Lambda} \rangle$ being negligible for all values of a_s of interest even when $\delta a_p = a_p(T = 1) - a_p(T = 0) \neq 0$. The second term is zero if the total density distributions are the same, i.e. $\delta a_p = 0$. It should be noted that (in LS coupling) any three-body forces will not essentially contribute to the differenced B_Λ. Results obtained for Δ using oscillator functions are shown in table 11 for $U(\mu_{2\pi}) = 1040$ and $U(\mu_K) = 780$ MeV fm^3, any uncertainties in U now being quite unimportant. Also shown for $\delta a_p = 0$ are the results obtained using square well p-nucleon wavefunctions for $B_p = 5.66$ MeV for both T = 0 and T = 1. For $\delta a_p > 0$ the second term in eqn.46 is positive and the value of Δ is less than for $\delta a_p = 0$. The value $\delta a_p = 0.4$ fm used in the table is appropriate for the isolated core nuclei and will be somewhat larger than for the hypernucleus since the reduction of a_p (T = 1) by the presence of the Λ is expected to be slightly larger than that of a_p (T = 0). Because of the large errors in the experimental value of δB_Λ and of further uncertainties due to the rearrangement energies it is clear that with $0 \lesssim \delta a_p \lesssim 0.4$ fm and for reasonable values of a_p (T = 0) there is no discrepancy with the values of Δ obtained from the s-shell hypernuclei. It is interesting to note that for such values a value of a_p (T = 0) $\approx$ 1.8 fm is indicated (especially if any reduction in the effective value δB_Λ due to rearrangement effects is taken into account) which is consistent with and supports the p-shell sizes used in previous discussions. The very similar values obtained for $\Delta(\mu_{2\pi})$ and $\Delta(\mu_K)$ is a consequence of the large p - shell sizes compared with both ranges.

Table 11 Results for Δ in MeV fm^3 from a

Comparison of the A = 7 Hypernuclei for $\delta B_\Lambda = 0.5 \pm 0.5$ MeV

$a_p(T=0)$	$\delta a_p = a_p(T=1) - a_p(T=0) = 0$			$\delta a_p = 0.4$ fm		Value of
	Oscillator Well	Square Well		Oscillator Well		$\delta a_p(T=1)$ for $\Delta = 0$
(fm)	$\Delta(\mu_{2\pi})$	$\Delta(\mu_K)$	$\Delta(\mu_{2\pi})$	$\Delta(\mu_{2\pi})$	$\Delta(\mu_K)$	(fm)
1.6	92 ± 92	78 ± 78	63 ± 63	0	0	$1.76^{+0.2}_{-0.16}$
2.0	148 ± 148	135 ± 135	100 ± 100	90	135	$2.3^{+0.35}_{-0.3}$
2.4	234 ± 234	222 ± 222	200 ± 200	150^{+234}_{-150}	175^{+222}_{-175}	$2.9^{+1.0}_{-0.5}$

5.6 A Three-Body Model of $_{\Lambda}Li^7$

The picture of the A = 7 hypernuclei suggested by the two-body model analysis of the preceeding sections requires that the motion of the Λ particle is strongly correlated to that of the "s" nucleons and that this core needs to be described by a density distribution which differs little from that of a free α particle. In addition it is also required that the nucleon re-arrangement energy is small and probably not in excess of 0.5 MeV. The purpose of the present section is to see to what extent the above conclusions are substantiated by a three-body calculation of the hypernucleus $_{\Lambda}Li^7$, assuming this to be composed of a free α- particle, a Λ particle and a free deuteron. It is then to be expected that the fact that the Λ is able to interact with a free α- particle is sufficient to account for the greater part of the observed Λ seperation energy. The remainder, which in $_{\Lambda}Li^7$ arises from the interactions of the Λ with the two "p" shell nucleons, may conveniently be split into a spin-independent and a spin dependent part. The observed seperation energy for the $T=1$ hypernuclei, for which the spin dependence has no importance, suggests that the former contribution is the larger and the two-body results obtained for these hypernuclei indicate that its magnitude is somewhat insensitive to the actual position of these nucleons.* In view of this the imposition of deuteron correlations on the two nucleons perhaps do not lead to much underestimation of this part of the seperation energy. The energy due to the spin dependence, which is expected to be of the order of 0.5 MeV, as witnessed by the difference $B_\Lambda(T=0) - B_\Lambda(T=1)$, may then possibly be obtained with a reasonable value of Δ (i.e. of the order suggested by analysis of the light hypernuclei). If, on the contrary, the nucleon system is strongly distorted by the addition of a Λ then one suspects that "unreasonable" forces may be needed to explain the magnitude of the seperation energy simply because the Λ interactions have also to account for the correspondingly large rearrangement energy of the nucleons.

The estimation of the energy B MeV with which the three particles are bound is performed variationally by choosing the trial wave function $\Psi(r_1, r_2, r_3)$ which gives a

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* See, for example, fig.6 from where it is clear that $\partial \bar{U}/\partial a_p)_{a_s}$ decreases quite strongly as a_s decreases from values associated with the form factor size to those of α- particle size. For the latter it is found that an uncertainty in $\bar{U}$ of 40 MeV fm^3 is equivalent to an uncertainty of 0.33 fm in a_p but of only 0.06 fm in a_s.

maximum value for B, such that

$$\int \sum_{i=1}^{3} \left\{ \frac{\hbar^2}{2m_i} \left(\frac{\partial \Psi}{\partial r_i} \right)^2 + \frac{\hbar^2}{M_i} \cos\theta_i \frac{\partial \Psi}{\partial r_j} \frac{\partial \Psi}{\partial r_k} + V_i(r_i) \Psi^2 \right\} d\tau = B \int \Psi^2 d\tau$$

$$\text{for } i \neq j \neq k. \tag{47}$$

The scalar coordinates r_i, $i = 1, 2, 3$ are the interparticle seperations shown in fig.8; M_i and θ_i are the mass and angle opposite the i^{th} side and $m_i = M_j M_k/(M_j + M_k)$ and $V_i(r_i)$ are the reduced mass and the interaction appropriate to this side. The left hand side of eq.(47) is just the expectation value of the three-body Hamiltonian, after removal of the centre of mass energy, expressed in terms of triangular coordinates.

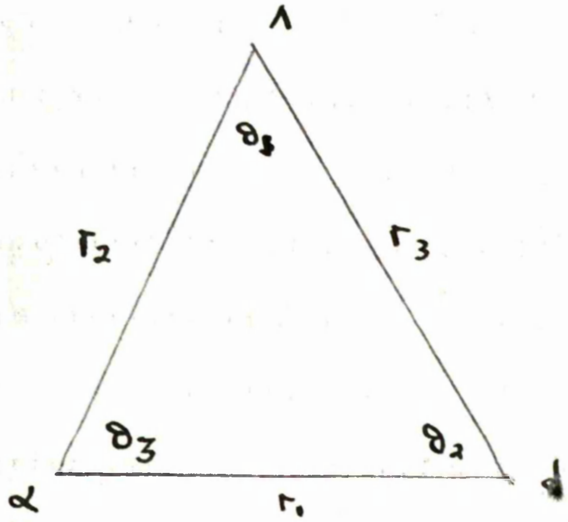

Figure 8.

The element of integration is accordingly $d\tau = 8\pi^2 \prod_{i=1}^{3} r_i \, dr_i$ and the domain of integration is restricted by the inequalities $r_i \leq r_j + r_k$, $i = 1, 2, 3$, $i \neq j \neq k$. The three-body binding energy B is defined as the sum $B_\Lambda({}_\Lambda Li^7) + B_d(Li^6)$ and the values $B_\Lambda({}_\Lambda Li^7)$

$= 5.5 \pm 0.25$ MeV and $B_d(Li^6) = 1.5$ MeV are obtained from experiment. The emphasis of the calculation is placed on the determination of the strength of the $\Lambda - N$ force necessary to reproduce this value of $B = 7.0 \pm 0.25$ MeV; the volume integrals obtained are then upper limits to the actual ones.

For computational simplicity the wave function $\Psi(r_1, r_2, r_3)$ is assumed to have the product form

$$\Psi(r_1, r_2, r_3) = \prod_{i=1}^{3} g_i(r_i) \tag{48}$$

and the functions $g_i(r_i)$ are taken to be sums of exponential terms, possibly multiplied by a polynomial in r_i. If the interactions $V_i(r_i)$ are also chosen to have this same form then the total integral of eq.(47) can be expressed in terms of the integrals

$$I_{lmn}(\alpha, \beta, \gamma) = \int_0^\infty r_1^{l} e^{-\alpha r_1} dr_1 \int_0^\infty r_2^{m} e^{-\beta r_2} dr_2 \int_{|r_1 - r_2|}^{r_1 + r_2} r_3^{n} e^{-\gamma r_3} dr_3$$

$$= (-)^{l+m+n} \left(\frac{\partial}{\partial \alpha}\right)^{l} \left(\frac{\partial}{\partial \beta}\right)^{m} \left(\frac{\partial}{\partial \gamma}\right)^{n} I_{000}(\alpha, \beta, \gamma). \tag{49}$$

By these means the calculation is reduced to computing the maximum value of a rather complicated algebraic expression. It is necessary, however, before the discussion of any results to consider the actual form of the functions $g_i(r_i)$ and the potentials $V_i(r_i)$ to be used in this calculation.

Although an α particle - deuteron structure, particularly one which consists of free particles, is not an extremely well justified model to take for the ground state of the Li^6 nucleus, it does predict a reasonable energy for this state together with a nucleon density distribution which has an r.m.s. radius only slightly greater than the one obtained from electron scattering experiments. It is interesting to digress a little and to discuss the main features of this model in order to fix the form of the function $g_1(r_1)$ and the potential $V_1(r_1)$.

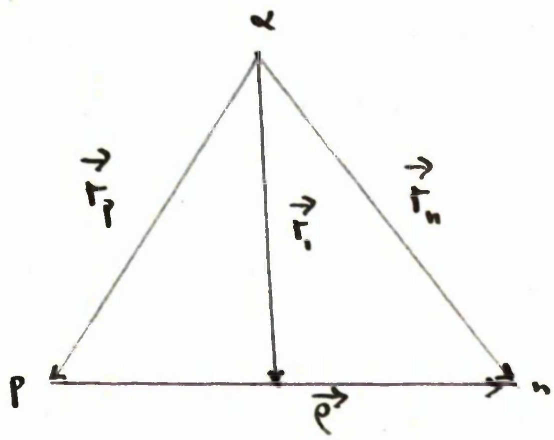

Figure 9.

Suppose that Li^6 consists of an α particle, a proton and a neutron, fig (9) ; then the intrinsic motion of these three particles may be described by the following Schrödinger equation, written in terms of the coordinates $\vec{r}_1$ and $\vec{\rho}$

$$\left[-\frac{\hbar^2}{2m}\left\{\frac{1}{2}\nabla_{r_1}^2 + 2\nabla_\rho^2\right\} + V_{\alpha-n}(|\vec{r}_1 + \tfrac{1}{2}\vec{\rho}|) + V_{\alpha-n}(|\vec{r}_1 - \tfrac{1}{2}\vec{\rho}|) + V_{np}(\rho)\right]\Phi = E\Phi \qquad (50)$$

where m is the nucleon mass, $V_{\alpha-n}$ represents the interaction of a nucleon and an α particle and V_{np} is the neutron-proton interaction. E is the total energy of the three particles. The alpha-deuteron model is obtained by assuming a wave function of the product form $\Phi = \psi(\vec{r}_1)\chi(\vec{\rho})$ where $\chi(\vec{\rho}) = \chi(\rho)$ is considered to be defined by

$$\left[-\frac{\hbar^2}{2m} 2\nabla_\rho^2 + V_{np}(\rho)\right]\chi(\rho) = E_d\,\chi(\rho) \qquad (51)$$

i.e. $\chi(\rho) = \frac{N}{\rho}(e^{-0.232\rho} - e^{-1.44\rho})$ is the free deuteron wave function and $-E_d$ is the deuteron binding energy. Taking expectation values with respect to $\chi(\rho)$ then gives an equation for the relative motion of the α particle and the deuteron, namely

$$\left[-\frac{\hbar^2}{2m}\overset{3/4}{\frac{1}{2}}\nabla_{r_1}^2 + V_{\alpha-d}(\vec{r}_1)\right]\psi(\vec{r}_1) = \varepsilon\,\psi(\vec{r}_1), \qquad (52)$$

with the effective α-d interaction given by

$$V_{\alpha-d}(\vec{r}_1) = \int d\vec{\rho}\, \chi^*(\rho) \left[V_{\alpha-n}(|\vec{r}_1 + \tfrac{1}{2}\vec{\rho}|) + V_{\alpha-n}(|\vec{r}_1 - \tfrac{1}{2}\vec{\rho}|) \right] \chi(\rho) \tag{53}$$

and $\varepsilon = E - E_d$ is the deuteron seperation energy from Li^6. Using the Gaussian phenomenological α-n interaction due to Sachs et.al.[28] the angular integration is easy to perform and one then obtains a central form for $V_{\alpha-d}(r_1)$. The interesting point is to decide which of the possible bound states of this well is most analogous to the ground state of Li^6. The total orbital angular momentum operator $\vec{L}$, expressed in terms of the above coordinates, is given by $\vec{L} = \vec{L}_{r_1} + \vec{L}_\rho$, where $\vec{L}_{r_1} = -i\hbar\, \vec{r}_1 \times \vec{\nabla}_{r_1}$ and $\vec{L}_\rho = -i\hbar\, \vec{\rho} \times \vec{\nabla}_\rho$. It then follows that as $L^2_\rho\, \chi(\rho) = 0$ one must have a relative S state of motion in order to produce a state with $L=0$ *. However, it must be noted, the wave function $\Psi(r_1, \rho)$ is not an eigen function of the angular momentum operators $\vec{\ell}_p$ and $\vec{\ell}_n$, where $\vec{\ell}_p = -i\hbar\, \vec{r}_p \times \vec{\nabla}_{r_p}$ etc. and thus contains some degree of configuration mixing.

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* The ground state of Li^6 is considered to be an eigenstate of the total orbital angular momentum $(L=0)$ and total spin $(S=1)$. Spin-orbit and tenser effects are thus being neglected in this approximation.

This wave function, used together with a phenomenological α-n interaction, allows the existence of a component which violates the Pauli Exclusion principle and the particular S state chosen for the relative α-d motion must be the onewhich contains a minimum amount of the spurious configurations. Clearly this can only be the $2S$ state, the $1S$ state corresponding to both nucleons being mainly found in the (spurious) $1s$ levels. That this is the correct choise is borne out by the energy spectrum of the potential well $V_{\alpha-d}(r_1)$ which possesses a $1S$ level at about 30 MeV and a $2S$ level at 3.4 MeV. The fact that the energy of the $2S$ state is lower than the measured seperation energy of the deuteron can readily be understodd in terms of the remaining (but small) spurious state component of this wave function. Further justification of the α-d model of Li^6 would require that this spurious part is removed and the energy recalculated. However, the main interest here is on the hypernucleus $_\Lambda Li^7$ and a much simpler procedure is adopted to obtain the required reduction in the binding energy. It is supposed that the main effects of the spurious state is taken into account simply by adjusting the strength of the potential well $V_{\alpha-d}(r_1)$ until the $2S$ state occurs with the correct energy. Thus it is found that a 13 %

decrease in the depth produces the required 55 % reduction in energy. In this manner some treatment of the spurious state is given whilst retaining the essential simplicity of formulating the problem. It is to be noted that the modified wave function so obtained gives rise to a nucleon density distribution which possesses an r.m.s. radius of only approximately 10 % greater than the experimental one. The α-d model used in this way then provides a simple two-body representation of the Li^6 nucleus which possesses the correct energy and very nearly the correct size.

