FOURIER ANALYSIS OF THE LIGHT CURVES OF ECLIPSING VARIABLES

by

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Being a thesis submitted in support of an application for the degree of Doctor of Philosophy in the Victoria University of Manchester.

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DECLARATION

I declare that to the best of my knowledge no part of the original work in this thesis has been submitted in support of an application for a degree at this or any other University of Institute of Learning.

BIOGRAPHICAL NOTE

The author graduated from the University of Ege, Izmir (TURKEY) in July, 1971, with a B. Sc. degree in Mathematics, Astronomy and Physics. He was then engaged in teaching, for one year, at the high school in Pasinler (Erzurum, TURKEY), which was followed by two years observational research in eclipsing binaries at Ege University Observatory under the supervision of Professor Dr. A. Kizilirmak. For this he was awarded the degree of M.Sc.; the subject was "The Apsidal Motion in Close Binary Systems".

In April, 1975, the author came to the University of Manchester to study under the auspices of Professor Z. Kopal. He was awarded another degree of M.Sc., in July 1976, for research on "The Light Curve Analysis of Partially Eclipsing Binary Systems, in the Frequency Domain".

In the following two years of research, in the same University, he worked on "The Fourier Analysis of the Light Curves of Eclipsing Variables". This work has culminated in the present thesis.

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ABSTRACT

This thesis deals with certain aspects of "The Fourier Analysis of Light Curves of Eclipsing Variables". The subject has been worked out mainly by Kopal in a series of papers starting in 1975, to devise swift and reliable methods for obtaining the elements of eclipsing binary systems from their observed photometric data.

When the fractional loss of light α was identified with a Hankel transform of zero order (Kopal, 1977a) it became possible to derive general expressions (Kopal, 1977b, c; Demircan, 1977b, 1978b) for the requisite basic quantities α, I and A_{2m} - integrals of the analysis. In this thesis a number of useful new algebraic expressions for the same basic quantities, in addition to those already referred to, have been presented. Their fast efficient computation has been put into practice. This has opened an easy way to practical applications of the frequency domain methods which have been constructed in the last four years of continuous effort.

Practical procedures (Kopal and Demircan, 1978) for obtaining the elements of any eclipsing system from observed photometric data by an analysis in the frequency domain have been reviewed. The methods for obtaining the elements of wide eclipsing binaries from a single minimum light curve have been automated and tested successfully on the light curves of YZ(21) Cassiopeiae and β Persei (Algol). A method has been developed for the iterative solution of the two fundamental

eclipse parameters a and c_o . Numerical tables of some new functions required in the analysis have been constructed.

It was noted that the determinacy of the unknown eclipse parameters depends on not only the accuracy of observations but also the nature of the employed g-functions. The choice of the most convenient g-function to obtain a good determinacy for the eclipse elements has been discussed. In this connection, i) the m dependence of the moments A_{2m} , and the errors in ~~the~~ their observational values have been considered, ii) different practical procedures for the solution of eclipse elements were introduced, and iii) different types of moments were tested.

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INTRODUCTION

The importance of the study of eclipsing variables in contemporary stellar astronomy is well known. Knowledge of all stars is increased by what eclipsing stars can tell us of the sizes, masses, densities, and luminous efficiencies of the stars of various spectral types. In fact, they try to tell us more than these, for example, about the nova phenomenon, X-ray emission mechanism, mass flow, extending atmospheres of Wolf-Rayet variables, etc.

Their fortunate characteristics are given below:

1. 0.1% of the stars in the vicinity of the Sun happen to be binary systems exhibiting eclipses. If a similar proportion were characteristic of our galaxy as a whole, the total number of eclipsing binary systems within it should be of the order of 10^8 (Kopal, 1975a, Introduction) and probably closer to 10^9.

2. From the physical point of view, close binary systems are found to occur amongst all types of stellar populations, in all regions of the Hertzsprung-Russell diagram occupied by the stars. They include the "dwarf stars", "neutron stars", "red giants", as well as the probable "black holes".

3. They represent the only kind of double stars whose nature can be recognized by their characteristic variation of light or radial velocity across the great distances in space, including external galaxies.

The characteristic light and radial velocity changes of eclipsing binaries exhibited in the course of each orbital cycle are the principle sources for their investigation. These changes can be observed by different methods (photometric or spectroscopic) with considerable accuracy. For the collection of the recent observations of eclipsing binaries see, e.g., Koch et al (1970), and Fracastoro (1972).

Many methods have been developed so far for the study and interpretation of observational data of eclipsing binaries, starting with the attempt by Pickering (1880) to determine the elements of Algol. Most investigators including Russell and Shapley (1912), Fetlaar (1923), Sharbe (1924), Krat (1934), Piotrowski (1937) and Schneller (1949) in this field worked in the direction of separating the specific parameters into groups, among which there is little or no correlation permitting determination separately. For example, it has long been known that the limb-darkening coefficient U_1, the ratio of the radii K, and the maximum geometrical depth P_o form one such group. Some others are the "rectification" parameters, the luminosities L_s and L_g, and finally i, r_s and r_g in their usual meaning.

Kopal, in a series of papers starting in 1941 and culminating in the monograph of 1959, introduced an iterative approach to the solution of the problem. He says, "As the problem proves to be highly nonlinear in quantities to be determined, its mathematical solution can be approached only by successive approximations."

On another line, Krat's (1940) suggestion about a minimization method was found to lead to the numerical experimentations of the iterative minimization procedures. Many investigators have worked on this line, since the job here is only the simple numerical experimentation to select the optimum theoretical light curve by using electronic computers. The required computations in this approach are too extensive even for the simplest physical models of binary systems with known initial parameters. Therefore, the analyst here has to work not with the large amounts of original observations, but judiciously manufactured small number of normal points. In spite of the above limitations, the methods in this line have been advanced, taking into account "reflection" and "distortion" effects as well as limb and gravity darkening. However, one should expect that, just as in other areas, this approach may yield useful and, very often, unexpected results.

In fact, it is known that the main problem here in the description of the forms and the luminosities of the components of eclipsing binaries is one of the most complicated problems in stellar astronomy. Close binaries are never the spherical, uniformly luminous bodies moving in circular orbits that would make for easy analysis. They are deformed in body and complicated in radiation intensity by rotation and gravitation as well as reflection. Further complexities come from other possible companions in the system, eccentric orbits, rotating apsides, mass flow, mass exchange, star spots, surrounding materials, equatorial rings and tides. But as it was noted, the labour in this field has been most profitable.

As the inevitable results of the above complexities one should not expect to find trustworthy solutions by using oversimplified models (use of spheres or two similar rotational ellipsoids for the shape of the components, and unrealistic "rectification" of the observational light curves). If the components of the system are sufficiently far apart for their mutual distortion to be ignored (therefore both components can be regarded as spheres) the methods for the solution of the eclipse elements seem to be adequate, and to make them more adequate attempts have been made to minimize the effects of observational errors by least-squares procedures developed by Wyse (1939) and Kopal (1959). When, however, the two components are brought closer so that the photometric proximity effects caused by the rotational and tidal distortion and radiative interaction of the components, become appreciable, the satisfactory situation with regard to solution for the elements of the system disappears. So, what could be done for the proper analysis of the light curves of at least uncomplicated systems?

With about twenty years of experience Kopal (1960) gradually came to a conclusion that a considerable alteration of the architectural style of the problem may be necessary for the proper analysis of the photometric observations of eclipsing binaries. Radically different methods should have been developed for the parallel (rather than consecutive) operations on the proximity and eclipse effects. This analysis would be possible, not in the time-domain, but in the frequency-domain, i.e., it is not the light curve itself which should be subject to orbital analysis, but rather its Fourier transform. Some of the underlying analysis in the frequency domain was carried out by Kopal in 1959-1960, but unexpected preoccupation with the problems of the solar system between 1961-1973 caused a postponement of the problem. Then, the "Fourier analysis of the light changes of eclipsing variables" attracted the attention of Mauder (1962, 1966) and Kitamura (1965, 1967) and they could develop in some respects the required techniques for the analysis. During these years, another novel integral approach was introduced to the solution of the problem by Cherepashcuk, Goncharskii and Yagola in a series of papers starting in 1966, by determining the light curve of an eclipsing system by a pair of Fredholm integral equations of the first kind.

After Kopal returned to the subject in 1973, the Fourier techniques have been developed to devise swift and reliable methods of deriving definite information on stars from eclipsing binary light curves. This work was chiefly instituted by Kopal (1975a,b,c,d,e, Papers I-V; Kopal, Markellos and Niarchos, 1976, Paper Vi; 1976a, b, Papers VIII and IX; 1977a,b,c, Papers X-XII, Kopal and Demircan, 1978, Paper XIV). Other contributions to the subject have been made by Kurutaç (1976), Smith (1976, Paper VII; 1977), Livaniou (1977; 1978), Al-Naimiy (1977a,b), Niarchos (1977a,b), Budding (1977), Tsouroplis (1977), Caracatsanis (1977), Najim (1977), Kaskambas (1977), Theokas (1977-78), Demircan (1977a; 1977b, Paper XIII; 1978a,b, Papers XV and XVI), and Edalati and Budding (1978, Paper XVII).

The present work is addressed to an outline of the certain aspects of this novel approach: "Fourier analysis of the light curves of eclipsing variables". The most parts of the work given here are unpublished original work of the author. For some parts which are quoted in the text the Papers I, X, XI, XII, XIII, XIV, XV and XVI will be the fundamental sources.

In the application of the methods a background knowledge of some special functions such as α - and I-integrals is still required. Extensive tables of these functions have been published by Tsesevich (1939, 1940), Kopal (1947), Merrill (1950), Irvine (1962) and Davis (1964), and so much effort has been exercised to approximate these tables by some interpolating polynomials. This was successful for α and inverse p-functions only in the case of linearly limb-darkened stars with the accuracy of almost four decimal places (cf. Jurkevich, 1970, and Fligel and Wilson, 1968). The I-integrals have been evaluated so far as elliptic integrals, just like α-functions, directly by the iterative solutions of the Landen type transforms (cf., e.g., Budding, 1974), and recently the author (Demircan, 1976) has re-tabulated these integrals for partial eclipses by making use of the hypergeometric functions ${}_2F_1$.

However, Kopal has recently shown in Paper XI that all these integrals can be defined as Hankel transforms in the frequency-domain. This new definition gave rise to a number of new useful expressions for the evaluation of respective integrals which will be the subject of Chapter I.

The α- and I-integrals are of fundamental importance for an analysis of the light curves of eclipsing variables in the time-domain. In the frequency-domain, however, they play but an auxiliary role, for the fundamental quantities used there as a basis for the solution of the elements of the respective system are the "moments of the light curves" A_{2m}. The second chapter is designed to provide an outline for these quantities. Some expressions for the theoretical moments A_{2m} which have been derived by employing the expressions given in Chapter I will be given.

Chapter III contains some discussions of the computational aspects for the quantities α's and A_{2m}'s as they are given in Chapters I and II. They have been checked numerically and the algorithms for their numerical computations for any set of parameters will be enclosed.

Chapter IV gives a review of the practical procedures (which are given in Paper XIV) for the solutions of the elements of any eclipsing system with the applications to two certain eclipsing binaries: YZ Cassiopiea (see Paper XV) and β Persei (Algol).

Chapter V is devoted to the conclusions and the accuracy of the Fourier analysis of the light curves of eclipsing variables. A discussion of the methods and the results is given, and various limitations in the practice and the further possible research are indicated.

In the Appendices the tables of a number of necessary functions for the analysis are given for grey plane-parallel stellar atmospheres up to four significant digits at intervals permitting linear interpolation.

CHAPTER 1

THE LOSS OF LIGHT

α AND THE RELATED INTEGRALS

The concept of "loss of light" is one of the most important concepts in the light curve analysis of the eclipsing binary systems. In this chapter we shall discuss this concept.

In section 1.1 we shall review briefly the certain aspects of the light changes of close binary systems. In section 1.2 a novel approach (see Kopal, 1977b; Paper XI) to the loss of light will be outlined and in the following sections a number of useful new expressions, developed by Kopal (1977b, c; Papers XI, XII) and the present author (cf., e.g., Demircan 1977b; Paper XIII), for the loss of light suffered by mutual eclipses of the components of close binary systems will be given.

1.1 Light Changes of Close Binary Systems.

A general task of determining the elements of eclipsing binary systems from an analysis of their light curves necessarily requires a knowledge of the light changes of close binary systems. This variation in light evidently depends on the form of both components as well as on the distribution of brightness over their apparent discs. Important contributions to this subject have been made by a number of authors, including von Zeipel (1924) Takeda (1934, 1937), Russell (1939) and Kopal (1942, 1947, 1954).

The surfaces of the components of close binary systems deviate from a sphere and take one of the equilibrium form under instantaneous rotational and tidal forces. In these systems the proximity effects upon the light variation involve two main influences of "ellipticity" and "reflection" which correspond to the gravitational effects and the radiative interactions. The description of "ellipticity" effects arising from rotational and tidal forces, in terms of spherical harmonics of up to and including fourth order has been given by Kopal (1942). We shall approach this description by considering the instantaneous luminosity l of a close binary system of the form

$$l = U - \Delta l \tag{1.1}$$

where U represents the sum of the uneclipsed luminosities $L_{1,2}$ of both components when the phase angle θ becomes 90° or 270°, which is presumed to be constant and usually set equal to unity, and is the luminosity correction due to both distortion and reflection effects which can be given more concisely (cf. Kopal 1976b, Eq. 2.6) as

$$\Delta l = -\sum_{j=1}^{n} c_j \cos^j\theta + L_1 \sum_{l=0}^{\Lambda} C^{(l)} \alpha_l^0 + L_1 \sum_{l=0}^{\Lambda} C^{(l)} \left\{ f_*^{(l)} + f_1^{(l)} + f_2^{(l)} \right\}, \tag{1.2}$$

where suffix 1 and 2 refer, respectively to the eclipsed and the eclipsing star and L stands for the luminosity of the respective sphere which represents the distorted component. The last two summations represent "eclipse effect" upon the light variation of the system which is decomposed in two parts: the first summation gives the "circular" part of the resulting loss of light, while the second summation stands for the "boundary corrections" arising from the distortion of the components, both summations vanish if there is no eclipse. The first summation on the r.h.s. of (1.2) as a power series in $\cos\theta$ represents the light variation of the system which is free from any eclipse effects. For the reduction procedures of this ellipticity effect, together with the reflection from the observations between minima, see Paper IX. However, an assumption is made here that "the proximity effect obtainable from the observations between minima can be extrapolated for the eclipse phases."

The $C^{(l)}$'s on the r.h.s. of (1.2) are associated with the law of limb darkening of degree Λ in powers of the cosine of the angle of foreshortening of the form

$$J(r) = J(0)\left\{1 - u_1 - u_2 - \cdots - u_\Lambda + u_1\cos\gamma + u_2\cos^2\gamma + \cdots u_\Lambda\cos^\Lambda\gamma\right\} \quad (1.3)$$

and given by Equations (2.4) and (2.5) of Paper II as rational fractions in terms of the coefficients of limb darkening $u_1, u_2, \ldots u_\Lambda$ as

$$C^{(0)} = \frac{1 - u_1 - u_2 - \cdots - u_\Lambda}{1 - \sum_{j=1}^{\Lambda} \frac{j\,u_j}{2+j}}, \quad \text{and for } j > 0 \quad C^{(l)} = \frac{u_l}{1 - \sum_{j=1}^{\Lambda} \frac{j\,u_j}{2+j}}. \quad (1.4)$$

The quantity f_* relates to the correction to the luminosity of the eclipsed area of the undergoing star and contains only the functions α_l^m, while $f_{1,2}$ represent the photometric contributions of the "boundary corrections" arising from the distortion of the primary and secondary components. These quantities $f_{1,2}$ contain only the $\mathcal{J}$ and I-integrals.

Thus, apart from the reflection effect, the whole complexity of the light changes of close binary systems between minima as well as within eclipses is evidently stored in the evaluation of α_l^m, $\mathcal{J}_{\beta,\gamma}^m$ and $I_{\beta,\gamma}^m$ -integrals. These eclipse functions, occurring on the r.h.s. of (1.2) have been extensively investigated (cf. Kopal, 1959, sections IV.4 and IV.5) by a purely geometrical approach and put into practice by a number of authors including Jurkevich (1970), Linnell and Proctor (1970, 1971), Budding (1973), Söderhjelm (1974) and Demircan (1977a).

The difficulty in carrying out the reductions for the above "photometric perturbations" by employing the formulae obtained by the geometrical approach is to work with different formulae of eclipse functions for different indices and different types of eclipses (partial, total, annular). For this reason computations are not only time-consuming, but also require extensive care.

However, recently, Kopal (1977b; Paper XI) proved that, in the Fourier approach, the "circular integrals" α_l^0 and "boundary corrections" f_* and $f_{1,2}$ of the theory of light curves of distorted eclipsing systems can be expressible in terms of Hankel transforms of the optical properties of the eclipsing and eclipsed components. Although this new approach may appear to be less "elementary" than the geometrical approach it possesses several distinct advantages. The following sections of the present chapter will be devoted to this novel approach and its advantages.

1.2 A New Definition for α_l^0 .

Kopal has recently shown in Paper XI that the fractional loss of light α due to mutual eclipses of the components of close binary systems can be expressed as a cross-correlation of two apertures representing the eclipsing and eclipsed discs in terms of Hankel transforms of the optical properties of the components. The advantages of such a strategy over the more conventional (geometrical) approach, as fully discussed by Kopal, are (a) greater symmetry of the respective expressions; (b) greater affinity of expressions arising from distortion with those expressing the light changes due to eclipses of spherical stars; and (c) greater freedom in dealing with the effects of particular distribution of brightness over the disc of the star undergoing eclipse (generalized limb-darkening), as well as of possible semi-transparency of the eclipsing component. In this approach the orders of respective Hankel transforms of the products of two Bessel functions depend on the physical characteristics (distribution of brightness, opacity) of the two components, and the geometry of the system (i.e., the fractional radii $r_{1,2}$ of the two components and the inclination i of the orbital plane) enter only through the arguments of those two Bessel functions.

We shall approach this new definition of the loss of light by considering first the stars as representing circular apertures and the function $f(x,y)$ the distribution of brightness within this aperture in the rectangular coordinates whose xy plane is tangent to the celestial sphere and the origin is at the centre of our aperture. If so, the two-dimensional Fourier transform $F(u,v)$ of the aperture function $f(x,y)$ is known to be given by

$$F(u,v) = \iint_{-\infty}^{+\infty} f(x,y)\, e^{-2\pi i(xu+yv)}\, dx\, dy \,. \tag{2.1}$$

By a resort to Jacobi's well-known expansion theorem (cf., e.g., Watson, 1945; p 22 or 368) which permits us to assert that

$$e^{-2\pi i q r \cos(\theta-\phi)} = e^{-2\pi i q r \sin(\frac{\pi}{2}+\theta-\phi)} =$$

$$= \sum_{n=0}^{\infty} \left\{ \epsilon_n J_{2n}(2\pi q r) \cos 2n\left(\tfrac{\pi}{2}+\theta-\phi\right) - 2i\, J_{2n+1}(2\pi q r) \sin(2n+1)\left(\tfrac{\pi}{2}+\theta-\phi\right) \right\}, \tag{2.2}$$

$$\epsilon_0 = 1 \quad \text{and} \quad \epsilon_n = 2 \quad \text{for } n > 0 \,,$$

where the symbols $J_n(x)$ denote the Bessel functions of the first kind, Eq. (2.1) can be easily rewritten in the form

$$F(q) = 2\pi \int_0^{r_1} f(r)\, J_0(2\pi q r)\, r\, dr \,, \tag{2.3}$$

in the spherical coordinates. If, moreover, we assume the distribution of brightness $f(x,y)$ is radially symmetrical and can be given by Eq. (1.3) of the previous section, then the Fourier transform $F(q)$ of the aperture function $f(r)$ can be defined (cf. Paper XI, Eq. 2.15) as a series given by

$$F(q) = L_1 \sum_{l=0}^{\Lambda} C^{(l)} 2^{\nu} \Gamma(\nu) \frac{J_{\nu}(2\pi q r_1)}{(2\pi q r_1)^{\nu}} , \tag{2.4}$$

where L_1 is the luminosity of the aperture, and the coefficients $C^{(l)}$ depend on the limb-darkening for the apertures which have been defined by Eq. (1.4) of the previous section, and

$$\nu = \frac{l+2}{2} . \tag{2.5}$$

Let us now turn our attention to an off-centre aperture (cf. Figure 1) which represents the eclipsing component, situated on the x-axis at a

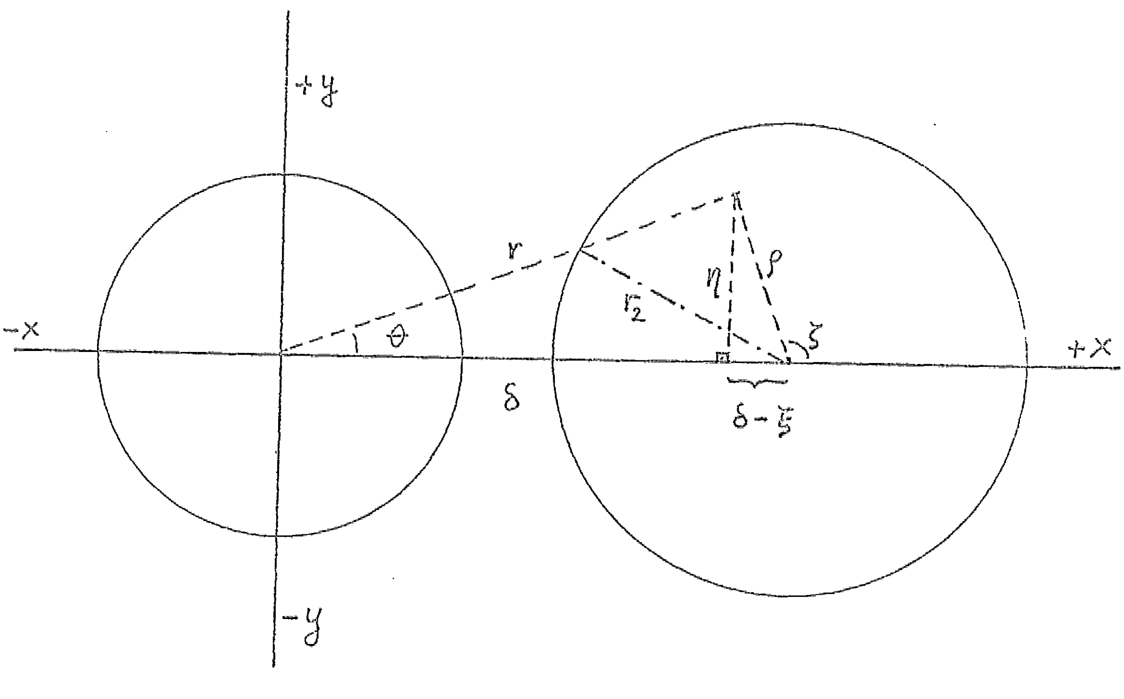

Figure 1. After Kopal (1977b)

distance δ from the origin of coordinates. If so, the Fourier transform G(u,v) of the "transparency function" $g(\xi, \eta)$ of this second aperture should – by analogy with (2.1) – be defined as

$$G(u,v) = \iint_{-\infty}^{+\infty} g(\xi,\eta)\, e^{-2\pi i[(\delta+\xi)u+\eta v]}\, d\xi\, d\eta\,. \tag{2.6}$$

If, in particular, this second aperture is also circular and the transparency function is unity within the aperture, i.e., it is wholly opaque for $\rho \leq r_2$ and wholly transparent for $\rho > r_2$, it can be shown that the above integrals can be evaluated in a closed form, by resorting to Eq. (2.2), as

$$G(q,\phi) = 2\pi\, e^{-2\pi i \delta q \cos\phi} \int_0^{r_2} J_0(2\pi q \rho)\, \rho\, d\rho$$

$$= 2\pi\, r_2^2\, e^{-2\pi i \delta q \cos\phi}\, \frac{J_1(2\pi q r_2)}{2\pi q r_2} \tag{2.7}$$

in the spherical coordinates.

A frequency convolution of two Fourier transforms of the respective aperture functions given by (2.4) and (2.7) should define the loss of light α ; thus, it would follow that

$$L_1 \alpha = \iint_{-\infty}^{+\infty} F(u,v)\, G(u,v)\, du\, dv\,, \tag{2.8}$$

which obviously is zero if the apertures do not overlap, and increases with an increase of their common intercept, weighted in accordance with the relative brightness of each element occulted. By making use of Equations (2.4) and (2.7) for the transforms F and G, and considering that

$$\alpha = \sum_{l=0}^{\Lambda} C^{(l)} \alpha_l^0 , \tag{2.9}$$

it can be shown that (cf. Paper XI, Eq. 2.32) the associated α functions α_l^0 of zero order are expressible as

$$\alpha_l^0(r_1, r_2, \delta) = 2^{\nu}\Gamma(\nu)\int_0^{\infty} \frac{J_{\nu}(2\pi q r_1)}{(2\pi q r_1)^{\nu}} J_1(2\pi q r_2) J(2\pi q \delta)\, d(2\pi q r_2). \tag{2.10}$$

This equation represents the associated alpha-functions of zero order as Hankel transforms of zero order of the products of two Bessel functions of orders ν and 1. This definition (2.10) holds good for any type of eclipse and any value of r_1, r_2 and δ. In other words, the α_l^0's as given by Eq. (2.10) represent real and continuous non-negative functions of δ between minima as well as within eclipses of any type. Let us remember that it holds good, to be sure, only if the eclipsing disc is wholly opaque. However, this result has been generalized in Paper XI to a case in which the light changes arise from occultations by discs which are increasingly transparent with the angle of foreshortening in the same manner as the limb-darkening of the eclipsed star. The generalized result (cf. Paper XI, Eq. 2-38) can be given by

$$\alpha_l^0(r_1, r_2, \delta) = 2^{\lambda+\nu}\Gamma(\lambda+1)\Gamma(\nu)\int_0^{\infty} \frac{J_{\nu}(2\pi q r_1)}{(2\pi q r_1)^{\nu}} \frac{J_1(2\pi q r_2)}{(2\pi q r_2)^{\lambda}} J_0(2\pi q \delta)\, d(2\pi q r_2) \tag{2.11}$$

for the radially-symmetric transparency function $g(\rho)$ given by

$$g(\rho) = \begin{cases} [1-(\rho/r_1)^2]^\lambda & , \rho \leq r_2 \\ 0 & , \rho > r_2 \end{cases} . \tag{2.12}$$

By the way, we shall not work on this latter form of the fractional loss of light α_l^o (due to the occultations by semi-transparent eclipsing stars) in the present work.

The right-hand sides of the results (2.10) and (2.11) contain three parameters as multiplicative factors of the arguments of the Bessel functions occurring in it, namely r_1, r_2 and δ. It is, however, possible by a convenient change of notation, by writing

$$2\pi q = \frac{x}{\mu} , \tag{2.13}$$

that the number of parameters can be reduced to two. For some particular values of μ, the arguments of the respective Bessel functions and the new parameters of the fractional loss of light α_l^o in this new notation have been listed in the accompanying Table I. Eq. (2.10) for $\alpha_l^o(r_1, r_2, \delta)$, for example, can be readily rewritten in terms of the above new parameters listed in Table I, as

$$\alpha_l^o\left(k=\frac{r_2}{r_1}, h=\frac{\delta}{r_1}\right) = 2^\nu \Gamma(\nu)\, k \int_0^\infty \frac{J_\nu(x)}{x^\nu} J_1(kx)\, J_0(hx)\, dx , \tag{2.14}$$

$$\alpha_l^o\left(k=\frac{r_1}{r_2}, h=\frac{\delta}{r_2}\right) = 2^\nu \Gamma(\nu) \int_0^\infty \frac{J_\nu(kx)}{(kx)^\nu} J_1(x)\, J_0(hx)\, dx ; \tag{2.15}$$

<u>Table I</u>

	$\mu = r_1$	$\mu = r_2$	$\mu = r_1 + r_2$	$\mu = \delta$	$\mu = \lvert r_1 - r_2 \rvert$
$2\pi q r_1$	X	$\frac{r_1}{r_2} X$	$\frac{r_1}{r_1+r_2} X$	$\frac{r_1}{\delta} X$	$\frac{r_1}{\lvert r_1-r_2\rvert} X$
$2\pi q r_2$	$\frac{r_2}{r_1} X$	X	$\frac{r_2}{r_1+r_2} X$	$\frac{r_2}{\delta} X$	$\frac{r_2}{\lvert r_1-r_2\rvert} X$
$2\pi q \delta$	$\frac{\delta}{r_1} X$	$\frac{\delta}{r_2} X$	$\frac{\delta}{r_1+r_2} X$	X	$\frac{\delta}{\lvert r_1-r_2\rvert} X$
Parameters of α_L^0	$k = \frac{r_2}{r_1}$ $h = \frac{\delta}{r_1}$	$k = \frac{r_1}{r_2}$ $h = \frac{\delta}{r_2}$	$a = \frac{r_1}{r_1+r_2}$ $c = \frac{\delta}{r_1+r_2}$	$k = \frac{r_1}{\delta}$ $h = \frac{r_2}{\delta}$	$a = \frac{r_1}{\lvert r_1-r_2\rvert}$ $c = \frac{\delta}{\lvert r_1-r_2\rvert}$

$$\alpha_L^0\left(a = \frac{r_1}{r_1+r_2},\ c = \frac{\delta}{r_1+r_2}\right) = 2^\nu \Gamma(\nu)(1-a) \int_0^\infty \frac{J_\nu(ax)}{(ax)^\nu} J_1([1-a]x)\, J_0(cx)\, dx, \quad (2.16)$$

$$\alpha_L^0\left(k = \frac{r_1}{\delta},\ h = \frac{r_2}{\delta}\right) = 2^\nu \Gamma(\nu)\, h \int_0^\infty \frac{J_\nu(kx)}{(kx)^\nu} J_1(hx)\, J_0(x)\, dx, \quad (2.17)$$

and

$$\alpha_L^0\left(a = \frac{r_1}{|r_1\ r_2|},\ c = \frac{\delta}{|r_1-r_2|}\right) = 2^\nu \Gamma(\nu)(|1-a|) \int_0^\infty \frac{J_\nu(ax)}{(ax)^\nu} J_1([|1-a|]x)\, J_0(cx)\, dx, \quad (2.18)$$

respectively. These integrals given by Equations (2.14) - (2.18) can be expanded in series of different types of hypergeometric functions and will mainly be the subject of the following sect ions. It is known that for the convergence of those hypergeometric functions their variables which will usually be the parameters of the above respective α_l^o's are required to be one or smaller than one. In this respect, the particular values of these parameters at certain critical points of both occultation and transit types of eclipses together with the notes on applicability for the respective expansions of the above integrals for α_l^o have been given in the accompanying Table 2. It is seen that in general Eq. (2.16) will be the most useful one for the expansions in series of hypergeometric functions, but it should be kept in mind that the convergence of the respective series to α_l^o's are slowed down with increasing values of μ which has been introduced by (2.13).