In order to reduce the calculation of ${}_\Lambda Li^7$ to an algebraic one it is necessary to replace the modified potential $V_{\alpha\text{-}d}(r_1)$ by a well having a simple analytic form. Except for small values of r_1 ($r_1 \lesssim 0.8$ fm) the modified well may be reasonably represented by $474(e^{-r_1} - e^{-1.3r_1})$ MeV and the corresponding $2S$ wave function by $(1-\rho r_1)(e^{-0.76r_1} - 0.92e^{-1.15r_1})$, where $\rho = 0.54$ has been chosen to give a node at r_1 = 1.85 fm, such as occurs in the eigen solution of eq.52 using the modified interaction. The $1S$ state is found to occur at 23 MeV and is described by the wave function $(e^{-1.23r_1} - 0.91e^{-1.58r_1})$. It is to be noted that the $1S$ and $2S$ variational wave functions obtained by the above procedure are orthogonal (i.e. the overlap is $\lesssim 0.2\%$) and this fact gives some confidence in the form of the function proposed for the $2S$ state. In the calculation of ${}_{\Lambda}Li^7$, which follows, it is assumed that the above $1S$ state is entirely spurious and a trial function $g_1(r_1)$ having the form above, namely $(1-\rho r_1)(e^{-\alpha_1 r_1} + \delta_1 e^{-\beta_1 r_1})$ is used. The value of the parameter ρ is always chosen to ensure orthogonality to the $1S$ state.

A potential well describing the interaction of an α-particle and a Λ may readily be obtained from the previously mentioned analysis of the ${}_{\Lambda}He^5$ hypernucleus. In the calculation of ${}_{\Lambda}Li^7$ two cases are considered, corresponding to different sizes of the α-particle core, and an excellent fit to these potential wells is found of the form

$$V_2(r_2) = \varpi_4 \left(e^{-\mu r_2} - e^{-\nu r_2} \right) \tag{54}$$

Thus for a $\Lambda - N$ interaction ~~of~~ with a 2π range one obtains $\varpi_4 = 1040$ MeV fm^3, $\mu = 1.9\ fm^{-1}$ and $\nu = 2.1154\ fm^{-1}$ coresponding to $a_s = 1.18$ fm and $\varpi_4 = 1111.5$ MeV fm^3, $\mu = 1.86\ fm^{-1}$ and $\nu = 2.049\ fm^{-1}$ corresponding to $a_s = 1.25$ fm and a larger α-particle radius. A variational calculation using the above interactions and a wave function of the form $g_2(r_2) = e^{-\alpha_2 r_2} + \delta_2 e^{-\beta_2 r_3}$ is found to give the correct Λ seperation energy together with the same expectation values of the kinetic and potential energies as obtained in the original analysis of $_\Lambda He^5$. For the two sizes of α- particle one has $\alpha_2 = 0.68\ fm^{-1}$ and $\beta_2 = 2.6\ fm^{-1}$ with $\delta_2 = -0.42$ for $a_s = 1.18$ fm and $\delta_2 = -0.44$ for $a_s = 1.25$ fm.

In much the same manner as the interaction of an α particle and a deuteron was obtained, eq.(53), an effective potential between the Λ and the deuteron may be generated. Details of this calculation are given in appendix 3; it is sufficient to note here that the potential well obtained is reasonably fit by a form

$$V_3(r_3) = U_2 A \left(e^{-\tilde{\mu} r_3} - e^{-\tilde{\nu} r_3}\right) \tag{55}$$

and for a Λ- N interaction with a ~~of 2π~~ range one finds that the values $A = 0.485$, $\tilde{\mu} = 1.8\ fm^{-1}$ and $\tilde{\nu} = 2.2\ fm^{-1}$ are suitable. In terms of the usual volume integrals of the Λ- N interaction $U_2 = \frac{1}{2} U + \Delta$. A single parameter function $g_3(r_3) = e^{-\alpha_3 r_3}$

is considered in the calculation of $_{\Lambda}Li^7$, the details of which are further discussed in appendix 4.

With given interactions the binding energy B was first maximised with respect to the single parameter α_3 using for the functions $g_1(r_1)$ and $g_2(r_2)$ the values appropriate to the $\alpha - d$ and the $\alpha - \Lambda$ isolated two-body systems respectively. At this stage variation of $g_1(r_1)$ was performed until a new and greater local maximum was obtained. Finally the function $g_2(r_2)$ was adjusted. For all interactions considered no further increase in the binding energy was now possible by readjustment of any of the parameters. The reason for this lies in the fact the functions $g_1(r_1)$ and $g_2(r_2)$ need little variation from their initial starting values to obtain the maximum value of B. In fact variation of these two functions produced improvements in the estimation of the binding energy of the order of 0.06 MeV at the most, frequently much less, and in general maximisation with respect to the single parameter of $g_3(r_3)$ gave an estimate only 0.1 MeV smaller than the maximum. The usefulness of these starting functions is very suggestive that the two systems concerned are little different in $_{\Lambda}Li^7$ than in isolation. It is also noteworthy that the inaccuracy which follows from using an alpha-deuteron model to provide an effective compressibility for the nucleons is also small; all that appears necessary is that the zero order energy is given correctly.

Fig.(10) shows the energy B plotted against the volume integral U_2 with different interactions $V_2(r_2)$. The experimental value $B = 7.0 \pm 0.35$ MeV is also shown together with the value $U_2 = 685 \pm 35$ MeV fm^3 reasonably obtained from analysis of the hypertriton (see chapter 1). The value of U_2 needed to reproduce the experimental value of B, obtained with $U_4 = 1040$ MeV fm^3 and a shape corresponding to an α- particle of size $a_s = 1.18$ fm, is found to be $U_2 = 730 \pm 50$ MeV fm^3, where the uncertainties arise only from those in B. This is somewhat larger than the value obtained from the hypertriton although the two values are consistent for smaller values of B. In part this discrepency may arise from inadequacies of the function $g_3(r_3)$. It has already been remarked that variation of α_3 produces by far the major increase in B (of the order of 2.3 MeV for $U_2 = 730$ MeV fm^3). It appears possible that a three-parameter form for this function may yield a further increase of the order of 5 % which would bring u_2 almost in alignment with that from ${}_\Lambda H^3$. This possibility is in the process of being followed up.

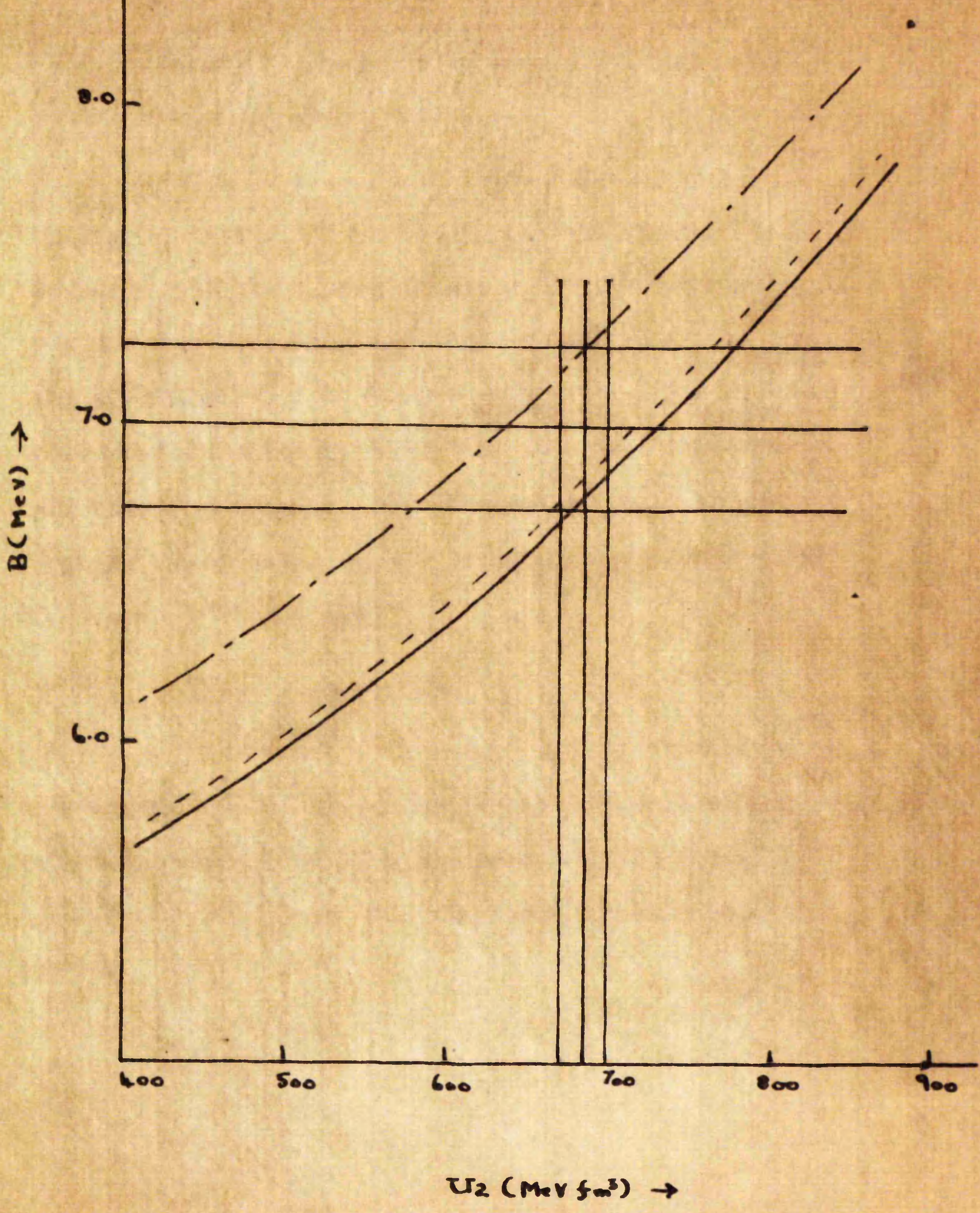
Uncertainties in the size of the alpha-particle core in ${}_\Lambda He^5$ give rise to uncertainties in u_4 of approximately $\pm$ 40 MeV fm^3 which, if the shape of the potential $V_2(r_2)$ is unchanged, in turn produces uncertainties of 90 MeV fm^3 in u_2. These figures, however, overestimate the effect as the α-Λ potential should be chosen to give the correct $B_\Lambda({}_\Lambda He^5)$ appropriate to whatever size is considered for the α- particle. The results obtained by using such a potential are seen to be much closer to the original curve (even though the value of u_4 is 70 MeV fm^3 larger) and a more realistic estimate of this source of error is of the order of 20 MeV fm^3.

The main conclusions of the two-body analyses of the A = 7 hypernuclei seem to be borne out quite well by the above three-body calculations. In particular the α-Λ correlations are probably not much different than in ${}_\Lambda He^5$ as evidenced by the little change necessary in $g_2(r_2)$ and in the smallness of the parameter α_3 (this being approximately 0.25 fm^{-1} for u_2 = 730 MeV fm^3). By a similar token the rearrangement effect

Fig.10 The Three-Body Binding Energy of $_{\Lambda}Li^7$

The maximum value of the three-body energy B, in MeV, is plotted against U_2 (MeV fm^3). The horizontal lines correspond to the experimental value B = 7.0 $\pm$ 0.25 MeV and the vertical ones to the value U_2 = 685 $\pm$ 15 fm^3 obtained from a similar analysis of $_{\Lambda}H^3$. The solid curve was obtained with $V_2(r_2) = 1040\,(e^{-1.9r_2} - e^{-2.1154r_2})$, the chain dotted curve with $V_2(r_2) = 1080\,(e^{-1.9r_2} - e^{-2.1154r_2})$ and the dashed one with $V_2(r_2) = 1111.5\,(e^{-1.86r_2} - e^{-2.049r_2})$.

fig.10.



of the nucleons is also probably small. Further even with the single parameter function $g_3(r_3)$ the correct $B_\Lambda({}_\Lambda Li^7)$ is obtained with $\Delta = 210 \pm 50$ MeV fm^3, which is more or less in good agreement with the value from the light hypernuclei. The calculation has, however, only been performed with respect to a $\Lambda - N$ interaction having a 2π range as the shell model results suggest that little information concerning the range dependence is likely to be obtained. Nevertheless it seems worthwhile to check this conclusion and the necessary modifications are in the process of being made.

A three parameter form $g_3(r_3) = e^{-\alpha_3 r_3} + \delta_3 e^{-\beta_3 r_3}$ for the Λ -d correlation function has been found to improve the estimate of B by 0.1 MeV. This is equivalent to reducing the estimates obtained for U_2 by approximately 20 MeV fm^3; agreement with the value from the hypertriton now being even better.

Chapter 6 The Hypernucleus $_{\Lambda}He^6$ *

6.1 Introduction

Various calculations and estimates [29,16] have been made to ascertain the stability of the $T = 1/2$ hypernucleus $_{\Lambda}He^6$. The conclusions obtained, however, are not in general agreement and it is at present uncertain whether the hypernucleus is expected to exist or not. In particular the calculation of Schick predicts that if $_{\Lambda}He^6$ is formed in some manner then it will very quickly decay via the mode $_{\Lambda}He^6 \rightarrow {}_{\Lambda}He^5 + n$, while that of Barsella and Rosati indicates that the system may be bound by as much as 0.5 MeV relative to this decay mode. It is thus of interest, as well as in keeping with the main theme of this thesis, to reconsider the above hypernucleus and the calculation is discussed in this chapter.

The above estimates have all one characteristic in common; no freedom is allowed for the Λ particle to have individual correlations with the 5-nucleon core and with the "p-shell" neutron respectively.

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* The work discussed here has been performed in collaboration with Dr.S. Rosati during his visit to Manchester. Dr.Rosati, however, bears no responsibility for this particular report of the work.

It is here proposed to examine the separation energy of the neutron assuming a three-body model of $_{\Lambda}He^6$ whose constituent particles are an α-particle, a Λ particle and a neutron. The calculation is performed by using an "equivalent" two-body method for the three-body problem.

The idea of an "equivalent" two-body formulation for a three-body system was first introduced by Wigner[30)] in order to discuss the triton and the method has received an interesting generalistion in two recent works[31,32)]. The approach of Delves and Derrick[31)] extends the method to include non-central interparticle forces with hard cores for both the bound and scattering states. Their formulation, which necessarily involves some approximations to be made to the "effective" three-body potential, gives a good starting trial wave function(in fact the best with respect to the approximations made) together with a prescription for systematically improving this wave function. In the work of Bodmer and Ali[32)] central interparticle forces only are considered and the method is applied to study particular bound S states*.

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* The states considered are just those appropriate to a wave function with the form $\prod_{i=1}^{3} g_i(r_i)$ where r_1, r_2 and r_3 are the interparticle separations.

It is proposed that the "effective"coupled, two-body Schrödinger equations obtained can be solved in a self-consistent manner through an iterative procedure whose progressive improvement originates in the variational basis of the equations. A useful modification of their method is to combine the use of explicit trial wave functions for two of the three pairs together with the numerical solution of the Schrödinger equation for the third pair. The "effective" potential appearing in this equation, and hence also the eigenvalue, are then functionals of the trial wave functions and, with respect to these, the best solution and energy are obtained by searching for the minimum eigenvalue.

In the present chapter the procedure of Bodmer and Ali is generalised to deal with states of arbitrary angular momenta, the necessary modifications being developed in the next two sections. The method is then applied to the hypernucleus $_{\Lambda}He^{6}$ in section 6.4.

6.2 Expectation Value of the Kinetic Energy.

To simplify the discussion the particles are considered to interact through central forces alone. It is then appropriate to use the L, S coupling scheme and the wave function of a state with total angular momentum J and z component J_z is written as the product of a spin-orbital and a radial function as follows:

$$\Psi^{J,J_z}_{(\ell_1,\ell_2)L,S} = \left[\sum_{L_z S_z} c(LSJ, L_z S_z J_z) \; \phi^{L,L_z}_{(\ell_1,\ell_2)} \; \chi^{S,S_z} \right]$$

$$u(r_1, r_2, r_3) \, , \tag{56}$$

where the coordinates r_i are the interparticle separations defined by $r_i = |\vec{r}_i| = |\vec{x}_j - \vec{x}_k|$ for $i \neq j \neq k$, $i = 1, 2$ and 3 and $c(a\,b\,c\,,\,\alpha\,\beta\,\gamma)$

implies the orbital function $\varphi^{L,L_z}_{(\rho_1,\rho_2)}$

with solid harmonics, $r^{\ell} Y_{\ell}^{m}(\theta,\varphi)$,

is a Clebs... ...ction

$\varphi^{L,L_z}_{(\rho_1,\rho_2)}$ is taken to be an eigenfunction of the

$\varphi_{(\rho_1,\rho_2)}$ is taken to be an eigenfunction of the

angular momentum operators $\vec{\ell}_1^{\,2}$, $\vec{\ell}_2^{\,2}$, $\vec{L}^2 = (\vec{\ell}_1 + \vec{\ell}_2)^2$ and L_z where $\vec{\ell}_j = -i\hbar\, \vec{r}_j \times \vec{\nabla}_{r_j}$ for $j = 1$ and 2 ; χ^{S,S_z} nce the radial function $u(r_1,r_2,r_3)$ transforms like a scalar with respect to rotations of the coordinate system, $\varphi^{L,L_z}_{(\rho_1,\rho_2)}$ specifies completely the total orbital angular momentum L. However $u(r_1,r_2,r_3)$, and hence Ψ , is not in general an eigenfunction of $\vec{\ell}_1^{\,2}$ and $\vec{\ell}_2^{\,2}$ due to its dependence on r_3. In fact these operators do not commute with that part of the Hamiltonian which describes the interaction of particles 1 and 2 and some degree of configuration mixing is present in the total wave function.

The form of the radial function is restricted by requiring that in the limit of no interaction between two of the particles $u(r_1,r_2,r_3)$ becomes an eigenfunction of angular momentum for the other two pairs, considered separately, belonging to the eigenvalue zero. It is to be noted that this implies the orbital function $\varphi^{L,L_z}_{(\rho_1,\rho_2)}$ is constructed with solid harmonics, $r^{\ell} Y_{\ell}^{m}(\theta,\varphi)$, rather than with the more usual surface spherical harmonics.

The expectation value of the total kinetic energy operator T with respect to the wave function Ψ, eq.(56) will be denoted by $\langle J J_z | T | J J_z \rangle$ and is given by

$$\langle J J_z | T | J J_z \rangle = \sum_{L_z, L_z'} c(LSJ, L_z S_z J_z)\, c(LSJ, L_z' S_z' J_z)\, \delta_{S_z S_z'}$$

$$\int d\vec{r}_1\, d\vec{r}_2 \left[\mathcal{P}^{L, L_z'}_{(\rho_1, \rho_2)}\, u \right]^* T \left[\mathcal{P}^{L, L_z}_{(\rho_1, \rho_2)}\, u \right] . \tag{57}$$

The appropriate form of T is

$$T = -2K_1 \nabla_{r_1}^2 - 2K_2 \nabla_{r_2}^2 + \frac{\hbar^2}{m_3} \vec{\nabla}_{r_1} \cdot \vec{\nabla}_{r_2} \tag{58}$$

with the mass factors given by $K_i = \frac{1}{4}\hbar^2 (m_j + m_k)/m_j m_k$ for $i \neq j \neq k$. Using the Wigner-Eckart theorem and the fact that T is a scalar operator allows the sum over the Clebsch-Gordan coefficients to be performed and one simply obtains the result

$$\langle J J_z | T | J J_z \rangle = \frac{1}{2L+1} \sum_{M=-L}^{L} \int d\vec{r}_1 d\vec{r}_2 \left[\mathcal{P}^{L,M}_{(\rho_1,\rho_2)} u \right]^* T \left[\mathcal{P}^{L,M}_{(\rho_1,\rho_2)} u \right] . \tag{59}$$

This expression greatly simplifies the discussion of the kinetic energy as it is only necessary to consider spin-zero particles, the expectation value depending only on the quantum number L. Introducing the explicit form of T into eq(59) and suitably integrating by parts enables one to write

$$\langle J J_z | T | J J_z \rangle$$

$$= \int d\vec{r}_1 d\vec{r}_2 \left[2K_1 (\vec{\nabla}_{r_1} u)^2 + 2K_2 (\vec{\nabla}_{r_2} u)^2 - \frac{\hbar^2}{m_3} (\vec{\nabla}_{r_1} u) . (\vec{\nabla}_{r_2} u) \right] \mathcal{L}_1$$

$$- \frac{\hbar^2}{m_3} \int d\vec{r}_1 d\vec{r}_2 \ \mathcal{L}_2 \ u^2 \tag{60}$$

with the functions $\mathcal{L}_1$ and $\mathcal{L}_2$ defined as follows

$$\mathcal{L}_1 = \frac{1}{2L+1} \sum_{M=-L}^{L} \left| \varphi^{L,M}_{(\rho_1,\rho_2)} \right|^2 \tag{61}$$

$$\mathcal{L}_2 = \frac{1}{2L+1} \sum_{M=-L}^{L} \varphi^{L,M*}_{(\rho_1,\rho_2)} \vec{\nabla}_{r_1} \cdot \vec{\nabla}_{r_2} \varphi^{L,M}_{(\rho_1,\rho_2)} . \tag{62}$$

The presence of the functions $\mathcal{L}_1$ and $\mathcal{L}_2$ in the expression for the kinetic energy represents the necessary modification to the work of Bodmer and Ali, who have considered the case $\rho_1 = \rho_2 = 0$. For these values of ρ_1 and ρ_2 one obtains $\mathcal{L}_1 = 1$ and $\mathcal{L}_2 = 0$ as may be seen from an inspection of eqs.(61) and (62), and expression (60) then corresponds with their result.

The functions $\mathcal{L}_1$ and $\mathcal{L}_2$ may be evaluated for arbitrary values of ρ_1 and ρ_2 by expressing the orbital function $\varphi^{L,M}_{(\rho_1,\rho_2)}$ as a sum of products of solid harmonics, i.e.

$$\mathcal{P}^{L,M}_{(\ell_1,\ell_2)} = r_1^{\ell_1} r_2^{\ell_2} \sum_m C(\ell_1 \ell_2 L, m\, M-m)\, Y_{\ell_1}^{m}(\theta_1,\varphi_1)\, Y_{\ell_2}^{M-m}(\theta_2,\varphi_2)$$

(63)

where θ_1, φ_1 and θ_2, φ_2 are the polar angles of the vectors $\vec{r}_1$ and $\vec{r}_2$ with respect to a fixed coordinate system. Under rotations of this coordinate system $\mathcal{P}^{L,M}_{(\ell_1,\ell_2)}$ transforms like an irreducible tensor of rank L and accordingly the quantity $\mathcal{L}_1$, obtained by contracting two such tensors, must be an invariant. $\mathcal{L}_1$ may then be evaluated in any convenient reference frame and the one chosen is such that $\theta_1 = 0$ and $\varphi_2 = 0$. It then follows that $\theta_2 = \pi - \alpha_3$, where $\alpha_3 < \pi$ is the angle between the vectors $\vec{r}_1$ and $\vec{r}_2$. Substituting these values in eq(61) and performing the sum over M then gives the result

$$\mathcal{L}_1 = \frac{(-)^L (2\ell_1+1)(2\ell_2+1)}{(4\pi)^2} \; r_1^{2\ell_1} \; r_2^{2\ell_2}$$

$$\sum_j \; C(\ell_1 \ell_1 j, 0 0) \; C(\ell_2 \ell_2 j, 0 0)$$

$$W(\ell_1 \ell_2 \ell_1 \ell_2 ; L j) \quad P_j(\cos \alpha_3), \tag{64}$$

where $W(a b c d ; e f)$ is a Racah coefficient and $P_j(\cos \alpha_3)$ is the Legendre polynomial of degree j. From the properties of the Clebsch-Gordan and Racah coefficients it can simply be seen that the values of j are restricted to the even integers $j = 0, 2, \ldots 2\ell_{min}$ where ℓ_{min} is the smaller of the pair ℓ_1 or ℓ_2.

The function $\mathcal{L}_2$ may be evaluated in a similar manner by using the fact that $\vec{\nabla}_{r_1} \cdot \vec{\nabla}_{r_2}$ is also a scalar operator. The quantity $\vec{\nabla} r^{\ell} Y_{\ell}^{m}$ is most simply evaluated by using the formula[34])

$$\vec{\nabla}\, r^{\ell} Y_{\ell}^{m}\left(\frac{\vec{r}}{r}\right) = \left[\ell(\ell+1)\right]^{1/2} r^{\ell-1}$$

$$\sum_{\mu} C(\ell\text{-}1\ 1\ \ell,\ m\text{-}\mu\ \mu)\, Y_{\ell}^{m-\mu}\left(\frac{\vec{r}}{r}\right) \vec{\xi}_{\mu}$$

(65)

where the spherical basis vectors $\vec{\xi}_{\mu}$, for $\mu = \pm 1$ and 0,

are orthogonal in the sense that

$$\vec{\xi}_{\mu} \cdot \vec{\xi}_{\nu} = (-)^{\mu}\, \vec{\xi}_{\mu}^{\,*} \cdot \vec{\xi}_{\nu} = (-)^{\mu}\, \delta_{\mu\,-\nu}\,. \tag{66}$$

We obtain

$$\mathcal{L}_2 = \frac{(-)^{\ell_1+\ell_2}}{(4\pi)^2}\, r_1^{2\ell_1-1}\, r_2^{2\ell_2-1}\, \ell_1 \ell_2 (2\ell_1+1)(2\ell_2+1)\left[(4\ell_1^2-1)(4\ell_2^2-1)\right]^{1/2}$$

$$W(\ell_1\,\ell_2\ \ell_1\text{-}1\ \ell_2\text{-}1; L\,1) \sum_{j} C(\ell_1\text{-}1\ \ell_1\ j,00)\, C(\ell_2\text{-}1\ \ell_2\ j,00)$$

$$W(\ell_1\,\ell_2\ \ell_1\text{-}1\ \ell_2\text{-}1, L\,j)\ \ P_j(\cos\alpha_3),$$

(67)

where the sum over j now comprises the odd integers

$j = 1, 3, \cdots\ 2\ell_{min} - 1.$ It is to be noted that

$\mathcal{L}_2$ is zero unless the total orbital angular momentum $L \leq \ell_1 + \ell_2 - 2$.

It is convenient at this stage to express the expectation value of the kinetic energy in terms of the triangular coordinates r_1, r_2 and r_3. Eq.(60) then takes the form

$$\langle J J_z | T | J J_z \rangle$$

$$= \sum_{i=1}^{3} \int \left[2K_i \left(\frac{\partial u}{\partial r_i} \right)^2 + \frac{\hbar^2}{m_i} \cos\alpha_i \frac{\partial u}{\partial r_j} \frac{\partial u}{\partial r_k} \right] \mathcal{L}_1 \, d\tau$$

$$- \frac{\hbar^2}{m_3} \int u^2 \mathcal{L}_2 \, d\tau$$

(68)

where the volume element $d\tau = 8\pi^2 r_1 r_2 r_3 \, dr_1 \, dr_2 \, dr_3$ and $i \neq j \neq k$. The domain of integration is just the usual one, being limited by the appropriate triangular inequalities $r_i \leq r_j + r_k$ and the angles α_i are those interior to the triangle.

From now onwards a radial function which has the product form

$$u(r_1, r_2, r_3) = \prod_{i=1}^{3} g_i(r_i) \tag{69}$$

is considered. For this type of function the first integral in eq.(68) can be further simplified, as is discussed more fully in appendix (5), and the final expression obtained is

$$\langle J J_z | T | J J_z \rangle$$

$$= \sum_{i=1}^{3} K_i \int \left[\left(\frac{\partial u}{\partial r_i} \right)^2 - u \frac{\partial^2 u}{\partial r_i^2} - 2 \frac{u}{r_i} \frac{\partial u}{\partial r_i} \right] \mathcal{L}_1 \, d\tau$$

$$+ \frac{1}{2} \int u^2 \mathcal{L}_3 \, d\tau \tag{70}$$

where the function $\mathcal{L}_3$ is defined in terms of $\mathcal{L}_1$ and $\mathcal{L}_2$ as

$$\mathcal{L}_3 = \sum_{i=1}^{3}\left[K_i\left(\frac{\partial^2 \mathcal{L}_1}{\partial r_i^2} + \frac{2}{r_i}\frac{\partial \mathcal{L}_1}{\partial r_i}\right) + \frac{\hbar^2}{2m_i}\cos\alpha_i \frac{\partial^2 \mathcal{L}_1}{\partial r_j \partial r_k}\right]$$

$$- 2\frac{\hbar^2}{m_3}\mathcal{L}_2 \; .$$

(71)

It is to be noted that for the case $\rho_1 = \rho_2 = 0$ then $\mathcal{L}_3 = 0$ and eq.(70) reduces to the expression quoted by Downs et al[6]) and used by ref.32).

6.3 The Variational Principle and the Effective Two-Body Equations.

The procedure of Bodmer and Ali is now followed, using as far as possible the same notations. The variational principle whichgives the best radial wave function with the product form of eq.(69) may be written

$$\delta_{g_i} \langle J J_z | T + \sum_{\substack{j \neq k=1 \\ j>k}}^{3} V_{jk} - NE | J J_z \rangle = 0 \qquad (72)$$

where δg_i denotes the functional variation with respect to g_i and V_{jk} is the potential energy for particles j and k. If the spin expectation value of this interaction is denoted by $v_i(r_i)$ then the potential energy ~~form~~ term of eq.(~~17~~ 72) just becomes

$$\langle J J_z | \sum V_{jk} | J J_z \rangle = \sum_{i=1}^{3} \int d\tau \, v_i(r_i) \mathcal{L}_1 \prod_{j=1}^{3} g_j^2(r_j) . \tag{73}$$

The kinetic energy integral is given by eq.(70) and the normalisation N is

$$N = \int d\tau \, \mathcal{L}_1 \prod_{j=1}^{3} g_j^2(r_j) . \tag{74}$$

Performing the variations with respect to the functions $g_i(r_i)$ then gives three coupled integro-differential Euler equations for these functions such that the eigenvalue is an upper bound to the total energy of the three particles. The equations obtained are

$$2K_i\left[g_i'' + \frac{(r_i N_i)'}{r_i N_i} g_i' + \frac{1}{4}\frac{N_i''}{N_i} g_i\right]$$

$$+\left[E-\left(v_i(r_i) + \frac{1}{N_i}\sum_{j\neq i}\left(K_j T_i^{(j)} + v_i^{(j)}\right) + \frac{v_i^{(L)}}{N_i}\right)\right] g_i = 0 \qquad (75)$$

for $i = 1, 2$ and 3 and where, with $d\tau^{(i)}$ denoting the volume elements $8\pi^2 \prod_{j\neq i} r_j\, dr_j$ the following notations have been introduced:

$$N_i(r_i) = \int d\tau^{(i)} \mathcal{L}_1 \prod_{j\neq i} g_j^2(r_j) \quad ,$$

$$T_i^{(j)}(r_i) = \int d\tau^{(i)} \mathcal{L}_1 \left\{(g_j')^2 - g_j'' g_j - \frac{2 g_j' g_j}{r_j}\right\} g_k^2 ,$$

$$v_i^{(j)}(r_i) = \int d\tau^{(i)}\, v_j(r_j)\, \mathcal{L}_1 \prod_{k\neq i} g_k^2(r_k) \quad , \qquad (76)$$

$$v_i^{(L)}(r_i) = \frac{1}{2}\int d\tau^{(i)} \mathcal{L}_3 \prod_{j\neq i} g_j^2(r_j) \; .$$

The difference between these equations and the ones of ref.[32] lies in the presence of the function $\mathcal{L}_1$ in the various integrals and in the additional term $v_i^{(L)}/N_i$. The transformation $f_i(r_i) = (r_i N_i)^{1/2} g_i(r_i)$ puts eq.(75) into normal form and the resulting Schrödinger equations are

$$2K_i f_i'' + \left[E - v_i(r_i) - W_i(r_i)\right] f_i = 0 \tag{77}$$

where the effective three-body potential energy term $W_i(r_i)$ is given by

$$W_i(r_i) = \frac{1}{N_i} \sum_{j \neq i} \left(K_j T_i^{(j)} + v_i^{(j)}\right) + \frac{v_i^{(L)}}{N_i} \tag{78}$$

$$- K_i \left[\frac{1}{2}\left(\frac{N_i'}{N_i}\right)^2 - \frac{1}{2}\frac{N_i''}{N_i} - \frac{1}{r_i}\frac{N_i'}{N_i} + \frac{1}{2r_i^2}\right].$$

When suitably normalised the function ρ_i/r_i can be interpreted as the probability amplitude for finding particles j and k separated by a distance r_i irrespective of the position of the third particle. The properties and self-consistency of the coupled equations may be discussed in a very similar manner to that of ref[3]. We will only note that in the limiting case $v_3(r_3) = 0$ and $m_3 \to \infty$ one obtains a self-consistent solution by taking $g_3(r_3) = 1$. The functions $g_1(r_1)$ and $g_2(r_2)$ then satisfy single particle Schrödinger equations with energies ε_1 and ε_2 respectively, such that $E = \varepsilon_1 + \varepsilon_2$, and with the appropriate centrifugal barrier potentials arising from the last two terms of eq.(78).