Table 2

parameters	OCCULTATION central eclipse	OCCULTATION second contact	OCCULTATION first contact	$r_1 = r_2$	TRANSIT central eclipse	TRANSIT second contact	TRANSIT first contact	NOTES
$k = \frac{r_2}{r_1}$	> 1			1	< 1			Applicable for Transit eclipses if $r_1 \gg r_2$
$h = \frac{\delta}{r_1}$	0	$\frac{r_2 - r_1}{r_1}$	$\frac{r_1 + r_2}{r_1}$	$0 < h < 2$	0	$\frac{r_1 - r_2}{r_1}$	$\frac{r_1 + r_2}{r_1}$	
$k = \frac{r_1}{r_2}$	< 1			1	> 1			Applicable for Occultation eclipses if $r_1 \ll r_2$
$h = \frac{\delta}{r_2}$	0	$\frac{r_2 - r_1}{r_2}$	$\frac{r_1 + r_2}{r_2}$	$0 < h < 2$	0	$\frac{r_1 - r_2}{r_2}$	$\frac{r_1 + r_2}{r_2}$	
$a = \frac{r_1}{r_1 + r_2}$	< 1			$\frac{1}{2}$	< 1			Applicable in all circumstances
$c = \frac{\delta}{r_1 + r_2}$	0	$\frac{r_2 - r_1}{r_1 + r_2}$	1	$0 < c < 1$	0	$\frac{r_1 - r_2}{r_1 + r_2}$	1	
$k = \frac{r_1}{\delta}$	∞	$\frac{r_1}{r_2 - r_1}$	$\frac{r_1}{r_1 + r_2}$	$\infty > k > \frac{1}{2}$	∞	$\frac{r_1}{r_1 - r_2}$	$\frac{r_1}{r_1 + r_2}$	Applicable for Partial eclipses if $i \ll 90^\circ$ and $r_1 \lesseqgtr r_2$
$h = \frac{r_2}{\delta}$	∞	$\frac{r_2}{r_2 - r_1}$	$\frac{r_2}{r_1 + r_2}$	$\infty > h > \frac{1}{2}$	∞	$\frac{r_2}{r_1 - r_2}$	$\frac{r_2}{r_1 + r_2}$	
$a = \frac{r_1}{r_1 - r_2}$	$\frac{r_1}{r_2 - r_1}$				$\frac{r_1}{r_1 - r_2}$			Applicable for complete phases if $-1 < \frac{r_{1,2}}{\lvert r_1 - r_2 \rvert} < 1$
$c = \frac{\delta}{r_1 - r_2}$	0	1	$\frac{r_1 + r_2}{r_2 - r_1}$	$0 < c < \infty$	0	1	$\frac{r_1 + r_2}{r_1 - r_2}$	

1.3 Expansions for α_l^o in Series of Appell Hypergeometric Functions of the First Kind.

In order to derive these expansions let us first consider the coefficients $a_l^{(n)}$ (cf. e.g., Paper IV, Eq.2.1) given by

$$a_l^{(n)} = -\int_{\delta_0}^{\delta_1} \delta^{2n} \frac{\partial \alpha_l^o}{\partial \delta} d\delta \; . \tag{3.1}$$

This equation for $n = 0$ makes it evident that, since $\alpha_l^o(\delta_1) = 0$ for any l,

$$a_l^{(o)} = -\int_{\delta_0}^{\delta_1} \frac{\partial \alpha_l^o}{\partial \delta} d\delta = \alpha_l^o(\delta_o) \tag{3.2}$$

as the respective obscuration at the time of maximum eclipse when $\delta = \delta_o$. If, moreover, we remember the derivative of α_l^o with respect to δ (cf. e.g., Paper II, Eq. 2.17) of the form

$$\left(\frac{\partial \alpha_l^o}{\partial \delta}\right)_{Part.} = -\frac{4^\nu}{2\pi\delta} B(\nu,\tfrac{1}{2})\left(\frac{r_2}{r_1}\right)^\nu\left(\frac{\delta}{r_1}\right)^\nu K^{2\nu-1}\,{}_2F_1\left({-\tfrac{1}{2}, \tfrac{3}{2} \atop \nu+\tfrac{1}{2}} \Big| K^2\right) \tag{3.3}$$

for partial eclipses, in which the modulus

$$K^2 = \frac{r_1^2 - (\delta - r_2)^2}{4\delta r_2} , \tag{3.4}$$

a beta function

$$B(\nu,\tfrac{1}{2}) = \sqrt{\pi}\,\frac{\Gamma(\nu)}{\Gamma(\nu+\tfrac{1}{2})} \tag{3.5}$$

as a numerical factor and the ${}_2F_1$ stands for the ordinary hypergeometric series. Now, on insertion of (3.3) in (3.2) it follows that, for partial eclipses

$$\alpha_i^o(\delta_o) = \int_{\delta_o}^{\delta_1} \frac{4^\nu}{2\pi\delta} B(\nu,\tfrac{1}{2}) \left(\frac{r_2}{r_1}\right)^\nu \left(\frac{\delta}{r_1}\right)^\nu K^{2\nu-1} {}_2F_1\left({-\tfrac{1}{2}, \tfrac{3}{2} \atop \nu+\tfrac{1}{2}} \middle| K^2\right) d\delta . \qquad (3.6)$$

Normalizing the limits by introduction of the auxiliary variable

$$u = \frac{\delta_1 - \delta}{\delta_1 - \delta_o} , \qquad (3.7)$$

we can write that

$$K^2 = \frac{(-\delta + \delta_1)(\delta + r_1 - r_2)}{4 r_2 \delta}$$

$$= \frac{1}{2} \frac{r_1}{r_2} y u (1 - yu)^{-1} \left(1 - \frac{\delta_1 y}{2 r_1} u\right) , \qquad (3.8)$$

and consequently it follows that

$$\alpha_i^o(\delta_o) = \frac{2^{\nu+\frac{1}{2}}}{2\pi} B(\nu,\tfrac{1}{2}) \left(\frac{y}{r_1}\right)^\nu \left(\frac{r_2}{y r_1}\right)^{\frac{1}{2}} \delta_1^{\nu-1} (\delta_1 - \delta_o) \cdot I \qquad (3.9)$$

with

$$I = \int_0^1 u^{\nu-\frac{1}{2}} (1 - yu)^{-\frac{1}{2}} \left(1 - \frac{\delta_1 y}{2 r_1} u\right)^{\nu-\frac{1}{2}} {}_2F_1\left({-\tfrac{1}{2}, \tfrac{3}{2} \atop \nu+\tfrac{1}{2}} \middle| K^2\right) du , \qquad (3.10)$$

where we have abbreviated

$$y = \frac{\delta_1 - \delta_o}{\delta_1} . \qquad (3.11)$$

The integral I on the right hand side of (3.9) given by (3.10) can be identified as an Appell generalized hypergeometric function of the first kind. To prove this let us first appeal to known expansion of ${}_2F_1$ which permits us to assert, by making use of Eq. (3.8) for the modulus K^2 that

$$ {}_2F_1\left(\begin{matrix}-\frac{1}{2}, \frac{3}{2} \\ \nu+\frac{1}{2}\end{matrix}\middle| K^2\right) = -\frac{\Gamma(\nu+\frac{1}{2})}{\pi} \sum_{m=0}^{\infty} \frac{\Gamma(m-\frac{1}{2})\Gamma(m+\frac{3}{2})}{2^m\, m!\, \Gamma(m+\nu+\frac{1}{2})} \left(\frac{r_1}{r_2} y\right)^m \times $$

$$ \times\, u^m (1-yu)^{-m} \left(1 - \frac{\delta_1 y}{2r_1} u\right)^m . \qquad (3.12) $$

Accordingly, inserting (3.12) in (3.10) we get

$$ I = -\frac{\Gamma(\nu+\frac{1}{2})}{\pi} \sum_{m=0}^{\infty} \frac{\Gamma(m-\frac{1}{2})\Gamma(m+\frac{3}{2})}{2^m\, m!\, \Gamma(m+\nu+\frac{1}{2})} \left(\frac{r_1}{r_2} y\right)^m \times $$

$$ \times \int_0^1 u^{\nu+m-\frac{1}{2}} (1-yu)^{-m-\frac{1}{2}} \left(1 - \frac{\delta_1 y}{2r_1} u\right)^{\nu+m-\frac{1}{2}} du . \qquad (3.13) $$

On the other hand, we have (cf. e.g., Erdelyi et al, 1953, Vol. I, p 231, Eq. 5)

$$ F_1(a, b, b', c; x, y) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 u^{a-1} (1-u)^{c-a-1} (1-xu)^{-b} (1-yu)^{-b'} du , $$

$$ a > 0 \quad \text{and} \quad c - a > 0 , \qquad (3.14) $$

as the integral definition of the Appell generalized hypergeometric functions of first kind. A combination of Equations (3.9), (3.13) and (3.14) discloses that for $\delta \equiv \delta_0$

$$\alpha_l^{\circ}(X,y) = \frac{1}{\pi\sqrt{\pi}}\, 2^{\nu}\Gamma(\nu)\, X^{\nu} y \sum_{m=0}^{\infty} \frac{\Gamma(m-\frac{1}{2})\,\Gamma(m+\frac{3}{2})}{2^{m-\nu+\frac{1}{2}}\, m!\,\Gamma(\nu+m+\frac{3}{2})} \times$$

$$\times \left[\frac{y^2}{2X-y}\right]^{m-\frac{1}{2}} F_1\left(\nu+m+\tfrac{1}{2}, -\nu-m+\tfrac{1}{2}, m+\tfrac{1}{2}, \nu+m+\tfrac{3}{2}; X, y\right), \quad (3.15)$$

as the desired expression for partial eclipses, in which y is as given by (3.11) and

$$X = \frac{\delta_1 - \delta}{2r_1} . \qquad (3.16)$$

This equation given by (3.15) represents the general expansion of the associated α-functions of zero order in series of Appell hypergeometric functions of the first kind. It is valid for any arbitrary degree l of the adopted law of limb-darkening in only partial phases, since the derivative (3.3) as it stands holds good, to be sure, only as long as the eclipse remains partial when $\delta_2 < \delta < \delta_1$. At the commencement of totality, when $\delta = \delta_2 = r_2 - r_1$, we find that

$$X = 1 \qquad \text{and} \qquad y = 1 - \frac{r_2 - r_1}{r_1 + r_2} < 1 \qquad (3.17)$$

At the moment of annular phase when $\delta = \delta_2 = r_1 - r_2$

$$X = \frac{r_2}{r_1} < 1 \qquad \text{and} \qquad y = 1 - \frac{r_1 - r_2}{r_1 + r_2} < 1 \;; \qquad (3.18)$$

while at the commencement of the eclipses of any kind, when $\delta = \delta_1 = r_1 + r_2$, both parameters vanish and so does α_l° as expected. Thus, above expansion (3.15) converges for partial eclipses - occultation or transit type - but not for total and annular phases. Fortunately, for total eclipses the α-functions take the simplest forms, as they are well-known from the geometrical approach (cf. e.g., Kopal, 1947) that, in general

$$\alpha_{2\nu}^{2\mu} = \frac{\nu!\,\Gamma(\mu+\frac{1}{2})}{\sqrt{\pi}\,(\mu+\nu+1)!}, \qquad \text{and} \qquad \alpha_{2\nu}^{2\mu+1} = 0 \tag{3.19}$$

where μ is an integer and ν may take both integral and half-integral values.

During annular phases of transit eclipses the modulus K^2 as defined by Eq. (3.4) becomes greater than one (which would make the hypergeometric series on the r.h.s. of (3.3) diverge) and, for central eclipses, it tends to infinity as $\delta \to 0$ (see Figure 2). In order to evaluate a similar expansion for annular eclipses Eq. (3.2) is to be replaced by

$$\alpha_l^o(\delta_0) = a_l^{(o)} + b_l^{(o)} = -\int_{\delta_2}^{\delta_1}\left(\frac{\partial \alpha_l^o}{\partial \delta}\right)_{Part.} d\delta - \int_{\delta_0}^{\delta_2}\left(\frac{\partial \alpha_l^o}{\partial \delta}\right)_{Ann.} d\delta, \tag{3.20}$$

where the integrand of the first integral will continue to be given by Eq. (3.3) and the integrand of the second integral will be given (cf. Paper II, Eq. 2.30) in the form

$$\left(\frac{\partial \alpha_l^o}{\partial \delta}\right)_{Ann.} = \frac{l\,4^{\nu-2}}{\delta}\left(\frac{r_2}{r_1}\right)^{\nu}\left(\frac{\delta}{r_1}\right)^{\nu} K^{2\nu-4}\, {}_2F_1\left({2-\nu,\ \frac{3}{2} \atop 3}\middle|\, K^{-2}\right), \tag{3.21}$$

with the same modulus K as it is given by (3.4). The remaining integrations in (3.20) can be performed in the same way as before to derive the expansion of the loss of light α_l^0 for the annular eclipses in series of Appell hypergeometric functions of the first kind.

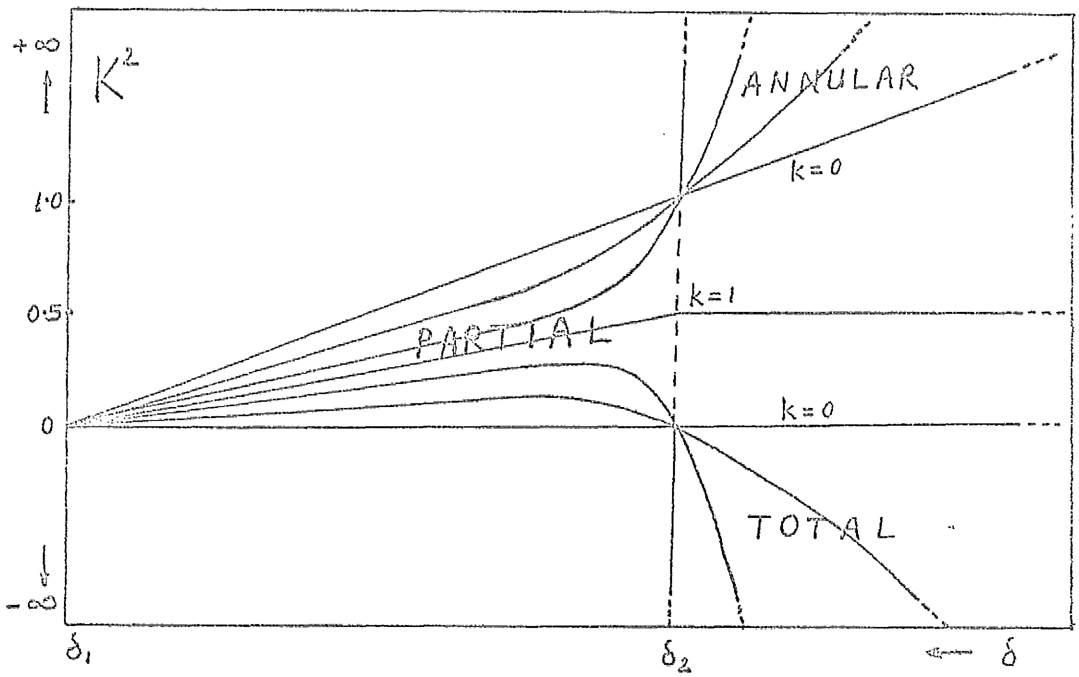

Figure 2

1.4 <u>Expansions for α_l^0 in Series of Appell Hypergeometric Functions of the Fourth Kind.</u>

Here, we shall make use of the expressions (2.14) - (2.18) established in section 2.2 of the present chapter, for the fractional loss of light α_l^0 of zero order. To do so, let us first avail ourselves of a theorem by Bailey (1936), asserting that

$$\int_0^\infty t^{\rho-1} J_K(\alpha t)\, J_\lambda(\beta t)\, J_\mu(\gamma t)\, dt = \frac{2^{\rho-1}\alpha^K \beta^\lambda \Gamma\left(\frac{K+\lambda+\mu+\rho}{2}\right)}{\gamma^{K+\lambda+\rho}\Gamma(K+1)\Gamma(\lambda+1)\Gamma\left(1-\frac{K+\lambda-\mu+\rho}{2}\right)} \times$$

$$\cdot F_4\left(\frac{K+\lambda-\mu+\rho}{2}, \frac{K+\lambda+\mu+\rho}{2}; K+1, \lambda+1; \frac{\alpha^2}{\gamma^2}, \frac{\beta^2}{\gamma^2}\right), \tag{4.1}$$

$$\rho+K+\lambda+\mu > 0\ , \quad \rho < \frac{5}{2}\ , \quad \alpha, \beta, \gamma > 0 \quad \text{and} \quad \gamma \geqslant \alpha+\beta\ ,$$

where F_4 stands for the Appell hypergeometric function of the fourth kind. For the outside eclipse phases when $\delta \geqslant r_1 + r_2$, it is easily seen from the combination of one of the equations (2.14) - (2.18) and (4.1) that outside eclipses α_l^0 becomes zero, since in this case

$$\Gamma\left(1-\frac{K+\lambda-\mu+\rho}{2}\right) = \Gamma(0) = \infty\ . \tag{4.2}$$

If we consider now the total eclipses ($\delta_0 \leqslant \delta \leqslant r_2 - r_1$) the conditions attached to (4.1) will be satisfied if $K \equiv \nu$, $\lambda \equiv 0$, $\mu \equiv 1$ and $\rho \equiv 1-\nu$, and the combination discloses that, for total eclipses α_l^0 becomes $1/\nu$ as known value from the geometrical approach. Should, however, the eclipse become annular ($\delta_0 < \delta < r_1 - r_2$) which conforms again to the necessary conditions for the application of (4.1) if $K \equiv 1$, $\lambda \equiv 0$, $\mu \equiv \nu$ and $\rho \equiv 1-\nu$ and the outcome of the respective combination for annular eclipses can be given as

$$\alpha_l^0 = \frac{r_2^2}{r_1^2}\, F_4\left(1-\nu, 1; 2, 1; \frac{r_2^2}{r_1^2}, \frac{\delta^2}{r_1^2}\right), \tag{4.3}$$

which reduces to $r_2^{\,2}/r_1^{\,2}$ only for uniformly bright discs ($\nu = 1$), and for even values of l it becomes a polynomial. If the inclination i is 90°, at the moment of maximum eclipse Eq. (4.3) becomes

$$\alpha_i^o = \frac{r_2^2}{r_1^2}\, {}_2F_1\left(\begin{matrix}1-\nu,\ 1\\ 2\end{matrix}\,\middle|\,\frac{r_2^2}{r_1^2}\right) = \frac{1}{\nu}\left[1-\left(1-\frac{r_2^2}{r_1^2}\right)^{\nu}\right] \tag{4.4}$$

in agreement with Eq. (3.9) of Paper III. Similarly, at the moment of internal tangency ($\delta = \delta_2 = r_1 - r_2$) Eq. (4.3) reduces to the ordinary hypergeometric series ${}_2F_1$ as

$$\alpha_i^o = \frac{r_2^2}{r_1^2}\,\frac{(2\nu)!}{\Gamma(\nu+1)\Gamma(\nu+2)}\left(\frac{r_2}{r_1}\right)^{\nu+1} {}_2F_1\left(\begin{matrix}1-\nu,\ \nu+1\\ \nu+2\end{matrix}\,\middle|\,\frac{r_2}{r_1}\right). \tag{4.5}$$

If, however, the eclipse becomes partial, Eq. (4.1) can no longer be applicable for the evaluation of the integrals (2.14) - (2.18). We shall, in what follows, extend the above results given in Paper XI by Kopal to cover all types of eclipses by introducing an expression for the Bessel functions of first kind (cf., e.g., Luke, 1969, Vol. II, p.49, Eq. 6) of the form

$$x J_0(x) = \sqrt{\frac{2}{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n (4n+1)\Gamma(n+\frac{1}{2})}{n!} J_{4n+1}(\sqrt{2}\, x). \tag{4.6}$$

If we now choose, for example, (2.16) for the fractional loss of light α_i^o to evaluate with the aid of (4.6), it can be written by setting $x = cy$ in (4.6) that

$$\alpha_i^o = \sqrt{\frac{2}{\pi}}\, 2^{\nu}\Gamma(\nu)\frac{b}{a^{\nu}c} \sum_{n=0}^{\infty} \frac{(-1)^n (4n+1)\Gamma(n+\frac{1}{2})}{n!} \times$$

$$\times \int_0^{\infty} y^{-\nu-1} J_{\nu}(ay) J_1(by) J_{4n+1}(\sqrt{2}\, cy)\, dy, \tag{4.7}$$

where we set $b = 1 - a$. It may be noted that the r.h.s. of (4.7) will be infinity for $c = 0$ which means this equation will no longer be applicable when the inclination i is nearly 90°, but it may be highly convergent for partial eclipses. If we resort to the Bailey's theorem given by (4.1) for the evaluation of the integral on the r.h.s. of (4.7) by setting $K \equiv \nu$, $\lambda \equiv 1$, $\mu = 4n+1$ and accordingly $\alpha \equiv a$, $\beta = b$ and $\gamma = \sqrt{2}c$, we get

$$\alpha_i^{o} = \frac{1}{2\nu\sqrt{\pi}}\left(\frac{b}{c}\right)^2 \sum_{n=0}^{\infty} \frac{(-1)^n(4n+1)\,\Gamma(n+\frac{1}{2})}{n!} \times$$

$$\times F_4\left(-2n, 2n+1; \nu+1, 2; \frac{a^2}{2c^2}, \frac{b^2}{2c^2}\right), \quad 2c^2 \geqq 1, \tag{4.8}$$

as the desired expression valid whenever $2c^2 \geqq 1$, i.e., $\cos i \geqq 0.71\,(r_1 + r_2)$. Note that the above Appell function F_4 is identified as a polynomial since first term of it is a negative integer and the coefficient of this polynomial does not depend on the degree l of the limb darkening.

Next, we shall derive three other expressions in terms of the F_4-functions for α_i^{o} alternative to (4.8). For the first expression let us resort to a Neumann-type expansion for Bessel functions (cf. Erdélyi et al, 1953, Vol. II, p.99, Eq. 3) in the form

$$J_{\mu}(x \sin\theta) = \sqrt{\frac{2}{\pi x}}\, \sin^{\mu}\theta\, \Gamma(\mu+\tfrac{1}{2}) \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{1}{2})(\mu+2n+\frac{1}{2})}{\Gamma(n+\mu+1)} \times$$

$$\times C_{2n}^{(\mu+\frac{1}{2})}(\cos\theta)\, J_{\mu+2n+\frac{1}{2}}(x), \tag{4.9}$$

where the $C_{2n}^{(\alpha)}$'s denote the ultraspherical polynomials of even orders. For $\mu = 0$ and by setting $\sin\theta \equiv c$, Eq. (4.9) will take the form

$$J_0(cx) = \sqrt{\frac{2}{x}} \sum_{n=0}^{\infty} (2n+\tfrac{1}{2}) \frac{\Gamma(n+\frac{1}{2})}{n!} P_{2n}(\sqrt{1-c^2})\, J_{2n+\frac{1}{2}}(x), \tag{4.10}$$

where the P_{2n} 's denote the Legendre polynomials of the argument $\sqrt{1-c^2}$. A combination of Equations (2.16) and (4.10) with the aid of the Bailey's formula (4.1) yields that

$$\alpha_l^0 = \frac{b^2}{\nu} \sum_{n=0}^{\infty} (2n+\tfrac{1}{2}) P_{2n}(\sqrt{1-c^2})\, F_4(-n+\tfrac{1}{2}, n+1; \nu+1, 2; a^2, b^2), \tag{4.11}$$

valid for any type of eclipse and any degree l of the limb darkening. The coefficients of the F_4 series are independent of limb darkening, but at this time F_4 which is the function of only variable a is not a polynomial.

The second expression for α_l^0 in terms of the F_4 series, alternative to (4.8) can be obtained by a resort to another Neumann-type expansion for the Bessel functions J_ν (cf., e.g., Erdélyi et al, 1953, Vol. II, p.64, Eq. 6) of the form

$$\frac{J_\nu(\alpha x)}{(\alpha x)^{\nu-\mu}} = \frac{\alpha^\mu}{2^{\nu-\mu}\Gamma(\nu+1)} \sum_{n=0}^{\infty} \frac{\Gamma(n+\mu)(2n+\mu)}{n!}\, {}_2F_1\left(\begin{matrix}-n, n+\mu \\ \nu+1\end{matrix}\middle| \alpha^2\right) J_{\mu+2n}(x), \tag{4.12}$$

$$\nu, \mu, \nu-\mu \neq 0, -1, -2, -3, \cdots$$

which for positive integral values of ($\nu-\mu$) transforms the Bessel functions of integral or fractional order ν into Bessel functions of order ($\mu + 2n$). However, it has been found that for $\mu = 0$ and the positive (including zero) values of real ν expansion given by (4.12) takes the form

$$\frac{J_\nu(\alpha x)}{(\alpha x)^\nu} = \frac{1}{2^\nu \Gamma(\nu+1)} \sum_{n=0}^{\infty} \epsilon_n \, {}_2F_1\left(\begin{matrix}-n, n\\ \nu+1\end{matrix}\middle| \alpha^2\right) J_{2n}(x), \tag{4.13}$$

where ϵ_n denotes the Neumann's number: $\epsilon_0 = 1$ and $\epsilon_n = 2$ for $n > 0$. For the Bessel functions of the zero order, (4.13) yields, if we replace α by c that

$$J_0(cx) = \sum_{n=0}^{\infty} \epsilon_n \, {}_2F_1\left(\begin{matrix}-n, n\\ 1\end{matrix}\middle| c^2\right) J_{2n}(x). \tag{4.14}$$

On inserting this result on the r.h.s. of Eq. (3.16), we find the latter to assume the form

$$\alpha_l^0 = \frac{2^\nu \Gamma(\nu) b}{a^\nu} \sum_{n=0}^{\infty} \epsilon_n \, {}_2F_1\left(\begin{matrix}-n, n\\ 1\end{matrix}\middle| c^2\right) \int_0^{\infty} y^{-\nu} J_\nu(ay) J_1(by) J_{2n}(y)\, dy. \tag{4.15}$$

Since $a + b = 1$, Bailey's formula given by (4.1) can help once more to evaluate the integral on the r.h.s. of Eq. (4.15). The outcome discloses that

$$\alpha_l^0 = \frac{b^2}{\nu} \sum_{n=0}^{\infty} \epsilon_n \, n \, {}_2F_1\left(\begin{matrix}-n, n\\ 1\end{matrix}\middle| c^2\right) F_4(1-n, 1+n; 2, \nu+1; b^2, a^2) \tag{4.16}$$

as the general expansion for the loss of light α_l^0 of zero order in terms of the products of two hypergeometric functions ${}_2F_1$ and F_4 both reduce to polynomials in this expansion and thus, can be easily evaluated. Moreover, let us note that the above expansion which converges under all circumstances is valid for any type of eclipse, occultation or transit, regardless of whether $r_1 \lessgtr r_2$ and for any degree l of the law of limb-darkening which does not occur in the first factor ${}_2F_1$. The variable parameters a and c of α_l^o occur in different constituent factors ${}_2F_1$ and F_4 of terms in the form ${}_2F_1(c)$ $F_4(a)$ which makes Eq. (4.16) more convenient for automatic computation.

For the third alternative expansion of α_l^o in series of F_4-functions use will be made of a formula due to Bailey (1935) of the form

$$\left(\tfrac{1}{2}x\right)^{K-\mu-\nu} J_\mu(ax)\, J_\nu(bx) = \frac{a^\mu b^\nu}{\Gamma(\mu+1)\Gamma(\nu+1)} \sum_{n=0}^{\infty} \frac{(K+2n)\,\Gamma(n+K)}{n!} \times$$

$$\times\, F_4\left(-n,\, n+K;\, \mu+1,\, \nu+1;\; a^2,\, b^2\right)\, J_{K+2n}(x)\,, \tag{4.17}$$

$$0 \leqslant a, b \leqslant 1 \quad , \quad a + b = 1\,.$$

With the aid of this formula we can write that

$$x^{-\nu} J_\nu(ax)\, J_1(bx) = \frac{2^{K-1} a^\nu b}{\Gamma(\nu+1)} \sum_{n=0}^{\infty} \frac{(K+2n)\,\Gamma(n+K)}{n!} \times$$

$$\times\, F_4\left(-n,\, n+K;\, 2,\, \nu+1;\; b^2,\, a^2\right) \frac{J_{2n+K}(x)}{x^{1-K}}\,. \tag{4.18}$$

And, moreover, let us consider (cf., e.g., Watson, 1952, p.439) that

$$\int_0^\infty J_\nu(\alpha t)\, J_\mu(\beta t)\, \frac{dt}{t^\lambda} = \frac{\alpha^\nu\, \Gamma\left(\frac{\nu+\mu-\lambda+1}{2}\right)}{2^\lambda \beta^{\nu-\lambda+1}\, \Gamma(\nu+1)\, \Gamma\left(\frac{-\nu+\mu+\lambda+1}{2}\right)} \times$$

$$\times\, {}_2F_1\left(\begin{matrix} \frac{\nu+\mu+\lambda+1}{2}, & \frac{\nu-\mu-\lambda+1}{2} \\ & \nu+1 \end{matrix} \;\middle|\; \frac{\alpha^2}{\beta^2}\right), \tag{4.19}$$

$$\nu+\mu-\lambda+1 > 0\,, \quad \lambda > -1 \quad \text{and} \quad 0 < \alpha < \beta\,.$$

Now, by substituting (4.17) in (2.16) and evaluating the remaining integral with the aid of (4.19) we are left with

$$\alpha_l^0 = \frac{2^{\nu+2K-2}\, b^2}{\nu} \sum_{n=0}^{\infty} \frac{(2n+K)\left[\Gamma(n+K)\right]^2}{(n!)^2} \times$$
$$\times\ {}_2F_1\left(\begin{matrix}-n,\ n+K\\ 1\end{matrix}\middle|\, c^2\right) \times F_4\left(-n,\ n+K;\ 2,\ \nu+1;\ b^2,\ a^2\right), \tag{4.20}$$

where K is a real number provided that $K \leq 2$. This general expansion has all the advantages of (4.8), (4.11) and (4.16), and moreover, the free parameter K in it can be used conveniently in the question of the speed of convergence of the expansion.

1.5 Expansions for α_l^0 in Fourier Cosine Series.

Let us first consider a summation theorem for the Bessel function J_0 (cf., e.g., Magnus and Oberhettinger, 1948, p.31) in the form

$$J_0(x \sin\theta) = \sum_{n=0}^{\infty} \epsilon_n\, J_n^2(\tfrac{1}{2}x)\, \cos 2n\theta\ , \tag{5.1}$$

where ϵ_n is the Neumann's number as it is given in the previous section. By substituting (5.1) into (2.16) after setting c as sine θ in (5.1) we are left simply with

$$\alpha_l^0 = 2^{\nu}\,\Gamma(\nu)\, b \sum_{n=0}^{\infty} \epsilon_n\, K_n^{(l)}\, \cos 2n\theta\ , \tag{5.2}$$

representing an expansion of the fractional loss of light α_l^0 in the Fourier cosine series, where we have abbreviated

$$\theta = \sin^{-1} c\ , \tag{5.3}$$

with $c = \delta/(r_1 + r_2)$ as in (2.16), and following from the combination of (2.16) and (5.1) to yield (5.2) the coefficients $K_n^{(l)}$ of the respective Fourier expansion can be given by

$$K_n^{(l)} = \int_0^\infty \frac{J_\nu(ax)}{(ax)^\nu} J_1(bx) J_n^2(\tfrac{1}{2}x)\, dx , \tag{5.4}$$

being functions of a only ($a = r_1/(r_1 + r_2)$). Our next task is to evaluate the above integral (5.4). For this, use will be made of an equation due to Bailey (1935) of the form

$$\begin{aligned} (\tfrac{1}{2}x)^{\frac{1}{2}} J_\mu(x\cos^2\phi)\, J_\nu(x\sin^2\phi) &= \frac{\cos^{2\mu}\phi\, \sin^{2\nu}\phi}{\Gamma(\nu+1)} \times \\ &\times \sum_{n=0}^{\infty} \frac{(-1)^n(\mu+\nu+2n+\frac{1}{2})\,\Gamma(\nu+n+\frac{1}{2})\,\Gamma(\mu+\nu+n+\frac{1}{2})}{n!\;\Gamma(\nu+\frac{1}{2})\,\Gamma(\mu+n+1)} \times \\ &\times {}_2F_1\left(\begin{matrix}-2n,\ 2\mu+2\nu+2n+1\\ 2\nu+1\end{matrix}\,\middle|\,\sin^2\phi\right) J_{\mu+\nu+2n+\frac{1}{2}}(x) . \end{aligned} \tag{5.5}$$

By replacing 1, ν, b and a as μ, ν, $\cos^2\phi$ and $\sin^2\phi$, respectively in Eq. (5.5), we get

$$\begin{aligned} J_\nu(ax)\, J_1(bx) &= \frac{a^\nu b}{(\frac{1}{2}x)^{\frac{1}{2}}\,\Gamma(\nu+1)} \sum_{m=0}^{\infty} (-1)^m \frac{(\nu+2m+\frac{3}{2})\,\Gamma(\nu+m+\frac{1}{2})\,\Gamma(\nu+m+\frac{3}{2})}{m!\,(m+1)!\,\Gamma(\nu+\frac{1}{2})} \times \\ &\times {}_2F_1\left(\begin{matrix}-2m,\ 2\nu+2m+3\\ 2\nu+1\end{matrix}\,\middle|\, a\right) J_{\nu+2m+\frac{3}{2}}(x) \end{aligned} \tag{5.6}$$

as the product of the first two Bessel functions on the r.h.s. of Eq. (5.4) for the coefficients $K_n^{(l)}$. Putting together these two results we have

$$K_n^{(l)} = \frac{\sqrt{2}\; b}{\Gamma(\nu+1)\,\Gamma(\nu+\frac{1}{2})} \sum_{m=0}^{\infty} (-1)^m \frac{(\nu+2m+\frac{3}{2})\,\Gamma(\nu+m+\frac{1}{2})\,\Gamma(\nu+m+\frac{3}{2})}{m!\,(m+1)!} \times$$

$$\times\, {}_2F_1\left(\begin{matrix}-2m,\, 2\nu+2m+3\\ 2\nu+1\end{matrix}\,\middle|\, a\right) \int_0^{\infty} \frac{J_n(\frac{1}{2}x)}{x^{\nu+\frac{1}{2}}}\, J_n(\tfrac{1}{2}x)\, J_{\nu+2m+\frac{3}{2}}(x)\, dx\,. \tag{5.7}$$

The integral on the r.h.s. of (5.7) does not contain the parameters a and c any more and can be easily evaluated with the aid of Bailey's formula given by (4.1), since $\gamma = 1 \geqslant (\alpha = \frac{1}{2}) + (\beta = \frac{1}{2})$. Consequently we have

$$K_n^{(l)} = \frac{b}{2^{2n+\nu}(n!)^2\,\Gamma(\nu+1)\,\Gamma(\nu+\frac{1}{2})} \sum_{m=0}^{\infty} (-1)^m \frac{(n+m)!\,(\nu+2m+\frac{3}{2})}{m!\,(m+1)!} \times$$

$$\times\, \frac{\Gamma(\nu+m+\frac{1}{2})\,\Gamma(\nu+m+\frac{3}{2})}{\Gamma(m+\nu+\frac{3}{2}-n)}\; {}_2F_1\left(\begin{matrix}-2m,\, 2\nu+2m+3\\ 2\nu+1\end{matrix}\,\middle|\, a\right) \times$$

$$\times\, F_4\left(n-m-\nu-\tfrac{1}{2},\; m+n+1;\; n+1,\; n+1;\; \tfrac{1}{4},\, \tfrac{1}{4}\right). \tag{5.8}$$

Alternatively, we can resort to another expression – instead of (5.5) – due to Bateman (1905) of the form

$$J_\mu(ax)\,J_\nu(bx) = \frac{2a\,b^\nu}{x} \sum_{n=0}^{\infty} (-1)^n (\mu+2n+\nu+1) \frac{\Gamma(\mu+\nu+n+1)\,\Gamma(\nu+n+1)}{n!\,\Gamma(\mu+n+1)\,[\Gamma(\nu+1)]^2} \times$$

$$\times \left[{}_2F_1\left(\begin{matrix}-n,\, \mu+\nu+n+1\\ \nu+1\end{matrix}\,\middle|\, b\right)\right]^2 J_{\mu+\nu+2n+1}(x)\,, \tag{5.9}$$

$$0 \leq a, b \leq 1\,, \quad a+b=1\,,$$

which permits us, for $\mu \equiv 1$ and $\nu \equiv \nu$, to write that

$$J_\nu(ax)\, J_1(bx) = \frac{2\, a^\nu b}{x} \sum_{m=0}^{\infty} (-1)^m \frac{(\nu+2m+2)\, \Gamma(\nu+m+1)\, \Gamma(\nu+m+2)}{m!\,(m+1)!\, [\Gamma(\nu+1)]^2} \times$$

$$\times \left[{}_2F_1\left({-m,\, m+\nu+2 \atop \nu+1} \middle| a \right) \right]^2 \times J_{\nu+2m+2}(x) . \tag{5.10}$$

It can be found by following the same way that Eq. (5.8) for the coefficients $K_n^{(l)}$ of the Fourier cosine series (5.2) of α_l^o takes slightly different form as

$$K_n^{(l)} = \frac{b}{2^{2n+\nu}(n!)^2[\Gamma(\nu+1)]^2} \sum_{m=0}^{\infty} (-1)^m \frac{(n+m)!\,(\nu+2m+2)\,\Gamma(\nu+m+1)\,\Gamma(\nu+m+2)}{m!\,(m+1)!\,\Gamma(\nu+m+2-n)} \times$$

$$\times \left[{}_2F_1\left({-m,\, m+\nu+2 \atop \nu+1} \middle| a \right) \right]^2 \times F_4\left(n-\nu-m-1,\, n+m+1;\, n+1,\, n+1;\, \tfrac{1}{4},\, \tfrac{1}{4} \right) . \tag{5.11}$$

Note that the terms of infinite summation on the r.h.s. of (5.8) for $K_n^{(l)}$ vanishes if $n \geq m+\nu+\frac{3}{2}$ for odd values of l ; while the similar ~~series~~ terms on the r.h.s. of (5.11) for the same coefficients $K_n^{(l)}$ vanishes if $n \geq \nu + m + 2$ for even values of l .

Next it will be shown that the fractional loss of light α_l^o assumes another form of expansion in Fourier cosine series. In order to illustrate this here we shall introduce another summation formula (cf. e.g., Watson, 1945, p.358) for the Bessel function J_o , given by

$$J_o(Rx) = \sum_{n=0}^{\infty} \epsilon_n\, J_n(rx)\, J_n(\rho x) \cos n\vartheta , \tag{5.12}$$

valid for all positive values of R , r and ρ if they are the sides of a triangle such that the angle between the sides r and ρ is equal to θ, i.e.,

$$R^2 = r^2 + \rho^2 - 2r\rho \cos\theta . \tag{5.13}$$

On insertion of the above expansion for $R \equiv c$, $r \equiv a$ and $\rho \equiv b$ in the r.h.s. of Eq. (2.16) we can rewrite the latter in the form of a Fourier cosine series as

$$\alpha_l^0 = 2^\nu \Gamma(\nu)\, b \sum_{n=0}^{\infty} \epsilon_n O_n^{(l)} \cos n\theta , \tag{5.14}$$

where it follows from (5.13) that

$$\theta = \cos^{-1} \frac{a^2 + b^2 - c^2}{2ab} , \tag{5.15}$$

and we have abbreviated the coefficients of the above Fourier expansion

$$O_n^{(l)} = \int_0^\infty \frac{J_\nu(ax)}{(ax)^\nu} J_1(bx)\, J_n(ax)\, J_n(bx)\, dx \tag{5.16}$$

being a function of $a = r_1/(r_1 + r_2)$ only so that it may be given in one column tables just as the $K_n^{(l)}$'s . For the evaluation of these integrals if we utilize Eq. (5.9) and the Bailey's theorem given by (4.1) in the same way as before we get

$$O_n^{(l)} = \frac{a\, b^{\nu+1}}{2^\nu \Gamma(\nu+1)} \sum_{m=0}^{\infty} (-1)^m \frac{(2n+2m+1)\, \Gamma(2n+m+1)}{m!\, (n!)^2} \times$$

$$\times \left[{}_2F_1\left(\begin{matrix} -m,\ 2n+m+1 \\ n+1 \end{matrix} \middle| b \right) \right]^2 \times F_4\left(-n-m, n+m+1; \nu+1, 2; a^2, b^2\right) , \tag{5.17}$$

in which both hypergeometric functions ${}_2F_1$ and F_4 turn out to be polynomials and, moreover, the ${}_2F_1$ is independent of limb-darkening.

Note that the Fourier expansions (5.2) and (5.14) of the fractional loss of light α_l^o can also be easily identified as the expansions in series of the Chebyshev polynomials

$$T_n(\cos\varphi) = \cos n\varphi \tag{5.18}$$

and the outcomes will be

$$\alpha_l^o = 2^\nu \Gamma(\nu)\, b \sum_{n=0}^{\infty} \epsilon_n K_n^{(l)} T_n^*(1-c^2), \tag{5.19}$$

and

$$\alpha_l^o = 2^\nu \Gamma(\nu)\, b \sum_{n=0}^{\infty} \epsilon_n O_n^{(l)} T_n^*\left(\frac{1-c^2}{4ab}\right), \tag{5.20}$$

respectively, with the respective coefficients $K_n^{(l)}$ and $O_n^{(l)}$ given by (5.8), (5.11) and (5.17). In (5.19) and (5.20) we employed the shifted Chebyshev polynomials

$$T_n^*(x) = T_n(2x-1) = T_{2n}(\sqrt{x}) = {}_2F_1\left(\begin{matrix}-n,\ n\\ \frac{1}{2}\end{matrix}\,\middle|\, 1-x\right) \tag{5.21}$$

since it is customary in our problem to work with the normalized arguments varying between zero and one.

1.6 Expansions for α_l^o in Series of Jacobi Polynomials.

To derive the expansions of the loss of light α_l^o, utilizing the integral form (2.16), in series of Jacobi polynomials we shall start with writing a formula (cf., e.g., Watson, 1952, p.413, Eq. 7) in the form

$$\int_0^\infty J_\mu(\alpha x)\, J_\nu(\beta x)\, J_\rho(\gamma x)\, \frac{dx}{x^{\lambda-1}} = \frac{\alpha^\mu \beta^\nu \gamma^\rho}{2^{\lambda-1}\,\Gamma(\rho+1)\,[\Gamma(\nu+1)]^2} \times$$

$$\times \sum_{n=0}^{\infty} (-1)^n \frac{(\mu+\nu+2n+1)\,\Gamma(\mu+\nu+n+1)\,\Gamma(\nu+n+1)\,\Gamma\left(\frac{\mu+\nu+\rho-\lambda}{2}+n+1\right)}{n!\,\Gamma(\mu+n+1)\,\Gamma\left(\frac{\mu+\nu-\rho+\lambda}{2}+n+1\right)} \times$$

$$\times {}_2F_1\left(\begin{matrix}\frac{\mu+\nu+\rho-\lambda}{2}+n+1, & \frac{\rho-\lambda-\mu-\nu}{2}-n \\ \rho+1 & \end{matrix}\middle| \gamma^2\right) \times \left[{}_2F_1\left(\begin{matrix}-n, \mu+\nu+n+1 \\ \nu+1\end{matrix}\middle| \beta\right)\right]^2, \tag{6.1}$$

$$\mu+\nu+\rho+2 \geqslant \lambda \geqslant -\tfrac{1}{2}, \quad 0 \leqslant \alpha, \beta \leqslant 1, \quad -1 \leqslant \gamma \leqslant 1, \quad \alpha+\beta = 1.$$

This formula can be applied to evaluate the integral (2.16) for α_l^o and the outcome, by setting $\mu = \nu$, $\nu = 1$, $\rho = 0$ and $\lambda = \nu+1$ and accordingly α = a, β = b and γ = c in (6.1), discloses that

$$\alpha_L^o = b^2\,\Gamma(\nu) \sum_{n=0}^{\infty} (-1)^n \frac{(\nu+2n+2)\,(n+1)!}{\Gamma(\nu+n+1)} \times$$

$$\times \left[{}_2F_1\left(\begin{matrix}-n, \nu+n+2 \\ 2\end{matrix}\middle| b\right)\right]^2 \times {}_2F_1\left(\begin{matrix}n+1, -n-\nu-1 \\ 1\end{matrix}\middle| c^2\right), \tag{6.2}$$

or, alternatively, by setting $\mu = 1$, $\nu = \nu$, $\rho = 0$, $\lambda = \nu+1$, $\alpha = b$, $\beta = a$ and $\gamma = c$ in (6.1) we get

$$\alpha_l^o = b^2 \sum_{n=0}^{\infty} (-1)^n \frac{(\nu+2n+2)\,\Gamma(\nu+n+1)}{\nu\,(n+1)!\;\Gamma(\nu+1)} \times$$

$$\times \left[{}_2F_1\left({-n,\; n+\nu+2 \atop \nu+1} \,\middle|\, a \right) \right]^2 \times {}_2F_1\left({-\nu-n-1,\; n+1 \atop 1} \,\middle|\, c^2 \right). \tag{6.3}$$

The last expansion turns out to be identical with the formula given by Eq. (2.34) of Paper XII. Next, following the same way as Kopal did in Paper XII, Eq. (6.2) and Eq. (6.3) can be rewritten in the simple forms in terms of Jacobi polynomials. To do so let us first introduce shifted Jacobi polynomials in hypergeometric form as

$$R_n^{(\alpha,\beta)}(x) = (-1)^n \frac{\Gamma(\beta+n+1)}{n!\,\Gamma(\beta+1)}\; {}_2F_1\left({-n,\; n+\lambda \atop \beta+1} \,\middle|\, x \right), \tag{6.4}$$

where $\lambda = \alpha+\beta+1$ and it is required that $0 \leqslant x \leqslant 1$. If, moreover, we consider that

$${}_2F_1\left({a,\; b \atop c} \,\middle|\, x \right) = (1-c)^{c-a-b}\; {}_2F_1\left({c-a,\; c-b \atop c} \,\middle|\, x \right), \tag{6.5}$$

then, Equations (6.2) and (6.3) take the following simple form

$$\alpha_l^o = b^2 (1-c^2)^{\nu+1}\,\Gamma(\nu) \sum_{n=0}^{\infty} \frac{n!\,(\nu+2n+2)}{(n+1)\,\Gamma(\nu+n+1)} \left[R_n^{(1,\nu)}(a) \right]^2 \times R_n^{(\nu+1,\,0)}(c^2), \tag{6.6}$$

which is valid for any type of eclipse and for any degree l of the law of limb-darkening. In this general expansion we used shifted Jacobi polynomials for the same reason as noted before that variation intervals of the parameters a and c which are now the arguments of the above polynomials are (0, 1) but not (−1, 1) as required in general. Eq. (6.6) for α_l^o can then be easily automated with the aid of known certain recursion formulae for shifted Jacobi polynomials given by

$$R_{n+1}(x) = (A_n x + B_n) R_n(x) - C_n R_{n-1}(x) \tag{6.7}$$

with the coefficients

$$\begin{aligned} A_n &= \frac{(2n+\lambda)(2n+\lambda+1)}{(n+1)(n+\lambda)}, \\ B_n &= \frac{(2n+\lambda)(\alpha^2-\beta^2+1) - (2n+\lambda)^3}{2(n+1)(n+\lambda)(2n+\lambda-1)}, \end{aligned} \tag{6.8}$$

and

$$C_n = \frac{(n+\alpha)(n+\beta)(2n+\lambda+1)}{(n+1)(n+\lambda)(2n+\lambda-1)}$$

where $\lambda = \alpha+\beta+1$ as given in definition (6.4). First a few values of R_n's for $n = 0, 1, 2, \ldots$ can be easily obtained from definition (6.4) as

$$\left.\begin{aligned} R_0^{(\alpha,\beta)}(x) &= 1 \\ R_1^{(\alpha,\beta)}(x) &= -(\beta+1) + (\lambda+1)\, x \\ R_2^{(\alpha,\beta)}(x) &= \frac{(\beta+1)(\beta+2)}{2} - (\beta+2)(\lambda+2)\, x + \frac{(\lambda+2)(\lambda+3)}{2}\, x^2 \end{aligned}\right\} \tag{6.9}$$

$$\vdots$$

etc. Finally, it should be noted that the numerical value of the coefficient in expansion (6.6) rapidly increases with increasing number of terms as a disadvantage in practice in spite of the simple structure of this expansion for α_l^0.

We shall in what follows present a number of alternative expressions (see Paper XIII) to (6.6) for the fractional loss of light α_l^0. In order to demonstrate how those expressions are derived let us first re-resort to Bailey's formula given by (5.5) for the expansion of the product of the first two Bessel functions on the r.h.s. of (2.16) as it is given by Eq. (5.6). On insertion of this result in integral (2.16) for α_l°, we find the latter to assume the form

$$\alpha_l^\circ = \frac{2^{\nu+\frac{3}{2}}}{\sqrt{\pi}}\,\Gamma(\nu)\, b \sum_{n=0}^{\infty} (-1)^n \frac{\left(\nu+2n+\frac{3}{2}\right)\Gamma\left(n+\frac{3}{2}\right)\Gamma\left(\nu+n+\frac{3}{2}\right)}{\Gamma(\nu+n+1)} \times$$

$$\times\, {}_2F_1\left(\begin{matrix}-2n,\ 2n+2\nu+3\\ 3\end{matrix}\,\middle|\, b\right) \int_0^\infty y^{-\nu-\frac{1}{2}} J_{\nu+2n+\frac{3}{2}}(x)\, J_0(cx)\, dx\,. \tag{6.10}$$

The integral on the r.h.s. of (6.10) can be evaluated with the aid of Eq. (4.19) by setting $\nu \equiv \nu+2n+\frac{3}{2}$, $\mu \equiv 0$, $\lambda \equiv \nu+\frac{1}{2}$ and accordingly $\alpha \equiv 1$ and $\beta \equiv c$ and the outcome can be given as

$$\int_0^\infty y^{-\nu-\frac{1}{2}} J_{2n+\nu+\frac{3}{2}}(x)\, J_0(cx)\, dx = \frac{n!}{2^{\nu+\frac{1}{2}}\,\Gamma\left(\nu+n+\frac{3}{2}\right)}\, {}_2F_1\left(\begin{matrix}n+1,\ -\nu-n-\frac{1}{2}\\ 1\end{matrix}\,\middle|\, c^2\right) \tag{6.11}$$

Putting together the results (6.10) and (6.11) and making use of the transformation formula (6.5) for ${}_2F_1$-functions we have

$$\alpha_l^\circ = \frac{2\,\Gamma(\nu)\, b^2 (1-c^2)^{\nu+\frac{1}{2}}}{\sqrt{\pi}} \sum_{n=0}^{\infty} (-1)^n \frac{\left(\nu+2n+\frac{3}{2}\right)\Gamma\left(n+\frac{3}{2}\right)}{\Gamma(\nu+n+1)} \times$$

$$\times\, {}_2F_1\left(\begin{matrix}-2n,\ 2n+2\nu+3\\ 3\end{matrix}\,\middle|\, b\right) {}_2F_1\left(\begin{matrix}-n,\ \nu+n+\frac{3}{2}\\ 1\end{matrix}\,\middle|\, c^2\right), \tag{6.12}$$

or alternatively

$$\alpha_l^o = \frac{b^2 (1-c^2)^{\nu+\frac{1}{2}}}{\nu \Gamma(\nu+\frac{1}{2})} \sum_{n=0}^{\infty} (-1)^n \frac{(\nu+2n+\frac{3}{2}) \Gamma(\nu+n+\frac{1}{2})}{(n+1)!} \times$$

$$\times {}_2F_1\left({-2n,\ 2n+2\nu+3 \atop 2\nu+1} \middle| a \right) {}_2F_1\left({-n,\ n+\nu+\frac{3}{2} \atop 1} \middle| c^2 \right). \tag{6.13}$$

It can be easily shown that the ordinary hypergeometric functions ${}_2F_1$ on the r.h.sides of (6.12) and (6.13) can be identified as shifted Jacobi polynomials (see Eq. 6.4) and (6.12) and (6.13) take the forms

$$\alpha_l^o = \frac{4\,\Gamma(\nu)\, b^2}{\sqrt{\pi}} (1-c^2)^{\nu+\frac{1}{2}} \sum_{n=0}^{\infty} \frac{(\nu+2n+\frac{3}{2})\, \Gamma(n+\frac{3}{2})}{(2n+1)(2n+2)\, \Gamma(\nu+n+1)} \times$$

$$\times R_{2n}^{(2\nu,2)}(b) \cdot R_n^{(\nu+\frac{1}{2},0)}(c^2), \tag{6.14}$$

and

$$\alpha_l^o = \frac{\Gamma(2\nu+1)\, b^2}{\nu\, \Gamma(\nu+\frac{1}{2})} (1-c^2)^{\nu+\frac{1}{2}} \sum_{n=0}^{\infty} \frac{(2n)!}{(n+1)!} \frac{(\nu+2n+\frac{3}{2})\, \Gamma(\nu+n+\frac{1}{2})}{\Gamma(2n+2\nu+1)} \times$$

$$\times R_{2n}^{(2,2\nu)}(a) \cdot R_n^{(\nu+\frac{1}{2},0)}(c^2) \tag{6.15}$$

respectively. These algebraic expressions for the fractional loss of light α_l^o are also valid for any type of eclipse and arbitrary degree l of limb darkening. They can be easily automated with the aid of (6.7) - (6.9). It is seen that the numerical value of the coefficient of the shifted Jacobi polynomials in (6.14) decrease rapidly with increasing number of terms which should be an important advantage for (6.14) over the other formulae of its kind.

Further alternative expressions to (6.6) can be obtained. For the sake of illustration, let us consider combination of (2.16) and (4.10) with the aid of (5.6), (4.19) and (6.4) (cf. Paper XIII, Eq. 2.21 and 2.22) given by

$$\alpha_i^o = b^2 \frac{\Gamma(\nu)\Gamma(2\nu+1)}{\Gamma(\nu+\frac{1}{2})} \sum_{n=0}^{\infty} (-1)^n \frac{(2n+\frac{1}{2})\Gamma(n+\frac{1}{2})}{n!} R_n^{(-\frac{1}{2},0)}(c^2) \times$$

$$\times \sum_{m=0}^{\infty} \frac{(-1)^m (2m)!(n+m)!(\nu+2m+\frac{3}{2})}{m!(m+1)!\Gamma(2\nu+2m+1)\Gamma(\nu+m-n+\frac{3}{2})} \times$$

$$\times \frac{\Gamma(\nu+m+\frac{1}{2})\Gamma(\nu+m+\frac{3}{2})}{\Gamma(\nu+m+n+2)\Gamma(n-m+\frac{1}{2})} R_{2m}^{(2,2m)}(a) . \tag{6.16}$$

A combination of (2.16) and (4.14) with the aid of (5.6), (4.19) and (6.4) (cf. Paper XIII, Eq. 2.25) discloses to a similar expansion of α_i^o as in (6.16):

$$\alpha_i^o = b^2 \frac{\Gamma(2\nu+1)}{\nu} \sum_{n=0}^{\infty} \epsilon_n (-1)^n R_n^{(-1,0)}(c^2) \sum_{m=0}^{\infty} \frac{(-1)^m (2m)!(\nu+2m+\frac{3}{2})}{m!(m+1)!(n-m-1)!} \times$$

$$\times \frac{\Gamma(\nu+m+\frac{1}{2})\Gamma(\nu+m+\frac{3}{2})}{\Gamma(2\nu+2m+1)\Gamma(\nu+m+n+\frac{3}{2})\Gamma(\nu+m-n+\frac{3}{2})} \times R_{2m}^{(2,2\nu)}(a) . \tag{6.17}$$

Both expansions (6.16) and (6.17) become again algebraic but in the form of double summations of the product of two shifted Jacobi polynomials. As the advantage here, the first series on the r.h. sides of (6.16) and (6.17) are independent of limb darkening, and it can be observed that second series in (6.16) terminates whenever $m = n - \nu -$

$^3/_2$ for odd values of the degree l of the law of limb darkening. The second series in (6.17) also terminates in any case whenever $m = n - 1$.

1.7 Expansions for α_l^o in Series of Kopal's J-Integrals.

Here, it will be shown (cf. Paper XIII) that the fractional loss of light α_l^o, as well as Kopal's (1947) modified associated α-functions A_l^o can be expanded in rapidly converging series of simple combinations of Kopal's $\mathfrak{J}$-integrals which are defined (see Eq. 7.30) in terms of well-known and best studied J-integrals of eclipsing binaries. For J-integrals refer chiefly to Kopal (1947) and Lanzano (1976).

First, let us consider, for example, Eq. (2.15) for α_l^o, replacing x by kx for $\alpha = 1$ in (4.13) we get

$$\frac{J_\nu(kx)}{(kx)^\nu} = \frac{1}{2^\nu} \sum_{n=0}^{\infty} \frac{\epsilon_n \Gamma(\nu+1)}{\Gamma(\nu-n+1)\,\Gamma(\nu+n+1)} J_{2n}(kx) , \tag{7.1}$$

$$\nu \geq 0 , \quad \epsilon_0 = 1 \quad \text{and} \quad \epsilon_n = 2 \quad \text{for} \quad n > 0 ,$$

which can be utilized for the first Bessel function on the r.h.s. of (2.15). Thus, a combination of (2.15) and (7.1) discloses that

$$\alpha_l^o = \nu\left[\Gamma(\nu)\right]^2 \sum_{n=0}^{\infty} \frac{\epsilon_n}{\Gamma(\nu-n+1)\,\Gamma(\nu+n+1)} \cdot K_{2n} \tag{7.2}$$

where we have abbreviated

$$K_{2n} = \int_0^\infty J_{2n}(kx)\, J_1(x)\, J_0(hx)\, dx , \quad k = \frac{r_1}{r_2} \quad \text{and} \quad h = \frac{\delta}{r_2} , \tag{7.3}$$

which is a function of k and h but independent of the degree l of the law of limb darkening. Note that the infinite series on the r.h.s. of (7.2) terminates whenever $n = \nu + 1$ for integral values of ν so that for even integer l , the loss of light α_l^o can be expressible in closed form. On the other hand, the above integrals given by (7.3) turn out to be expressible in terms of the derivatives of α_l^o with respect to k which are simpler in evaluation. To prove this, let us first write the derivative of α_l^o from (2.15) as

$$k\frac{\partial \alpha_l^o}{\partial k} = -2^{\nu}\Gamma(\nu)\int_0^{\infty}\frac{J_{\nu+1}(kx)}{(kx)^{\nu-1}}\,J_1(x)\,J_0(hx)\,dx\,. \tag{7.4}$$

By a resort to Eq. (4.12) for $x \equiv kx$ and $\alpha \equiv 1$, the first Bessel function on the r.h.s. of (7.4) can be given as

$$\frac{J_{\nu+1}(kx)}{(kx)^{\nu-1}} = \frac{\Gamma(\nu)}{2^{\nu-2}}\sum_{n=0}^{\infty}\frac{(n+1)^2}{\Gamma(\nu-n)\,\Gamma(\nu+n+2)}\,J_{2n+2}(kx)\,, \quad \nu \geqslant 1\,. \tag{7.5}$$

Substituting this result in (7.4) we have

$$k\frac{\partial \alpha_l^o}{\partial k} = -4\left[\Gamma(\nu)\right]^2\sum_{n=0}^{\infty}\frac{(n+1)^2}{\Gamma(\nu-n)\,\Gamma(\nu+n+2)}\cdot K_{2n+2} \tag{7.6}$$

in terms of same integrals K_{2n} given by (7.3). Hence, by inverting this series, which also reduces to a sum of (ν) terms for positive integral values of ν , the integrals K_{2n} on the r.h.s. of (7.2) will be expressible in terms of the derivatives of α_l^o with respect to k . Furthermore, the above derivatives can be identified as Kopal's $\mathfrak{J}$-integrals, since (cf. Kopal, 1959; Chapter IV.5, Eq. 5.30)

$$2 J^{o}_{-1,l} = -r_1 \frac{\partial \alpha^{o}_{l}}{\partial r_1} = r_1 \frac{\partial \alpha^{1}_{l}}{\partial r_2} . \tag{7.7}$$

Thus, we can invert from Eq. (7.6) for even values of l that

$$\left.\begin{aligned}
K_2 &= J^{o}_{-1,0} \\
K_4 &= 3 J^{o}_{-1,2} - J^{o}_{-1,0} \\
K_6 &= 10 J^{o}_{-1,4} - 8 J^{o}_{-1,2} + J^{o}_{-1,0} \\
K_8 &= 35 J^{o}_{-1,6} - 45 J^{o}_{-1,4} + 15 J^{o}_{-1,2} - J^{o}_{-1,0} \\
K_{10} &= 126 J^{o}_{-1,8} - 224 J^{o}_{-1,6} + 126 J^{o}_{-1,4} - 24 J^{o}_{-1,2} + J^{o}_{-1,0} \\
&\vdots
\end{aligned}\right\} \tag{7.8}$$

etc. By making use of equation

$$\alpha^{o}_{o} = K_o + K_2 \tag{7.9}$$

which can be deduced from (7.2) for $l = 0$, the fractional loss of light α^{o}_{l} for some particular values of l can be given from expansion (7.2) as follows

$$\alpha^{o}_{-1} = 2\alpha^{o}_{o} - \frac{2}{3} K_2 - \frac{4}{15} K_4 + \frac{4}{35} K_6 - \frac{4}{63} K_8 + \frac{4}{99} K_{10} + \cdots \tag{7.10}$$

$$\alpha^{o}_{o} = \frac{1}{\pi}\left[\cos^{-1} s + \left(\frac{r_2}{r_1}\right)^2 \cos^{-1}\mu - \frac{\delta}{r_1}\sqrt{1-s^2}\right] \tag{7.11}$$

$$\alpha^{o}_{1} = \frac{2}{3}\alpha^{o}_{o} + \frac{2}{15} K_2 + \frac{4}{35} K_4 - \frac{4}{315} K_6 + \frac{4}{1155} K_8 + \cdots \tag{7.12}$$

$$\alpha_2^0 = \frac{1}{2}\alpha_0^0 + \frac{1}{6}K_2 + \frac{1}{6}K_4 \tag{7.13}$$

$$\alpha_3^0 = \frac{2}{5}\alpha_0^0 + \frac{6}{35}K_2 + \frac{4}{21}K_4 + \frac{4}{231}K_6 - \frac{4}{3003}K_8 + \cdots \tag{7.14}$$

$$\alpha_4^0 = \frac{1}{3}\alpha_0^0 + \frac{2}{3}K_2 + \frac{1}{5}K_4 + \frac{1}{30}K_6 \tag{7.15}$$

$$\alpha_5^0 = \frac{2}{7}\alpha_0^0 + \frac{10}{63}K_2 + \frac{20}{99}K_4 + \frac{20}{429}K_6 + \frac{4}{1287}K_8 + \cdots \tag{7.16}$$

$\vdots$

etc., where

$$s = \frac{r_1^2 - r_2^2 + \delta^2}{2\delta} \quad \text{and} \quad \mu = \frac{\delta - s}{r_2} = \frac{r_2^2 - r_1^2 + \delta^2}{2\delta\, r_2} \tag{7.17}$$

which are used in Eq. (7.11) for α_0^0 (cf. e.g., Kopal, 1959, Chapter IV.4, and many other sources). Note from equations (7.10) - (7.16) that the fractional loss of light α_l^0 for even indices such as α_0^0, α_2^0, α_4^0, etc. were all expressible in closed form, in general

$$\alpha_{2l}^0 = \frac{1}{l+1}\alpha_0^0 + A_{2l}^0 \tag{7.18}$$

where we have abbreviated

$$A_{2l}^0 = (l+1)\left[\Gamma(l+1)\right]^2 \sum_{n=1}^{l+1} \frac{2}{\Gamma(l+2-n)\,\Gamma(l+2+n)} K_{2n} - \frac{1}{(l+1)} K_2 . \tag{7.19}$$

For odd indices the α_l^0's remain to be infinite series form in the foregoing formulation. However, if we replace equations (7.1) and (7.5) by the forms

$$\frac{J_\nu(kx)}{(kx)^{\nu-\frac{1}{2}}} = \frac{\Gamma(\nu+\frac{1}{2})}{2^{\nu-\frac{1}{2}}} \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{1}{2})\,(2n+\frac{1}{2})}{n!\,\Gamma(\nu+\frac{1}{2}-n)\,\Gamma(\nu+n+1)}\, J_{2n+\frac{1}{2}}(kx)\,, \tag{7.20}$$

$\nu \geqslant \frac{1}{2}$,

and

$$\frac{J_{\nu+1}(kx)}{(kx)^{\nu-\frac{3}{2}}} = \frac{\Gamma(\nu-\frac{1}{2})}{2^{\nu-\frac{3}{2}}} \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{5}{2})\,(2n+\frac{5}{2})}{n!\,\Gamma(\nu-\frac{1}{2}-n)\,\Gamma(\nu+n+2)}\, J_{2n+\frac{5}{2}}(kx)\,, \tag{7.21}$$

$\nu \geqslant \frac{3}{2}$,

which can be deduced from (4.12) by setting $x \equiv kx$ and $\alpha \equiv 1$, then the fractional loss of light α_l° takes a different form

$$\alpha_l^\circ = \sqrt{2}\,\Gamma(\nu)\,\Gamma(\nu+\tfrac{1}{2}) \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{1}{2})\,(2n+\frac{1}{2})}{n!\,\Gamma(\nu+\frac{1}{2}-n)\,\Gamma(\nu+n+1)} \times M_{2n} \quad , \quad \nu \geqslant \tfrac{1}{2} \tag{7.22}$$

where we have abbreviated

$$M_{2n} = \int_0^\infty J_{2n+\frac{1}{2}}(kx)\, J_1(x)\, J_0(hx)\, \frac{dx}{(kx)^{\frac{1}{2}}}\,. \tag{7.23}$$

On the other hand, if we consider that (see Eq. 7.4 and 7.7)

$$\mathfrak{J}_{-1,l}^{\circ} = 2^{\nu-1}\,\Gamma(\nu) \int_0^\infty \frac{J_{\nu+1}(kx)}{(kx)^{\nu-1}}\, J_1(x)\, J_0(hx)\, dx\,. \tag{7.24}$$

a combination of (7.21) and (7.24) will become

$$\mathfrak{J}_{-1,l}^{\circ} = \sqrt{2}\,\Gamma(\nu)\,\Gamma(\nu-\tfrac{1}{2}) \sum_{n=0}^{\infty} \frac{(2n+\frac{5}{2})\,\Gamma(n+\frac{5}{2})}{n!\,\Gamma(\nu-n-\frac{1}{2})\,\Gamma(\nu+n+2)}\, M_{2n+2}\,, \quad \nu \geqslant \tfrac{3}{2}, \tag{7.25}$$

in terms of the same integrals as given by (7.23). The foregoing formulation of the problem enables us to rewrite equations (7.10) – (7.16) in terms of the M_{2n} –integrals given by (7.23) which are also expressible in terms of the $\mathfrak{J}$–integrals by inversion from (7.25). This time it can be observed that the fractional loss of light α_l^0 for odd values of l turns out to be expressible in closed form, i.e.,

$$\alpha_{2l-1}^0 = \frac{3}{2l-1}\,\alpha_1^0 + A_{2l-1}^0 \; , \tag{7.26}$$

where we have abbreviated

$$\begin{aligned} A_{2l-1}^0 = {} & \sqrt{2}\,\Gamma(l+1)\,\Gamma(l+\tfrac{1}{2}) \sum_{n=1}^{l} \frac{(2n+\frac{1}{2})\,\Gamma(n+\frac{1}{2})}{n!\,\Gamma(l+1-n)\,\Gamma(l+n+\frac{3}{2})}\, M_{2n} - \\ & - \frac{\sqrt{2\pi}}{2l-1}\, M_2 \; , \end{aligned} \tag{7.27}$$

since both equations (7.22) and (7.25) reduce to a closed form for positive odd values of l . Furthermore, it may be observed that the quantities A_l^0 given by (7.19) and (7.27) are identical with the Kopal's modified associated α-functions of zero order.