6.4 Application to $_{\Lambda}He^6$

The formalism of the preceeding sections is applied to the three-body model of $_{\Lambda}He^6$, fig.(11), by using trial wave functions $g_2(r_2)$ and $g_3(r_3)$, for the α - Λ and Λ - n sides respectively, to generate the effective three-body potential $W_1(r_1)$ given by eq.(78). The resulting single particle Schrödinger equation for the relative motion of the α -particle and the neutron is then numerically integrated.

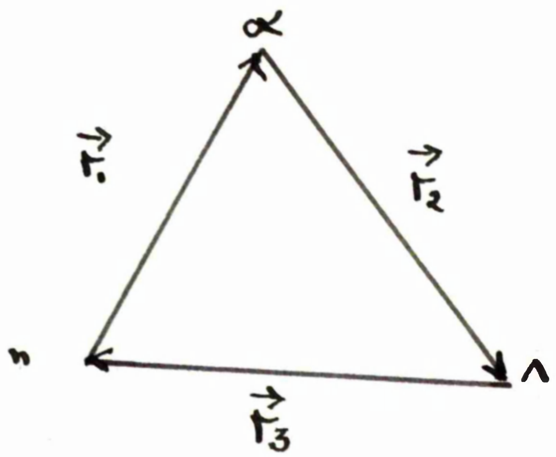


figure 11

If the three particles interact through central forces alone then a suitable wave function is obtained from eq.(56) by considering $\ell_1 = 1$ and $\ell_2 = 0$, corresponding to total orbital angular momentum $L = 1$, and $S = 0$ corresponding to the interaction of the Λ particle and the neutron being more attractive when the two particles are found in a singlet spin state. However phase shift analysis of the scattering of protons off the α particle indicates the presence of a small spin-orbit force in the phenomonological interaction so obtained. In particular the interaction considered here and due to Sack et al[28] is given by

$$V_{n\alpha}(r_1) = -V_c\, e^{-k^2 r_1^2} - V_{S.O.}\, e^{-k^2 r_1^2}\, (\vec{\ell}_1 . \vec{\sigma}_1) \tag{79}$$

where V_c = 47.32 MeV, $V_{S.O.}$ = 5.86 MeV and k = 0.4348 fm^{-1}. Such a force does not commute with the operator for the total spin $\vec{S}^2 = 1/4\,(\vec{\sigma}_1 + \vec{\sigma}_2)^2$, where $\vec{\sigma}_1$ and $\vec{\sigma}_2$ are the Pauli spin operators for the neutron and the Λ particle respectively, and a more appropriate wave function for ${}_\Lambda He^6$ consists of a linear superposition of the two possible spin states. The wave function considered is given by

$$\Psi^{1,J_z} = \left[x\, \Phi^{1,J_z}_{(1,0)1,0} + y\, \Phi^{1,J_z}_{(1,0)1,1} \right] u(r_1, r_2, r_3), \tag{80}$$

where the notation $\Psi^{J,J_z}_{(\ell_1,\ell_2)L,S} = \Phi^{J,J_z}_{(\ell_1,\ell_2)L,S}\, u(r_1,r_2,r_3)$ has been introduced and the normalisation implied is such that

$x^2 + y^2 = 1$ and the phase is chosen such that

$x > 0$ and $y > 0$.

If the wave function, eq(80) did not contain any configuration mixing, i.e. $g_3(r_3) = 1$ and the entire dependence on the polar angles of the vectors $\vec{r}_1$ and $\vec{r}_2$ was specified by the functions $\Phi^{1,J_z}_{(1,0)1,S}$ then the expectation value of the operator $\vec{\ell}_1 . \vec{\sigma}_1$ may simply be obtained by expanding the functions $\Phi^{1,J_z}_{(1,0)1,S}$ in terms of the eigen states of the angular momentum operators $\vec{j}_1 = \vec{\ell}_1 + 1/2\,\vec{\sigma}_1$ and $\vec{j}_2 = \vec{\ell}_2 + 1/2\,\vec{\sigma}_2$. In this case the procedure of taking spin expectation values of the α-n interaction, indicated by eq.(73), is just equivalent to considering a central interaction given by

$$v_1(r_1) = -\left(V_c + V_{S.O.} \langle \vec{\ell}_1 . \vec{\sigma}_1 \rangle\right) e^{-k^2 r_1^2}, \tag{81}$$

where $\langle \vec{\ell}_1 . \vec{\sigma}_1 \rangle = x^2 - 1 + 2\sqrt{2}\, xy$. In general a correlation function of the form $g_3(r_3) = e^{-d r_3}$ is considered and, with $\alpha \neq 0$, the above potential should be supplemented by an additional non-central term which arises from the implicit dependence of the s part of the wave function on the polar angles of $\vec{r}_1$. Such a contribution possesses a rather complicated shape but its strength is proportional to the product of $\alpha V_{S.O.}$. Anticipating somewhat the results of the calculation, where α is found to be reasonably small ($\lesssim$ 0.15) the modification to eq.(81) is not thought to be important and is neglected.

The prescription $\rho_1 = 1$ and $\rho_2 = 0$ enables the functions $\mathcal{L}_1$ and $\mathcal{L}_3$ to be calculated from eqs.(64) and (71) respectively. Thus one obtains $\mathcal{L}_1 = r_1^2/(4\pi)^2$ and $\mathcal{L}_3 = 6K_1/(4\pi)^2$. Assuming also trial wave functions of the form $g_2(r_2) = e^{-ar_2} + c\,e^{-br_2}$ and $g_3(r_3) = e^{-dr_3}$ and the following central interactions for the α-Λ and Λ-n systems:

$$v_2(r_2) = -U_4\left(e^{-\mu r_2} - e^{-\nu r_2}\right)$$

$$v_3(r_3) = -\frac{\mu_2^2}{4\pi}\left[x\,U_s + y\,U_t\right] e^{-\mu_2 r_3}/r_3 \tag{82}$$

$$= -\frac{\mu_2^2}{4\pi}\left[\frac{1}{4}U_4 + (x^2 - 1/4)\,\Delta\right] e^{-\mu_2 r_3}/r_3 \,,$$

the various integrals of eq.(76) may be expressed in terms of the functions $I_{nm}(\alpha, \beta)$, which are defined

$$I_{nm}(\alpha,\beta) = \int_0^\infty r_2^n e^{-\alpha r_2} dr_2 \int_{|r_1-r_2|}^{r_1+r_2} r_3^m e^{-\beta r_3} dr_3 . \tag{83}$$

Here the $\alpha - \Lambda$ interaction used has been previously discussed in chapter 5 and the $\Lambda - n$ interaction is just that of eq.(~~38~~ 1) after taking the appropriate spin expectation values. Thus for the effective three-body potential one obtains

$$W_1(r_1) = \frac{K_2 \mathcal{J}_2 + v_2}{\omega_1} + \frac{K_3 \mathcal{J}_3 + v_3}{\omega_1} + \frac{2K_1\, \ell_1(\ell_1+1)}{r_1^2}$$

$$- K_1\left[\frac{1}{2}\left(\frac{\omega_1'}{\omega_1}\right)^2 - \frac{1}{2}\frac{\omega_1''}{\omega_1} - \frac{1}{r_1}\frac{\omega_1'}{\omega_1} + \frac{1}{2r_1^2}\right] \tag{84}$$

where the notations $\mathcal{J}_j = 2\,T_1^{(j)}/r_1^2$, $v_j = 2\,v_1^{(j)}/r_1^2$, for $j = 2$ and 3, and $\omega_1 = 2\,N_1/r_1^2$ have been introduced; primes denoting differentiation with respect to r_1. Expression 84 is given in appendix 6 in terms of the standard integrals of eq. 83 . It is to be noted that in the case $g_3(r_3) = 1$ then the last term of eq.84 is zero and the ordinary centrifugal barrier term is obtained without modification.

It is normal procedure when integrating a Schrödinger equation to consider a potential function $\mathcal{V}(r_1)$ chosen such that $\mathcal{V}(r_1) \to 0$ as $r_1 \to \infty$

If $\lim_{r_1 \to \infty} W_1(r_1) = \varepsilon$ then such a potential for ${}_\Lambda He^6$ is given by $\mathcal{V}(r_1) = v_1(r_1) + W_1(r_1) - \varepsilon$ and the equation describing the relative motion of the α particle and the neutron may be written

$$2K_1 f'' - (\mathcal{V}(r_1) + \mathcal{B}') f = 0 \tag{85}$$

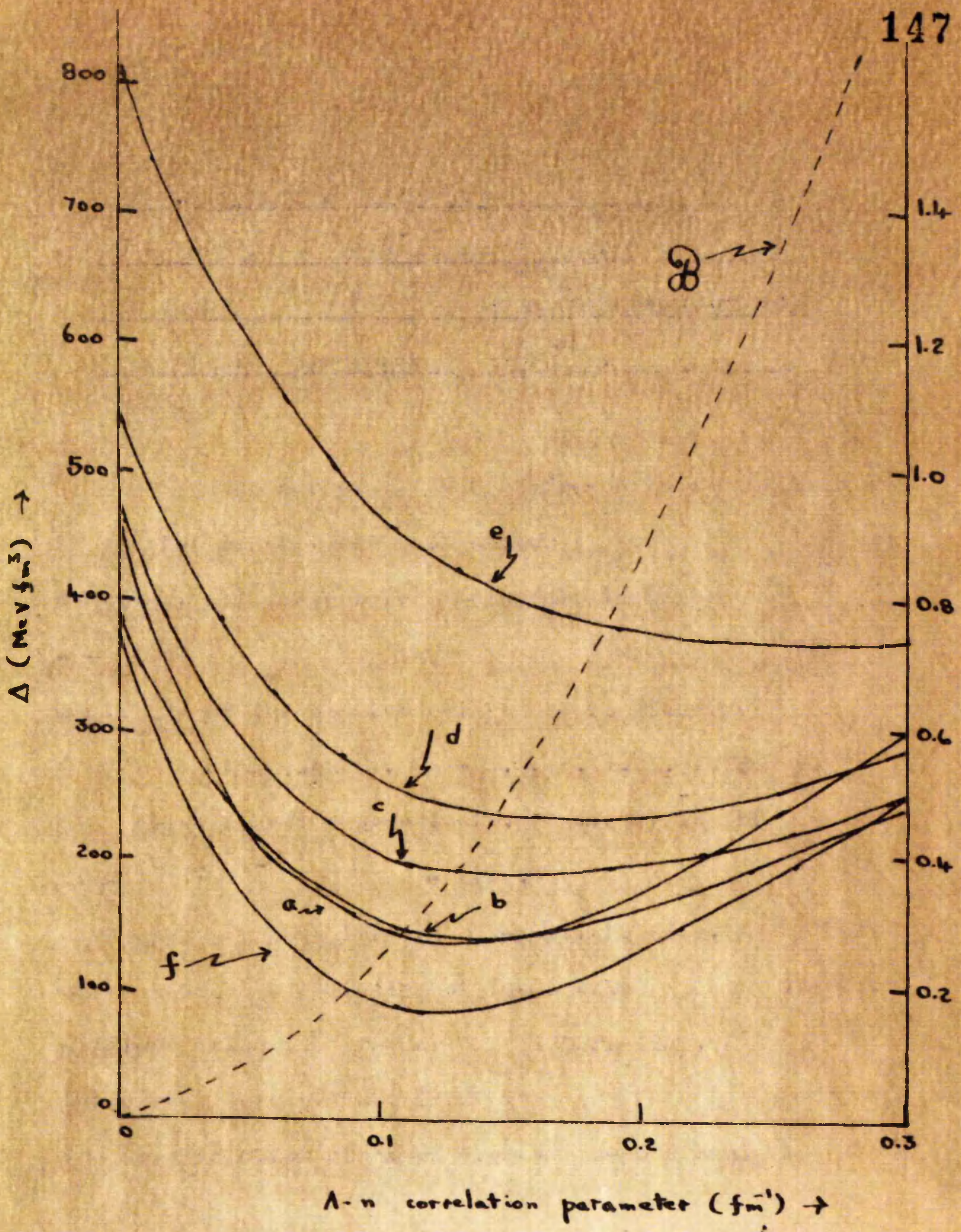
with $\mathcal{B}' = B + \varepsilon$ and $B = B_n({}_\Lambda He^6) + B_\Lambda({}_\Lambda He^5)$ is the total three-body binding energy. For the hypernucleus to be stable with respect to the decay mode ${}_\Lambda He^6 \to {}_\Lambda He^5 + n$ it is required that $B_n({}_\Lambda He^6) > 0$ and thus $\mathcal{B}' > 3.1 + \varepsilon$. The energy ε is found to arise entirely from the first term of eq. 84 and its calculation is further discussed in appendix 6. It is sufficient to note that for a given $\alpha - \Lambda$ interaction ε depends essentially on the parameters of the trial wave functions $g_2(r_2)$ and $g_3(r_3)$ and for the case $g_3(r_3) = 1$ it attains its minimum value of -3.1 MeV when $g_2(r_2)$ is chosen to be the best variational wave function appropriate to an isolated $\alpha - \Lambda$ system. For any other choice of wave functions we must then have $\mathcal{E} > -3.1$ MeV. The stability of ${}_{\Lambda}He^6$ may then be discussed by numerically integrating eq.85 to ascertain whether a bound state exists for energies $\geqslant \mathcal{B} = 3.1 + \mathcal{E}$. This is most simply achieved by inspection of the logarithmic derivative of the function $f(r_1)$ evaluated at some suitable radius ($r_1 = r_0$, say) If the discontinuity D is defined $D = f'/f)_{r_0<} - f'/f)_{r_0>}$ then $D = 0$ specifies an eigenstate at energy $\mathcal{B}$ characterised by the number of nodes n of the eigenfunction $f(r_1)$; the node at $r_1 = 0$ of the function $f(r_1)$ is not counted. If, however, $D > 0$ and $n = 0$ then no bound state exists such that $B_n({}_{\Lambda}He^6) > 0$ and accordingly the hypernucleus is not stable. On the contrary if $D < 0$ then there are at most $n+1$ bound states possible of which only the most deeply bound is interesting.

The results presented here have mostly been obtained using an $\alpha - \Lambda$ interaction given by $U_4 = 1040$ MeV fm^3, $\mu = 1.9$ fm^{-1} and $\nu = 2.1154$ fm^{-1} which, as previously discussed in chapter 5, is itself obtained from a two-body analysis of ${}_{\Lambda}He^5$ by assuming an α particle of r.m.s. radius 1.44 fm together with a Λ-n interaction of Yukawa form and a 2 π range parameter. This potential, together with the best variational wave function, $a = 0.68\ fm^{-1}$, $b = 2.6\ fm^{-1}$ and $c = -0.42$, gives an excellent representation of ${}_{\Lambda}He^5$. In the calculation of ${}_{\Lambda}He^6$, for given forces, the parameters obtained which minimise the discontinuity in the logarithmic derivitive D evaluated at the energy $\mathcal{B}$ may, of course, differ from the "zero" order ones quoted above. However, it is to be noted, that this minimum value of D differs only insignificantly from the value obtained with the zero order parameters due to the fact that $\mathcal{B}$ itself increases as the parameters are changed from these. Thus, for example, with $\Delta = 150$ MeV fm^3, $d = 0.15\ fm^{-1}$ and $x = 1.0$ one obtains $D = 0.13115$ for $\mathcal{B} = 0.5162$ MeV with the zero order wave function and the minimum value $D = 0.13036$ for $\mathcal{B} = 0.5449$ MeV with $a = 0.67\ fm^{-1}$, $b = 2.36\ fm^{-1}$ and $c = -0.42$. Accordingly the results given were obtained using the α-Λ wave function obtained from ${}_{\Lambda}He^5$. Some results obtained by using $v_2(r_2) = -1111.5\,(e^{-1.86 r_2} - e^{-2.049 r_2})$ MeV, corresponding to the larger α -particle r.m.s. radius of 1.54 fm, are also given.

Fig (12) shows the value of the spin dependance Δ necessary to produce a bound state, with $n=0$, at the energy $\mathcal{B}$ (i.e. with B_n (${}_\Lambda He^6$) = 0) expressed as a function of the correlation parameter of λ fm^{-1}. Also given is the variation of the energy $\mathcal{B}$ itself. The different curves were obtained by changing the amplitude of the singlet spin state in the ${}_\Lambda He^6$ wave function, eq. 80. The amplitude varies from x = 1.0, when Ψ^{1,J_z} is an eigenstate of the total spin belonging to $S=0$ (assuming $\lambda=0$) and the contribution from the Λ-n interaction is most attractive, to $x=\sqrt{2/3}$ when Ψ^{1,J_z} is likewise an eigenstate of $\vec{j}_1$ belonging to $j_1=3/2$ and the α-n interaction is most attractive. It is clearly seen that the latter structure is the one most favourable to the existence of ${}_\Lambda He^6$. If one supposed a value for the spin dependence, say Δ = 150 $\pm$ 50 MeV fm^3 as is appropriate to the light hypernuclei, it is possible to estimate the energy with which the neutron is bound. Thus a value $B_n({}_\Lambda He^6)$ = 0.04 $^{+0.3}_{-0.04}$ MeV is obtained corresponding to the $j_1=3/2$ structure.

Fig.12 The spin dependence Δ (MeV fm^3) necessary for a bound state at energy $\mathcal{B}$ and the energy $\mathcal{B}$ (MeV) shown as functions of the correlation parameter d (fm^{-1}).

The curves a) to e) differ only in the amounts of singlet and triplet spin states in the $_\Lambda He^6$ wave function, all these curves being obtained with $v_2(r_2)$ $= -1040\,(e^{-1.9r_2} - e^{-2.1154r_2})$ MeV. The amplitude of the singlet spin state is given by $x = \sqrt{2/3}$ ($j_1 = 3/2$ eigen state), x = 0.9, 0.95. 0.975 and 1.0 ($S=0$ eigen state) respectively. Curve f) is for $x = \sqrt{2/3}$ with $v_2(r_2) = -1111.5\,(e^{-1.86r_2} - e^{-2.049r_2})$ corresponding to a larger α-particle size. The dotted curve shows the variation of the energy $\mathcal{B}$ in MeV, obtained using the former α-Λ interaction.
The curves were all obtained with $g_2(r_2) = e^{-0.68r_2} - 0.42e^{-2.6r_2}$, the wave function obtained from a two-body analysis of $_\Lambda He^5$.



800
700
600
500
400
300
200
100
0
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0
0.1
0.2
0.3
e
d
c
b
a
f
$\mathcal{B}$
Δ (MeV fm^3) →
$\mathcal{B}$ (MeV) →
Λ-n correlation parameter (fm^{-1}) →

Fig. 12.

Uncertainties in $B_n({}_\Lambda He^6)$ of this order also arise from those in ϖ_4 as may be seen from the results relating to the larger α -particle size.

In conclusion, it is not yet possible, on the basis of the above calculations, to form a definite opinion as tothe existence of ${}_\Lambda He^6$ due to the lack of detailed knowledge of the Λ- N interaction. Even with the simple forces considered here the situation is not clear cut but it appears that the neutron is either only just bound or only just unbound. However, one serious objection to the above calculation can be made, namely, for $\alpha \neq 0$ the wave function of eq. 80 contains a component which violates the Pauli Exclusion principle in allowing $1s$ state of motion of the α- particle and the neutron to be possible. In neglecting the spurious state so far the neutron seperation energy has been overestimated and it is quite possible that the neutron will not be bound, for reasonable values of the spin dependence, when this effect is taken into account. The spurious state correction is at present being undertaken.

Chapter 7 Summary and Conclusions.

The studies of the p-shell hypernuclei reported here show that the simple assumption of a soft, central, spin-dependent and charge-independent, two-body Λ-N interaction is at present adequate to account for all the known values of B_Λ, including those of the s-shell hypernuclei as well as those of the heavier hypernuclei relevant to a determination of the well depth D. In particular, if there are assumed to be no three-body forces, the values of the spin-averaged volume integral of the two-body interaction that are obtained from the hypernuclei ${}_\Lambda He^5$, ${}_\Lambda Be^9$ 23) and ${}_\Lambda C^{13}$ agree very well with each other. As a reasonable average value for the hypernuclei one has $U = 1020 \pm 60$ MeV fm^3 and 800 ± 50 MeV fm^3 for ordinary Yukawa interactions of ranges $\mu_{2\pi}^{-1}$ and μ_K^{-1} respectively.

Further, it has been shown that not only are central three-body forces not necessary but that any that may exist must in fact be weak: they cannot contribute more than 20% of the total interaction energy of ${}_\Lambda He^5$. The most probable sign depends on the range assumed for the two-body force.

Thus for $\mu_{2\pi}$ the three-body interaction turns out to be most probably attractive while for μ_K it is most probably repulsive. These results for the three-body interaction depend on the validity of a shell model description for C^{12} with the values of the size parameters having been determined by electron scattering experiments. Uncertainties in the core sizes account for most of the errors in the values of u as well as for the upper limits which it is at present possible to place on the strength of the three-body forces. These size uncertainties furthermore preclude the possibility of deciding between the ranges spanned by $\mu_{2\pi}^{-1}$ and μ_K^{-1} (although an intermediate range is marginally favoured and a range as long as μ_π^{-1} is most probably excluded). More accurate determinations of the relevant density distributions, especially that of C^{12}, is thus very desirable.

The uncertainty about the range, quite apart from other possible uncertainties in the form and/or due to higher order effects of the interaction, implies a correspondingly large uncertainty in the value predicted for the well depth D felt by the Λ in nuclear matter since this is proportional to u in first order. Thus for ordinary forces and with $\rho = 0.168 f^{-3}$ one has $D(\mu_{2\pi}) = 43 \pm 2.5$ MeV and $D(\mu_K)$ $= 34.5 \pm 2$ MeV; for exchange forces of range μ_K^{-1} the latter value would be reduced by a factor 0.88. The corresponding values of B_Λ for $A \approx 80$, relevant to the heavier emulsion nuclei, are $B_\Lambda(\mu_{2\pi}) = 31 \pm 2$ MeV and $B_\Lambda(\mu_K) = 25 \pm 2$ MeV, the latter being reduced to 22 ± 2 MeV for exchange forces. None of these values of B_Λ is very inconsistent with the experimental evidence.

It is to be noted that the effective Λ- nucleus potential $V_\Lambda(r)$ at or near the centre of C^{12} depends only slightly on the range of the interaction and on the details of the shape of the density distribution of the core, but that $|V_\Lambda(0)|$ can differ very substantially from the corresponding value of D unless the range of the interaction is quite short. Thus for $\mu_{2\pi}$ the value of $|V_\Lambda(0)|$ is considerably less than the corresponding value of D whereas for μ_K there is only a small difference. Only if one believes the range of the Λ-N interaction to be quite short, as would be the case if the interaction is dominated by single exchanges of heavy mesons, would $|V_\Lambda(0)|$ also be a reasonable guide to the value of D.

The difference between ordinary and exchange interactions is even smaller for the p-shell hypernuclei than for a Λ

in nuclear matter, the proportion of the interaction energy in relative p states being quite small for soft forces even for the long range $\mu_{2\pi}^{-1}$. Then, at least for soft forces, it does not seem possible to say anything about the interactions in relative p states from the values of B_Λ for the p shell hypernuclei. This may be possible for forces with a hard core, however, since for such forces interactions in relative p states are expected to be enhanced as compared with soft forces.

The spin dependence Δ obtained from the p shell hypernuclei seems, unfortunately, to be at the mercy of quite small uncertainties in the core sizes and to a lesser extent, also of uncertainties in the rearrangement energies. Assuming no differences in the sizes and rearrangement energies of the members of each of the three pairs of hypernuclei, ${}_\Lambda Be^9 - {}_\Lambda Li^8$, ${}_\Lambda Be^9 - {}_\Lambda Li^9$, ${}_\Lambda C^{13} - {}_\Lambda B^{12}$, for which the B_Λ values are reasonably well known, one obtains very considerably larger values of Δ than are obtained from the s-shell hypernuclei. However, for very reasonable and quite small size differences and for reasonable rearrangement energies, there is not the slightest difficulty in obtaining values of Δ that are completely consistent with the s-shell ones.

Indeed the B_Λ of just the p shell hypernuclei are quite consistent with a spin-independent interaction (and also one which has a considerably larger spin dependence than required for the s shell hypernuclei). It seems quite probable than at appreciable part of the fluctuations in the values of B_Λ of the p-shell hypernuclei about the average trend is attributable merely to variations in the spin averaged interaction energy, which arise from quite small size differences, and from differences in the rearrangement energies. In particular the rearrangement energy for ${}_\Lambda Be^9$ is expected to be quite appreciable[23)] (somewhat less than 1 MeV). Small differences in the spin averaged interaction energy can be comparable with the whole of the spin-dependent part of the interaction energy because the latter is approximately only of order $^1/A$ times the spin-averaged interaction energy.

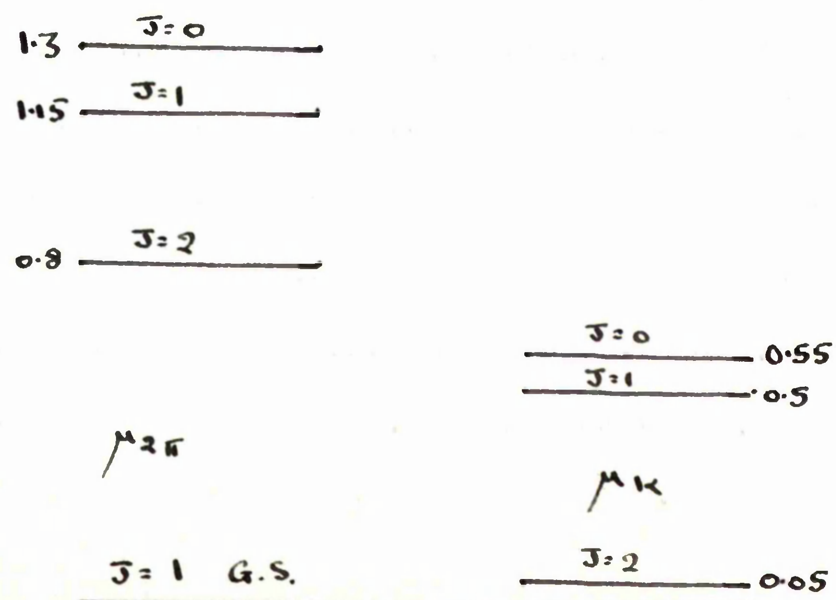
Central three-body forces have been shown not to contribute appreciably to the differences in B_Λ. Of course non-central forces (both two-body and three-body) could in principle contribute significantly - but because of the uncertainties mentioned it seems most unlikely that anything could be deduced about such non-central forces. In fact, assuming the effect of these to be reasonably small and the effective spin-dependence to be about that for the s-shell hypernuclei implies that the Λ is a sensitive probe into small size differences for neighbouring core nuclei having similar or small rearrangement energies. Quite generally it seems clear that the information obtainable about the $\Lambda - N$ interaction is at present limited at least as much by uncertainties in the core sizes, and to a lesser extent by uncertainties in rearrangement energies, as it is by uncertainties in B_Λ.

In view of the above conclusions the only meaningful determination of Δ, and hence U_t, must at present therefore be still from the B_Λ of the s-shell hypernuclei. If three-body forces are neglected and one uses the results of ref.1) for ${}_\Lambda H^3$, then one obtains $\Delta = 170 \pm 35$ MeV fm^3 and 25 ± 20 MeV fm^3 for $\mu_{K\pi}$ and μ_K respectively. If the results obtained for three-body forces (table 8) and also in ref3). for ${}_\Lambda H^3$ are taken into account one obtains somewhat larger values of Δ for $\mu_{K\pi}$ ($\approx 200 \pm 40$ MeV fm^3)., and slightly smaller values for μ_K ($\approx 5\,{}^{-5}_{+30}$ MeV fm^3). If the indications of a range shorter than $\mu_{K\pi}^{-1}$ are taken seriously then this implies that the spin dependence may be correspondingly smaller.

Perhaps the only direct determination of Δ from the p-shell hypernuclei may be possible subsequent to the observation of excited states. Of particular interest would be a measurement of the excitation energy of a state for which the Λ has flipped its spin relative to the ground state. This energy could be obtained from a measurement of the energy of the γ-ray of the relevant transition. It is reasonable, for states which differ only in the fact that the Λ has flipped its spin, to expect that the core wave functions are quite similar and therefore uncertainties due to differences in the core size and rearrangement energy will be quite small.

A favourable case seems to be that of $_{\Lambda}Li^7$ for which the excitation energy of the first excited state ($J = 3/2$) relative to the ground state ($J = 1/2$) is $E = 3/2\, \Delta\, F_{2p}^{(0)}(a_p)$. The assumptions on which this value of E is obtained are that the p-nucleons can be represented by oscillator wave functions and that the wave functions for s and p nucleons are the same for both states. Then with $\Delta = 150 \pm 50$ MeV fm^3 one gets for $\mu_{2\pi}$ the values $E = 0.75 \pm 0.25$ MeV and 1.5 ± 0.5 MeV for $a_p = 2$ fm and 1.6 fm respectively. For μ_K the values of E are correspondingly smaller. It is to be noted that especially for the larger values of a_p the value of $F_{2p}^{(0)}(a_p)$ is not very dependent on μ^{-1}. Clearly the accuracy with which Δ can be determined from a measurement of the excitation energy E will be limited by the accuracy with which one knows the extension of the p- nucleons in $_{\Lambda}Li^7$.

Observation of the low lying levels of the A = 8 hypernuclei would also be very interesting for these purposes, although the situation is more complicated. The level schemes to be expected are illustrated below for assumed values of Δ consistent with the s-shell hypernuclei and a core density distribution with a_p = 1.6 fm, i.e. with $\Delta F_{2p}^{(0)} \approx$ 0.88 MeV and 0.16 MeV for $\mu_{2\pi}$ and μ_K respectively (even quite substantial uncertainties in the core size, say $\pm$ 0.2 fm only move these levels by $\mp$ 0.3 MeV for $\mu_{2\pi}$ and $\mp$ 0.2 MeV for μ_K). Three excited states are now to be found



with energies less than 1.5 MeV and observation of the first excited state, which has $J = 2$, could lead to a determination of Δ as outlined above. The second pair of spin flip states, namely the first excited $J = 1$ and the $J = 0$ states, are not at all useful in this respect, their energy separation being almost independent of $\Delta F_{2p}^{(0)}$ (see Appendix 1). However the observation of a pair of states lying close together may yield some information about the range of the $\Lambda - N$ interaction. In particular it seems likely that one or other of the two ranges $\mu_{2\pi}^{-1}$ and μ_K^{-1} will be favoured, possibly with the exclusion of the other. These conclusions follows directly from the range dependence obtained for Δ from the s-shell hypernuclei. It is also worth noting that the observation of a single state, without determination of its spin, will give very little information (for example a state at 0.6 MeV could be a $J = 2$ state with a long range interaction or equally likely one of the pair with $J = 1$ or 0 and a short range interaction); this is to be contrasted with ${}_{\Lambda}Li^7$ where any state observed at less than 1 MeV excitation is almost certainly the $J = {}^3/2$ spin slip state. Observation of three low lying states, without spin measurements, would, of course, be extremely useful as it is strongly indicated that the excited states appear in the order $J = 2, 1, 0^?$, independent of $\Delta F_{2p}^{(0)}$.