In the foregoing expressions the fractional loss of light α_l^0 has been established in a general form in terms of known associated series (terminating or infinite) which converge rapidly to the desired results for any type of eclipse and are easily programmable for automatic computations. The expansion given by (7.2) gives the fractional loss of light α_l^0 for any degree l of the law of limb darkening, in terms of Kopal's $\mathfrak{J}$-integrals and α_0^0 which can be easily evaluated with the aid of explicit formula (7.11). Thus, the whole complexity of α_l^0 is evidently stored in α_0^0 and the $\mathfrak{J}_{-1,2\gamma}^0$ -integrals if we use Eq. (7.2).

It may also be added that general formula given by (7.2) for α_l^0 can be further simplified for occultation type of eclipses. To illustrate this let us combine Eq. (2.15) with (4.13) for $\alpha \equiv k$, when we get

$$\alpha_1^o = \frac{1}{\nu} \sum_{n=0}^{\infty} \epsilon_n \, {}_2F_1\left(\begin{matrix}-n, n \\ \nu+1\end{matrix} \middle| k^2\right) Q_{2n}(h), \quad k \leqslant 1 \tag{7.28}$$

which reduces to (7.2) for $k = 1$ and diverges if $k > 1$, where we have abbreviated

$$Q_{2n}(h) = \int_0^{\infty} J_{2n}(x)\, J_1(x)\, J_0(hx)\, dx \, , \tag{7.29}$$

which is a function of h (or δ) only. Note that the Q_{2n}'s are simply the K_{2n}'s for $k = 1$, in other words the Q_{2n}'s can also be defined as a linear combination of the $\mathfrak{J}^o_{-1,2\gamma}$ $(k = 1, h)$ integrals with the aid of (7.4) and (7.7). Furthermore, the ordinary hypergeometric series on the r.h.s. of (7.28) is a Jacobi polynomial of n^{th} degree in k^2 (see the definition given by 6.4). It seems this simpler structure of the r.h.s. of Eq. (7.28) for occultation type of eclipses lends itselt more conveniently for automatic computations.

The constituent $\mathfrak{J}$-integrals in (7.2), (7.21) and (7.28) have long been known (cf. Kopal, 1947) from the theory of the light changes that

$$\mathfrak{J}^o_{-1,\gamma} = \left(\frac{r_2}{r_1}\right)^{\gamma+2} \left(\frac{\delta}{r_2}\right)^{\frac{\gamma}{2}} \left[J^o_{-1,\gamma} - \frac{\delta}{r_2} J^1_{-1,\gamma} \right] \tag{7.30}$$

and the J-integrals $J^o_{-1,2\gamma}$ and $J^1_{-1,2\gamma}$ which are required to evaluate the K_{2n}'s and Q_{2n}'s can be readily automated by taking advantage of the recursion formulae

$$J^o_{-1,\gamma+2} = 2 \left[J^1_{-1,\gamma} - \mu \, J^o_{-1,\gamma} \right] \tag{7.31}$$

and

$$J^{1}_{-1,\gamma+2} = \frac{2(\gamma+2)}{\gamma+4}\left[J^{0}_{-1,\gamma} - \mu J^{1}_{-1,\gamma} \right] \tag{7.32}$$

since the starting values for $\gamma = 0$ are known as

$$J^{0}_{-1,0} = \frac{1}{\pi}\cos^{-1}\mu \qquad \text{and} \qquad J^{1}_{-1,0} = \frac{1}{\pi}\sqrt{1-\mu^2} \tag{7.33}$$

if the eclipse is partial, and

$$J^{0}_{-1,0} = 1 \qquad \text{and} \qquad J^{1}_{-1,0} = 0 \tag{7.34}$$

if it is annular, where μ continues to be given by (7.17).

Finally, another alternative approach of the foregoing type will be presented for obtaining the fractional loss of light α_l^o. To do so, let us first combine equations (2.16) and (4.13) replacing α by a in the latter. The outcome of this combination yields that

$$\alpha_l^o = \frac{b}{\nu}\sum_{n=0}^{\infty} \epsilon_n \, {}_2F_1\left({-n,\, n \atop \nu+1} \,\middle|\, a^2\right) I_{2n} \,, \tag{7.35}$$

$$\epsilon_0 = 1 \quad \text{and} \quad \epsilon_n = 2 \quad \text{for} \quad n > 0 \,,$$

where we have abbreviated

$$I_{2n} = \int_0^{\infty} J_{2n}(x)\, J_1(bx)\, J_0(cx)\, dx \,. \tag{7.36}$$

This expansion opens the alternative way to approach the fractional loss of light α_l^o, but the series on the r.h.s. of (7.35) does not terminate in any circumstance. The ordinary hypergeometric series in this equation is also of Jacobi polynomials. As for the evaluation of the integrals I_{2n}, we can introduce another expansion of the form

$$\mathcal{J}^{o}_{-1,l} = \frac{2b}{\nu(\nu+1)} \sum_{n=0}^{\infty} (n+1)^2 \, {}_2F_1\left(\begin{matrix}-n,\ n+2\\ \nu+1\end{matrix} \,\middle|\, a^2\right) \cdot I_{2n+2} \tag{7.37}$$

with the same integrals I_{2n} as given by (7.36). This last expansion can be obtained by deriving (2.16) with respect to a and utilizing the definition (7.7) of the $\mathcal{J}$-integrals and the expansion given by (4.12) for $\alpha \equiv a$. Consequently, if we take into account N terms in (7.37), N independent equations can be constructed for N different values of l (remember that I_{2n}'s do not depend on l at all), and this set of linear equations can be solved numerically or otherwise for I_{2n} integrals. Hence, the α^{o}_{l} integrals given by (7.35) become expressible again in terms of Kopal's $\mathcal{J}$-integrals which can be easily automated as it has been outlined before.

1.8 Other Expressions for α^{o}_{l}.

Here we shall present two expressions for the fractional loss of light α^{o}_{l}. The first expression may be stated by substituting $J_\nu(ax)$ from (4.9) for $\mu \equiv \nu$ and $\sin\theta \equiv a$ in integral (2.16) of the α^{o}_{l} that

$$\alpha^{o}_{l} = \sqrt{\frac{2}{\pi}}\, 2^{\nu}\, \Gamma(\nu)\, \Gamma(\nu+\tfrac{1}{2}) \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{1}{2})(\nu+2n+\frac{1}{2})}{\Gamma(\nu+n+1)} \times$$

$$\times\, C_{2n}^{(\nu+\frac{1}{2})}\left(\sqrt{1-a^2}\right) \cdot E_n^{(l)} \tag{8.1}$$

where we have abbreviated

$$E_n^{(l)} = \int_0^\infty \frac{J_{2n+\nu+\frac{1}{2}}(x)}{x^{\nu+\frac{1}{2}}} \, J_1(bx) \, J_0(cx) \, dx \, , \tag{8.2}$$

which remains yet to be evaluated. The C_{2n}'s in (8.1) stand for the ultraspherical polynomials.

For the second expression we shall introduce a Neumann series for J_ν (cf. e.g., Erdelyi et al, 1953, Vol. II, p.66, Eq. 16) of the form

$$J_\nu(\lambda x) = \lambda^\nu \sum_{n=0}^{\infty} \frac{\left[\frac{1}{2}x(1-\lambda^2)\right]^n}{n!} J_{\nu+n}(x) \, . \tag{8.3}$$

On insertion from this result by setting $\nu \equiv 0$ and $\lambda \equiv c$ in Eq. (2.16) we find the latter to assume the form

$$\alpha_l^0 = 2^\nu \Gamma(\nu) \, b \sum_{n=0}^{\infty} \frac{(1-c^2)^n}{n! \, (2a)^n} \, H_n \, , \tag{8.4}$$

where we have abbreviated

$$H_n = \int_0^\infty \frac{J_\nu(ax)}{(ax)^{\nu-n}} \, J_1(bx) \, J_n(x) \, dx \, , \tag{8.5}$$

which also remains yet to be evaluated. Note that the H_n's are independent of c (or δ) and thus can be presented in the form of univariate tables.

1.9 Differential Equations Satisfied by α_l^0 .

1) $$\delta \frac{\partial \alpha_l^0}{\partial \delta} + r_1 \frac{\partial \alpha_l^0}{\partial r_1} + r_2 \frac{\partial \alpha_l^0}{\partial r_2} = 0 \tag{9.1}$$

2) $$\delta \frac{\partial \alpha_l^0}{\partial \delta} = -2\left(\frac{\delta}{r_2}\right)\left(\frac{r_2}{r_1}\right)^{l+2} \times I_{-1,l}^{1} \tag{9.2}$$

3) $$r_2 \frac{\partial \alpha_l^0}{\partial r_2} = 2\left(\frac{r_2}{r_1}\right)^{l+2} \times I_{-1,l}^{0} \tag{9.3}$$

4) $$r_1 \frac{\partial \alpha_l^0}{\partial r_1} = 2\, J_{-1,l}^{0} \tag{9.4}$$

5) $$\frac{\partial^2 \alpha_l^0}{\partial \delta^2} - \frac{l}{2\delta}\frac{\partial \alpha_l^0}{\partial \delta} = l\,\frac{\delta}{r_2}\left(\frac{r_2}{r_1}\right)^{l}\frac{\delta^2+r_1^2-r_2^2}{\delta^2 r_1^2} \times I_{-1,l-2}^{1} \tag{9.5}$$

6) $$\left\{\begin{aligned} &\delta^2 \frac{\partial^2 \alpha_l^0}{\partial \delta^2} + \delta \frac{\partial \alpha_l^0}{\partial \delta} = -\left(\frac{2}{k}\right)^{\nu}\Gamma(\nu)\,h^2\int_0^\infty x^{2-\nu} J_\nu(kx)\, J_1(x)\, J_0(hx)\,dx, \\ &\delta^2 \frac{\partial \alpha_l^0}{\partial \delta^2} + \delta \frac{\partial \alpha_l^0}{\partial \delta} + \alpha_l^0 = 2^{\nu}\,\Gamma(\nu)\int_0^\infty \frac{(1-h^2x^2)}{(kx)^{\nu}}\, J_\nu(kx)\, J_1(x)\, J_0(hx)\,dx, \end{aligned}\right. \tag{9.6}$$

$$k = \frac{r_1}{r_2} \quad \text{and} \quad h = \frac{\delta}{r_2},$$

7) $$\left\{1 + \sum_{n=1}^{N\to\infty} (-1)^n \frac{t(t-2)(t-4)\ldots(t-2n+2)}{2^n\; n!}\right\}\alpha_l^0 = \alpha_l^0(k, h=0), \qquad t \equiv h\frac{\partial}{\partial h}. \tag{9.7}$$

We have given a number of known differential equations for α_l^0. For first five equations which have long been known from the geometrical approach of the problem see Kopal (1959, Chapters IV.4 and IV.5). The last two equations have been presented in Paper XII by Kopal. For the solution of the second differential equation given by (9.2), refer to section 1.3 in the present chapter. In these foregoing differential equations, in general,

$$I^{m}_{\beta,\gamma} = \left(\frac{\delta}{r_2}\right)^{\frac{\gamma}{2}} J^{m}_{\beta,\gamma} \tag{9.8}$$

and they can be expressed in terms of ordinary hypergeometric functions for $\beta = -1$, m = 1 and $\gamma \equiv 1$ of the form

$$I^{1}_{-1,1} = \frac{4^{\nu-1}}{\pi} B(\nu,\tfrac{1}{2})\left(\frac{r_1}{r_2}\right)^{\nu}\left(\frac{r_2}{\delta}\right)\left(\frac{\delta}{r_1}\right)^{\nu} K^{2\nu-1} {}_2F_1\left({-\tfrac{1}{2}, \tfrac{3}{2} \atop \nu+\tfrac{1}{2}} \middle| K^2\right) \tag{9.9}$$

if the eclipse is partial, i.e., $\delta_2 \leqslant \delta \leqslant \delta_1$, and

$$I^{1}_{-1,1} = \frac{l\, 4^{\nu}}{32}\left(\frac{r_1}{r_2}\right)^{\nu-1} K^{2\nu-4} {}_2F_1\left({2-\nu, \tfrac{3}{2} \atop 3} \middle| K^{-2}\right) \tag{9.10}$$

if it is annular, i.e., $\delta_0 < \delta < \delta_2$ and $r_2 < r_1$.

For m = 0 we have

$$I^{0}_{-1,1} = \frac{4^{\nu-1}}{\pi}\left(\frac{r_1}{r_2}\right)^{\nu}\left(\frac{r_2}{\delta}\right)\left(\frac{\delta}{r_1}\right)^{\nu} K^{2\nu-1} {}_2F_1\left({\tfrac{1}{2}, \tfrac{1}{2} \atop \nu+\tfrac{1}{2}} \middle| K^2\right) \tag{9.11}$$

if the eclipse is partial, and

$$I^{0}_{-1,1} = 4^{\nu-1}\left(\frac{r_1}{r_2}\right)^{\nu}\left(\frac{r_2}{\delta}\right)\left(\frac{\delta}{r_1}\right)^{\nu} K^{2\nu-2} {}_2F_1\left({1-\nu, \tfrac{1}{2} \atop 1} \middle| K^{-2}\right) \tag{9.12}$$

if it is annular, where K continues to be given by (3.4) of the present chapter. The $\mathcal{J}$-integrals for m = 0 and $\beta = -1$ which appear in (9.4) are as given by Eq. (7.30) and they can be rearranged by making use of Eq. (9.8) ih terms of the I-integrals in the form

$$\mathcal{J}^{0}_{-1,1} = \left(\frac{r_2}{r_1}\right)^{2\nu}\left[I^{0}_{-1,1} - \frac{\delta}{r_2} I^{1}_{-1,1} \right] . \tag{9.13}$$

Substituting directly from (9.9) - (9.12) for I-integrals in (9.13) we find that the $\mathcal{J}$'s assume the following forms in terms of hypergeometric functions:

$$J^{0}_{-1,l} = -\frac{4^{\nu-1}}{\pi} B(\nu,\tfrac{1}{2})\left(\frac{r_2}{r_1}\right)^{\nu}\left(\frac{\delta}{r_1}\right)^{\nu} K^{2\nu-1}\left[{}_2F_1\left({-\tfrac{1}{2},\tfrac{3}{2} \atop \nu+\tfrac{1}{2}} \middle| K^2\right) - \right.$$

$$\left. - \frac{r_2}{\delta}\, {}_2F_1\left({\tfrac{1}{2},\tfrac{1}{2} \atop \nu+\tfrac{1}{2}} \middle| K^2\right)\right], \quad \delta_2 \leqslant \delta \leqslant \delta_1 , \tag{9.14}$$

$$J^{0}_{-1,l} = 4^{\nu-1}\left(\frac{r_2}{r_1}\right)^{\nu}\left(\frac{\delta}{r_1}\right)^{\nu} K^{2\nu-2}\left[\frac{r_2-\delta^2}{\delta}\, {}_2F_1\left({1-\nu,\tfrac{1}{2} \atop 1} \middle| K^{-2}\right) - \right.$$

$$\left. - \delta\, {}_2F_1\left({1-\nu,\tfrac{1}{2} \atop 2} \middle| K^{-2}\right)\right], \quad \delta_0 < \delta < \delta_2 , \quad r_2 < r_1 . \tag{9.15}$$

In all the foregoing equations ν continues to be given as $\nu = (l+2)/2$ and $B(x, y)$ stands for a beta function which can be expressed as in (3.5) for $x \equiv \nu$ and $y = \frac{1}{2}$. In the last differential equation given by (9.7) the fractional loss of light $\alpha^{0}_{l}(k, h = 0)$ at the moment of central eclipse when $\delta = 0$ is known that for occultation eclipses

$$\alpha^{0}_{l}(k, h=0) = \frac{1}{\nu} \tag{9.16}$$

which is equal to the maximum value of α^{0}_{l} attained during totality (see Eq. 3.19); while if the eclipse is a transit type

$$\alpha^{0}_{l}(k, h=0) = \frac{1-\nu(1-k^{-2})^{\nu}}{\nu} \tag{9.17}$$

(see Eq. 4.4). If, moreover, we set

$$\alpha_l^{\circ}(k,h) = \begin{cases} \frac{1}{\nu} - f_l(k,h)\,, & \text{occultation} \\ \frac{1-\nu(1-k^{-2})^{\nu}}{\nu} - f_l(k,h)\,, & \text{transit} \end{cases} \tag{9.18}$$

it follows that the differential equation (9.7) can be rewritten as

$$\begin{aligned} N=0:&\quad f_l = 0 \\ N=1:&\quad \left[1 - \frac{1}{2^1\,1!}\,t\right] f_l = 0 \\ N=2:&\quad \left[1 - \frac{1}{2^1\,1!}\,t + \frac{t(t-2)}{2^2\,2!}\right] f_l = 0 \\ &\vdots \\ N=n:&\quad \left[1 - \frac{t}{2^1\,1!} + \frac{t(t-2)}{2^2\,2!} - \dots (-1)^n \frac{t(t-2)(t-4)\dots(t-2n+2)}{2^n\,n!}\right] f_l = 0 \end{aligned} \tag{9.19}$$

and more concisely

$$\begin{aligned} N=0 &\qquad f_l = 0 \\ N=1 &\qquad (t-2)\,f_l = 0 \\ N=2 &\qquad (t-2)(t-4)\,f_l = 0 \\ &\vdots \\ N=n &\qquad (t-2)(t-4)\dots(t-2n)\,f_l = 0 \end{aligned} \tag{9.20}$$

with the same t defined by (9.7). Consequently, the differential Eq. (9.7) takes the following form:

$$(t-2)(t-4)\dots(t-2n)\,f_l = 0 \tag{9.21}$$

which is exact only if n is allowed to approach infinity; but remains approximate for finite values of $n < \infty$: the approximations it represents being the more accurate, the larger the value of n. It can be shown that the solution of this equation gives us

$$f_l(k,h) = \sum_{n=1}^{\infty} C_n h^{2n} \tag{9.22}$$

as the expansion of f_l in power series of the quantity h only. Unfortunately, this expansion does not converge if $\delta > r_2$. However, it can be proved by making use of Eq. (2.16) instead of (2.15) in the derivation of (9.7) that Eq. (9.22) can be replaced by

$$f_l(a,c) = \sum_{n=1}^{\infty} K_n c^{2n} \tag{9.23}$$

which converges under all circumstances, since c varies between zero and one. The quantity a which depends on the sizes of the components enters only through the coefficients K_n which can be defined by the boundary conditions of the problem. These boundary conditions in addition to (9.16) and (9.17) can be listed as follows:

$$\alpha_l^o(a, c=1) = 0 \tag{9.24}$$

$$\left(\delta \frac{\partial \alpha_l^o}{\partial \delta}\right)_{\delta=\delta_1} = 0 \tag{9.25}$$

$$\left(\delta \frac{\partial \alpha_l^o}{\partial \delta}\right)^{Occ.}_{\delta=\delta_2} = 0 \tag{9.26}$$

$$\left(\delta \frac{\partial \alpha_l^o}{\partial \delta}\right)^{Tra.}_{\delta=\delta_2} = -2\left(\frac{\delta_2}{r_2}\right)\left(\frac{r_2}{r_1}\right)^{2\nu} I^1_{-1,1}{}'(a,\delta_2) \neq 0 \tag{9.27}$$

As for the higher derivatives, it can be shown (cf. Paper XI, Equations 3.40 - 3.42) that

$$\left(\frac{\partial^n \alpha_l^0}{\partial \delta^n}\right)_{\delta=\delta_1} = \begin{cases} 0 & \text{for } n < \frac{l+1}{2} \\ \infty & \text{for } n > \frac{l+1}{2} \end{cases} \tag{9.28}$$

for even values of l, and

$$\left(\frac{\partial^n \alpha_l^0}{\partial \delta^n}\right)_{\delta=\delta_1} = \begin{cases} 0 & \text{for } n \neq \frac{l+3}{2} \\ -\frac{4^\nu}{\delta_1}\left(\frac{l+1}{2}\right)!\left(\frac{\partial K^2}{\partial \delta}\right)^{\frac{l+1}{2}}_{\delta=\delta_1}\left(\frac{r_2}{r_1}\right)^\nu\left(\frac{\delta_1}{r_1}\right)^\nu & \text{for } n = \frac{l+3}{2} \end{cases} \tag{9.29}$$

for odd values of l. On the other hand, at the moment of inner contact for transit eclipses, these higher derivatives will exhibit a pole whenever $n \geq \frac{l+3}{2}$. In the derivation of the foregoing boundary conditions given by (9.25) - (9.29), it should be remembered that (see Figure 2)

$$(K^2)_{\delta_1} = 0, \quad (K^2)^{Occ.}_{\delta_2} = 0, \quad (K^2)^{Tra.}_{\delta_2} = 1 \quad \text{and} \quad (K^2)_{\delta=0} = \infty \tag{9.30}$$

1.10 <u>The Alpha-Functions α_l^m of Higher Orders (m > 0).</u>

Kopal in the final section of Paper XI has shown how to proceed to develop similar expressions for the associated alpha-functions α_l^m of higher orders ($m > 0$) in terms of the Hankel transforms. The associated α -functions of the type α_l^m($m > 0$) are required to represent the photometric effects of "gravity darkening" over the surface of a distorted star undergoing eclipse.

We develop in this section the requisite expressions for these functions by mainly following the known strategy from Paper XI. Let us note in the beginning that for $m \geqslant 2$ the relations between the resulting expressions and the α_l^m-integrals become complicated and make it difficult to write the explicit Hankel transforms for the α_l^m's.

In order to derive the requisite expressions we shall proceed as follows: First, it is known that in the presence of distortion the aperture function f(x, y) in Eq. (2.1) will be of the form (Paper XI, Eq. 4.10)

$$f(x, y) = x^m z^n \tag{10.1}$$

or, in spherical coordinates

$$f(r, \theta) = r^m \cos^m\theta \, \cos^n\gamma \tag{10.2}$$

where γ stands for the angle of foreshortening. By remembering that

$$L_1 = \pi r_1^2 f(0) \left[1 - \sum_{l=0}^{\Lambda} \frac{l\, u_l}{2+l} \right] , \tag{10.3}$$

and, hence (see Eq. 1.3)

$$f(r) = \frac{L_1}{\pi r_1^2} \sum_{l=0}^{\Lambda} C^{(l)} \cos^l\gamma , \qquad \cos^l\gamma = \frac{(r_1^2 - r^2)^{\nu-1}}{r_1^{2\nu-2}} , \tag{10.4}$$

the Fourier transform of the aperture function f(r, Θ) will then be of the form (Paper XI, Eq. 4.11)

$$F(q, \phi) = \int_0^{\infty} f(r) \left[\int_{-\pi}^{\pi} e^{-2\pi i q r \cos(\theta-\phi)} \cos^m\theta \, d\theta \right] r^{m+1} dr . \tag{10.5}$$

In order to evaluate the integral with respect to Θ in (10.5) we resort not to the Jacobi expansion (2.2) but for simplicity to another expansion (cf. e.g., Magnus and Oberhettinger, 1948; p.27) of the form

$$e^{i\zeta \cos\varphi} = \sum_{k=0}^{\infty} \epsilon_k \, i^k J_k(\zeta) \cos k\varphi \, , \tag{10.6}$$

where the ϵ_k's denote Neumann's numbers as defined before. Substituting from (10.6), by setting $\cos\varphi \equiv -\cos(\varphi+\pi)$ and $\zeta = 2\pi q r$ in the respective integral we have

$$\int_{-\pi}^{\pi} e^{-2\pi i q r \cos(\theta-\phi)} \cos^m\theta \, d\theta = \sum_{k=0}^{\infty} \epsilon_k \, i^k J_k(2\pi q r) \, \times$$

$$\times \int_{-\pi}^{\pi} \cos k(\theta-\phi-\pi) \cos^m\theta \, d\theta$$

$$= \frac{\pi \, m!}{2^m} \sum_{k=0}^{m} \frac{\epsilon_k \, i^k \left[1-(-1)^{m+k+1}\right] \cos k(\phi+\pi)}{\Gamma\left(\frac{m+k+2}{2}\right) \Gamma\left(\frac{m-k+2}{2}\right)} J_k(2\pi q r) \tag{10.7}$$

since

$$\int_{-\pi}^{\pi} \cos k(\theta-\phi-\pi) \cos^m\theta \, d\theta = \frac{\left[1-(-1)^{m+k+1}\right] \pi \, m! \cos k(\phi+\pi)}{2^m \, \Gamma\left(\frac{m+k+2}{2}\right) \Gamma\left(\frac{m-k+2}{2}\right)} \, . \tag{10.8}$$

It may be noted that the terms of the summation on the r.h.s. of (10.7) vanish for odd values of (m + k). Moreover, if (m + k) is an odd number (m − k) also becomes an odd number; thus, if we consider only even values of (m − k) the infinite summation on the r.h.s. of (10.7) terminates whenever m = k . From a combination of (10.5) and (10.7) it follows that

$$F(q,\phi) = \frac{\pi\, m!}{2^m} \sum_{k=0}^{m} \frac{\epsilon_k (-i)^k \left[1-(-1)^{m+k+1}\right] \cos k\phi}{\Gamma\left(\frac{m+k+2}{2}\right) \Gamma\left(\frac{m-k+2}{2}\right)} \times$$

$$\times \int_0^\infty f(r)\, J_k(2\pi q r)\, r^{m+1}\, dr\,, \tag{10.9}$$

which for $m = 0$ reduces to (2.3). If, moreover, we assume that the function $f(r)$ continues to obey the law of limb-darkening of the form (1.3), for the evaluation of the remaining integral on the r.h.s. of (10.9), we can employ a formula (cf. e.g., Erdelyi et al, 1954; Vol. II, p.26, Eq. 34) of the form

$$\int_0^\infty x^\mu (a^2-x^2)^\lambda J_\nu(xy)\, dx = \frac{a^{2\lambda+\mu+\nu+1}\, y^\nu\, \Gamma(\lambda+1)\, \Gamma\left(\frac{\mu+\nu+1}{2}\right)}{2^{\nu+1}\, \Gamma(\nu+1)\, \Gamma\left(\lambda+1+\frac{\mu+\nu+1}{2}\right)} \times$$

$$\times {}_1F_2\left(\begin{matrix} \frac{\mu+\nu+1}{2} \\ \nu+1\,, \ \frac{\mu+\nu+3}{2}+\lambda \end{matrix} \,\middle|\, -\frac{a^2 y^2}{4} \right)\,, \tag{10.10}$$

by considering the definition (10.4) for $f(r)$ and setting $\mu \equiv m+1$, $\lambda \equiv \nu-1$, $\nu \equiv k$ and accordingly $x \equiv r$, $y \equiv 2\pi q$ and $a \equiv r_1$; thus, the outcome yields that

$$\int_0^\infty f(r)\, J_k(2\pi q r)\, r^{m+1}\, dr = \frac{L_1}{\pi} \sum_{l=0}^{\Lambda} C^{(l)} \frac{(2\pi q)^k\, r_1^{m+k}\, \Gamma(\nu)\, \Gamma\left(\frac{m+k+2}{2}\right)}{2^{k+1}\, k!\, \Gamma\left(\nu+\frac{m+k+2}{2}\right)} \times$$

$$\times {}_1F_2\left(\begin{matrix} \frac{m+k+2}{2} \\ k+1\,, \ \frac{m+k+2}{2}+\nu \end{matrix} \,\middle|\, -\left(\frac{2\pi q r}{2}\right)^2 \right). \tag{10.11}$$

Thus, the evaluation of the Fourier transform for the aperture function f(x, y) is completed. On the other hand, the Fourier transform G(u, v) of the occulting disc remains the same as in Eq. (2.7). The convolution integral (given by 2.8) of the transforms F and G , then furnishes the requisite form

$$M = \frac{m!\, L_1 r_1^m}{2\pi} \sum_{l=0}^{\Lambda} C^{(l)} \Gamma(\nu) \sum_{k=0}^{m} \frac{\epsilon_k (-i)^k \left[1-(-1)^{m+k+1}\right]}{2^{m+k+1}\, k!\, \Gamma\left(\frac{m+k+2}{2}\right) \Gamma\left(\frac{m+k+2}{2}+\nu\right)} \times$$

$$\times \int_0^\infty (2\pi q r_1)^k J_1(2\pi q r_2)\, {}_1F_2\left(\begin{matrix} \frac{m+k+2}{2} \\ k+1,\ \frac{m+k+2}{2}+\nu \end{matrix} \middle| -\left(\frac{2\pi q r_1}{2}\right)^2 \right) \times$$

$$\times \left[\int_{-\pi}^{\pi} e^{-2\pi i \delta q \cos\phi} \cos k\phi \, d\phi \right] d(2\pi q r_2) . \qquad (10.12)$$

The integral with respect to ϕ here can be easily evaluated with the aid of (10.6) for $z \equiv 2\pi q \delta$ and $\cos\varphi \equiv -\cos\phi$, and we have

$$\int_{-\pi}^{\pi} e^{-2\pi i q \delta \cos\phi} \cos k\phi \, d\phi = \sum_{n=0}^{\infty} \epsilon_n (-i)^n J_n(2\pi q \delta) \times$$

$$\times \int_{-\pi}^{\pi} \cos n\phi \cos k\phi \, d\phi =$$

$$= 2\pi (-i)^k J_k(2\pi q \delta) \qquad (10.13)$$

as the result, since

$$\int_{-\pi}^{\pi} \cos n\phi \cos k\phi \, d\phi = \begin{cases} 0 & \text{for } n \neq \pm k \\ \pi & \text{for } n = m \\ 0 & \text{for } k = 0 \text{ and } n = 1,2,3,\ldots \\ 2\pi & \text{for } n = 0 \text{ and } k = 0 \ . \end{cases} \qquad (10.14)$$