Due to uncertainties in the core sizes, rearrangement energies and in the values of B_Λ it is found that no information about the $\Lambda - N$ interaction can be deduced from studying the $A = 7$ hypernuclei using the two-body approach. However, if (most plausibly) the effective $\Lambda - N$ interaction in these hypernuclei is assumed to be about the same as that obtained from the s-shell hypernuclei (and therefore the value for u also as obtained from $_\Lambda Be^9$ and $_\Lambda C^{13}$), then the Λ particle becomes a quite effective probe into the structure of the $A = 6$ nuclei. For the $T = 1$ hypernuclei $_\Lambda He^7$ and $_\Lambda Be^7$ (for which only u enters) as well as for the $T = 0$ hypernucleus $_\Lambda Li^7$, the experimental values of B_Λ then strongly imply that the Λ effectively sees the s- nucleons with an extension close to that of a free α-particle instead of the considerably greater extension appropriate to the isolated $A = 6$ nuclei. This is most naturally interpreted in terms of, and indeed strongly suggests, a structure for the core nuclei consisting of an α-particle and two nucleons. If the Λ is strongly correlated with the α-particle it will then effectively see the α as having its free size, whereas for the isolated core nuclei the more extended s-shell distribution is to be interpreted as a consequence of the recoil motion of the α-particle. A strong α-Λ correlation seems quite natural because of the predominance of the interaction of the Λ with the s-nucleons of the A = 6 cores. The p-nucleon distribution is probably only slightly compressed by the presence of the Λ; also the rearrangement energy is most probably small ($\lesssim$ 0.5 MeV). These conclusions are further supported by calculations for p nucleons moving in a square well. In particular, reasonable parameters for such a well are obtained only for s-nucleon distributions close to that of a free α-particle.

As by-products of the studies of the A = 7 hypernuclei two further points are worth note. Firstly, due to the fact that B_Λ always increases as the core nucleons are compressed, it appears most unlikely that the low experimental values obtained for $B_\Lambda({}_\Lambda He^7)$ as compared with $B_\Lambda({}_\Lambda Be^7)$ are caused by rearrangement effects. Thus support is obtained for the interpretation of these low values as due to an isomeric state of $_{\Lambda}He^7$. Secondly, assuming the binding energy difference between He^6 and the corresponding $T = 1$ state of Li^6 to be entirely due to the Coulomb interaction, the r.m.s. radius obtained with square well wave functions for the p-nucleons in He^6 is very close to the r.m.s. radius which was obtained with oscillator functions[23)].

In the main the conclusions obtained from the two-body studies of the $A = 7$ hypernuclei are further justified by the α-Λ-d model calculation for $_{\Lambda}Li^7$. Notwithstanding the approximations made to describe the nucleon structure of this hypernucleus, the experimental value for the Λ separation energy is readily explained with a volume integral U_2 little different from the corresponding value obtained from $_{\Lambda}He^3$. In addition it seems likely that the degree of agreement obtained can be still further improved by consideration of a more elaborate function to describe the Λ-d correlations. This, together with the fact that the wave functions obtained for the other two pairs of particles in isolation give a very reasonable estimation of the binding energy, strongly suggests both that the Λ particle effectively interacts with a free α-particle and that the rearrangement energy of the nucleons is small.

Finally, it is found that the results obtained from the three-body calculation of ${}_{\Lambda}He^{6}$, taken together with values of the spin dependence consistent with the s-shell hypernuclei, do not allow a definite conclusion to be drawn as to whether the hypernucleus is stable or not. Dependent upon the particular value considered for Δ it is indicated that the hypernucleus is either only just bound or only just unbound. However the fact that the wave function used to obtain these results contained a component corresponding to spurious motion of the neutron relative to the s-shell core nucleus, even though this component is likely to be small, suggests that the latter event is possibly the more likely and that the hypernucleus will not be observed. The effect of the spurious state will be partially offset by any inadequacy of the single parameter Λ -neutron correlation function considered and it is clear that further calculation is necessary before any definite conclusion can be reached, although the situation is qualitatively determined.

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Appendix 1

The Quantity κ

In analogy with the nuclear shell model, the hypernuclear wave functions $| J \rangle$ should be chosen to provide a representation in which both the residual nucleon-nucleon interaction and the Λ-N interaction are diagonal. However, rather than performing this calculation, it is more illustrative to consider the representation provided by the functions $| \alpha ; J_n , 1/2 ; J \rangle$ of eqn. 15, in which the nuclear part of the Hamiltonian is already diagonal. The only non-diagonal terms are then due to the matrix elements of the Λ-N interaction taken between states which differ only in the quantum number J_N; these elements being proportional to the spin dependent volume integral Δ. For many cases of interest the energy associated with the off-diagonal part is much smaller than the energy separation of the nuclear states involved. (i.e. one obtains a matrix A such that $|A_{ij}| \ll |A_{ii} - A_{jj}|$) and accordingly the eigenfunctions can differ but little from the original basis functions $| \alpha ; J_n , 1/2 ; J \rangle$. Table 1.1 gives the values of the quantity κ

$$\kappa = - \langle J | \sum_{i=1}^{N_p} \vec{\sigma}_i \cdot \vec{\sigma}_\Lambda | J \rangle$$

Table 1.1 Values of κ

Hypernucleus	Nuclear State	J	κ	Nuclear Wave Function
$_\Lambda Li^7$	ground state	$^1/_2$	1	$^{13}S_1$
		$^3/_2$	$-\frac{1}{2}$	
	$J_N=1$ T=0	$^1/_2$	0.98	i.c.
	1st excited $J_N=3$ T=0 2.18 MeV	$^5/_2$	$^2/_3$	$^{13}D_3$
		7/2	$-\frac{1}{2}$	
A = 8	ground state $J_N=3/2$ T=1/2	1	$^5/_{12}$	$^{22}P_{3/2}$
		2	$-\frac{1}{4}$	
		1	$5.08/_{12}$	i.c.
	1st excited $J_N=1/2$ T=1/2 0.478 MeV	0	$-\frac{1}{4}$	$^{22}P_{\frac{1}{2}}$
		1	$1/_{12}$	
		1	$0.96/_{12}$	i.c.
	2nd excited $J_N=7/2$ T=1/2 4.63 MeV	3	$^9/_{28}$	$^{22}F_{7/2}$
		4	$-\frac{1}{4}$	
$_\Lambda Li^9$	ground state $J_N=2$ T=1	$^3/_2$	$\frac{3}{4}$	$^{33}P_2$
		$^5/_2$	$-\frac{1}{2}$	
		$^3/_2$	$2.84/_4$	i.c.
$_\Lambda B^{12}$	ground state $J_N=3/2$ $T=\frac{1}{2}$	1	$^5/12$	jj coupled
		1	$3.03/_{12}$	i.c.

for variant hypernuclear states, calculated by neglecting the non-diagonal contributions. It is to be noticed that the use of intermediate coupled nuclear wave functions for the lighter hypernuclei, produces only very small changes in κ ; the wave functions being largely dominated by a single LS coupled state. For the heavier hypernuclei intermediate coupling effects are more important as may be witnessed by the 40% reduction in the value of κ obtained for ${}_{\Lambda}B^{12}$. For these hypernuclei which have spin-zero cases ($J_N = 0$) the value $\kappa = 0$ is obtained. This is independent of whatever coupling scheme is adopted and the only possible source of spin dependent effects in the ground state of these hypernuclei must arise from admixtures of the excited nuclear states with $J_n = 1$. For Be^8 and C^{12} these states occur at energies of 18 MeV and 12 MeV respectively and the value $\kappa = 0$ for these hypernuclei is thus well justified.

The above procedure applied to the expected J = 1 ground state for the A = 8 hypernuclei leads to an underestimation of κ . For this hypernucleus, as mentioned in the text, it is necessary to include also the contribution due to the $J_N = \frac{1}{2}$ excited state. No other states lie low enough in energy to produce any further significant effects and the energy matrix $\mathcal{E}$ to be diagonalised is

$$\mathcal{E} = \begin{bmatrix} -K_{3/2\,3/2}\,\Delta F_{2p}^{(0)} & -K_{3/2\,1/2}\,\Delta F_{2p}^{(0)} \\ & \\ -K_{1/2\,3/2}\,\Delta F_{2p}^{(0)} & E_{1/2} - K_{1/2\,1/2}\,\Delta F_{2p}^{(0)} \end{bmatrix}$$

where the energy is measured relative to the ground state of the nucleus and $E_{1/2} > 0$ is the excitation energy of the $J_N = \frac{1}{2}$ state (the contribution from the spin independent part of the Λ -N interaction produces a change in the origin and is taken to be zero without loss of generality). Diagonalisation of this matrix then gives for the eigenvalues $\lambda^{(\pm)}$ ([illegible]) the expression

$$\lambda^{(\pm)} = \frac{1}{2}\,\mathrm{trace}\,\mathcal{E} \pm \frac{1}{2}\sqrt{(\mathrm{trace}\,\mathcal{E})^2 - 4\det\mathcal{E}}$$

with

$$\mathrm{trace}\,\mathcal{E} = E_{1/2} - (K_{3/2\,3/2} + K_{1/2\,1/2})\,\Delta F_{2p}^{(0)}$$

$$\det\mathcal{E} = K_{\frac{3}{2}\frac{3}{2}}\,\Delta F_{2p}^{(0)}\left(K_{\frac{1}{2}\frac{1}{2}}\,\Delta F_{2p}^{(0)} - E_{1/2}\right) - \left(K_{3/2\,1/2}\,\Delta F_{2p}^{(0)}\right)^2.$$

For large excitation energies $E_{\frac{1}{2}}$ it is clear that $\lambda^{(+)} \rightarrow -K_{3/2\,3/2}\,\Delta F_{2p}^{(0)}$ and $\lambda^{(-)} \rightarrow E_{1/2} - K_{1/2\,1/2}\,\Delta F_{2p}^{(0)}$ as previously stated. The eigenfunction $\varphi^{(+)}$ corresponding to the eigenvalue lying lowest in energy may be written

$$\varphi^{(+)} = \cos\epsilon\,|3/2, 1/2; 1\rangle + \sin\epsilon\,|1/2, 1/2; 1\rangle$$

where

$$\tan\epsilon = \frac{\lambda^{(+)} + K_{3/2\,3/2}\,\Delta F_{2p}^{(0)}}{-K_{3/2\,1/2}\,\Delta F_{2p}^{(0)}} .$$

The value of K relevant to the J = 1 hypernuclear ground state is then

$$K = \cos^2\epsilon\,K_{3/2\,3/2} + 2\cos\epsilon\sin\epsilon\,K_{3/2\,1/2} + \sin^2\epsilon\,K_{1/2\,1/2} .$$

In figure 1.1 the energies $\lambda^{(\pm)}$ MeV and the value of K are shown as functions of $\Delta F_{2p}^{(0)}$ for $E_{\frac{1}{2}}$ = 0.478 MeV and with $K_{3/2\,3/2} = 5/12$, $K_{1/2\,1/2} = 1/12$ and $K_{3/2\,1/2} = -\sqrt{2}/3$ appropriate to LS coupling. Also shown are the energies of the J = 0 and the J = 2 states and the two J = 1 states (chain dotted) obtained by neglecting the admixture. It is to be noticed that diagonalising the Λ -N interaction in this manner effectively

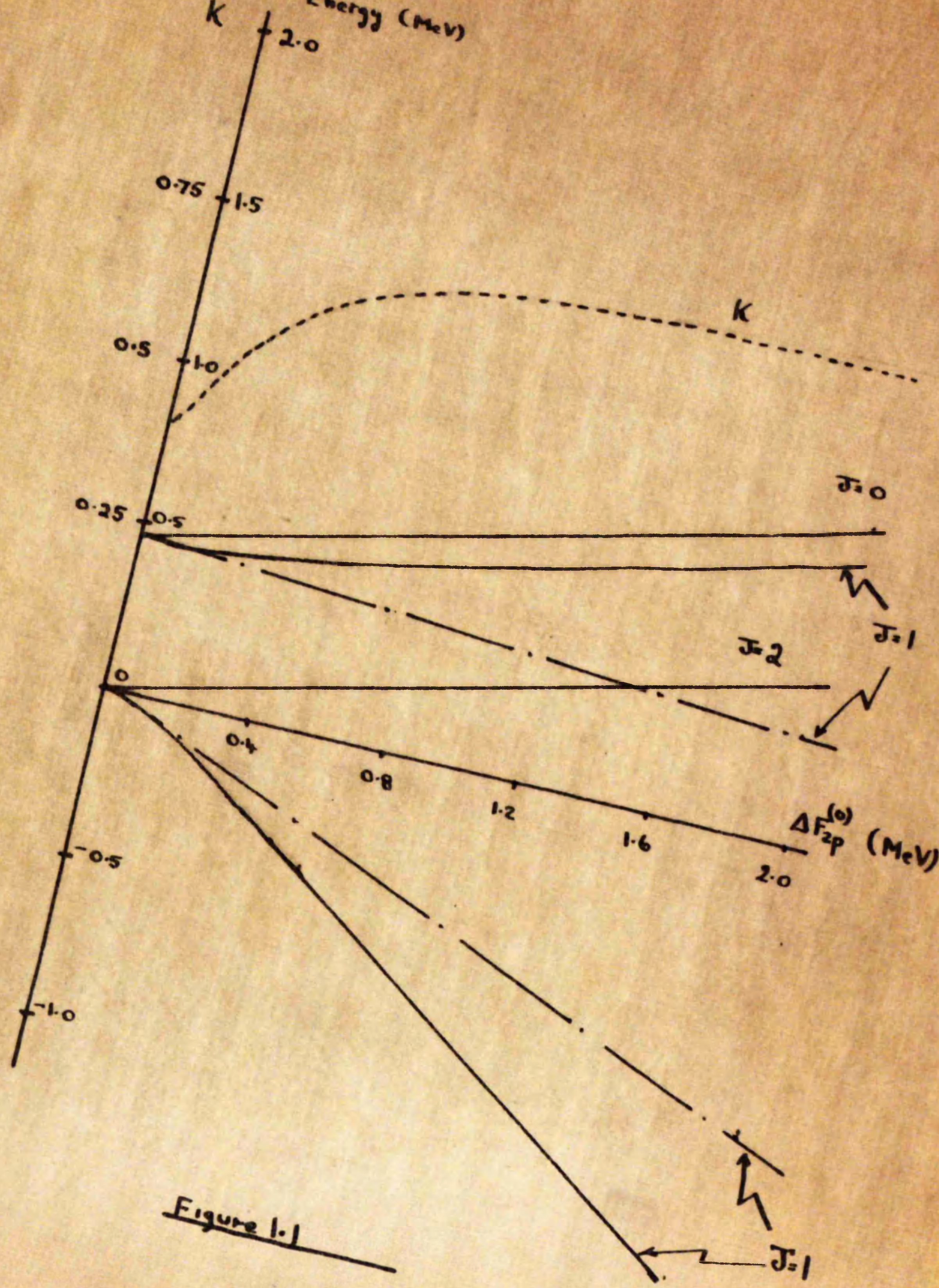
K
Energy (MeV)
2.0
0.75
1.5
0.5
1.0
K
0.25
0.5
J=0
J=1
J=2
0
0.4
0.8
1.2
1.6
2.0
$\Delta F_{2p}^{(0)}$ (MeV)
-0.5
-1.0
J=1

Figure 1.1

depresses the J = 1 ground state at the expense of the first J = 1 excited state; in fact, this latter state now behaves in a similar manner to its J = 0 partner obtained by flipping the Λ spin.

Assuming the values $\Delta = 150$ MeV fm^3 and $F_{2p}^{(0)} = 5.5 \times 10^{-3}$ fm^{-3}, as typical for faces with a range $\mu_{2\pi}^{-1}$, one obtains $\kappa \approx 0.7$ whereas for $\Delta = 25$ MeV fm^3 and $F_{2p}^{(0)} = 7.0 \times 10^{-3}$ fm^{-3}, relevant to μ_{K}, one finds $\kappa \approx 0.52$; the apparent range dependence arises principally from considering values of Δ obtained from the s-shell hypernuclei and appropriate to forces of the above ranges.

Appendix 2

The Potential Shape Functions with Two-Body Forces

The potential shape functions $v_{2\ell}^{(k)}(r)$ defined by eqn. (22) can be written

$$v_{2\ell}^{(k)}(r) = \frac{2k+1}{2} \int_0^\infty t^2 dt \, v(t) \int_{-1}^{1} u_\ell^2(r_1) P_k \left(\frac{r - tz}{r_1} \right) dz \tag{2.1}$$

where $\vec{t} = \vec{r}_1 - \vec{r} = \vec{r}_{1n}$ and z is the cosine of the angle between the vectors $\vec{t}$ and $\vec{r}$. For normalised Yukawa interactions, $v(t) = \mu^2 \exp[-\mu t]/4\pi t$, and with harmonic oscillator radial wave functions $v_{2s}^{(0)}$ and $v_{2p}^{(0)}$ can further be expressed in terms of of the integrals [14]

$$Hh_n(x) = \frac{1}{n!} \int_0^\infty y^n \exp\left[-\tfrac{1}{2}(y+x)^2\right] dy. \tag{2.2}$$

Thus with $x_\ell = \mu a_\ell / \sqrt{2}$ and $z_\ell = \sqrt{2}\, r / a_\ell$, the suffix ℓ = s or p referring to the appropriate oscillator for the s and p nucleons respectively, one obtains

$$v_{2s}^{(0)}(r) = \frac{\mu^2}{4\sqrt{2}\pi^{3/2} r} \exp\left[\tfrac{1}{2}x_s^2\right]\left[e^{-\mu r} Hh_0(x_s - z_s) - e^{\mu r} Hh_0(x_s + z_s)\right] \tag{2.3}$$

and

$$v_{2p}^{(0)}(r) = \frac{\mu^2}{6\sqrt{2}\pi^{3/2} r} \exp\left[\tfrac{1}{2}x_p^2\right]\left[e^{-\mu r}\left\{Hh_2(x_p - z_p) - z_p Hh_1(x_p - z_p) + \tfrac{1}{2}(2 + z_p^2) Hh_0(x_p - z_p)\right\}\right.$$

$$\left. - e^{\mu r}\left\{Hh_2(x_p + z_p) + z_p Hh_1(x_p + z_p) + \tfrac{1}{2}(2 + z_p^2) Hh_0(x_p + z_p)\right\}\right]. \tag{2.4}$$

At the origin, r = 0, the explicit range dependence of the interaction can be exhibited and one has

$$v_{2s}^{(0)}(0) = \rho_s(0)\, \mathcal{J}_0\left(\frac{\mu a_s}{\sqrt{2}}\right) \tag{2.5}$$

and

$$v_{2p}^{(0)}(0) = 3\exp[1]\,\rho_p(a_p)\,\mathcal{J}_p\left(\frac{\mu a_p}{\sqrt{2}}\right) \tag{2.6}$$

where

$$\mathcal{J}_s(x) = x^2 \exp\left[\tfrac{1}{2}x^2\right] Hh_1(x) \tag{2.7}$$

and

$$\mathcal{J}_p(x) = x^2 \exp\left[\tfrac{1}{2}x^2\right] Hh_3(x). \tag{2.8}$$

The function $\mathcal{J}_s(x)$ is such that $\mathcal{J}_s \to 0$ as $x \to 0$ and $\mathcal{J}_s \to 1$ as $x \to \infty$. Thus in the limit of short range forces or extensive density distributions the contribution to the potential shape is just given by the single nucleon density distribution. On the other hand, $\mathcal{J}_p \to 0$ both as $x \to 0$ and $x \to \infty$, the short range limit reflecting the fact for δ function interactions no contributions from the p-shell nucleons are expected for r = 0.

Appendix 3

The Λ -deuteron interaction

The interaction of the Λ particle with a free deuteron can be obtained by the same procedure as is given in section 5 for the α -d interaction. Thus one can write

$$V_3(r_3) = \bar{U}_2 \, v_3(r_3) \tag{3.1}$$

where

$$v_3(r_3) = \frac{1}{2}\int d\vec{r}\, |\chi(r)|^2 \left\{ v(|\vec{r_3} + \tfrac{1}{2}\vec{r}|) + v(|\vec{r_3} - \tfrac{1}{2}r|) \right\} \tag{3.2}$$

and $\bar{U}_2 = \frac{1}{2}U + \Delta$ is the appropriate volume integral of the Λ -N interaction. The free deuteron wave function is taken to be

$$\chi(r) = \frac{N(e^{-\alpha r} - e^{-\beta r})}{r} \quad ; \quad N = \left[\frac{\alpha\beta(\alpha+\beta)}{2\pi(\alpha-\beta)^2}\right]^{1/2} \tag{3.3}$$

where N is the normalisation constant. A typical wave function of this form is obtained by choosing α = 0.232 fm^{-1} and β = 6.2 α . Substituting for $v(r_{\Lambda n})$ the normalised Yukawa interaction of eqn. 1 enables expression 3.2 to be written as

$$v_3(r_3) = \frac{\mu^3}{4\pi} \int d\vec{r}\, |\chi(r)|^2 \, \frac{e^{-\mu|\vec{r}_3 - \frac{1}{2}\vec{r}|}}{|\vec{r}_3 - \frac{1}{2}\vec{r}|} . \tag{3.4}$$

The Yukawa factor in this integral can be expanded into a series of products of the Bessel functions of the first and second kinds with imaginary arguments, I_s and K_s respectively. Thus one has

$$\frac{e^{-\mu|\vec{a}-\vec{b}|}}{|\vec{a}-\vec{b}|} = \sum_{s=0}^{\infty} \frac{(2s+1)}{(ab)^{1/2}} K_{s+1/2}(\mu b)\, I_{s+1/2}(\mu a)\, P_s(\cos\Theta) \tag{3.5}$$

where Θ is the angle between the vectors $\vec{a}$ and $\vec{b}$ and the expression is valid if $0 \leq b \leq a$. Use of the spherical harmonics addition theorem then enables the angular integrations to be performed. The contribution to $v_3(r_3)$ which arises from that part of the radial integration for which $r/2 \leq r_3$ is then given by $\frac{\mu N^2}{r_3} e^{-\mu r_3} \, \mathcal{J}_1(2r_3)$, where

$$\mathcal{J}_1(\rho) = \int_0^{\rho} \frac{dr}{r} (e^{-\alpha r} - e^{-\beta r})^2 (e^{\frac{\mu r}{2}} - e^{-\frac{\mu r}{2}}) . \tag{3.6}$$

In a like manner the contribution for $r/2 > r_3$ is $\frac{\mu N^2}{r_3}(e^{\mu r_3} - e^{-\mu r_3}) \, \mathcal{J}_2(2r_3)$ where

$$\mathcal{J}_2(\rho) = \int_\rho^\infty \frac{dr}{r}\left(e^{-\alpha r} - e^{-\beta r}\right)^2 e^{-\frac{\mu r}{2}} . \tag{3.7}$$

Thus one finally obtains

$$v_3(r_3) = \frac{\mu\alpha\beta(\alpha+\beta)}{2\pi(\alpha-\beta)^2}\frac{1}{r_3}\left[e^{-\mu r_3}\left(\mathcal{J}_1(2r_3) - \mathcal{J}_2(2r_3)\right) + e^{\mu r_3}\mathcal{J}_2(2r_3)\right]. \tag{3.8}$$

Appendix 4

The $\alpha - \Lambda$ -d Model of ${}_\Lambda Li^7$

The details of the calculation of the three-body binding energy B are presented in this appendix. For convenience the notation is somewhat changed from that in the main text. The interactions considered are denoted by

$$V_i(r_i) = U_i\,(e^{-\mu_i r_i} - e^{-\nu_i r_i}) \tag{4.1}$$

and the wave functions are

$$\begin{aligned} g_1(r_1) &= (1-\rho r_1)(e^{-\alpha_1 r_1} + s_1 e^{-\beta_1 r_1}) \\ g_j(r_j) &= e^{-\alpha_j r_j} + s_j e^{-\beta_j r_j} \quad \text{for } j = 2 \text{ and } 3. \end{aligned} \tag{4.2}$$

In addition the following notations are introduced :

$$\begin{aligned}
x_i^{(j)} &\rightarrow (x_1^{(j)},\ x_2^{(j)},\ x_3^{(j)}) = (2\alpha_j,\ \alpha_j+\beta_j,\ 2\beta_j) \\
A_i^{(j)} &\rightarrow (1,\ 2s_j,\ s_j^2) \\
B_i^{(1)} &\rightarrow ((\rho+\alpha_1)^2,\ 2s_1(\rho+\alpha_1)(\rho+\beta_1),\ s_1^2(\rho+\beta_1)^2) \\
B_i^{(2)} &\rightarrow (-2\rho\alpha_1(\rho+\alpha_1),\ -2s_1\rho\{\alpha_1(\rho+\beta_1)+\beta_1(\rho+\alpha_1)\},\ -2s_1^2\rho\beta_1(\rho+\beta_1)) \\
B_i^{(3)} &\rightarrow (\rho^2\alpha_1^2,\ 2s_1\rho^2\alpha_1\beta_1,\ s_1^2\rho^2\beta_1^2) \\
C_i^{(j)} &\rightarrow (\alpha_j^2,\ 2s_j\alpha_j\beta_j,\ s_j^2\beta_j^2) \\
D_i^{(1)} &\rightarrow (\rho+\alpha_1,\ s_1(2\rho+\alpha_1+\beta_1),\ s_1^2(\rho+\beta_1)) \\
D_i^{(2)} &\rightarrow (-\rho(\rho+2\alpha_1),\ -2\rho s_1(\rho+\alpha_1+\beta_1),\ -\rho s_1^2(\rho+2\beta_1)) \\
D_i^{(3)} &\rightarrow (\rho^2\alpha_1,\ \rho^2 s_1(\alpha_1+\beta_1),\ -\rho s_1^2(\rho+2\beta_1)) \\
E_i^{(j)} &\rightarrow (\alpha_j,\ s_j(\alpha_j+\beta_j),\ s_j^2\beta_j) \\
t_i &= k^2/2m_i \quad \text{and} \quad T_i = k^2/2M_i .
\end{aligned} \tag{4.3}$$

In the following expressions the integrals $I_{\rho m n}(\alpha, \beta, \gamma)$ always have the argument $(x_i^{(1)}, x_j^{(2)}, x_k^{(3)})$, unless otherwise indicated.

The normalisation integral $\mathcal{N}$ is given by

$$\mathcal{N} = \int_0^\infty r_1 dr_1 \int_0^\infty r_2 dr_2 \int_{|r_1-r_2|}^{r_1+r_2} \Psi^2 r_3 dr_3$$

$$= \sum_{i,j,k=1}^{3} A_i^{(1)} A_j^{(2)} A_k^{(3)} \left[I_{111} - 2\rho I_{211} + \rho^2 I_{311} \right] \tag{4.4}$$

$$= \sum_{i,j,k=1}^{3} A_i^{(1)} A_j^{(2)} A_k^{(3)} \mathcal{J}(x_i^{(1)}, x_j^{(2)}, x_k^{(3)}),$$

say,

and the potential energy by

$$\mathcal{V} = \int_0^\infty r_1 dr_1 \int_0^\infty r_2 dr_2 \int_{|r_1-r_2|}^{r_1+r_2} \left(\sum_{i=1}^{3} V_i(r_i)\right) \Psi^2 r_3 dr_3$$

$$= \sum_{i,j,k=1}^{3} A_i^{(1)} A_j^{(2)} A_k^{(3)} \Big[U_1 \left\{ \mathcal{J}(x_i^{(1)}+\mu_1, x_j^{(2)}, x_k^{(3)}) - \mathcal{J}(x_i^{(1)}+\nu_1, x_j^{(2)}, x_k^{(3)}) \right\}$$

$$+ U_2 \left\{ \mathcal{J}(x_i^{(1)}, x_j^{(2)}+\mu_2, x_k^{(3)}) - \mathcal{J}(x_i^{(1)}, x_j^{(2)}+\nu_2, x_k^{(3)}) \right\}$$

$$+ U_3 \left\{ \mathcal{J}(x_i^{(1)}, x_j^{(2)}, x_k^{(3)}+\mu_3) - \mathcal{J}(x_i^{(1)}, x_j^{(2)}, x_k^{(3)}+\nu_3) \right\} \Big].$$

(4.5)

Similarly the kinetic energy K becomes

$$K = \int_0^\infty r_1 dr_1 \int_0^\infty r_2 dr_2 \int_{|r_1-r_2|}^{r_1+r_2} \left(\sum_{i=1}^{3} \left\{ t_i \left(\frac{\partial \Psi}{\partial r_i}\right)^2 + T_i \frac{(r_j^2+r_k^2-r_i^2)}{r_j r_k} \frac{\partial \Psi}{\partial r_j} \right\} \right) r_3 dr_3$$

$$= \sum_{i,j,k=1}^{3} \Big[t_1 A_j^{(2)} A_k^{(3)} \left\{ B_i^{(1)} I_{111} + B_i^{(2)} I_{211} + B_i^{(3)} I_{311} \right\}$$

$$+ A_i^{(1)} (t_2 C_j^{(2)} A_k^{(3)} + t_3 A_j^{(2)} C_k^{(3)}) \left\{ I_{111} - 2\rho I_{211} + \rho^2 I_{311} \right\}$$

$$+ T_1 E_j^{(2)} A_k^{(3)} \Big\{ D_i^{(1)} (I_{201} + I_{021} - I_{003})$$

$$+ D_i^{(2)} (I_{301} + I_{121} - I_{103}) + D_i^{(3)} (I_{401} + I_{221} - I_{203}) \Big\}$$

$$+ T_2 A_j^{(2)} E_k^{(3)} \Big\{ D_i^{(1)} (I_{210} + I_{012} - I_{030})$$

$$+ D_i^{(2)} (I_{310} + I_{112} - I_{130}) + D_i^{(3)} (I_{410} + I_{212} - I_{230}) \Big\}$$

$$+ T_3 A_i^{(1)} E_j^{(2)} E_k^{(3)} \Big\{ I_{120} + I_{102} - I_{300}$$

$$- 2\rho (I_{220} + I_{202} - I_{400}) + \rho^2 (I_{320} + I_{302} - I_{500}) \Big\} \Big]. \qquad (4.6)$$

The three-body binding energy B is now simply given by

$$B = \frac{\mathcal{V} - K}{\mathcal{N}} \tag{4.7}$$

and it is this expression which is to be maximised.

In performing the variations of the parameters of the wave function Ψ, eqn. (48), it is necessary to prevent the production of that maximum of B which corresponds to the value $\ell = 0$. This value of B is just the one which would result from using a trial wave function with a form appropriate to the 1S state of motiОn of the alpha and the deuteron and will grossly overestimate the binding energy. It is consistent with the assumption already made, namely that the 2S wave function does not contain spa*ious components, to treat this 1S state as being wholly spurious and accordingly the position of the node in $g_1(r_1)$ is chosen to minimise the amplitude $\mathcal{A}$ of the 1S state, $\Psi_{1S}(r_1) = e^{-\alpha_1 r_1} + \tilde{\delta}_1 e^{-\beta_1 r_1}$, in the wave function Ψ.

The amplitude $\mathcal{A}$ is given by

$$\mathcal{A} = \int d\vec{r}_1 \, \Psi^*_{1s}(r_1)\, \Psi(r_1, r_2, r_3)$$

$$= g_2(r_2) \int_0^\infty r_1^2\, \Psi^*_{1S}(r_1)\, g_1(r_1)\, dr_1 \int d\Omega\, g_3(r_3) \,. \tag{4.8}$$

Expanding $g_3(r_3) = \sum_{K=0}^{\infty} a_K(r_1, r_2)\, P_K(\cos\Theta)$, where Θ is the angle between the vectors $\vec{r_1}$ and $\vec{r_2}$ and using the spherical harmonics addition theorem enables the angular integration to be performed.