Furthermore, by making use of a formula (cf. e.g., Luke, 1975; p.58, Eq. 1) of the form

$$ {}_1F_2\left(\begin{matrix} a \\ b,\, c \end{matrix}\,\middle|\, -\frac{z^2}{4}\right) = \Gamma(d)\left(\frac{z}{2}\right)^d \sum_{n=0}^{\infty} \frac{(2n+d)\,(d)_n\,(b-a)_n\,(c-a)_n}{n!\,(b)_n\,(c)_n} \times $$

$$ \times\, J_{2n+d}(z) \quad , \tag{10.15} $$

where $d = b + c - a - 1$ is not a negative integer, and $(a)_j = a(a+1)(a+z) \ldots (a+j-1)$, $(a)_0 = 1$ are Pochhammer symbols. The hypergeometric functions ${}_1F_2$ on the r.h.s. of (10.12) can be rewritten in terms of the Bessel functions J_ν , as

$$ {}_1F_2\left(\begin{matrix} \frac{m+k+2}{2} \\ k+1,\, \frac{m+k+2}{2}+\nu \end{matrix}\,\middle|\, -\left(\frac{2\pi q r_1}{2}\right)^2\right) = \left(\frac{2\pi q r_1}{2}\right)^{\nu+k} \times \sum_{n=0}^{\infty} \frac{k!\,(2n+\nu+k)}{n!\,(n+k)!} \times $$

$$ \times\, \frac{\Gamma(\nu+n)\,\Gamma(\nu+n+k)\,\Gamma\left(\nu+\frac{m+k+2}{2}\right)\left(\frac{k-m}{2}\right)_n}{\Gamma(\nu)\;\Gamma\left(n+\nu+\frac{m+k+2}{2}\right)} \times J_{2n+\nu+k}(2\pi q r_1) \,. \tag{10.16} $$

Consequently, by a substitution directly from (10.16) and (10.13) in (10.12) we have the convolution integral M of the transforms F and G , in the form

$$ M = m!\,L_1 r_1^m \sum_{l=0}^{\Lambda} C^{(l)} \sum_{k=0}^{m} \frac{\epsilon_k (-1)^k \left[1-(-1)^{m+k+1}\right]}{2^{m+1-\nu}\,\Gamma\left(\frac{m-k+2}{2}\right)} \times $$

$$ \times \sum_{n=0}^{\infty} \frac{(2n+\nu+k)\,\Gamma(n+\nu)\,\Gamma(n+\nu+k)\,\left(\frac{k-m}{2}\right)_n}{n!\,(n+k)!\;\Gamma\left(n+\nu+\frac{m+k+2}{2}\right)} \times $$

$$ \times \int_0^{\infty} \frac{J_{2n+\nu+k}(2\pi q r_1)}{(2\pi q r_1)^{\nu}}\, J_1(2\pi q r_2)\, J_k(2\pi q S)\, d(2\pi q r_2) \,, \tag{10.17} $$

where the summations terminate in many cases: remember that

$$\lambda_{m,k} \equiv \left[1 - (-1)^{m+k+1} \right] = \begin{cases} 0 & \text{for } m+k = 1, 3, 5, \ldots \\ 2 & \text{for } m+k = 0, 2, 4, \ldots \end{cases} \tag{10.18}$$

and

$$\left(\frac{k-m}{2}\right)_n = \begin{cases} 0 & \text{for } m = k,\ n \neq 0 \\ 1 & \text{for } n = 0 \\ 0 & \text{for } \frac{m-k}{2} + 1 \leqslant n,\ m \neq k \text{ and } \\ & \qquad m - k = 2, 4, 6, \ldots \\ (-1)^n \dfrac{\Gamma\left(\frac{m-k}{2}+1\right)}{\Gamma\left(\frac{m-k}{2}+1-n\right)} & \text{for } m - k = 1, 3, 5, \ldots \end{cases} \tag{10.19}$$

Hence, we can rewrite (10.17) more concisely as

$$M = m!\, L_1 r_1^m \sum_{l=0}^{\Lambda} C^{(l)} \sum_{k=0}^{m} \frac{(-1)^k \epsilon_k \lambda_{m,k}}{2^{m-\nu}} \times$$
$$\times \sum_{n=0}^{N_1} \frac{(-1)^n (2n+\nu+k)\, \Gamma(n+\nu)\, \Gamma(n+\nu+k)}{n!\, (n+k)!\, \Gamma(n+\nu+N_2+1)\, \Gamma(N_1+1-n)} \times$$
$$\times \int_0^\infty \frac{J_{2n+\nu+k}(2\pi q r_1)}{(2\pi q r_1)^\nu} J_l(2\pi q r_2)\, J_k(2\pi q \delta)\, d(2\pi q r_2) , \tag{10.20}$$

where we have abbreviated

$$N_1 = \frac{m-k}{2} , \qquad \text{and} \qquad N_2 = \frac{m+k}{2} . \tag{10.21}$$

The quantity $\lambda_{m,k}$ denotes a number defined by (10.18). This result representing the convolution integral M of the transforms F and G

should somehow lead us to the associated α -functions. For $m = 0$ and 1 we find that this is really the case, i.e.,

$$(M)_{m=0} \equiv L_1 \sum_{l=0}^{\Lambda} C^{(l)} \alpha_l^0 \tag{10.22}$$

and

$$(M)_{m=1} \equiv -L_1 r_1 \sum_{l=0}^{\Lambda} C^{(l)} \alpha_l^1 \tag{10.23}$$

But, for the higher values of m the relation between M and α_l^m becomes complicated. For example, for $m = 2$ we have

$$(M)_{m=2} = \frac{L_1 r_1^2}{2} \sum_{l=0}^{\Lambda} C^{(l)} \Big[\frac{1}{\nu+1} \alpha_l^0 - \frac{1}{\nu+1} \mathcal{J}_{-1,l+2}^0 +$$

$$+ 2^\nu \Gamma(\nu) \int_0^\infty \frac{J_{\nu+2}(2\pi q r_1)}{(2\pi q r_1)^\nu} J_1(2\pi q r_2) J_2(2\pi q \delta)\, d(2\pi q r_2) \Big], \tag{10.24}$$

where for the identification of the integrals as α_l^0 and $\mathcal{J}_{-1,l+2}^0$ see the Hankel transforms (2.10) and (7.24) for these functions. The last term involving the integral on the r.h.s. of (10.24) may define the function α_l^m for $m = 2$. The physical significance of why the relations between the convolution M and the associated α-functions α_l^m become complicated with increasing values of m is not perfectly clear at the moment.

On the other hand, these functions for $m > 1$ may well be obtainable in terms of the α_l^0 , α_l^1 and $I_{-1,l}^0$ -integrals by making use of the recursion formulae established by Kopal (1959, Chapter IV.4-5).

Some of the results from Kopal's (1978) work are given below:

$$\alpha_l^1 = \frac{2\delta}{r_1}\left(\frac{r_2}{r_1}\right)^{l+2} I^0_{+1,\,l} = \frac{2}{l+2}\left(\frac{r_2}{r_1}\right)^{l+3} I^1_{-1,\,l+2} \tag{10.25}$$

$$\begin{aligned} 2\,\alpha_l^2 &= \alpha_l^0 - \alpha_{l+2}^0 + \left(\frac{r_2}{r_1}\right)^{l+4}\left[\frac{2\delta s}{r_2^2} I^0_{1,\,l} - I^0_{1,\,l+2}\right] \\ &= \frac{2}{l+4}\alpha_l^0 + 2\,\frac{l+3}{l+4}\,\frac{s}{r_1}\,\alpha_l^1 - \frac{l+5}{l+4}\,\frac{r_1}{\delta}\,\alpha_{l+2}^1 \end{aligned} \tag{10.26}$$

$$\begin{aligned} 3\,\alpha_l^3 &= \left[2 + \left(\frac{s}{r_1}\right)^2\right]\alpha_l^1 - \left[2 + \frac{s}{\delta}\right]\alpha_{l+2}^1 + \left(\frac{r_1}{2\delta}\right)^2 \alpha_{l+4}^1 \\ \alpha_l^4 &= \frac{3}{(l+4)(l+6)}\alpha_l^0 + \frac{s}{4r_1}\left[\frac{3(l^2+10l+20)}{(l+4)(l+6)} + \left(\frac{s}{r_1}\right)^2\right]\alpha_l^1 - \end{aligned} \tag{10.27}$$

$$\begin{aligned} &- \frac{3}{8}\,\frac{r_1}{\delta}\left[\frac{l+6}{l+4} + \frac{l+5}{l+6}\left(\frac{2\delta s}{r_1^2}\right) + \left(\frac{s}{r_1}\right)^2\right]\alpha_{l+2}^1 + \\ &+ \frac{3}{8}\,\frac{r_1}{\delta}\left[\frac{l+7}{l+6} + \frac{s}{2\delta}\right]\alpha_{l+4}^1 - \frac{1}{4}\left(\frac{r_1}{2\delta}\right)^3 \alpha_{l+6}^1 \,, \end{aligned} \tag{10.28}$$

where s and μ continue to be given by (7.17).

1.11 The $\mathcal{J}$ and I-Integrals.

The $\mathcal{J}$ and I-integrals additional to α_l^m must be invoked to describe the photometric effects of the mutual eclipses of distorted close binaries. These integrals are connected with the distortion of the projected boundaries of the two components, and they are purely geometrical. The aim of the present section will be to give a number of necessary expressions for the evaluation of these integrals in addition to the expressions given in Sections 1.7 and 1.8.

It can be shown (cf. Paper XII, Equations 4.8 and 4.9) by a partial differentiation of (2.10) with respect to r_2 and δ and considering Equations (9.2) and (9.3) that

$$I^{0}_{-1,l} = 2^{\nu-1}\Gamma(\nu)\, k^{\nu}\int_0^\infty x^{1-\nu} J_\nu(kx)\, J_0(x)\, J_0(hx)\, dx \tag{11.1}$$

and

$$I^{1}_{-1,l} = 2^{\nu-1}\Gamma(\nu)\, k^{\nu}\int_0^\infty x^{1-\nu} J_\nu(kx)\, J_1(x)\, J_1(hx)\, dx \tag{11.2}$$

as Hankel transforms, where $k = r_1/r_2$ and $h = \delta/r_2$ as before. For the similar expression for the $\mathfrak{J}$-integrals see Eq. (7.24). The above integrals may well be evaluated by employing the same methods described in the previous sections. If, for example, we utilize the formula (7.1) by changing over from k and h to a and c for the parameters of the respective integrals, we have

$$I^{0}_{-1,l} = \left(\frac{a}{b}\right)^{2\nu} b^2 (1-c^2)^\nu\, \Gamma(\nu) \sum_{n=0}^{\infty} (-1)^n \frac{n!\,(\nu+2n+1)}{\Gamma(\nu+n+1)} \left[R_n^{(0,\nu)}(a)\right]^2 \times R_n^{(0,\nu)}(1-c^2), \tag{11.3}$$

and

$$I^{1}_{-1,l} = \left(\frac{a}{b}\right)^{2\nu} b^3 c\, (1-c^2)^\nu\, \Gamma(\nu) \sum_{n=0}^{\infty} (-1)^n \frac{n!}{n+1} \frac{(\nu+n+1)(\nu+2n+2)}{\Gamma(n+\nu+1)} \left[R_n^{(1,\nu)}(a)\right]^2 \times R_n^{(1,\nu)}(1-c^2), \tag{11.4}$$

where all the notations are in their usual meaning. For the closed form expressions in terms of the ordinary hypergeometric functions see Equations (9.9) - (9.12). The above expansions for the respective I-integrals in series of shifted Jacobi polynomials - unlike the close form expressions for the same integrals – are valid for any type of eclipse and any degree l of the limb-darkening. A similar expansion for the $\mathcal{J}^{0}_{-1,l}$ –integrals can be set up simply by substituting (11.3) and (11.4) in (9.13).

All these functions for higher values of $m > 1$ may also be generated successively by making use of the well-known recursion relations established by Kopal (1959; Chapter IV.4–5) and Lanzano (1976).

CHAPTER 2

THE MOMENTS OF LIGHT CURVES

The basic data for the analysis of the light curves of eclipsing variables in the frequency domain are represented by a set of the quantities A_{2m} defined (cf. Paper I, Eq. 3.1) by

$$A_{2m} = \int_0^{\theta'} (1-l)\, d(\sin^{2m}\theta) \;, \tag{0.1}$$

as the areas subtended by the lines $l = 1$, $\sin^{2m}\theta = 0$, and the actual shape of the light curve of the respective spherical system in the $l - \sin^{2m}\theta$ coordinates (see Figure 3). The certain relations between the moments A_{2m} and the eclipse elements, for $m = 1, 2$ and 3, have been developed in Papers I–IV and XIII under the following assumptions: i) Distribution of brightness over the apparent discs of spherical components of the system is radially symmetric, ii) components of the system revolve about their common centre of gravity in circular orbits. For the explicit expressions in the case of annular and partial eclipses refer also to Kurutaç (1976), and Demircan (1976). In Papers I and II it has been shown that the respective expressions turn out to be algebraic for $m = 1, 2$ and 3, but this was not the case for annular and partial eclipses.

When, however, we utilize the expressions for the fractional loss of light α_l^o given in the preceding chapter in the evaluation of the integral (0.1) of the moments A_{2m}, it becomes possible to derive general expressions for A_{2m} in the simple forms of series expansions which are valid for any type of eclipse, any degree l of limb darkening and for any positive real value of m.

The main aim of the present chapter will be to represent such expressions developed by Kopal (1977c) in Paper XII and the present author (partly in Paper XV).

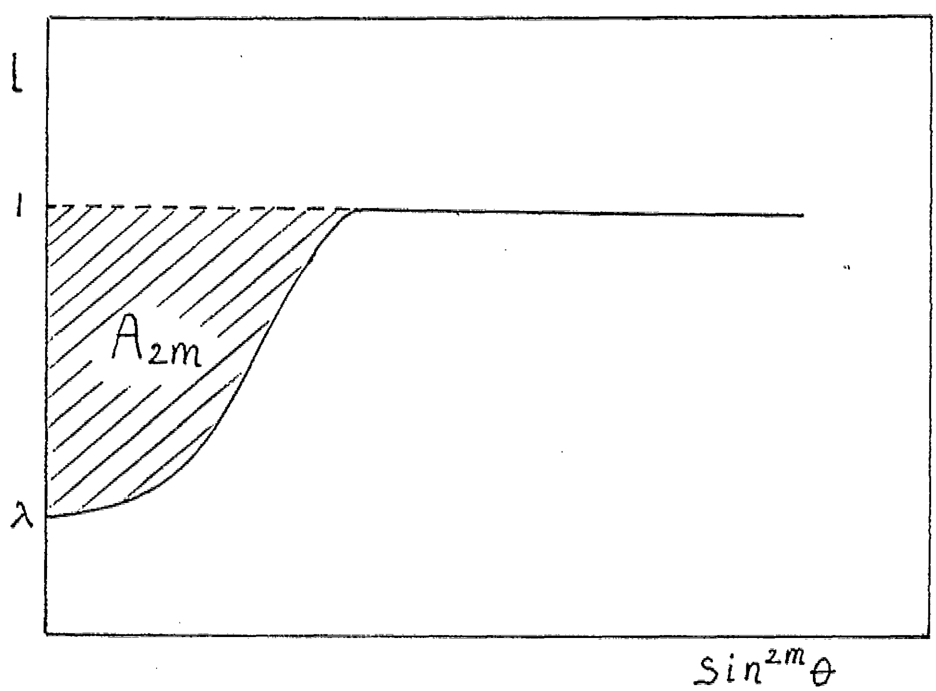

Figure 3.

2.1 The Moments A_{2m} as Hankel Transforms.

Firstly, if we consider that

$$1 - l = L_1 \sum_{l=0}^{\Lambda} C^{(l)} \alpha_l^{\circ} \tag{1.1}$$

and

$$d(\sin^{2m}\theta) = m \csc^{2m} i \,(\delta^2 - \delta_0^2)^{m-1} d\delta^2, \quad \delta_0 \equiv \cos i \tag{1.2}$$

which can be obtained from the well-known formula given by

$$\delta^2 = \sin^2\theta \sin^2 i + \cos^2 i, \tag{1.3}$$

the moments of the light curves A_{2m} can then be easily rewritten as

$$A_{2m} = m L_1 \csc^{2m} i \sum_{l=0}^{\Lambda} C^{(l)} \int_{\delta_0^2}^{\delta_1^2} (\delta^2 - \delta_0^2)^{m-1} \alpha_l^0 \, d\delta^2 \tag{1.4}$$

or, by making use of the Binom expansion for $(\delta^2 - \delta_0^2)^{m-1}$ given by

$$(\delta^2 - \delta_0^2)^{m-1} = \sum_{j=0}^{m-1} \frac{(m-1)!}{j!(m-j-1)!} (-\delta_0^2)^{m-j-1} \delta^{2j}, \tag{1.5}$$

we have

$$A_{2m} = m L_1 \csc^{2m} i \sum_{l=0}^{\Lambda} C^{(l)} \sum_{j=0}^{m-1} \frac{(m-1)!}{j!(m-j-1)!} (-\delta_0^2)^{m-j-1} \int_{\delta_0^2}^{\delta_1^2} \delta^{2j} \alpha_l^0 \, d\delta^2. \tag{1.6}$$

Next, by a direct substitution from the Hankel transform (2.10) of Chapter 1 for α_l^0 in Eq. (1.6) we are left with

$$A_{2m} = m L_1 \csc^{2m} i \sum_{l=0}^{\Lambda} C^{(l)} \sum_{j=0}^{m-1} \frac{(m-1)!}{j!(m-j-1)!} (-\delta_0^2)^{m-j-1} \times K, \tag{1.7}$$

where we have abbreviated

$$K = 2^\nu \Gamma(\nu) \int_0^\infty \frac{J_\nu(2\pi q r_1)}{(2\pi q r_1)^\nu} J_1(2\pi q r_2) \left[\int_{\delta_0^2}^{\delta_1^2} \delta^{2j} J_0(2\pi q \delta) \, d\delta^2 \right] d(2\pi q r_2). \tag{1.8}$$

In order to evaluate the above integral with respect to δ, use can be made of an Equation (cf. Erdelyi et al, 1953; Vol. II, p.90, Eq. 7) of the form

$$\int z^{\mu} J_{\nu}(z)\, dz = (\mu+\nu-1) z\, J_{\nu}(z)\, S_{\mu-1,\nu-1}(z) - z\, J_{\nu-1}(z)\, S_{\mu,\nu}(z), \qquad (1.9)$$

where the symbol $S(z)$ denotes the Lommel functions. Consequently, Eq. (1.7) can be rewritten in terms of Hankel transforms, as

$$A_{2m} = 2m L_1 \csc^{2m} i \sum_{l=0}^{\Lambda} C^{(l)} \sum_{j=0}^{m-1} \frac{(m-1)!}{j!\,(m-j-1)!} (-\delta_0^2)^{m-j-1} \times$$

$$\times \left[I_1(k,h_1) - I_1(k,h_0) + I_2(k,h_1) - I_2(k,h_0) \right], \qquad (1.10)$$

where I_1 and I_2 are two Hankel transforms given by

$$I_1(k,h) = 2j\, 2^{\nu} \Gamma(\nu) \frac{r_2^{2j+2} h}{k^{\nu}} \int_0^{\infty} \frac{J_{\nu}(kx)}{x^{\nu+2j+1}} J_1(x) J_0(hx) S_{2j,-1}(hx)\, dx \qquad (1.11)$$

and

$$I_2(k,h) = 2^{\nu} \Gamma(\nu) \frac{r_2^{2j+2} h}{k^{\nu}} \int_0^{\infty} \frac{J_{\nu}(kx)}{x^{\nu+2j+1}} J_1(x) J_1(hx) S_{2j+1,0}(hx)\, dx \qquad (1.12)$$

being yet to be evaluated, where $k = r_1/r_2$, $h_0 = \delta_0/r_2$ and $h_1 = \delta_1/r_2$. The Lommel functions $S(hx)$ occurring on the r.h.sides of (1.11) and (1.12) can be easily obtained by the known recursion relations

$$S_{2j+2,-1}(hx) = (hx)^{2j+1} - \left[(2j+1)^2 - 1\right] S_{2j,-1}(hx), \qquad (1.13)$$

and

$$S_{2j+3,0}(hx) = (hx)^{2j+2} - (2j+2)^2 S_{2j+1,0}(hx) \tag{1.14}$$

since

$$S_{0,-1}(hx) = \frac{1}{hx} , \qquad \text{and} \qquad S_{1,0}(hx) = 1 . \tag{1.15}$$

It may be observed that (1.10) is valid for only positive integer values of m , and reduces to Equations (3.11) - (3.13) of Paper XII for m = 1, 2 and 3. Furthermore it may be verified that the integrals (1.11) and (1.12) can be expressed in terms of α_l^0 and its derivatives with respect to r_1, r_2 and δ . Alternatively, known methods from Chapter 1 for the evaluation of similar Hankel transforms can be utilized to evaluate these integrals. This work to develop explicit forms of the integrals (1.11) and (1.12) has been left to be investigated.

2.2 Expansions for A_{2m} in Series of Polynomials.

A general expansion for the moments A_{2m} has been given by Kopal in Paper XII. We shall approach his result by substituting directly from Eq. (6.3) of Chapter 1 in Eq. (1.4) for the moments. This substitution gives us

$$A_{2m} = m L_1 \csc^{2m} i \sum_{l=0}^{\Lambda} \frac{C^{(l)}}{\nu \Gamma(\nu+1)} \sum_{n=0}^{\infty} (-1)^n (\nu+2n+2) \frac{\Gamma(\nu+n+1)}{(n+1)!} \times$$

$$\times \left[G_{n+1}(\nu, \nu+1, a) \right]^2 \times \int_{\delta_0^2}^{\delta_1^2} (\delta^2 - \delta_0^2)^{m-1} (1-c^2)^{\nu+1} G_n(\nu+2, 1, c^2) \, d\delta^2 , \tag{2.1}$$

where the same notations have been adopted from Paper XII. The symbol G_n stands for the Jacobi polynomials defined by

$$G_n(\alpha,\beta,x) = {}_2F_1\left(\begin{matrix}-n,\ n+\alpha\\ \beta\end{matrix}\,\middle|\,x\right)$$

$$= \frac{n!\,\Gamma(n+\alpha)}{\Gamma(2n+\alpha)}\,P_n^{[(\alpha-\beta),(\beta-1)]}(2x-1)$$

$$= \frac{n!\,\Gamma(n+\alpha)}{\Gamma(2n+\alpha)}\,R^{[(\alpha-\beta),(\beta-1)]}(x)\ . \tag{2.2}$$

In Eq. (2.1), replacing δ by $(r_1 + r_2)c$ as the variable of integration, and normalizing the limits by introducing an auxiliary variable

$$u = \frac{c^2 - c_0^2}{1 - c_0^2}\ , \tag{2.3}$$

we can rewrite (2.1) as

$$A_{2m} = m\,L_1\left[(r_1+r_2)\csc i\right]^{2m}\sum_{l=0}^{\Lambda}\frac{C^{(l)}\,(1-c_0^2)^{m+\nu+1}}{\nu\,\Gamma(\nu+1)}\,\times$$

$$\times\sum_{n=0}^{\infty}(-1)^n(\nu+2n+2)\frac{\Gamma(\nu+n+1)}{(n+1)!}\left[G_{n+1}(\nu,\nu+1,a)\right]^2\times$$

$$\times\int_0^1 u^{m-1}(1-u)^{\nu+1}\,G_n(\nu+2,1,c^2)\,du$$

$$= m!\,L_1(1-a)^2\left[(r_1+r_2)^2\csc^2 i - \cot^2 i\right]^m\sum_{l=0}^{\Lambda}\frac{1}{(\nu)_{m+2}}\,C^{(l)}\,\times \tag{2.4}$$

$$\times \sum_{n=0}^{\infty} \frac{(\nu+n+1)(\nu+2n+2)(n-1)!}{n(n+1)} \left[\frac{G_n(\nu+2,\nu+1,a)}{B(n,\nu+1)} \right]^2 \times$$

$$\times \sum_{j=0}^{n} (-1)^j \frac{(n+\nu+2)_j \, (1-c_0^2)^{\nu+j+1}}{(m+\nu+2)_j \; j! \, (n-j)!} ,$$

where $(\alpha)_j = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+j-1)$, $(\alpha)_0 = 1$ are Pochhammer symbols. This result representing a general expansion for the moments A_{2m} in polynomials is due to Kopal (1977c).

However, we can rewrite this result given by (2.4) more concisely as

$$A_{2m} = L_1 \frac{b^2}{a^{2m}} \left(\frac{r_1}{\sin i} \right)^{2m} (1-c_0^2)^{m+1} \times f_{2m}(a, c_0) , \tag{2.5}$$

where we have abbreviated

$$f_{2m} = \Gamma(m+1) \sum_{l=0}^{\Lambda} C^{(l)} \Gamma(\nu) \sum_{n=0}^{\infty} \frac{n! \, (\nu+2n+2)}{(n+1)\,\Gamma(\nu+n+1)} \left[R_n^{(1,\nu)}(a) \right]^2 \times$$

$$\times \sum_{j=0}^{n} (-1)^j \frac{\Gamma(n+\nu+2+j)}{\Gamma(m+\nu+2+j)} \frac{(1-c_0^2)^{\nu+j}}{j!\,(n-j)!} , \tag{2.6}$$

in terms of shifted Jacobi polynomials (since $0 \leqslant c \leqslant 1$) defined by Eq. (6.4) of Chapter 1. This expression (2.5) in algebraic form, representing a general expansion for the moments A_{2m} of the light curve, is valid for every type of eclipse for any positive real value of m and for any arbitrary degree l of the adopted law of limb darkening.

An alternative expression for the same moments has been given by the present author (Demircan, 1978b) in Paper XV. We shall in what follows outline the development of this alternative general expansion for the moments A_{2m}. To do so, let us first consider the expansion (6.14) in Chapter 1 for the fractional loss of light α_l^o (see also Demircan, 1977b, Paper XIII for this expansion). If we now substitute for α_l^o directly from this result in Eq. (1.4), a general series expansion for the moments A_{2m} can be obtained:

$$A_{2m} = \frac{2m L_1 b^2}{\sqrt{\pi}} \csc^{2m} i \sum_{l=0}^{\Lambda} C^{(l)} \Gamma(\nu) \sum_{n=0}^{\infty} (-1)^n \frac{(\nu+2n+\frac{3}{2}) \Gamma(n+\frac{3}{2})}{\Gamma(\nu+n+1)} \times$$

$$\times {}_2F_1\left({-2n, 2n+2\nu+3 \atop 3} \middle| b\right) \int_{\delta_0^2}^{\delta_1^2} (\delta^2-\delta_0^2)^{m-1} (1-c^2)^{\nu+\frac{1}{2}} {}_2F_1\left({-n, n+\nu+\frac{3}{2} \atop 1} \middle| c^2\right) d\delta^2, \qquad (2.7)$$

where $a = r_1/(r_1 + r_2)$, $b \equiv 1 - a = r_2/(r_1 + r_2)$ and $C = \delta/(r_1 + r_2)$ in their usual meaning. Normalizing the limits of the integral on the r.h.s. of (2.7) by making use of a new variable u as defined by (2.3) we can rewrite (2.7) as

$$A_{2m} = \frac{2m L_1 b^2}{\sqrt{\pi}} \left[(r_1+r_2) \csc i\right]^{2m} \sum_{l=0}^{\Lambda} C^{(l)} \Gamma(\nu) (1-c_0^2)^{m+\nu+\frac{1}{2}} \times$$

$$\times \sum_{n=0}^{\infty} (-1)^n \frac{(\nu+2n+\frac{3}{2}) \Gamma(n+\frac{3}{2})}{\Gamma(\nu+n+1)} {}_2F_1\left({-2n, 2n+2\nu+3 \atop 3} \middle| b\right) \times$$

$$\times \int_0^1 u^{m-1} (1-u)^{\nu+\frac{1}{2}} {}_2F_1\left({-n, n+\nu+\frac{3}{2} \atop 1} \middle| c^2\right) du . \qquad (2.8)$$

It has been shown by the present author (Demircan, 1978a, also refer to Demircan, 1978b) that the above type of integrals can, in general, be given as

$$\int_0^1 u^{\rho-1}(1-u)^{\lambda-1}\,{}_2F_1\left(\begin{matrix}-n,\ n+\lambda\\ \beta+1\end{matrix}\,\middle|\,1-z(1-u)\right)du =$$

$$= (-1)^n\,\Gamma(\rho)\sum_{j=0}^{n}\frac{(-1)^j\,\Gamma(\lambda+j)\,\Gamma(n+\lambda+j)}{j!\,(n-j)!\,\Gamma(\alpha+j+1)\,\Gamma(\rho+\lambda+j)}\,z^j\,, \qquad (2.9)$$

where $\lambda = \alpha+\beta+1$. For $\beta=0$ the r.h.s. of (2.9) reduces to an ordinary hypergeometric form:

$$\int_0^1 u^{\rho-1}(1-u)^{\alpha}\,{}_2F_1\left(\begin{matrix}-n,\ n+\alpha+1\\ 1\end{matrix}\,\middle|\,1-z(1-u)\right)du =$$

$$= (-1)^n\frac{\Gamma(\rho)\,\Gamma(n+\alpha+1)}{n!\,\Gamma(\rho+\alpha+1)}\,{}_2F_1\left(\begin{matrix}-n,\ n+\alpha+1\\ \rho+\alpha+1\end{matrix}\,\middle|\,z\right). \qquad (2.10)$$

A combination of Equations (2.8) and (2.10) discloses that

$$A_{2m} = \frac{2\,\Gamma(m+1)}{\sqrt{\pi}}\,L_1\,b^2\left[(r_1+r_2)\csc i\right]^{2m}\sum_{l=0}^{\Lambda} C^{(l)}\,\Gamma(\nu)\,\times$$

$$\times\sum_{n=0}^{\infty}\frac{\left(\nu+2n+\frac{3}{2}\right)\Gamma\left(n+\frac{3}{2}\right)}{\Gamma(n+\nu+1)}\,{}_2F_1\left(\begin{matrix}-2n,\ 2n+2\nu+3\\ 3\end{matrix}\,\middle|\,b\right)\times$$

$$\times\sum_{j=0}^{n}(-1)^j\frac{\Gamma\left(n+\nu+\frac{3}{2}+j\right)\left(1-c_0^2\right)^{m+\nu+j+\frac{1}{2}}}{\Gamma\left(m+\nu+\frac{3}{2}+j\right)\,j!\,(n-j)!}\,, \qquad (2.11)$$

or

$$A_{2m} = \frac{2\,\Gamma(m+1)}{\sqrt{\pi}} L_1 b^2 \left[(r_1+r_2)\csc i\right]^{2m} \sum_{l=0}^{\Lambda} C^{(l)} \Gamma(\nu)(1-c_0^2)^{m+\nu+\frac{1}{2}} \times$$

$$\times \sum_{n=0}^{\infty} \frac{(\nu+2n+\frac{3}{2})\,\Gamma(n+\frac{3}{2})\,\Gamma(n+\nu+\frac{3}{2})}{n!\,\Gamma(\nu+n+1)\,\Gamma(m+\nu+\frac{3}{2})} \times$$

$$\times {}_2F_1\left(\begin{matrix}-2n,\ 2n+2\nu+3\\ 3\end{matrix}\middle| b\right) {}_2F_1\left(\begin{matrix}-n,\ n+\nu+\frac{3}{2}\\ m+\nu+\frac{3}{2}\end{matrix}\middle| 1-c_0^2\right). \tag{2.12}$$

Note that both hypergeometric functions on the r.h.s. of (2.11) are polynomials. Thus, this expression, in simple algebraic form, constitutes an alternative to that given by Kopal, and represents another general expansion for the moments A_{2m} of the light curves, valid for any type of eclipse, any arbitrary degree l of limb darkening, and any positive real value of m. For $m = 0$ it reduces to

$$A_0 = L_1 \sum_{l=0}^{\Lambda} C^{(l)} \alpha_l^0 \quad . \tag{2.13}$$

It may also be noted that first hypergeometric function can be identified as shifted Jacobi polynomials (see Eq. 6.4 of Chapter 1) so that the moments A_{2m} can be given more concisely as

$$A_{2m} = \frac{4 L_1 b^2}{\sqrt{\pi}\; a^{2m}} (1-c_0^2)^{m+\frac{1}{2}} \left(\frac{r_1}{\sin i}\right)^{2m} \times f_{2m}(a, c_0) \tag{2.14}$$

where

$$f_{2m}(a,c_0) = \Gamma(m+1)\sum_{l=0}^{\Lambda} C^{(l)}\Gamma(\nu)\sum_{n=0}^{\infty}\frac{(\nu+2n+\frac{3}{2})\ \Gamma(n+\frac{3}{2})}{(2n+1)(2n+2)\,\Gamma(n+\nu+1)}\times$$

$$\times\ R_{2n}^{(2,2\nu)}(a)\sum_{j=0}^{n}(-1)^j\frac{\Gamma(n+\nu+\frac{3}{2}+j)\ (1-c_0^2)^{\nu+j}}{\Gamma(m+\nu+\frac{3}{2}+j)\ j!\,(n-j)!}\ ,$$

$$= \Gamma(m+1)\sum_{l=0}^{\Lambda} C^{(l)}\Gamma(\nu)(1-c_0^2)^{\nu}\sum_{n=0}^{\infty}\frac{(\nu+2n+\frac{3}{2})\Gamma(n+\frac{3}{2})\Gamma(n+\nu+\frac{3}{2})}{n!(2n+1)(2n+2)\Gamma(n+\nu+1)\Gamma(m+\nu+\frac{3}{2})}\times$$

$$\times\ R_{2n}^{(2,2\nu)}(a)\ {}_2F_1\left(\begin{matrix}-n,\ n+\nu+\frac{3}{2}\\ m+\nu+\frac{3}{2}\end{matrix}\ \middle|\ 1-c_0^2\right). \qquad (2.15)$$

On the other hand, if we use the expression (6.15) for α_l° given in Chapter 1 in the same way as we followed in this section, another similar expression for the moments A_{2m} can be obtained in the form

$$A_{2m} = \frac{L_1 b^2}{a^{2m}}(1-c_0^2)^{m+\frac{1}{2}}\left(\frac{r_1}{\sin i}\right)^{2m}\times\ f_{2m}(a,c_0)\ , \qquad (2.16)$$

where we have abbreviated

$$f_{2m}(a,c_0) = \Gamma(m+1)\sum_{l=0}^{\Lambda} C^{(l)}\frac{\Gamma(2\nu+1)}{\nu\,\Gamma(\nu+\frac{1}{2})}\sum_{n=0}^{\infty}\frac{(2n)!(2n+\nu+\frac{3}{2})\ \Gamma(\nu+n+\frac{1}{2})}{(n+1)!\ \Gamma(2n+2\nu+1)}\times$$

$$\times\ R_{2n}^{(2,2\nu)}(a)\sum_{j=0}^{n}(-1)^j\frac{\Gamma(n+\nu+\frac{3}{2}+j)\ \ (1-c_0^2)^{\nu+j}}{\Gamma(m+\nu+\frac{3}{2}+j)\ \ j!\,(n-j)!}\ ,$$

$$= \Gamma(m+1)\sum_{l=0}^{\Lambda} C^{(l)}\frac{\Gamma(2\nu+1)}{\nu\,\Gamma(\nu+\frac{1}{2})}\ (1-c_0^2)^{\nu}\times$$

$$\times \sum_{n=0}^{\infty} \frac{(2n)!\,(2n+\nu+\frac{3}{2})\,\Gamma(\nu+n+\frac{1}{2})\,\Gamma(\nu+n+\frac{3}{2})}{n!\,(n+1)!\,\Gamma(2n+2\nu+1)\,\Gamma(m+\nu+\frac{3}{2})} \times$$

$$\times\, R_{2n}^{(2,2\nu)}(a)\ {}_2F_1\left(\begin{matrix}-n,\ n+\nu+\frac{3}{2}\\ m+\nu+\frac{3}{2}\end{matrix}\ \middle|\ 1-c_0^2\right) . \tag{2.17}$$

For the evaluation of the shifted Jacobi polynomials which occur in the foregoing expressions in this section, see certain three-term recursion relation (6.7) with (6.8) and (6.9) given in Chapter 1.