One obtains

$$\mathcal{A} = \frac{g_2(r_2)}{2r_2}\Big[\Big\{ I_{11}(\tilde{\alpha}_1+\alpha_1,\alpha_3) + \tilde{D}_1 I_{11}(\tilde{\beta}_1+\alpha_1,\alpha_3) + D_1 I_{11}(\tilde{\alpha}_1+\beta_1,\alpha_3) + D_1\tilde{D}_1 I_{11}(\tilde{\beta}_1+\beta_1,\alpha_3)\Big\}$$

$$- \rho\Big\{ I_{21}(\tilde{\alpha}_1+\alpha_1,\alpha_3) + \tilde{D}_1 I_{21}(\tilde{\beta}_1+\alpha_1,\alpha_3) + D_1 I_{21}(\tilde{\alpha}_1+\beta_1,\alpha_3) + D_1\tilde{D}_1 I_{21}(\tilde{\beta}_1+\beta_1,\alpha_3)\Big\}\Big] \tag{4.9}$$

where the radial integral is given by

$$I_{nm}(a,b) = \int_0^{\infty} r_1^{\,n} e^{-ar_1} dr_1 \int_{|r_1-r_2|}^{r_1+r_2} r_3^{\,m} e^{-br_3} dr_3 . \tag{4.10}$$

For the case of no Λ -d correlations, $g_3(r_3) = 1$ and $\alpha_3 = 0$, one has the relation $I_{n1}(a,0) = 2r_2\, I_{n+1}(a)$, where $I_n(a) = \int_0^{\infty} r_1^{\,n} e^{-ar_1} dr_1$, and this allows a constant value of ρ to be chosen such that the overlap integral $\mathcal{A}$ is zero for all values of r_2. (This is just equivalent to making the requirement $\int d\vec{r}_1 \Psi^*_{1s}(r_1)\, g_1(r_1) = 0$) . For general calculations ($\alpha_3 \neq 0$) the integrals $I_{nm}(a, \alpha_3)$ are not linear functions of r_2 and $\mathcal{A}$ can only be made zero for one particular value of r_2 (at least if ρ is to be a constant). This value is arbitrarily chosen to be $\bar{r}_2$, the r.m.s. radius of $g_2(r_2)$ where

$$\bar{r}_2 = \frac{I_4(2\alpha_2) + 2\delta_2 I_4(\alpha_2+\beta_2) + \delta_2^2\, I_4(2\beta_2)}{I_2(2\alpha_2) + 2\delta_2(\alpha_2+\beta_2) + \delta_2^2\, I_2(2\beta_2)} \tag{4.11}$$

The value of ρ used is then given by

$$\rho = \frac{I_{11}(\tilde{\alpha}_1+\alpha_1,\alpha_3) + \tilde{\delta}_1 I_{11}(\tilde{\beta}_1+\alpha_1,\alpha_3) + \delta_1 I_{11}(\tilde{\alpha}_1+\beta_1,\alpha_3) + \delta_1\tilde{\delta}_1 I_{11}(\tilde{\beta}_1+\beta_1,\alpha_3)}{I_{21}(\tilde{\alpha}_1+\alpha_1,\alpha_3) + \tilde{\delta}_1 I_{21}(\tilde{\beta}_1+\alpha_1,\alpha_3) + \delta_1 I_{21}(\tilde{\alpha}_1+\beta_1,\alpha_3) + \delta_1\tilde{\delta}_1 I_{21}(\tilde{\beta}_1+\beta_1,\alpha_3)} \tag{4.12}$$

where the integrals are calculated with $r_2 = \bar{r}_2$. For all the various wave functions considered the value of the overlap integral

$$\varepsilon = \int d\tau\, \Psi^*_{1s}(r_1)\, g_1(r_1)\, g_2^2(r_2)\, g_3^2(r_3) \tag{4.13}$$

was found to be $\lesssim 2 \times 10^{-3}$, a fact largely to be expected in view of the smallness of the parameter α_3 ($\lesssim$ 0.3 fm^{-1}) required to maximise B; the amplitude $\mathcal{A}$ being everywhere small.

The integrals $I_{\ell m n}(\alpha, \beta, \gamma)$ defined by eqn. (49) in the text are now given. A basic set of integrals is obtained by the specification of those integrals corresponding to all the partitions $\ell \geq m \geq n$ associated with the integer $J = \ell + m + n$. Every possible integral can then be obtained by using the symmetry relation

$$I_{\ell m n}(\alpha, \beta, \gamma) = I_{\widetilde{\ell m n}}(\widetilde{\alpha \beta \gamma}) \qquad (4.14)$$

where $\widetilde{\ell m n}$ is some permutation of the integers $l\ m\ n$ and $\widetilde{\alpha\beta\gamma}$ is the same permutation of α, β, γ. The table shows the expressions for $J \leq 5$ and the notation $a = \alpha + \beta$, $b = \beta + \gamma$, $c = \gamma + \alpha$ has been introduced.

The other integrals required are

$$I_{11}(x,y) = 4\left[\left(4xy + r_2 y(x^2-y^2)\right)e^{-xr_2} - \left(4xy + r_2 x(y^2-x^2)\right)e^{-yr_2}\right]/(x^2-y^2)^3,$$

$$I_{21}(x,y) = 4\Big[\left(20x^2y + 4y^3 + 8xyr_2(x^2-y^2) + r_2^2 y(x^2-y^2)\right)e^{-xr_2}$$

$$- \left(20x^2y + 4y^3 + (3x^2+y^2)(y^2-x^2)r_2\right)e^{-yr_2}\Big]/(x^2-y^2)^4,$$

$$I_2(x) = 2/x^3 \quad ; \quad I_4(x) = 24/x^5.$$

Table. The Integrals $I_{lmn}(x, y, z)$

J	Integral	Expression
0	I_{000}	$2/abc$
1	I_{100}	$2[a+c]/a^2bc^2$
2	I_{200}	$4[a^2+ac+c^2]/a^3bc^3$
	I_{110}	$2[a^2+a(b+c)+2bc]/a^3b^2c^2$
3	I_{300}	$12[a^3+a^2c+ac^2+c^3]/a^4bc^4$
	I_{210}	$4[a^3+a^2(b+c)+ac(2b+c)+3bc^2]/a^4b^2c^3$
	I_{111}	$4[ab(a+b)+bc(b+c)+ac(a+c)+abc]/a^3b^3c^3$
4	I_{400}	$48[a^4+a^3c+a^2c^2+ac^3+c^4]/a^5bc^5$
	I_{310}	$12[a^4+a^3(b+c)+a^2c(2b+c)+ac^2(3b+c)+4bc^3]/a^5b^2c^4$
	I_{220}	$8[a^4+a^3(b+c)+a^2(b^2+2bc+c^2)+3a(b^2c+bc^2)+6b^2c^2]/a^5b^3c^3$
	I_{211}	$4[3b(a^3+c^3)+(2ac+3b^2)(a^2+c^2)+3abc(a+c)+2ac(ac+2b^2)]/a^4b^3c^4$
5	I_{500}	$240[a^5+a^4c+a^3c^2+a^2c^3+ac^4+c^5]/a^6bc^6$
	I_{410}	$48[a^5+a^4(b+c)+a^3c(2b+c)+a^2c^2(3b+c)+ac^3(4b+c)+5bc^4]/a^6b^2c^5$
	I_{320}	$24[a^5+a^4(b+c)+a^3(b+c)^2+a^2c(3b^2+3bc+c^2)+2abc^2(3b+2c)+10b^2c^3]/a^6b^3c^4$
	I_{311}	$24[2b(a^4+c^4)+(ac+2b^2)(a^3+c^3)+2abc(a^2+c^2)+(3ab^2c+a^2c^2)(a+c)+2a^2bc^2]/a^5b^3c^5$
	I_{221}	$8[3a^4(b+c)+a^3(3b^2+3c^2+4bc)+a^2(3b^3+5b^2c+5bc^2+3c^2)+6abc(b^2+bc+c^2)$ $+6(b^3c^2+b^2c^3)]/a^5b^4c^4$

Appendix 5

Simplification of the Kinetic Energy

In order to simplify the expression for the expectation value of the kinetic energy, eqn. (68), consider the integral

$$\mathcal{J} = \int \cos\alpha_i \frac{\partial u}{\partial r_j} \frac{\partial u}{\partial r_k} h(r_1, r_2, r_3) d\tau \tag{5.1}$$

where $i \neq j \neq k = 1, 2, 3.$ This may be written in a more convenient manner if use is made of the following relationships :

$$(\vec{\nabla}_{r_j} u)^2 = \left(\frac{\partial u}{\partial r_j}\right)^2 + \left(\frac{\partial u}{\partial r_k}\right)^2 + 2\cos\alpha_i \frac{\partial u}{\partial r_j}\frac{\partial u}{\partial r_k} \tag{5.2}$$

$$\nabla^2_{r_j} u = \frac{\partial^2 u}{\partial r_j^2} + \frac{\partial^2 u}{\partial r_k^2} + \frac{2}{r_j}\frac{\partial u}{\partial r_j} + \frac{2}{r_k}\frac{\partial u}{\partial r_k} + 2\cos\alpha_i \frac{\partial^2 u}{\partial r_j \partial r_k}. \tag{5.3}$$

Using expression (5.2) in eqn. (5.1) and suitably integrating by parts then gives

$$\mathcal{J} = \frac{1}{2}\int\left[\frac{1}{2}u^2\nabla_j^2 h - u h \nabla_j^2 u - \left\{\left(\frac{\partial u}{\partial r_j}\right)^2 + \left(\frac{\partial u}{\partial r_k}\right)^2\right\} h\right] d\tau$$

If the expression for $\nabla_j^2 u$, eqn. (5.3), is now used and note is taken of the fact that if the function $u(r_1, r_2, r_3)$ has the product form $\prod_{i=1}^{3} g_i(r_i)$ then $u\frac{\partial^2 u}{\partial r_j \partial r_k} = \frac{\partial u}{\partial r_j}\frac{\partial u}{\partial r_k}$, one simply obtains

$$\mathcal{J} = \frac{1}{8}\int u^2 \nabla_j^2 h(r_1, r_2, r_3)\, d\tau$$

$$-\frac{1}{4}\int\left[\left(\frac{\partial u}{\partial r_j}\right)^2 + \left(\frac{\partial u}{\partial r_k}\right)^2 + \frac{2u}{r_j}\frac{\partial u}{\partial r_j} + \frac{2u}{r_k}\frac{\partial u}{\partial r_k} + u\frac{\partial^2 u}{\partial r_j^2} + u\frac{\partial^2 u}{\partial r_k^2}\right] h\, d\tau$$

(5.4)

where $\nabla_j^2 h(r_1, r_2, r_3)$ may be evaluated by using eqn. (5.3). This result then leads directly to eqns. (70) and (71).

Appendix 6

The Effective Three-Body Potential for $_{\Lambda}He^6$

For completeness the various quantities involved in eqn. (84) are here expressed in terms of the integrals $I_{nm}(\alpha,\beta)$ and the parameters of the trial wave functions. One has

$$\mathcal{J}_2 = -c(a-b)^2 I_{11}(a+b,2d) + 2a I_{10}(2d,2a) + 2c(a+b) I_{10}(2d,a+b) + 2bc^2 I_{10}(2d,2b) \quad ,$$

$$\mathcal{J}_3 = 2d\left\{ I_{10}(2a,2d) + 2c\, I_{10}(a+b,2d) + c^2 I_{10}(2b,2d) \right\},$$

$$v_2 = -U_4\Big[I_{11}(2a+\mu,2d) - I_{11}(2a+\nu,2d) + 2c\left\{ I_{11}(a+b+\mu,2d) - I_{11}(a+b+\nu,2d)\right\} + c^2\left\{ I_{11}(2b+\mu,2d) - I_{11}(2b+\nu,2d)\right\}\Big],$$

$$v_3 = -\frac{\mu_2^2}{4\pi}\left(\tfrac{1}{4}U_4 + (x^2-\tfrac{1}{4})\Delta\right)\Big[I_{10}(2a,2d+\mu_2) + 2c\, I_{10}(a+b,2d+\mu_2) + c^2 I_{10}(2b,2d+\mu_2)\Big],$$

$$v_1 = I_{11}(2a,2d) + 2c\, I_{11}(a+b,2d) + c^2 I_{11}(2b,2d),$$

$$v_1' = I_{11}'(2a,2d) + 2c\, I_{11}'(a+b,2d) + c^2 I_{11}'(2b,2d),$$

$$v_1'' = I_{11}''(2a,2d) + 2c\, I_{11}''(a+b,2d) + c^2 I_{11}''(2b,2d).$$

The standard integrals required are given by

$$I_{10}(\alpha,\beta) = \frac{2}{(\alpha^2-\beta^2)^2}\left[2\alpha e^{-\beta r_1} - \left(2\alpha + r_1(\alpha^2-\beta^2)\right)e^{-\alpha r_1}\right],$$

$$I_{11}(\alpha,\beta) = \frac{4}{(\alpha^2-\beta^2)^3}\left[\left\{4\alpha\beta + r_1\beta(\alpha^2-\beta^2)\right\}e^{-\alpha r_1} - \left\{4\alpha\beta + r_1\alpha(\beta^2-\alpha^2)\right\}e^{-\beta r_1}\right],$$

$$I_{11}'(\alpha,\beta) = \frac{4}{(\alpha^2-\beta^2)^3}\left[\alpha\left\{3\beta^2+\alpha^2+r_1\beta(\beta^2-\alpha^2)\right\}e^{-\beta r_1} - \beta\left\{3\alpha^2+\beta^2+r_1\alpha(\alpha^2-\beta^2)\right\}e^{-\alpha r_1}\right],$$

$$I_{11}''(\alpha,\beta) = \frac{4\alpha\beta}{(\alpha^2-\beta^2)^3}\left[\left\{2(\alpha^2+\beta^2)+r_1\alpha(\alpha^2-\beta^2)\right\}e^{-\alpha r_1} - \left\{2(\alpha^2+\beta^2)+r_1\beta(\beta^2-\alpha^2)\right\}e^{-\beta r_1}\right].$$

The behaviour of these integrals for large values of r_1 is given by

$$I_{10}(\alpha,\beta) \sim K(\alpha,\beta)\frac{(\beta^2-\alpha^2)}{2\alpha} r_1 e^{-\alpha r_1} \qquad \text{for } \alpha<\beta$$

$$\sim K(\alpha,\beta) e^{-\beta r_1} \qquad \text{for } \beta<\alpha$$

$$\left.\begin{aligned} I_{11}(\alpha,\beta) &\sim K(\alpha,\beta) r_1 e^{-\beta r_1} \\ I_{11}'(\alpha,\beta) &\sim -K(\alpha,\beta)\beta r_1 e^{-\beta r_1} \\ I_{11}''(\alpha,\beta) &\sim K(\alpha,\beta)\beta^2 r_1 e^{-\beta r_1} \end{aligned}\right\} \text{for } \beta<\alpha$$

where $K(\alpha,\beta) = \frac{4\alpha}{(\alpha^2-\beta^2)^2} > 0$ if $\alpha > 0$

and the behaviour of the integral I_{11} and its derivatives for $\alpha<\beta$ is obtained by using these interchanging α and β. Using these asymptotic expansions in eqn. (30) then gives

$$\lim_{r_1 \to \infty} W_1(r_1) = \mathcal{E}$$

$$= 2k_2\Big[(a^2-d^2)K(2a,2d) + 2c\,(ab-d^2)K(a+b,2d) + c^2(b^2-d^2)K(2b,2d)\Big]$$

$$+ U_4\Big[K(2a+\mu,2d) - K(2a+\nu,2d)$$

$$+ 2c\big\{K(a+b+\mu,2d) - K(a+b+\nu,2d)\big\} + c^2\big\{K(2b+\mu,2d) - K(2b+\nu,2d)\big\}\Big]$$

$$\div \Big[K(2a,2d) + 2c\,K(a+b,2d) + c^2K(2b,2d)\Big].$$

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