2.3 Further Expansions for A_{2m} .

Here, we shall present some further expressions for the moments A_{2m} . To do so, let us first resort to the expansion (5.19) in Chapter 1 for the fractional loss of light α_L^o (see also Paper XIII for this expansion). Substituting this result in (1.4) for A_{2m}'s we are left with

$$A_{2m} = m L_1 \csc^{2m} i \sum_{l=0}^{\Lambda} C^{(l)}\, 2^{\nu}\, \Gamma(\nu) \sum_{n=0}^{\infty} \epsilon_n (-1)^n K_n^{(l)} \times$$

$$\times \int_{\delta_0^2}^{\delta_1^2} (\delta^2 - \delta_0^2)^{m-1}\, T_n^*(c^2)\, d\delta^2 , \tag{3.1}$$

where T_n^*'s denote shifted Chebyshev polynomials, the coefficients $K_n^{(l)}$ which are the function of only quantity $a = r_1/(r_1 + r_2)$ have been given by (5.8) and (5.11) in Chapter 1. In order to evaluate the integrals on the r.h.s. of (3.1) if we change over the variable of integration from δ to u with the aid of (2.3) and remembering that $c = \delta/(r_1 + r_2)$, we have

$$I = \int_{\delta_0^2}^{\delta_1^2} (\delta^2 - \delta_0^2)^{m-1}\, T_n^*(c^2)\, d\,\delta^2 = (r_1 + r_2)^{2m} (1 - c_0^2)^m \int_0^1 u^{m-1}\, T_n^*(c^2)\, du. \tag{3.2}$$

Let us moreover utilize the result (5.21) from Chapter 1 which can be easily rewritten by making use of the series expansion for ${}_2F_1$, as

$$\begin{aligned} T_n^*(c^2) &= {}_2F_1\left(\begin{matrix} -n,\ n \\ \frac{1}{2} \end{matrix} \,\middle|\, 1 - c^2\right) \\ &= \sum_{j=0}^{n} \frac{(-n)_j\,(n)_j}{(\frac{1}{2})_j\ j!} (1 - c^2)^j \\ &= \sqrt{\pi} \sum_{j=0}^{n} \frac{(-1)^j\,(n+j-1)!}{j!\,(n-j)!\,\Gamma(j+\frac{1}{2})} (1 - c^2)^j \end{aligned} \tag{3.3}$$

since

$$(\alpha)_j = \frac{\Gamma(j+\alpha)}{\Gamma(\alpha)}, \qquad \text{and} \qquad (-\alpha)_j = (-1)^j \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-j+1)} \tag{3.4}$$

This result permits us to rewrite the integral (3.2) as

$$I = \sqrt{\pi}\, \delta_1^{2m} \sum_{j=0}^{n} \frac{(-1)^j\,(n+j-1)!}{j!\,(n-j)!\,\Gamma(j+\frac{1}{2})} (1 - c_0^2)^{m+j} \int_0^1 u^{m-1} (1-u)^j\, du. \tag{3.5}$$

The integral on the r.h.s. is a beta function B(m, j+1). Thus, consequently, we have

$$A_{2m} = \Gamma(m+1)\sqrt{\pi}\ L_1\, (\delta_1 \csc i)^{2m} \sum_{l=0}^{\Lambda} C^{(l)}\, 2^{\nu}\, \Gamma(\nu)\, \times$$

$$\times \sum_{n=0}^{\infty} \epsilon_n (-1)^n K_n^{(l)} \sum_{j=0}^{n} \frac{(-1)^j (n+j-1)!\ (1 - c_0^2)^{m+j}}{(n-j)!\,(m+j)!\,\Gamma(j+\frac{1}{2})}, \tag{3.6}$$

which can also be written in the same form as (2.13) and (2.15). Note that a similar expression for the moments A_{2m} can be obtained by making use of Eq. (5.20) from Chapter 1.

Next, we shall represent another similar expression for A_{2m} due to Kopal (1977c). This expression can be obtained if we replace (4.11) of Chapter 1 in (1.4) for the moments, leading to

$$A_{2m} = m L_1 b^2 \csc^{2m} i \sum_{l=0}^{\Lambda} \frac{1}{\nu} C^{(l)} \sum_{n=0}^{\infty} (2n+\tfrac{1}{2}) \times$$

$$\times F_4\left(-n+\tfrac{1}{2}, n+1; \nu+1, 2; a^2, b^2\right) \int_{\delta_0^2}^{\delta_1^2} (\delta^2-\delta_0^2)^{m-1} P_{2n}\left(\sqrt{1-c^2}\right) d\delta^2. \tag{3.7}$$

In changing over (as before) from δ to U as the variable of integration, we can evaluate (3.7) as

$$A_{2m} = m L_1 b^2 (\delta_1 \csc^{2m} i)^{2m} (1-c_0^2)^m \sum_{l=0}^{\Lambda} \frac{1}{\nu} C^{(l)} \sum_{n=0}^{\infty} (2n+\tfrac{1}{2}) \times$$

$$\times F_4\left(-n+\tfrac{1}{2}, n+1; \nu+1, 2; a^2, b^2\right) \int_0^1 u^{m-1} P\left(\sqrt{1-c^2}\right) du$$

$$= L_1 b^2 \left(\delta_1^2 \csc^2 i - \cot^2 i\right)^m \sum_{l=0}^{\Lambda} \frac{1}{\nu} C^{(l)} \sum_{n=0}^{\infty} (-1)^n \frac{(2n)!\,(4n+1)}{2^{2n+1}(n!)^2} \times$$

$$\times F_4\left(-n+\tfrac{1}{2}, n+1; \nu+1, 2; a^2, b^2\right) \sum_{j=0}^{n} \frac{(-n)_j (n+\tfrac{1}{2})_j}{(\tfrac{1}{2})_j (m+1)_j} (1-c_0^2)^j, \tag{3.8}$$

where P_n's stand for the Legendre polynomials, the F_4 is the Appell function of fourth kind, and for the symbol $(\alpha)_j$ see Eq. (3.4)

Finally, we wish to present an expansion for A_{2m} in series of polynomials. This can be derived on insertion for α_l^0 directly from (4.20) of Chapter 1 in (1.4):

$$A_{2m} = 2^{2K-2} b^2 m L_1 \csc^{2m} i \sum_{l=0}^{\Lambda} \frac{2^{\nu}}{\nu} C^{(l)} \sum_{n=0}^{\infty} \frac{(2n+K)[\Gamma(n+K)]^2}{(n!)^2} \times$$

$$\times F_4(-n, n+K; 2; \nu+1; b^2, a^2) \int_{\delta_0^2}^{\delta_1^2} (\delta^2 - \delta_0^2)^{m-1} {}_2F_1\left({-n, n+K \atop 1} \middle| c^2\right) d\delta^2 ,$$

$$\mathrm{Re}\, K \leqslant 2 . \tag{3.9}$$

For $K = 1$, for example, if we use (2.3) and (2.10) for the evaluation of the above integral, results enable us to rewrite (3.9) in the form

$$A_{2m} = L_1 b^2 (\delta_1 \csc i)^{2m} (1 - c_0^2)^m \sum_{l=0}^{\Lambda} \frac{2^{\nu}}{\nu} C^{(l)} \sum_{n=0}^{\infty} (-1)^n (2n+1) \times$$

$$\times {}_2F_1\left({-n, n+1 \atop m+1} \middle| 1 - c_0^2\right) F_4(-n, n+1; 2, \nu+1; b^2, a^2) . \tag{3.10}$$

All the foregoing expressions for the moments A_{2m} in this section hold good likewise for any type of eclipse, any positive real value of m and any arbitrary degree l of the adopted law of limb-darkening.

It should be observed that many additional expressions are obtainable for the moments A_{2m} of the light curves by simply replacing the expansions for α_l^0 directly from Chapter 1 into the Eq. (1.4) of the present chapter for the moments.

2.4 Closed Form Expressions for Integral Values of m.

We begin with integrating Eq. (1.4) by parts to yield

$$A_{2m} = -L_1 \csc^{2m} i \sum_{l=0}^{\Lambda} C^{(l)} \int_{\delta_0^2}^{\delta_1^2} (\delta^2 - \delta_0^2)^m \frac{\partial \alpha_l^0}{\partial \delta} d\delta , \tag{4.1}$$

remembering that

$$\alpha_l^0 \, d\left[(\delta^2-\delta_0^2)^m\right] = d\left[\alpha_l^0 (\delta^2-\delta_0^2)^m\right] - (\delta^2-\delta_0^2)^m \frac{\partial \alpha_l^0}{\partial \delta} d\delta ,$$

$$\alpha_l^0(\delta_1) = 0 . \tag{4.2}$$

By making use of the Binom expansion for $(\delta^2 - \delta_0^2)^m$, Eq. (4.1) discloses that

$$A_{2m} = L_1 \csc^{2m} i \sum_{l=0}^{\Lambda} C^{(l)} \sum_{j=0}^{m} \frac{m!}{j!(m-j)!} (-\delta_0^2)^{m-j} \, a_l^{(j)} , \tag{4.3}$$

which leads us to the desired closed form expressions* for the moments A_{2m} when m is a positive integer number, where we have abbreviated

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* These expressions have been given by the present author in Paper XV for m = 0 , 1, 2 and 3. The methods will be developed in this section to generalise these results.

$$\alpha_l^{(j)} = -\int_{\delta_0}^{\delta_1} \delta^{2j} \frac{\partial \alpha_l^0}{\partial \delta} d\delta \tag{4.4}$$

which is yet to be evaluated and obviously reduces to $\alpha_l^0(\delta_0)$ for $j = m = 0$ (see Eq. 3.2 in Chapter 1) so that we have (2.13) for the moments A_0 which can be deduced from observations as $U - \lambda$ (where U is the total maximum light received on the quadratures when $\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$, while the quantity λ in the same scale represents the total light received at the time of maximum eclipse when $\delta = \delta_0 \equiv \cos i$, or $\theta = 0$ or π).

It may be proved that the quantities $\alpha_l^{(j)}$ given by (4.4) (which are valid for $0 \leqslant j \leqslant m$ and $0 \leqslant l \leqslant \Lambda$) can be expressed in terms of the α_l^0 and I-integrals. Our next task will be to illustrate this and to utilize the results in the evaluation of the desired closed form expressions for the moments A_{2m}, by simply substituting the results for $\alpha_l^{(j)}$ in Eq. (4.3).

In order to approach our aim we resort to a basic recursion formula derived by Kopal (1975c) in Paper III which permits us to assert that

$$(l+4j+2)\,\alpha_l^{(j)} - \frac{l}{r_1^2}\,\alpha_{l-2}^{(j+1)} - l\left[1-\left(\frac{r_2}{r_1}\right)^2\right]\alpha_{l-2}^{(j)} = 2\delta_0^{2j+1}\left(\frac{\partial \alpha_l^0}{\partial \delta}\right)_{\delta_0}, \tag{4.5}$$

which holds good for any type of eclipse. By re-arranging the terms in this recursion formula and remembering from the geometrical approach that

$$\left(\frac{\partial \alpha_l^\circ}{\partial \delta}\right) = -\frac{2}{r_2}\left(\frac{r_2}{r_1}\right)^{l+2} I^{\,1}_{-1,\,l} \quad , \tag{4.6}$$

then (4.5) can be readily rewritten as

$$a_l^{(j)} = \frac{l+4j}{l+2}\, r_1^2\, a_{l+2}^{(j-1)} - \left[1-\left(\frac{r_2}{r_1}\right)^2\right] r_1^2\, a_l^{(j-1)} +$$

$$+ \frac{4}{l+2}\left(\frac{r_2}{r_1}\right)^{l+4}\left(\frac{r_1^2}{r_2}\right)\delta_0^{2j-1}\, I^{\,1}_{-1,\,l+2}(\delta_0) \quad , \quad j > 0 \ . \tag{4.7}$$

This result permits us to construct successively all the requisite values of $a_l^{(j)}$ if the functions α_l° and I can be regarded as known. Thus, the closed form expressions for the moments A_{2m} may easily be written in terms of α_l° and I-integrals for any positive integer value of m . To illustrate this here analytically, for the firs few values of m , let us proceed as follows: as for the $a_l^{(j)}$, if for example j = 0 , from (4.4) we have

$$a_l^{(0)} = \alpha_l^\circ(\delta_0) \quad , \tag{4.8}$$

and for j = 1 , 2 and 3 it can be deduced from (4.7) that

$$\frac{a_l^{(1)}}{r_1^2} = \left(\frac{r_2}{r_1}\right)^2 \alpha_l^\circ + \frac{2}{l+2}\left(\frac{r_2}{r_1}\right)^{l+4} Q_{l+2} \tag{4.9}$$

$$\frac{a_l^{(2)}}{r_1^4} = \frac{l+8}{l+2}\,\frac{1}{r_1^2}\, a_{l+2}^{(1)} - \left[1-\left(\frac{r_2}{r_1}\right)^2\right]\frac{1}{r_1^2}\, a_l^{(1)} + \frac{4}{l+2}\left(\frac{r_2}{r_1}\right)^{l+4}\left(\frac{\delta_0^3}{r_1^2 r_2}\right) I^{\,1}_{-1,\,l+2}$$

$$= \left[\left(\frac{r_2}{r_1}\right)^4 + \frac{4}{l+4}\left(\frac{r_2}{r_1}\right)^2 \right] \alpha_l^{\circ} + \frac{2}{l+2}\,\frac{l+8}{l+4}\left(\frac{r_2}{r_1}\right)^2 J^{\circ}_{-1,\,l+2} +$$

$$+ \frac{2}{l+2}\left(\frac{r_2}{r_1}\right)^{l+4}\left[\left(\frac{r_2}{r_1}\right)^2 - 1\right] Q_{l+2} + \frac{2}{l+4}\,\frac{l+8}{l+2}\left(\frac{r_2}{r_1}\right)^{l+6} Q_{l+4} +$$

$$+ \frac{4}{l+2}\left(\frac{r_2}{r_1}\right)^{l+4}\left(\frac{\delta_0^3}{r_1^2 r_2}\right) I^{1}_{-1,\,l+2} \tag{4.10}$$

and

$$\frac{a_l^{(3)}}{r_1^6} = \frac{l+12}{l+2}\,\frac{1}{r_1^4}\,a_{l+2}^{(2)} - \left[1 - \left(\frac{r_2}{r_1}\right)^2\right]\frac{1}{r_1^4}\,a_l^{(2)} + \frac{4}{l+2}\left(\frac{r_2}{r_1}\right)^{l+4}\left(\frac{\delta_0^5}{r_1^4 r_2}\right) I^{1}_{-1,\,l+2}$$

$$= \left[\left(\frac{r_2}{r_1}\right)^6 + \frac{12}{l+4}\left(\frac{r_2}{r_1}\right)^4 + \frac{24}{(l+4)(l+6)}\left(\frac{r_2}{r_1}\right)^2\right]\alpha_l^{\circ} +$$

$$+ \left[\frac{4(l+10)}{(l+2)(l+4)}\left(\frac{r_2}{r_1}\right)^4 + \frac{8(l+12)}{(l+2)(l+4)(l+6)}\left(\frac{r_2}{r_1}\right)^2 - \frac{2(l+8)}{(l+2)(l+4)}\left(\frac{r_2}{r_1}\right)^2\right] J^{\circ}_{-1,\,l+2} +$$

$$+ \frac{2(l+10)(l+12)}{(l+2)(l+4)(l+6)}\left(\frac{r_2}{r_1}\right)^2 J^{\circ}_{-1,\,l+4} + \frac{2}{l+2}\left(\frac{r_2}{r_1}\right)^{l+4}\left[\left(\frac{r_2}{r_1}\right)^2 - 1\right]^2 Q_{l+2} +$$

$$+ \frac{4(l+10)}{(l+2)(l+4)}\left[\left(\frac{r_2}{r_1}\right)^2 - 1\right]\left(\frac{r_2}{r_1}\right)^{l+6} Q_{l+4} + \frac{2(l+10)(l+12)}{(l+2)(l+4)(l+6)}\left(\frac{r_2}{r_1}\right)^{l+8} Q_{l+6} +$$

$$+ \frac{4}{(l+2)}\left(\frac{r_2}{r_1}\right)^{l+4}\left(\frac{\delta_0^2 + r_2^2 - r_1^2}{r_1^2}\right)\left(\frac{\delta_0^3}{r_1^2 r_2}\right) I^{1}_{-1,\,l+2} + \tag{4.11}$$

$$+ \frac{4(l+12)}{(l+2)(l+4)}\left(\frac{r_2}{r_1}\right)^{l+6}\left(\frac{\delta_0^3}{r_1^2 r_2}\right) I^{1}_{-1,\,l+4} \;,$$

where for the symbol $\mathcal{J}^{0}_{-1,\gamma}$ see Eq. (7.30) or (9.13) in Chapter 1, and we have abbreviated

$$Q_n = I^{0}_{-1,n} + \frac{\delta_0}{r_2} I^{1}_{-1,n} \quad . \tag{4.12}$$

If, now, we rewrite Eq. (4.3) in the form

$$A_{2m} = L_1 \left(\frac{r_1}{\sin i}\right)^{2m} f'_{2m}(a_1, c_0) \tag{4.13}$$

it can be shown for m = 0, 1, 2 and 3 , that

$$f'_0 = \sum_{l=0}^{\Lambda} C^{(l)} \alpha_l^{0} \tag{4.14}$$

(see Eq. 2.13)

$$\begin{aligned} f'_2 &= \sum_{l=0}^{\Lambda} C^{(l)} \left[-\frac{\delta_0^2}{r_1^2} a_l^{(0)} + \frac{1}{r_1^2} a_l^{(1)} \right] \\ &= \left(\frac{b^2 - c_0^2}{a^2}\right) f'_0 + \left(\frac{b}{a}\right)^2 \sum_{l=0}^{\Lambda} \frac{2}{l+2} C^{(l)} \times S_1 \end{aligned} \tag{4.15}$$

$$\begin{aligned} f'_4 &= \sum_{l=0}^{\Lambda} C^{(l)} \left[\frac{\delta_0^4}{r_1^4} a_l^{(0)} - 2\frac{\delta_0^2}{r_1^4} a_l^{(1)} + \frac{1}{r_1^4} a_l^{(2)} \right] \\ &= \left(\frac{b^2 - c_0^2}{a^2}\right)^2 f'_0 + \left(\frac{b}{a}\right)^2 \sum_{l=0}^{\Lambda} \frac{4}{l+4} C^{(l)} \alpha_l^{0} + \\ &\quad + \left(\frac{b}{a}\right)^4 \sum_{l=0}^{\Lambda} \frac{2}{(l+2)(l+4)} C^{(l)} \times S_2 \ , \end{aligned} \tag{4.16}$$

$$f_6' = \sum_{l=0}^{\Lambda} C^{(l)} \left[-\frac{\delta_0^6}{r_1^6} a_l^{(0)} + 3\frac{\delta_0^4}{r_1^6} a_l^{(1)} - 3\frac{\delta_0^2}{r_1^6} a_l^{(2)} + \frac{1}{r_1^6} a_l^{(3)} \right]$$

$$= \left(\frac{b^2 - c_0^2}{a^2}\right)^3 f_0' + \left(\frac{b}{a}\right)^2 \left(\frac{b^2 - c_0^2}{a^2}\right) \sum_{l=0}^{\Lambda} \frac{12}{l+4} C^{(l)} \alpha_l^0 +$$

$$+ \left(\frac{b}{a}\right)^2 \sum_{l=0}^{\Lambda} \frac{24}{(l+4)(l+6)} C^{(l)} \alpha_l^0 + \left(\frac{b}{a}\right)^6 \sum_{l=0}^{\Lambda} \frac{2}{(l+2)(l+4)} C^{(l)} \times S_3 \,, \qquad (4.17)$$

where α_l^0, a, b and c_0 are all in their usual meaning as in Chapter 1. The quantities S_1, S_2 and S_3 are again the functions of a and c_0. They can be given in terms of Kopal's I-integrals (cf. Demircan, 1978b) as

$$S_1 = \left(\frac{b}{a}\right)^{l+2} Q_{l+2} \,, \qquad (4.18)$$

$$S_2 = (l+8)\left(\frac{a}{b}\right)^2 I_{-1,l+2}^0 + (l+4)\left(\frac{b}{a}\right)^l \left(\frac{b^2 - a^2 - 2c_0^2}{a^2}\right) Q_{l+2} +$$

$$+ (l+8)\left(\frac{b}{a}\right)^{l+2} Q_{l+4} + 2(l+4)\left(\frac{b}{a}\right)^{l+2} \left(\frac{c_0}{b}\right)^3 I_{-1,l+2}^1 \,, \qquad (4.19)$$

and

$$S_3 = \frac{(l+10)(l+12)}{(l+6)} \left(\frac{b}{a}\right)^{l+2} Q_{l+6} + \left(\frac{b}{a}\right)^l \left[\frac{2(l+10)(b^2 - a^2) - 3(l+8)c_0^2}{a^2}\right] Q_{l+4} +$$

$$+ (l+4)\left(\frac{b}{a}\right)^{l-2} \left[\left(\frac{b^2 - a^2}{a^2}\right)\left(\frac{b^2 - a^2 - 3c_0^2}{a^2}\right) + 3\left(\frac{c_0}{a}\right)^4\right] Q_{l+2} +$$

$$+ \frac{1}{l+6}\left(\frac{a}{b}\right)^4 \left[\frac{2(l+6)(l+10)\,b^2 - 3(l+6)(l+8)\,c_0^2}{a^2} - l(l+10)\right] \mathfrak{J}^{0}_{-1,\,l+2} +$$

$$+ \frac{(l+10)(l+12)}{(l+6)}\left(\frac{a}{b}\right)^4 \mathfrak{J}^{0}_{-1,\,l+4} + 2(l+12)\left(\frac{b}{a}\right)^{l+2}\left(\frac{c_0}{b}\right)^3 I^{0}_{-1,\,l+4} +$$

$$+ \; 2(l+4)\left(\frac{b}{a}\right)^{l}\left(\frac{c_0}{b}\right)^3\left(\frac{b^2 - a^2 - 2c_0^2}{a^2}\right) I^{1}_{-1,\,l+2} \quad . \tag{4.20}$$

The quantities $\mathfrak{J}^{0}_{-1,\gamma}$ and Q_n are defined in terms of the I-integrals (see Equations 7.30 or 9.13 in Chapter 1 for $\mathfrak{J}^{0}_{-1,\gamma}$'s and (4.12) for Q_n's). For these well-known and best studied functions of eclipsing binaries refer chiefly to Kopal (1947) and Lanzano (1976).

The f'_{2m}'s given by (4.14) - (4.17) hold good for any type of eclipse and any degree l of the adopted law of limb-darkening. Let us consider now what happens when the eclipses become total. It is known that all the I-integrals vanish for total eclipses, so do S_m's since all the Q's and $\mathfrak{J}$'s are zero there. Thus, for total eclipses Equations (4.14) - (4.17) reduce to

$$f'_0 = \sum_{l=0}^{\Lambda} \frac{2}{l+2}\, C^{(l)} = 1 \tag{4.21}$$

$$f'_2 = \frac{b^2 - c_0^2}{a^2} \tag{4.22}$$

$$f_4' = \left(\frac{b^2 - c_0^2}{a^2}\right)^2 + \left(\frac{b}{a}\right)^2 \sum_{l=0}^{\Lambda} \frac{8}{(l+2)(l+4)} \, C^{(l)} \tag{4.23}$$

and

$$f_6' = \left(\frac{b^2 - c_0^2}{a^2}\right)^3 + \left(\frac{b}{a}\right)^2 \left(\frac{b^2 - c_0^2}{a^2}\right) \sum_{l=0}^{\Lambda} \frac{24}{(l+2)(l+4)} \, C^{(l)} +$$

$$+ \left(\frac{b}{a}\right)^2 \sum_{l=0}^{\Lambda} \frac{48}{(l+2)(l+4)(l+6)} \, C^{(l)} , \tag{4.24}$$

for any degree l of the law of limb darkening. The summations on the r.h. sides of (4.23) and (4.24) become

$$\frac{15 - 7u_1}{5(3 - u_1)} , \quad \frac{3(15 - 7u_1)}{5(3 - u_1)} \quad \text{and} \quad \frac{3(35 - 19u_1)}{35(3 - u_1)} \tag{4.25}$$

respectively, for the linear law of limb darkening, and

$$\frac{2(15 - 7u_1 - 10u_2)}{5(6 - 2u_1 - 3u_2)} , \quad \frac{6(15 - 7u_1 - 10u_2)}{5(6 - 2u_1 - 3u_2)} \quad \text{and} \quad \frac{3(140 - 76u_1 - 105u_2)}{70(6 - 2u_1 - 3u_2)} \tag{4.26}$$

for the quadratic law of limb darkening. If the distribution of brightness over the disc of totally eclipsed components is uniform, then, Equations (4.21) - (4.24) further reduce to

$$f_0' = 1 \tag{4.27}$$

$$f_2' = \frac{b^2 - c_o^2}{a^2} \tag{4.28}$$

$$f_4' = \left[f_2'\right]^2 + \left(\frac{b}{a}\right)^2 \tag{4.29}$$

and

$$f_6' = \left[f_2'\right]^3 + 3\left(\frac{b}{a}\right)^2 f_2' + \left(\frac{b}{a}\right)^2 . \tag{4.30}$$

It may be noted from (4.21) and (4.22) that f_o' and f_2' do not depend on limb darkening for total eclipses.

CHAPTER 3

NUMERICAL COMPUTATION OF α'S, I'S AND A_{2m}'S

The requisite basic quantities for the Fourier analysis of the light curves of eclipsing variables are known to be the moments A_{2m} and the α and I-integrals. In the foregoing chapters all these quantities have been redefined as Hankel transforms and these transform integrals have been expanded to a number of convergent series hopefully to gain new properties of the respective quantities and utilize them for the proper and fast analysis.

The present chapter is devoted to the numerical computation of the above functions. This will show us the way to the practical applications of the certain methods which have been constructed in the last four years of continuous effort.

The numerical integration of the respective Hankel transforms were performed and the results were quoted in the first section. In section two the computation of the same requisite quantities were made by employing their series expansions. Section three is devoted to the development of some certain recursion formulae to reduce the round-off error and speed up the computations. The revised expressions for the moments A_{2m} were also given in this section. Finally, in the last section fast and general fortran programs were enclosed for the numerical computations of these functions.

3.1 Numerical Integrations.

For the application of numerical quadrature methods we considered here two integrals, namely the fractional loss of light α for uniform limb-darkening:

$$\alpha_0^0 = 2 \int_0^\infty \frac{J_1(kx)}{kx} J_1(x) J_0(hx)\, dx$$

$$= 2b \int_0^\infty \frac{J_1(ax)}{ax} J_1(bx) J_0(cx)\, dx \tag{1.1}$$

and for linear limb-darkening:

$$\alpha_1^0 = \sqrt{2\pi} \int_0^\infty \frac{J_{\frac{3}{2}}(kx)}{(kx)^{\frac{3}{2}}} J_1(x) J_0(hx)\, dx$$

$$= 2 \int_0^\infty \frac{1}{(kx)^3}\left[\sin kx - kx \cos kx\right] J_1(x) J_0(hx)\, dx$$

$$= 2b \int_0^\infty \frac{1}{(ax)^3}\left[\sin ax - ax \cos ax\right] J_1(bx) J_0(cx)\, dx \tag{1.2}$$

as Hankel transforms which can be easily deduced from the general results (2.15) and (2.16) of Chapter 1, where the parameters k, h, a, b and c are as given in (2.15) and (2.16). We have these two functions in tabular form (cf. e.g., Tsesevich, 1939, 1940) which enable us to check our results. Here, it will be useful to note the following correlations between the notations used in the present work and in the tables of Tsesevich for these functions:

$$\alpha_0^\circ \left(a \leq \tfrac{1}{2}\right) = \overset{\circ}{\alpha}$$

$$\alpha_0^\circ \left(a \geq \tfrac{1}{2}\right) = \overset{\circ}{\alpha} \times k^2$$

$$\alpha_1^\circ \left(a \leq \tfrac{1}{2}\right) = \overset{1}{\alpha}{}' \times \tfrac{2}{3} \times k^2$$

$$\alpha_1^\circ \left(a \geq \tfrac{1}{2}\right) = \overset{1}{\alpha}{}'' \times \phi(k) \times k^2 \qquad (1.3)$$

where $k = r_s/r_g$ and

$$\phi(k) = \alpha_1^\circ \left(a \geq \tfrac{1}{2},\ c = \frac{r_1 - r_2}{r_1 + r_2}\right)$$

$$= \frac{4}{3\pi}\left[\sin^{-1}\sqrt{k} + \frac{1}{3}(4k-3)(2k+1)\sqrt{k(1-k)}\right] \qquad (1.4)$$

(cf. Kopal, 1959; Chapter IV.4) and see also Eq. (4.5) in Chapter 1 for more general expression.

For the numerical integration of (1.1) and (1.2), first the simple trapezoidal rule was employed for some particular values of k and h. In Table 3 we present one such computation of (1.1) for $r_1/r_2 = 0.6$ and $\delta/r_2 = 0.76$ (or $p = -0.4$).

It is seen from Table 3 that the number n of the trapezoids should be large even though the upper limit of infinity in (1.1) may be reduced to a number as low as twenty (see column two of Table 3) owing to the strongly oscillating character of the integrand (see Figure 4). In the

Table 3

n	x	f(x)	$\alpha_0^\circ(0.6, 0.76)$
5	0.1	0.024 92150	0.00299
10	0.2	0.049 37449	0.01092
20	0.4	0.095 07400	0.04091
40	0.8	0.162 98179	0.14802
80	1.6	0.168 00908	0.43601
160	3.2	-0.001 10851	0.66224
320	6.4	-0.000 03498	0.70149
640	12.8	0.000 57792	0.68750
1280	20	0.000 06773	0.68670
	Exact value		0.68697

beginning the faster approximation to the true result is also observable from Table 3. For example, the first zero of the integrand is about 3.17 in the above example and the integration up to this point with only 159 trapezoids gives us almost two significant decimals (0.6623). If steps in x apsisas are taken to be larger than that of the example, better accuracy for the result cannot be secured even if the integration is performed until much higher limits. For example, if the upper limit is taken to be 50 and the step is 0.1, we get $\alpha_0^\circ = 0.68649$, on the other hand, if we use 500 trapezoids between 0 and 5, and 1000 trapezoids between 5 and 25, we have $\alpha_0^\circ = 0.68691$ for the same parameters k and h .

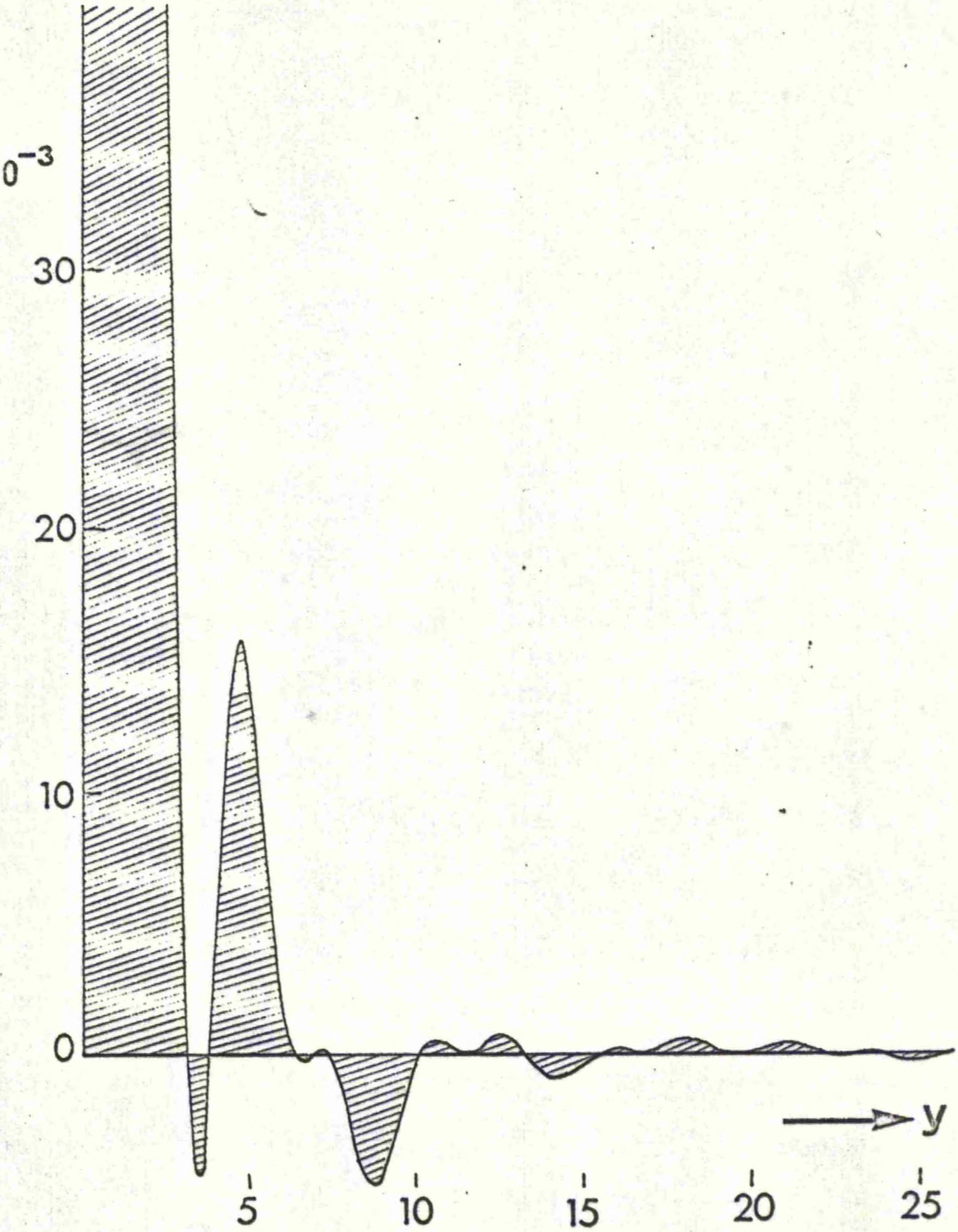

Figure 4. The behaviour of the integrand $f(x)$ of the loss of light α_i° for $l = 0$, $k = 0.6$ and $h = 0.76$ ($p = -0.4$). It is zero in the origin and reaches its first maximum value ($f = 0.187$) for $x \doteq 1.20$. It is estimated that $\int_{20}^{\infty} f(x)dx \sim 10^{-4}$;

In what follows, we give some important results which are revealed from the application of the trapezoidal rule for the numerical integration of the Hankel transforms (1.1) and (1.2).

i) The numerical integration of the Hankel transforms (1.1) and (1.2) by means of primitive formulae are possible with the high-speed computing machines by using very large values of n .

ii) The integrands of the respective Hankel transforms are strongly oscillatory (see Figure 4), which requires very large number of divisions in the numerical integrations by means of primitive formulae.

iii) The rate of convergence to the true results is usually very fast in the beginning - for small values of x - but slowes down rapidly with increasing values of the variable x of integration. As a result of this property we may truncate the summation at a certain point, depending on the requisite accuracy for the approximation.

Next, it was intended to apply some of the more efficient numerical methods. The Gaussian-type formulae have been employed for some different values of the parameters k and h . For comparison purposes we give here a table. In this example we chose the same parameters k and h as in the earlier example in Table 3 and the upper limit B of integration was taken as 30.

It is obvious from Table 4 that we gain better accuracy by employing Gaussian-type formulae. It may also be noted that G24 is enough to secure four figures in this case. The error of 10^{-4} in the result is obviously caused by the neglected part of the integration beyond 30.

Table 4

Rule	$\alpha_0^o(0.6,\ 0.76)$
G4	0.64268
G16	0.67637
G24, 32, 48, 64	0.68705
Exact value	0.68697

G4 = Gaussian 4-point formula;

G16 = Gaussian 16-point formula, etc.

The Chebyshev and Laguerre integration formulae have also been employed for different values of the parameters k and h. It was found that in most cases 3-4 significant figures can be secured easily in the results. But, unfortunately, in the beginning of eclipses, i.e., if $h \sim 1 + kp$ for occultation type eclipses and $h \sim k + p$ for transit type eclipses, almost all the accuracy vanishes and 2 significant figures can hardly be achieved even if much higher point formulae are employed. For example, Gauss - 32 - point formula gives $\alpha_0^o(0.2,\ 1.2) = -0.093$ instead of true value zero in this first contact point.

Expansions (5.19), (5.20), (7.2) and (8.4) from Chapter 1 have been automated by employing Gaussian 32-point formula (G32) for the evaluation of their coefficients. It may be useful to note that the numerical values of the coefficients H_n in (8.4) rapidly increases with increasing n, but contrary $(1 - c^2)^n/2^n/a^\nu/n!$ diminish more rapidly so that we, in most cases (for large values of c), have the convergent results. If $c \sim 0$ (or $\delta \sim 0$) the respective expansion diverges.

It was found from the practical applications in addition to the above results (which are also true for the applications of these expansions) that approximations to the true results are also highly oscillatory with irregularly diminishing amplitudes by increasing n . For this see the accompanying Table 5 which was constructed by employing (5.20) from Chapter 1 in the first contact point ($\delta = \delta_1 \equiv r_1 + r_2$) for $r_2/r_1 = 0.6$. In this case

$$\alpha_0^0(a, \delta_1) = 2b \sum_{n=0}^{\infty} \epsilon_n (-1)^n O_n^{(1)} = 0 \quad . \tag{1.3}$$

The coefficients $O_n^{(1)}$ were evaluated by G32 where the upper limits of respective integrals were taken to be 30. This example reveals that the present numerical method for (1.3) is unable to secure more than three figures in the respective points.

In all the foregoing numerical integrations we reduced the infinite interval to a finite one by ignoring the "tail" of the integral, as

$$\int_0^{\infty} f(x)\,dx \doteq \int_0^{B} f(x)\,dx\,, \quad \text{'Tail'} = \int_B^{\infty} f(x)\,dx \leq \textit{Tolerance} \tag{1.4}$$

where it is difficult to fix the quantity B owing to its dependence on the values of parameters k and h in addition to the tolerance in the result. On the other hand, since the integrand is highly oscillatory in our case, summation of the positive and negative contributions to the results

Table 5

n	$O_n^{(o)}(\frac{r_2}{r_1} = 0.6)$	$\alpha_o^o(\frac{r_2}{r_1} = 0.6, \delta_1)$
0	0.186 451	0.139 838
1	0.119 640	-0.039 622
2	0.027 696	0.001 922
3	0.001765	-0.000 727
4	0.000 215	-0.000 405
5	-0.000 873	0.000 906
6	-0.000 569	0.000 052
7	-0.000 557	0.000 889
8	-0.000 407	0.000 280
9	-0.000 261	0.000 673
10	-0.000 249	0.000 299
11	-0.000 149	0.000 522
12	-0.000 093	0.000 382
13	-0.000 083	0.000 506
14	-0.000 036	0.000 452
15	0.000 015	0.000 430
16	0.000 035	0.000 483
17	0.000 032	0.000 435
18	0.000 019	0.000 463
19	0.000 005	0.000 455
20	-0.000 001	0.000 452
Exact value		0.000 000

cause some loss of accuracy even if B is properly fixed, especially if the successive contributions are almost equal in absolute value. This happens in the case of first contact points when $\delta = r_1 + r_2$ in our problem. There are, of course, numerous devices to speed up the convergence of series summation. It is the author's expectation that these devices would not give satisfactory improvement in our problem, besides their application will require some additional time-consuming operations. However, they may be worth applying.

In the following section we shall compute the respective Hankel transforms without any numerical integration, but by series of hypergeometric functions which have been developed in Chapter 1 for the α and I-integrals, and in Chapter 2 for the moments A_{2m}.

3.2 Approximations by Series Summations.

In this section the numerical evaluation of the requisite functions will be examined by series summation from their expansions in series of hypergeometric functions (see Chapter 1). In these expansions terms consist of the product of two or three hypergeometric functions of the type $_2F_1$ and F_4. As it is well-known (cf. Erdelyi et al, 1953; Vol.1, Chapters 1 and 3)

$$ {}_2F_1\left({A, B \atop C} \middle| Q \right) = \sum_{n=0}^{\infty} \frac{(A)_n \, (B)_n}{(C)_n \; n!} \, Q^n \tag{2.1} $$

and

$$F_4(A1, B1; C1, C2; X, Y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(A1)_{m+n} (B1)_{m+n}}{m!\, n!\, (C1)_m (C2)_n} X^m Y^n. \tag{2.2}$$

where $(\alpha)_n$ is the Pochhammer symbol as defined before. Below we produce Fortran programs for these two hypergeometric functions.

Fortran program for ${}_2F_1$

```
      FUNCTION F21(A,B,C,Q)
      TA=A
      TB=B
      TC=C
      P=1.
      IA=0
      T1=TA
      T2=TB
      T3=TC
      G    =1.+((A*B)/C)*Q
102   IA=IA+1
      P=P*(IA+1)
      T1=T1*(A+IA)
      T2=T2*(B+IA)
      T3=T3*(C+IA)
      AA=IA+1
      Q1=Q**AA
      W=(T1/T3)*(T2/P)*Q1
      G=G+W
      AW=ABS(W)
      IF(AW-0.00000001)101,101,102
101   F21=G
      RETURN
      END
```

Fortran program for F_4

```
      FUNCTIØN F4(A1,B1,C1,C2,X,Y)
      G=1.+A1*B1*X/C1+A1*B1*Y/C2
      T1=A1
      T2=B1
      N=0
    6 N=N+1
      T1=T1*(A1+FLØAT(N))
      T2=T2*(B1+FLØAT(N))
      MM=N+2
      DØ 2 II=1,MM
      I=II-1
      M=MM-II
      T3=GAM(C2+FLØAT(I))/GAM(C2)
      T4=GAM(C1+FLØAT(M))/GAM(C1)
      RM=FLØAT(M)
      RI=FLØAT(I)
      XX=X**RM
      YY=Y**RI
      W=(T1/T3)*(T2/T4)*(XX/GAM(RI+1.))*(YY/GAM(RM+1.))
      G=G+W
    2 CØNTINUE
      AW=ABS(W)
      IF(AW-0.0000001)12,12,6
   12 F4=G
      RETURN
      END
```

Thus, with the aid of these programs ${}_2F_1$ and F_4-functions can be evaluated efficiently if the respective series given by (2.1) and (2.2) are rapidly convergent. If they are not rapidly convergent, the above Fortran programs may give inaccurate results owing to the growth of round-off error with n . However, the following transformation formula (see above reference, p.108) may be used to speed up the convergence of ${}_2F_1$-functions if $Q > \frac{1}{2}$

$$ {}_2F_1\left(\begin{matrix}A, B\\ C\end{matrix}\middle| Q\right) = \frac{\Gamma(C)\,\Gamma(C-A-B)}{\Gamma(C-A)\,\Gamma(C-B)}\, {}_2F_1\left(\begin{matrix}A, B\\ A+B-C+1\end{matrix}\middle| 1-Q\right) + $$

$$ + (1-Q)^{C-A-B}\frac{\Gamma(C)\,\Gamma(A+B-C)}{\Gamma(A)\,\Gamma(B)}\, {}_2F_1\left(\begin{matrix}C-A, C-B\\ C-A-B+1\end{matrix}\middle| 1-Q\right) \tag{2.3} $$

which reduces to

$$ {}_2F_1\left(\begin{matrix}-n, B\\ C\end{matrix}\middle| Q\right) = \frac{(C-B)_n}{(C)_n}\, {}_2F_1\left(\begin{matrix}-n, B\\ 1-n+B-C\end{matrix}\middle| 1-Q\right) \tag{2.4} $$

if ${}_2F_1$ concerned is a polynomial of degree n .

To terminate the summation routine in the above programs we make the following assumption: if

$$ W = \frac{(A)_N\,(B)_N}{(C)_N\,N!}\,Q^N \leq SIG\,, \quad \text{then} \sum_{n=N}^{\infty}\frac{(A)_n\,(B)_n}{(C)_n\,n!}\,Q^n \leq \text{tolerance} \tag{2.5} $$

for ${}_2F_1$ and similarly if

$$ W = \frac{(A1)_{M+N}\,(B1)_{M+N}}{M!\,N!\,(C1)_M\,(C2)_N}\,X^M Y^N \leq SIG, \text{ then} $$

$$ \sum_{n=N}^{\infty}\sum_{m=M}^{\infty}\frac{(A1)_{m+n}\,(B1)_{m+n}}{m!\,n!\,(C1)_m\,(C2)_n}\,X^m Y^n \leq \text{tolerance} \tag{2.6} $$

for F_4-functions, where arbitrarily small real number SIG has been taken to be 10^{-8} just as an example. Thus, once W becomes equal to or smaller than SIG the routine terminates. In the "FUNCTION F4", GAM(α) stands for the Gamma functions $\Gamma(\alpha)$. Computer software for $\Gamma(\alpha)$ can be found in any subroutine library.

Gauss' relations between ${}_2F_1$ and its two contiguous functions (see above reference, p.103) can be used as the three term recursion relations for the evaluation of the n dependent ${}_2F_1$-functions. This procedure i) considerably shortens the computing time, and ii) reduces the round-off error. Moreover, if the quantity A or B in ${}_2F_1$ happens to be a negative integer, then the above procedure leads us to utilize the three-term certain recursion relation for the shifted Jacobi polynomials (see Equations 6.4, 6.7 and 6.8 in Chapter 1).

Expressions (4.3), (4.11), (6.3), (6.6), 6.12), (6.13), (6.14), and (6.15) for the fractional loss of light α_l^o , (11.3), (11.4) for $I_{-1,l}^0$ and $I_{-1,l}^1$-integrals all from Chapter 1, and in addition (2.5), (2.11), (2.13), (2.14) and (2.16) for the moments A_{2m} from Chapter 2 have been automated. For this the above programs of ${}_2F_1$ and F_4 , and the certain recursion relation (6.7) with (6.8) (from Chapter 1) for the shifted Jacobi polynomials have been utilized. The tables of α_l^o , $I_{-1,l}^0$, $I_{-1,l}^1$ and f_{2m}-functions with large intervals in a and c have been constructed for small values of l and m by employing the above automated expressions. An inspection of the numerical tables leads to the following conclusions:

1. In general, all the approximations to true values show irregularly and slowly damping oscillations (see Table 5). This gives rise to a question: How and where to stop the summation of the respective series? It is obvious that here a similar criterion to (2.5) cannot be applied.

When a term for $n = N$ becomes smaller than an arbitrarily small SIGMA, the summation beyond N may well be large amount. We decided from the applications that the criterion can be put on the maximum number of the considered terms to achieve the wanted accuracy. For example, in general, the summation of about the first 20 terms is enough to secure three significant figures for α_l^o from expansions (6.3), (6.6), (6.12), (6.13), (6.14) and (6.15).

2. If the recursion relations are used in the re-computation of terms in the series summation of the respective expansions, in general, the results with 4-5 significant figures are obtainable, but in the first and second contact points (when $\delta = r_1 + r_2$ and $= |r_1 - r_2|$) the convergence of the expansions is considerably slowed down. Therefore, in this case 4 figures are hardly obtainable by considering first 80 terms for α_l^o and more than 100 terms for I-integrals. As it was noted, 3 figures for α_l^o are usually obtainable by the summation of only the first 20 terms.

3. If the expressions include a F_4-function in double infinite series form, in many points round-off error starts propagating before we achieve three figures in the results. Moreover, these expressions are time consuming, since we have no existing recursion relation for the re-computation of n dependent F_4-functions for successive values of n.

4. All the expansions (6.12) - (6.15) show identical behaviour in approaching to the exact values. In convergence they seemed to be slightly faster than the expansions (6.3) and (6.6) for the same α_l^o -

functions, but the former expansions take more computing time than the latter, since they include a shifted Jacobi polynomial of even order for which the recursion relation is algebraically complicated and has not been employed in our computations.

5. The elgebraic expansions (2.6), (2.15) and (2.17) for the functions f_{2m} hardly secure two significant figures up to $n \sim 15$. After this point round-off error starts propagating as it has been shown by the present author (Demircan, 1978a) and consequently the respective series diverge. This loss of significance in computations is caused by the successive subinstructions of almost equal quantities in the last summations in the form

$$M_n^{(m,\alpha)}(x) = \sum_{j=0}^{n} (-1)^j \frac{\Gamma(n+\alpha+j) \quad x^j}{\Gamma(m+\alpha+j)\, j!\,(n-j)!} =$$

$$= \frac{\Gamma(n+\alpha)}{n!\,\Gamma(m+\alpha)}\, {}_2F_1\left(\begin{matrix}-n,\ n+\alpha \\ m+\alpha\end{matrix}\,\middle|\, x\right) \tag{2.7}$$

of the respective expansions. Here, for the sake of illustrating the way in which the round-off error propagates with n, we extracted the following numerical example (see Table 7) from Demircan (1978a). This example gives the machine results of the summation of the l.h.s. of (2.7) for $X = 1$, $m = 0$ and $\alpha = 3$. It can be shown that for these elements (2.7) reduces to

$$P_n^{(2,0)}(-1) = (-1)^n \tag{2.8}$$

as exact values.

Table 7

n	$\sum_{j=0}^{n}$	n	$\sum_{j=0}^{n}$
0	1.000 00000	12	1.000 00054
1	-1.000 00000	13	-0.999 99332
2	1.000 00000	14	1.000 01144
3	-1.000 00000	15	-1.000 19455
4	1.000 00000	16	0.999 19128
5.	-1.000 00000	17	-0.994 75098
6.	1.000 00000	18	1.051 14746
7.	-1.000 00000	19	-1.275 39063
8	1.000 00000	20	0.476 56250
9	-0.999 99999	21	-5.515 62500
10	0.999 99998	22	-36.906 25000
11	-1.000 00027	23	639.750 00000
		⋮	⋮

In order to improve the numerical results for the functions f_{2m}, it is necessary to get rid of the propagating round-off error in the computations so that we could consider larger number of terms in the respective expansions. Numerically it is known that nothing can be done, for example, use of a double precision arithmetic would merely delay, but not prevent the growth of error. However, the analytical methods as the development of a recursion relation for the successive numerical evaluation of the summation (2.7) for any set of parameters may solve the problem.

3.3 Recursion Relations and the Revised Formulae for A_{2m}'s.

The basic quantities of the problem are α_l^0, $I^0_{-1,l}$, $I^1_{-1,l}$ and f_{2m}. All the other requisite quantities may be defined in terms of these basic ones. In point of fact α_l^0 constitutes merely a particular case of f_{2m} if $m = 0$ (see Eq. 2.13 in Chapter 2) and $I^1_{-1,l}$ may be written in terms of $I^0_{-1,l}$ and $I^0_{-1,\,l+2}$ by means of a recursion relation provided by Kopal (1959, p.215, Eq. 5.52) in the form

$$I^{m+1}_{\beta,\gamma} = \mu\, I^{m}_{\beta,\gamma} + \frac{1}{2}\frac{r_2}{\delta}\, I^{m}_{\beta,\gamma+2} \tag{3.1}$$

where μ is as given by (7.17) of Chapter 1. Thus, actual basic quantities are only the $I^0_{-1,l}$ and f_{2m}. The $I^0_{-1,l}$ may be easily and efficiently evaluated by means of its algebraic but slowly converging general expansion (11.3) given in Chapter 1. Its Fortran program will be produced in the following section. As for the f_{2m}-functions, it will be more convenient - as it has been discussed in the previous section - to work with the algebraic expressions (2.6), (2.15), and (2.17) whose simple analytic structure lend themselves readily for automatic computation by means of computers of very modest size. The computing time may be considerably shortened by using the recursion relation (6.7) for the shifted polynomials R_n from Chapter 1. However, a serious problem in the computation of these f_{2m}-functions is the slow convergence of the respective series and propagating round-off error with increasing n (see the conclusion 5 in the previous section) where in the computation of summation (2.7) the following procedures have been employed:

$$
\begin{aligned}
&\text{i)} \quad \text{Set} \quad M_n = \sum_{j=0}^{n} a_j \ , \quad t_0 = a_0 \\
&\text{ii)} \quad \text{Find} \quad t_{i+1} = t_i + a_{i+1} \ , \quad i = 0,1,2,\ldots,n-1 \\
&\text{iii)} \quad M_n = t_n \ .
\end{aligned}
\tag{3.2}
$$

Moreover, we have utilized the recursion relation

$$
a_{j+1} = \frac{(j-n)(n+\alpha+j)}{(j+1)(m+\alpha+j)} \cdot a_j \cdot X \ , \qquad a_0 = \frac{\Gamma(n+\alpha)}{n!\,\Gamma(m+\alpha)} \tag{3.3}
$$

in the successive evaluation of the respective terms. This recursion relation may be easily deduced by a combination of well-known formula

$$
\Gamma(A+n+1) = (A+n)^n \, \Gamma(A+1) \tag{3.4}
$$

for the Gamma functions, and (2.7) with the definition i) in Eq. (3.2).

We shall, in what follows, represent two other methods (cf., e.g., Davis and Robinowitz, 1975, p.216) to reduce the round-off error in the computation of the summation (2.7) which is a factor in Equations

(2.6), (2.15) and (2.17) for the functions f_{2m} of Chapter 2.

METHOD 1.

i) Set $M_n = \sum_{j=0}^{n} a_j \quad , \quad t_0 = a_0$

ii) Find $t_{i+1} = t_i + a_{i+1} \quad , \quad i = 0, 1, 2, \ldots, n-1$

$$\eta_{i+1} = t_{i+1} - t_i \quad ; \quad \varepsilon_{i+1} = \eta_{i+1} - a_{i+1} \quad ; \tag{3.5}$$

$$w_{i+1} = w_i - \varepsilon_{i+1} \quad , \quad w_0 = 0$$

iii) $M_n = t_n + w_n$.

METHOD 2.

i) Set $M_n = \sum_{j=0}^{n} a_j \; , \; t_0 = a_0 \; , \; t_{-1} = 0 \; , \; \varepsilon_{-1} = 0$

ii) Find $\overline{a}_i = a_i + \varepsilon_{i-1} \; ; \; u_i = t_{i-1} + \overline{a}_i \; ;$

$$\varepsilon_i = (t_{i-1} - u_i) + \overline{a}_i \tag{3.6}$$

iii) $M_n = u_n$

These methods are based upon the fact that when a small number is added to a large number a part of the accuracy inherent in the former will be lost.

Next, we shall represent the derivation of a recursion relation (cf. Demircan, 1978a, b) by which all the sequence of the summation (2.7) for the successive values of n can be generated recursively in a very simple manner. The process is not only fast but also prevents the round-off error from propagating. It is well adapted to modern computing machinery, and may be useful even if only one member of the sequence is desired.

In order to illustrate the derivation of this recursion relation let us first write the well-known three-term recurrence relation satisfied by the orthogonal polynomials

$$M_{n+1}(X) = (A_n X + B_n) M_n(X) + C_n M_{n-1}(X) . \tag{3.7}$$

It may be shown that summation (2.7) as polynomials of degree n in X satisfies this recurrence relation (cf. Demircan, 1978a). In order to derive the respective coefficients A_n, B_n and C_n let us write (2.7) more explicitly for n - 1, n and n + 1, as

$$M_{n-1}^{(m,\alpha)}(X) = a_{n-1,0} + a_{n-1,1} X + \cdots + a_{n-1,n-1} X^{n-1} \tag{3.8}$$

$$M_{n}^{(m,\alpha)}(X) = a_{n,0} + a_{n,1} X + \cdots + a_{n,n} X^{n} \tag{3.9}$$

$$M_{n+1}^{(m,\alpha)}(X) = a_{n+1,0} + a_{n+1,1} X + \cdots + a_{n+1,n+1} X^{n+1} \tag{3.10}$$

where

$$a_{n,j} = (-1)^j \frac{\Gamma(n+\alpha+j)}{\Gamma(m+\alpha+j)} \frac{1}{j!\,(n-j)!} , \quad n \geqslant j . \tag{3.11}$$

It may be easily shown that

$$a_{n,\,j+1} = \frac{(j-n)\,(n+\alpha+j)}{(j+1)\,(m+\alpha+j)}\ a_{n,\,j} \quad , \quad j \leq n-1 \tag{3.12}$$

(see Eq. 3.3) and

$$a_{n+1,\,j} = \frac{(n+\alpha+j)}{(n+1-j)}\ a_{n,\,j} \quad , \quad j \leq n\,. \tag{3.13}$$

Now, if we construct the r.h.s. of (3.6) with the aid of (3.7) and (3.8) we get

$$\begin{aligned} M_n^{(m,\alpha)}(x) = {} & \left[B_n\, a_{n,0} + C_n\, a_{n-1,0} \right] + \\ & + \left[A_n\, a_{n,0} + B_n\, a_{n,1} + C_n\, a_{n-1,1} \right] x + \cdots + \\ & + \left[A_n\, a_{n,n-2} + B_n\, a_{n,n-1} + C_n\, a_{n-1,n-1} \right] x^{n-1} + \\ & + \left[A_n\, a_{n,n-1} + B_n\, a_{n,n} \right] x^n + A_n\, a_{n,n}\, x^{n+1} \,. \end{aligned} \tag{3.14}$$

Thus, one can easily write by comparing Equations (3.10) and (3.14) that

$$\begin{aligned} a_{n+1,0} &= B_n\, a_{n,0} + C_n\, a_{n-1,0} && , \quad n \geqslant 1 \\ a_{n+1,1} &= A_n\, a_{n,0} + B_n\, a_{n,1} + C_n\, a_{n-1,1} && , \quad n \geqslant 2 \\ &\ \vdots \\ a_{n+1,n-1} &= A_n\, a_{n,n-2} + B_n\, a_{n,n-1} + C_n\, a_{n-1,n-1} && , \quad n \geqslant 2 \end{aligned}$$

$$a_{n+1,n} = A_n \, a_{n,n-1} + B_n \, a_{n,n} \quad , \; n \geqslant 1 \tag{3.15}$$

$$a_{n+1,n+1} = A_n \, a_{n,n} \quad , \; n \geqslant 0$$

Consequently, we find from these results by using definition (3.11) and recursion relations (3.12) and (3.13) that

$$A_n = -\frac{(2n+\alpha)(2n+\alpha+1)}{(n+1)(m+n+\alpha)}$$

$$B_n = (2n+\alpha) + \frac{n(m+n+\alpha-1)}{(2n+\alpha-1)} A_n \tag{3.16}$$

and

$$C_n = \frac{(n+\alpha)(n+\alpha-1)}{n(n+1)} - \frac{(n+\alpha-1)}{n} B_n \, .$$

Thus, the derivation of the requisite recurrence relation for computation of the sequence $M_n^{(m,\alpha)}(X)$ given by (2.7) is completed. This procedure not only shortens the computing time, but also reduces the round-off error in the evaluation of (2.7) for any set of parameters m, α and n. Therefore, the functions $f_{2m}(a, c_o)$ can now be evaluated easily and effectively by means of the recursion relations (6.7) for the shifted Jacobi polynomials from Chapter 1, and newly derived (3.7) with (3.16) for the new polynomials.

Finally, we want to give a few special values of summation (2.7) for $n = 0$, 1 and 2

$$M_0^{(m,\alpha)}(X) = \frac{\Gamma(\alpha)}{\Gamma(m+\alpha)}$$

$$M_1^{(m,\alpha)}(X) = M_0^{(m,\alpha)}(X)\left[\alpha - \frac{\alpha(\alpha+1)}{(m+\alpha)} X\right] \tag{3.17}$$

and

$$M_2^{(m,\alpha)}(X) = M_0^{(m,\alpha)}(X)\left[\frac{1}{2}\alpha(\alpha+1) - \frac{\alpha(\alpha+1)(\alpha+2)}{(m+\alpha)} X + \right.$$

$$\left. + \frac{\alpha(\alpha+1)(\alpha+2)(\alpha+3)}{2(m+\alpha)(m+\alpha+1)} X^2 \right]$$

which will be necessary to start the computation of the sequence $M_n^{(m,\alpha)}(X)$ by using recurrence relations derived.

According to the above development, Equations (2.5), (2.6), (2.14), (2.15), (2.16) and (2.17) for the moments may be revised as follows:

$$A_{2m} = L_1 \frac{b^2}{a^{2m}}\left(\frac{r_1}{\sin i}\right)^{2m}(1-c_0^2)^{m+1} f_{2m}(a, c_0) \tag{3.18}$$

where

$$f_{2m} = \Gamma(m+1)\sum_{l=0}^{\Lambda} C^{(l)}\Gamma(v)(1-c_0^2)^{v}\sum_{n=0}^{\infty}\frac{n!\,(v+2n+2)}{(n+1)\,\Gamma(v+n+1)} \times$$

$$\times\left[R_n^{(l,v)}(a)\right]^2 M_n^{(m,v+2)}(1-c_0^2) \tag{3.19}$$

$$= \frac{4}{\sqrt{\pi}}\Gamma(m+1)\sum_{l=0}^{\Lambda} C^{(l)}\Gamma(v)(1-c_0^2)^{v-\frac{1}{2}}\sum_{n=0}^{\infty}\frac{(v+2n+\frac{3}{2})\,\Gamma(n+\frac{3}{2})}{(2n+1)(2n+2)\,\Gamma(v+n+1)} \times \tag{3.20}$$

$$\times\ R_{2n}^{(2,2\nu)}(a)\ M_n^{(m,\nu+\frac{3}{2})}(1-c_0^2)$$

or

$$= \Gamma(m+1)\sum_{l=0}^{\Lambda} C^{(l)} \frac{\Gamma(2\nu+1)}{\nu\Gamma(\nu+\frac{1}{2})}(1-c_0^2)^{\nu-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{(2n)!(2n+\nu+\frac{3}{2})\Gamma(\nu+n+\frac{1}{2})}{(n+1)!\,\Gamma(2n+2\nu+1)} \times$$

$$\times\ R_{2n}^{(2,2\nu)}(a)\ M_n^{(m,\nu+\frac{3}{2})}(1-c_0^2)\ . \tag{3.21}$$

3.4 Algorithms for α_l^0, $I_{-1,l}^0$ and f_{2m}-Functions.

1. FUNCTION ALFAO

Purpose

ALFAO calculates approximate values (accuracy almost 4 figures) of the fractional loss of light α_l^0 for any type of eclipse and any degree l of the adopted law of limb-darkening (see Eq. 1.3 in Chapter 1).

Method

It uses the formula (6.6) for α_l^0 with the recursion relation (6.7) for the shifted Jacobi polynomials from Chapter 1. The coefficients

$$F_n = \Gamma(\nu)\frac{n!\,(\nu+2n+2)}{(n+1)\,\Gamma(\nu+n+1)} \tag{4.1}$$

in (6.6) are also evaluated recursively without any resort to gamma functions, as

$$F_n = FI_n \times \frac{(\nu+2n+2)}{(n+1)}\ , \tag{4.2}$$

where

$$FI_{n+1} = FI_n \times \frac{(n+1)}{(\nu+n+1)}\ , \tag{4.3}$$

with

$$Fl_0 = \frac{1}{\nu} \text{ for } n=0, \; Fl_1 = \frac{1}{\nu(\nu+1)} \text{ for } n=1, \text{ etc.} \quad (4.4)$$

Usage

It can be used solely through the argument list.

It requires no subprograms.

Description of parameters

L - An integer specifying the adopted degree l of the law of limb-darkening (L = 0 for α_0^0, L = 1 for α_1^0, etc.).

NM - Number of terms to be considered from the infinite series summation (6.6) of Chapter 1. It is related with the accuracy of the results for NM $\lesssim$ 100.

A11 - Stands for the quantity $a = r_1/(r_1 + r_2)$.

C11 - Stands for the quantity $c = \delta/(r_1 + r_2)$.

Accuracy

For NM = 15, ~ 2 significant figures are accurate.

For NM = 25, ~ 3 significant figures are accurate.

For NM = 80, ~ 4 significant figures are accurate.

```
      FUNCTION ALFAO(L,NM,A11,C11)
      DIMENSION R1(80),R2(80),F1(80)
      NX=NM
      NX=NX-1
      V=FLOAT(L+2)/2.
      CH=ABS(C11-1.)
      IF(CH-.0001)20,20,11
20    ALFAO=0.
      RETURN
```

```
 11     IF(A11-.5)42,42,43
 42     XA=1.-2.*A11-C11
        IF(XA+.0001)43,41,41
 41     ALFA0=1./V
        RETURN
 43     PP=V+1.
        Q=V+2.
        Q1=V*(Q+1.)
        B11=1.-A11
        A12=A11**2
        B12=B11**2
        C12=C11**2
        C22=1.-C12
        E=B12*C22**PP
        F10=1./V
        F1(1)=1./(V+1.)/V
        F1(2)=2./(V+2.)/(V+1.)/V
        FP=F10*(V+2.)
        FQ= F1(1)*(V+4.)/2.
        FR=F1(2)*(V+6.)/3.
        R1(1)=-(V+1.)+(Q+1.)*A11
        R2(1)=Q-(Q+1.)*C22
        R1(2)=(V+1.)*Q/2.-Q*(Q+2.)*A11+(Q+2.)*(Q+3.)*A12/2.
        R2(2)=Q1/2.-(Q+1.)*(Q+2.)*C22+(Q+2.)*(Q+3.)/2.*C22**2.
        AL= FP+FQ*R1(1)**2.*R2(1)+FR*R1(2)**2.*R2(2)
        DO 100 N=2,NX
        NN=N+1
        AN=FLOAT(N)
        BN=2.*AN
        RN=FLOAT(NN)
        PN=2.*RN
        F1(NN)=F1(N)*RN/(V+RN)
        F=F1(NN)*(V+PN+2.)/(RN+1.)
        D11=(BN+Q)*(BN+Q+1.)/RN/(AN+Q)
        D1A=(BN+Q)*(2.-V**2.)-(BN+Q)**3.
        D1B=2.*RN*(AN+Q)*(BN+Q-1.)
        D1C=(AN+1.)*(AN+V)*(BN+Q+1.)
        D1D=RN*(AN+Q)*(BN+Q-1.)
        D12=D1A/D1B
        D13=D1C/D1D
        D21=-D11
        D22=AN*(AN+Q-1.)*D21/(BN+Q-1.)+BN+Q
        D23=(AN+Q-1.)*D22/AN-(AN+Q)*(AN+Q-1.)/AN/RN
        R1(NN)=(D11*A11+D12)*R1(N)-D13*R1(N-1)
        R2(NN)=(D21*C22+D22)*R2(N)-D23*R2(N-1)
        G=F*R1(NN)**2.*R2(NN)
        AL=AL+G
100     CONTINUE
        ALFA0=AL*E
        RETURN
        END
```

2. FUNCTION AINTO

Purpose

AINTO calculates approximate values (accuracy almost 4 figures) of the $I^{0}_{-1,l}$ -integrals for any type of eclipse and for any value of real $l \geqslant 0$.

Method

It uses the formula (11.3) for $I^{0}_{-1,l}$-integrals with the recursion relation (6.7) for the shifted Jacobi polynomials from Chapter 1. A recursion relation similar to (4.2) is also employed for the successive evaluation of the coefficients

$$F_n = \Gamma(\nu) \frac{n!(\nu+2n+1)}{\Gamma(\nu+n+1)} \tag{4.5}$$

in (11.3) for the respective integrals, as

$$F_n = Fl_n \times (\nu+2n+1) \tag{4.6}$$

where

$$Fl_{n+1} = Fl_n \times \frac{n+1}{n+\nu+1} \tag{4.7}$$

with

$$Fl_0 = \frac{1}{\nu} \text{ for } n=0, \quad Fl_1 = \frac{1}{\nu(\nu+1)} \text{ for } n=1 \text{ etc. } . \tag{4.8}$$

Usage

It can be used solely through the argument list.

It requires no subprograms.

Description of parameters.

Same as in ALFAO.

Accuracy

Slightly lower than that of ALFAO.

```
      FUNCTIØN AINTO(L,NM,A11,C11)
      DIMENSIØN R1(80),R2(80),F1(80)
      NX=NM
      NX=NX-1
      V=FLØAT(L+2)/2.
      CH=ABS(C11-1.)
      IF(CH-.0001)20,20,11
20    AINTO=0.
      RETURN
11    IF(A11-.5)42,42,43
42    XA=1.-2.*A11-C11
      IF(XA+.0001)43,41,41
41    AINTO=0.
      RETURN
43    VV=2.*V
      Q=V+1.
      Q1=V*(Q+1.)
      B11=1.-A11
      A12=A11**2
      B12=B11**2
      C12=C11**2
      C22=1.-C12
      F10=1./V
      F1(1)=1./(V+1.)/V
      F1(2)=2./(V+2.)/(V+1.)/V
      FP=F10*(V+1.)
      FQ=F1(1)*(V+3.)
      FR=F1(2)*(V+5.)
      E=(A11/B11)**VV*B12*C22**V
      R1(1)=-(V+1.)+(Q+1.)*A11
      R2(1)=Q-(Q+1.)*C22
      R1(2)=(V+2.)*Q/2.-(Q+1.)*(Q+2.)*A11+(Q+2.)*(Q+3.)*A12/2.
      R2(2)=(V+2.)*Q/2.-(Q+1.)*(Q+2.)*C22+(Q+2.)*(Q+3.)*C22**2/2.
      AL=FP+FQ*R1(1)**2.*R2(1)+FR*R1(2)**2.*R2(2)
      DØ 100 N=2,NX
      NN=N+1
      AN=FLØAT(N)
      BN=2.*AN
      RN=FLØAT(NN)
      PN=2.*RN
```

```
      F1(NN)=F1(N)*RN/(V+RN)
      F=F1(NN)*(V+PN+1.)
      D11=(BN+Q)*(BN+Q+1.)/RN/(AN+Q)
      D1A=(BN+Q)*(1.-V**2.)-(BN+Q)**3.
      D1B=2.*RN*(AN+Q)*(BN+Q-1.)
      D1C=AN*(AN+V)*(BN+Q+1.)
      D1D=RN*(AN+Q)*(BN+Q-1.)
      D12=D1A/D1B
      D13=D1C/D1D
      D21=-D11
      D22=AN*(AN+Q-1.)*D21/(BN+Q-1.)+BN+Q
      D23=(AN+Q-1.)*D22/AN-(AN+Q)*(AN+Q-1.)/AN/RN
      R1(NN)=(D11*A11+D12)*R1(N)-D13*R1(N-1)
      R2(NN)=(D21*C22+D22)*R2(N)-D23*R2(N-1)
      G=F*R1(NN)**2.*R2(NN)
      AL=AL+G
100   CONTINUE
      ATNTO=AL*E
      RETURN
      END
```

3. FUNCTION F2M

Purpose

F2M calculates approximate values (accuracy almost 4 figures) of the constituent $f'_{2m}(a, c_o)$-functions (see Eq. 4.13 in Chapter 2) for the moments A_{2m} of the light curves for any type of eclipse and for the quadratic law of limb-darkening ($\lambda = 2$).

Method

It uses (3.18) of the previous section to construct f'_{2m}'s:

$$f'_{2m} = \frac{b^2}{a^{2m}} (1 - c_o^2)^{m+1} f_{2m} \qquad (4.9)$$

The recursion relation (6.7) with (6.8) from Chapter 1, and (3.7) with (3.16) from the previous section have been utilized to evaluate the f_{2m}'s recursively. Remember that the general form

$$f_{2m} = \sum_{l=0}^{\Lambda} C^{(l)} f_{2m}^{(l)} \tag{4.10}$$

reduces to

$$\alpha = \sum_{l=0}^{\Lambda} C^{(l)} \alpha_l^0 \tag{4.11}$$

for $m = 0$, where the coefficients $C^{(l)}$ are associated with the law of limb-darkening of degree Λ. T hey can be given by (see Eq. 1.4 in Chapter 1)

$$C^{(0)} = \frac{3(1-u_1)}{3-u_1} , \quad C^{(1)} = \frac{3u_1}{3-u_1} \tag{4.12}$$

for a linear law of limb-darkening (i.e., when $\Lambda = 1$) and

$$\left.\begin{aligned} C^{(0)} &= \frac{6(1-u_1-u_2)}{(6-2u_1-3u_2)} \\ C^{(1)} &= \frac{6u_1}{(6-2u_1-3u_2)} \\ C^{(2)} &= \frac{6u_2}{(6-2u_1-3u_2)} \end{aligned}\right\} \tag{4.13}$$

for a quadratic law of limb darkening ($\Lambda = 2$). Thus the "FUNCTION F2M" for $m = 0$ permits us to evaluate the α_l^0-functions as well.

FUNCTION F2M also utilizes the closed form expressions in the case of total eclipses for integer values of m (see Equations 4.21 - 4.24 in Chapter 2).

Usage

It can be used through the argument list. The user must supply one additional function program: GAM(A) for the gamma functions $\Gamma(A)$. Moreover, the following coefficients have to be provided by the main program or by another subprogram. This is done to save the computing time since the F2M is called repeatedly with the same coefficients given below.

i) The coefficients C denoting the $C^{(l)}$'s for the quadratic law of limb-darkening (see Eq. 4.13).

ii) F_n as given by (4.1) for every adopted l (for the quadratic law $l = 0$, 1 and 2) and considered n .

iii) The coefficients D11, D12 and D13 (for every l and n) of the recursion relation for the shifted Jacobi polynomials as given by (6.8) of Chapter 1.

iv) The coefficients D21, D22 and D23 (for every l , m and n) of the recursion relation for the new polynomials $M_n(X)$ as given by (3.7) and (3.16) in the previous section.

Description of parameters

NM - Same as in ALFAO and AINTO.

M - An integer specifying the index m of f_{2m} .

A11, C11 - Same as in ALFAO and AINTO.

D11, D12, D13 – The coefficients A_n, B_n and C_n of the recursion relation for the $R_n(X)$-polynomials.

D21, D22, D23 – The coefficients A_n, B_n and C_n of the recursion relation for the $M_n(X)$-polynomials.

Restriction

The parameter M cannot be any integer but 1, 2, 3, 4, 5, 6, 7 and 8 specifying m = 0, 1, 2, 3, ½, 3/2, ¼ and ¾ respectively.

Accuracy

Almost the same as in ALFAO.

```
      FUNCTION F2M(NM,M,A11,C11)
      EXTERNAL GAM
      COMMON C(3),F(3,100),D11(3,100),D12(3,100),D13(5,100),
     1D21(8,3,100),D22(8,3,100),D23(8,3,100)
      DIMENSION FM(3),R1(80),R2(80)
      B11=1.-A11
      A12=A11**2
      B12=B11**2
      Z1=B12/A12
      C12=C11**2
      C22=1.-C12
      NX=NM-1
      IF(A11-.5)42,42,43
42    XA=1.-2.*A11-C11
      IF(XA)63,41,41
41    X=(B12-C12)/A12
      GO TO (44,45,46,47)M
44    F2M=1.
      RETURN
45    F2M=X
      RETURN
46    F2M=X**2+Z1*(C(1)+8./15.*C(2)+1./3.*C(3))
      RETURN
47    F2M=X**3+X*Z1*(3.*C(1)+8./5.*C(2)+C(3))+
     1Z1*(C(1)+48./105.*C(2)+1./4.*C(3))
      RETURN
43    GO TO(11,12,13,14,15,16,17,18)M
11    RM=0.
      GO TO 20
12    RM=1.
      GO TO 20
13    RM=2.
      GO TO 20
```

```
14    RM=3.
      GO TO 20
15    RM=.5
      GO TO 20
16    RM=1.5
      GO TO 20
17    RM=.25
      GO TO 20
18    RM=.75
20    DO 30 II=1,3
      IF(C(II))51,50,51
 50   FM(II)=0.
      GO TO 30
 51   IL=II-1
      V=FLOAT(IL+2)/2.
      PP=RM+V+1.
      E=B12*C22**PP/A12**RM*GAM(RM+1.)*C(II)
      A=V+2.
      R1(1)=-(V+1.)+(V+3.)*A11
      R1(2)=A*(V+1.)/2.-A*(A+2.)*A11+(A+2.)*(A+3.)/2.*A12
      RMA=RM+A
      GA=GAM(A)
      GMA=GAM(RMA)
      S=GA/GMA
      R20=S
      XP=A-A*(A+1.)*C22/RMA
      R2(1)=XP*S
      XR=A*(A+1.)/2.-A*(A+1.)*(A+2.)*C22/RMA+A*(A+1.)*(A+2.)*
     1C22**2/2./RMA/(RMA+1.)*(A+3.)
      R2(2)=XR*S
      F0=A/(A-2.)
      F1=(A+2.)/(A-1.)/(A-2.)/2.
      F2=2.*(A+4.)/(A-1.)/(A-2.)/A/3.
      AL=F0*R20+F1*R1(1)**2*R2(1)+F2*R1(2)**2*R2(2)
      DO 10 N=2,NX
      NN=N+1
      R1(NN)=(D11(II,N)*A11+D12(II,N))*R1(N)-D13(II,N)*R1(N-1)
      R2(NN)=(D21(MR,II,N)*C22+D22(MR,II,N))*R2(N)-D23(MR,II,N)
     1*R2(N-1)
      G=F(MR,II,NN)*R1(NN)**2*R2(NN)
      AL=AL+G
10    CONTINUE
      FM(II)=AL*E
30    CONTINUE
      F2M=FM(1)+FM(2)+FM(3)
      RETURN
      END
```

CHAPTER 4

THE PRACTICAL PROCEDURES FOR OBTAINING THE ECLIPSE ELEMENTS

In the preceding chapters the fundamental quantities for the new approach to the problem of an analysis of the light changes of eclipsing binary systems in the frequency-domain have been given as the simple algebraic formulae and their fast efficient computation in practice ~~has~~ have been discussed. The algorithms in Fortran have also been enclosed for the numerical evaluation of these quantities.

The moments A_{2m} of the light curves (for the spherical model) have been presented (Chapter 2) i) in the form of general closed expressions in terms of the Kopal's J-integrals for integer values of the quantity m , and ii) in the form of infinite series expansions. In these latter expressions, the terms have been given as the product of two different polynomials which satisfy certain three-term recursion formulae. Thus, the numerical values of the theoretical moments A_{2m} can be generated recursively up to four significant figures for any given set of eclipse elements.

When we evaluate the observational values of these moments with the aid of definition (0.1) in Chapter 2 for any positive value of real m , they constitute simple algebraic relations between the unknown elements of the eclipses and the observed characteristics of the light curves. This can be utilized to solve the eclipse elements in two ways: i) with a direct method ~~as~~ for minimization ~~to~~ of errors of the observational moments A_{2m} (area fitting), and ii) with an indirect method (for the procedures see Paper XIV) as a suitable elimination of the unknown parameters and solving the remaining two non-linear independent equations for the remaining two unknown parameters a and c_o .

The aim of the present chapter will be to utilize the results obtained in the preceding chapters for the development of practical procedures (cf. Paper XIV) for obtaining the elements of any eclipsing system from the observed photometric data by their analysis in the frequency domain for any type of eclipse, any proximity of the two components, and any degree of the law of limb-darkening of the eclipsed star.

In the first section the wide binaries with ~~the~~ spherical components will be considered. The generalized procedures ~~to the~~ for systems consisting of arbitrarily distorted stars will be given in section 2. In these procedures, the distribution of surface brightness for the apparent discs of the stars will be assumed to be radially symmetrical and the respective limb darkening coefficients will be taken to be known from the theory of the stellar atmospheres. Section 3 is devoted to the development of a method to the solution of the two respective simultaneous non-linear equations for the unknown parameters a and c_o . The methods from section 1 for obtaining the eclipse elements of wide binaries from one observed minimum alone have been automated and tested on the light curves of YZ(21) Cassiopeiae (cf. Paper XV) and β Persei (Algol). The results of these applications will be given in the final section.

4.1 Systems Consisting of Spherical Stars.

The aim of the present section will be to outline the practical methods (cf. Paper XIV) by which an analysis of the light curves for the elements of eclipsing systems which consist of spherical stars can be performed in the frequency-domain for any type of eclipse. At first sight, the adoption of spherical shape for both components of such systems may seem to be unduly restrictive, and limit the applicability of a model based upon it only to a very small class of systems which are sufficiently wide for mutual distortion of both stars to be negligible.

However, by virtue of their relatively low probability of discovery (narrow eclipses) alone, such systems are likely to constitute but a very small fraction of eclipsing systems known to us at the present time. As the probability of their discovery increases with increasing proximity of their components, so does their mutual distortion; and the proximity effects arising from it are bound to cease to be negligible. However, as it will be outlined in Section 2, a solution for the elements of distorted eclipsing systems can always be reduced to one based on a spherical model. This fact should make the subject matter of the present section fundamental for an analysis of any light curve in the frequency-domain - regardless of whether or not the form of the components whose mutual eclipses give rise to the observed light changes is spherical or distorted: in the former case, it represents the final solution; in the latter, a necessary prerequisite for subsequent developments.

As has been pointed out already in Chapter 2, the fundamental observed quantities which will serve as a basis for a determination of the elements of the eclipse giving rise to the observed light curve are the moments A_{2m} of such light curves defined by Eq. (0.1) in Chapter 2. The basic data from which we depart in quest of our solution are represented by a set of these moments A_{2m}, the number of which must not be less than that of the unknowns sought for. The zeroth moment (see Eq. 2.13 in Chapter 2)

$$A_0 = 1-\lambda = L_1 \sum_{l=0}^{\Lambda} C^{(l)} \alpha_l^0 \tag{1.1}$$

$$= L_1 \alpha(a, c_0)$$

where $\lambda \equiv l(0)$ stands for the light of the system at the moment of conjunction of the respective minimum (caused by an eclipse of the star of fractional luminosity L_1). The $\alpha(a, c_0) \equiv \alpha_0$ signifies the maximum obscuration of the star undergoing eclipse, of luminosity L_1, depending on a and c_0. The fractional luminosities $L_{1,2}$ are defined so that

$$L_1 + L_2 = 1 \,. \tag{1.2}$$

Even in the absence of a knowledge of the foregoing parameters a, c_0 and L_1, the value of A_0 can be ascertained from Eq. (1.1) in terms of the observed depth λ of the respective minimum;while all higher moments (corresponding to $m > 0$) can be established (cf. Figure 3) by a quadrature (or planimetry) of its observed light curve. As could be expected from the fact that, within eclipses, $\sin\theta < 1$ and therefore, $\sin^{2m}\theta \ll 1$ for increasing values of m, it follows that, numerically, $A_{2m} > A_{2\mu}$ for $m < \mu$.

On the other hand, the theoretical values of the moments A_{2m} for any type of eclipse, expressed in terms of the elements L_1, a and c_o, have been established in the previous Chapters 2 and 3.

Suppose, now, that the coefficients u_l of limb-darkening can be estimated from the theory of stellar atmospheres, and that our aim is to determine the four elements r_1, r_2, i and L_1 of the eclipse from the moments A_{2m} (the number of which must not be less than four). In general, if we form the ratios

$$\mathcal{B} \equiv \frac{A_{x_1}^{k_1} A_{x_2}^{k_2} \cdots A_{x_n}^{k_n}}{A_{y_1}^{j_1} A_{y_2}^{j_2} \cdots A_{y_m}^{j_m}}, \tag{1.3}$$

we can obtain that

$$\mathcal{B} = \frac{f_{x_1}^{k_1} f_{x_2}^{k_2} \cdots f_{x_n}^{k_n}}{f_{y_1}^{j_1} f_{y_2}^{j_2} \cdots f_{y_m}^{j_m}} \equiv g(\alpha, c_o) \tag{1.4}$$

provided that

$$k_1 x_1 + k_2 x_2 + \cdots + k_n x_n = j_1 y_1 + j_2 y_2 + \cdots + j_m y_m, \tag{1.5}$$

with real powers k_n's and j_m's, and the positive orders x_n's and

y_m's $(m, n = 1, 2, 3, \ldots)$. At least two values of y_m's should be different from x_n's. For two different sets of parameters k_n, x_n, j_m and y_m, Eq. (1.4) constitutes two independent relations between the observed quantities and the eclipse parameters a and c_o, since the quantities B given by (1.3) can be established from the observations as ratios of the respective powers of the respective moments of the light curves. In particular, if we suppose that $n = m = 2$; $j_1 = j_2 = k_1 = k_2 = 1$; $x_1 = x_2 = 2$ and $y_i = 0, 2$; $y_2 = 4, 6$, we obtain

$$\frac{A_2^2}{A_0 A_4} = \frac{f_2^2}{f_0 f_4} \equiv g_2(\alpha, c_0) \tag{1.6}$$

and

$$\frac{A_4^2}{A_2 A_6} = \frac{f_4^2}{f_2 f_6} \equiv g_4(\alpha, c_0) \tag{1.7}$$

These two simultaneous nonlinear functions can be solved for the parameters a and c_o (this will be the subject of the Section 3 of this chapter). Once the values of the unknown parameters a and c_o have been determined we can use any one of the ratios A_{x_i}/A_{x_j} $(x_i \neq x_j)$ to determine the fractional radius r_1 of the star undergoing eclipse. If, for example, we use the ratio A_2/A_0, with the aid of (3.18) of Chapter 3 it may be inverted to yield r_1, as

$$r_1^2 = \frac{a^2 f_0 A_2}{(1-c_0^2) f_2 A_0 + c_0^2 f_0 A_2} , \tag{1.8}$$

or with the aid of Eq. (4.9) of Chapter 3

$$r_1^2 = \frac{f_0' A_2}{\left(\frac{c_0}{a}\right)^2 f_0' A_2 + f_2' A_0} , \tag{1.9}$$

since in (3.18) of Chapter 3

$$\sin i = \left[1 - \left(\frac{c_0}{a}\right)^2 r_1^2\right]^{\frac{1}{2}} . \tag{1.10}$$

The other parameters can be obtained very easily; r_2 from the definition of a, as

$$r_2 = \frac{r_1 (1-a)}{a} , \tag{1.11}$$

i from the definition of c_0, as

$$i = \cos^{-1}\left[(r_1 + r_2)\, c_0\right] , \tag{1.12}$$

and, finally, L_1 from the moment $A_0 = 1 - \lambda$ (see Eq. 1.1),

$$L_1 = \frac{1}{b^2(1-c_0^2)} \frac{A_0}{f_0} = \frac{A_0}{f_0'} . \tag{1.13}$$

In more specific terms, the fortran program F2M given in the previous chapter can be used as a basis of the construction of appropriate tables of the quantities $g(a, c_o)$ for two different sets of parameters; and these tables, for example, in the case of g_2, g_4 may be inverted to yield the unknown parameters $a(g_2, g_4)$ and $c_o(g_2, g_4)$ in terms of the observable quantities g_2 and g_4 for any adopted values of the coefficients u_l of limb-darkening. This work has been done for g_2 and g_4 for a quadratic law of limb-darkening, characterized by the coefficients $u_1 = 0.6500$ and $u_2 = -0.0226$, which reproduces the solution of the equation of radiative transfer in grey plane-parallel stellar atmospheres within errors graphically shown on the accompanying Figure 5. The corresponding tables of $g_2(a, c_o)$, $g_4(a, c_o)$ were constructed (see the Appendices 4 and 5) with the aid of a CDC 7600 electronic computer of the University of Manchester, and their values accurate to almost four significant digits (tabulated at intervals permitting linear interpolation between neighbouring entries within errors not in excess of 0.0004). The general behaviour of these functions is illustrated on the accompanying Figures 6-11, and the graphically inverted almost two digit accurate $a(g_2, g_4)$ and $c_o(g_2, g_4)$ tables are given in the accompanying Tables 8 and 9.

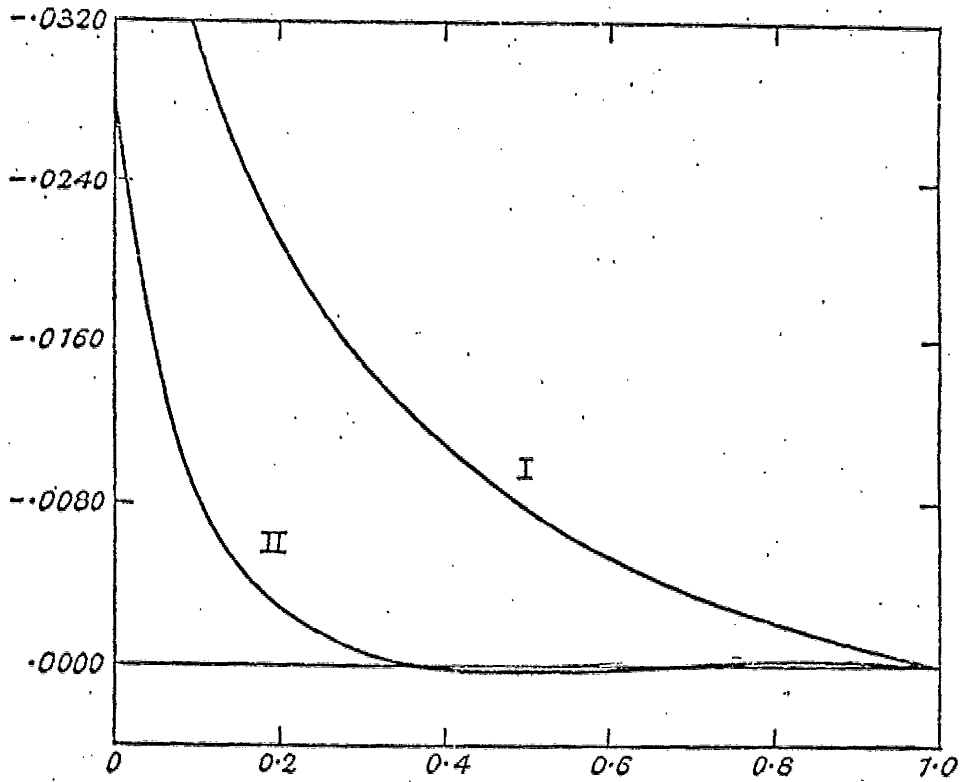

Figure 5. Deviations of the linear (I) and quadratic (II) approximations of the law of limb-darkening from the exact solution of radiative transfer in grey plane-parallel atmospheres (abscissae) plotted against cos γ of the angle of foreshortening (ordinate). Curve I represents the deviations of the exact solution of the problem from a linear approximation of the form $J(\gamma) = 0.4 + 0.6 \cos\gamma$; curve II, the corresponding deviations from the quadratic approximation $J(\gamma) = 0.373 + 0.650 \cos\gamma - 0.023 \cos^2\gamma$ (after Kopal, 1949).

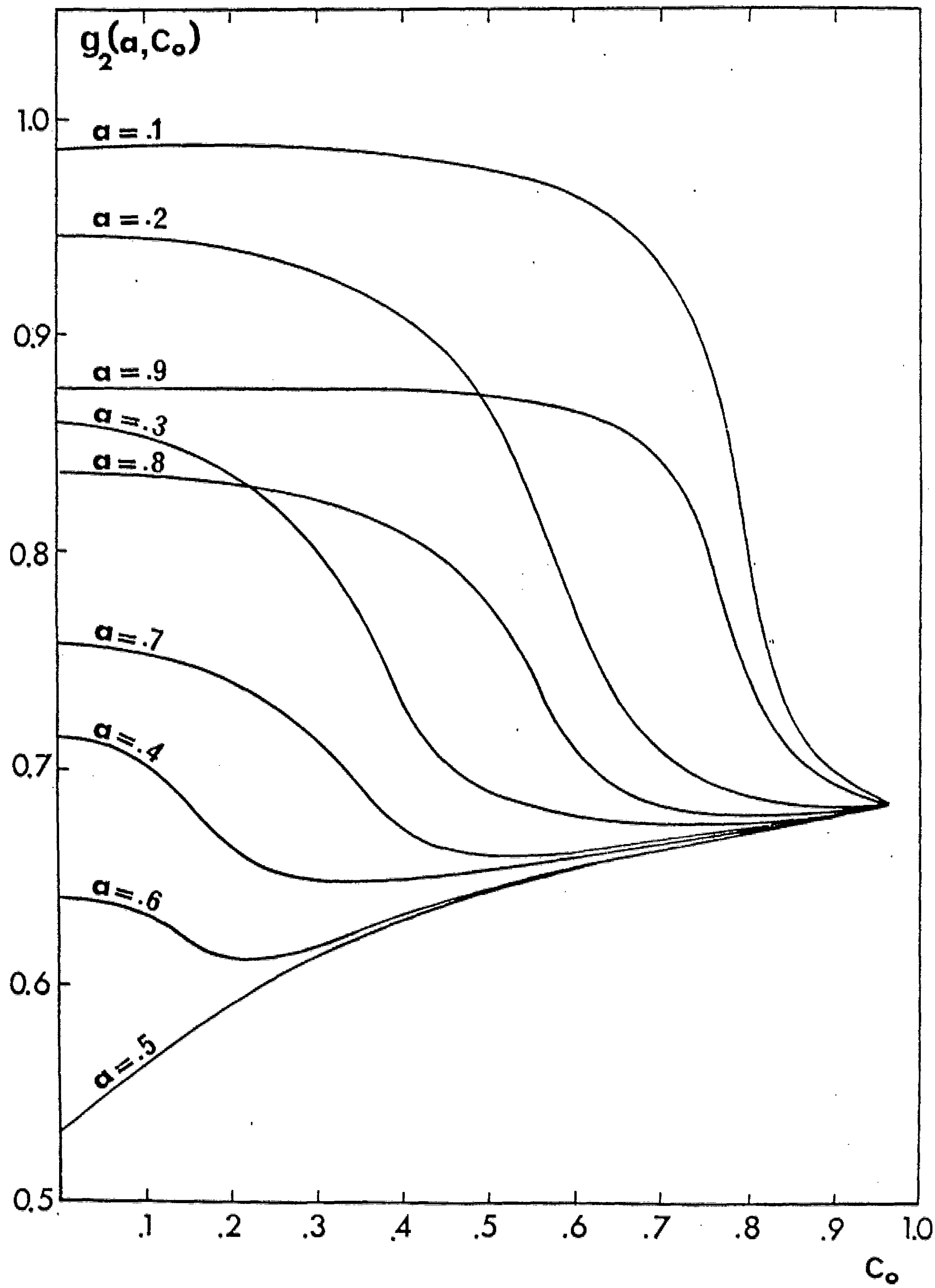

Figure 6. A plot of the function $g_2(a, c_o)$ versus c_o for fixed values of a .

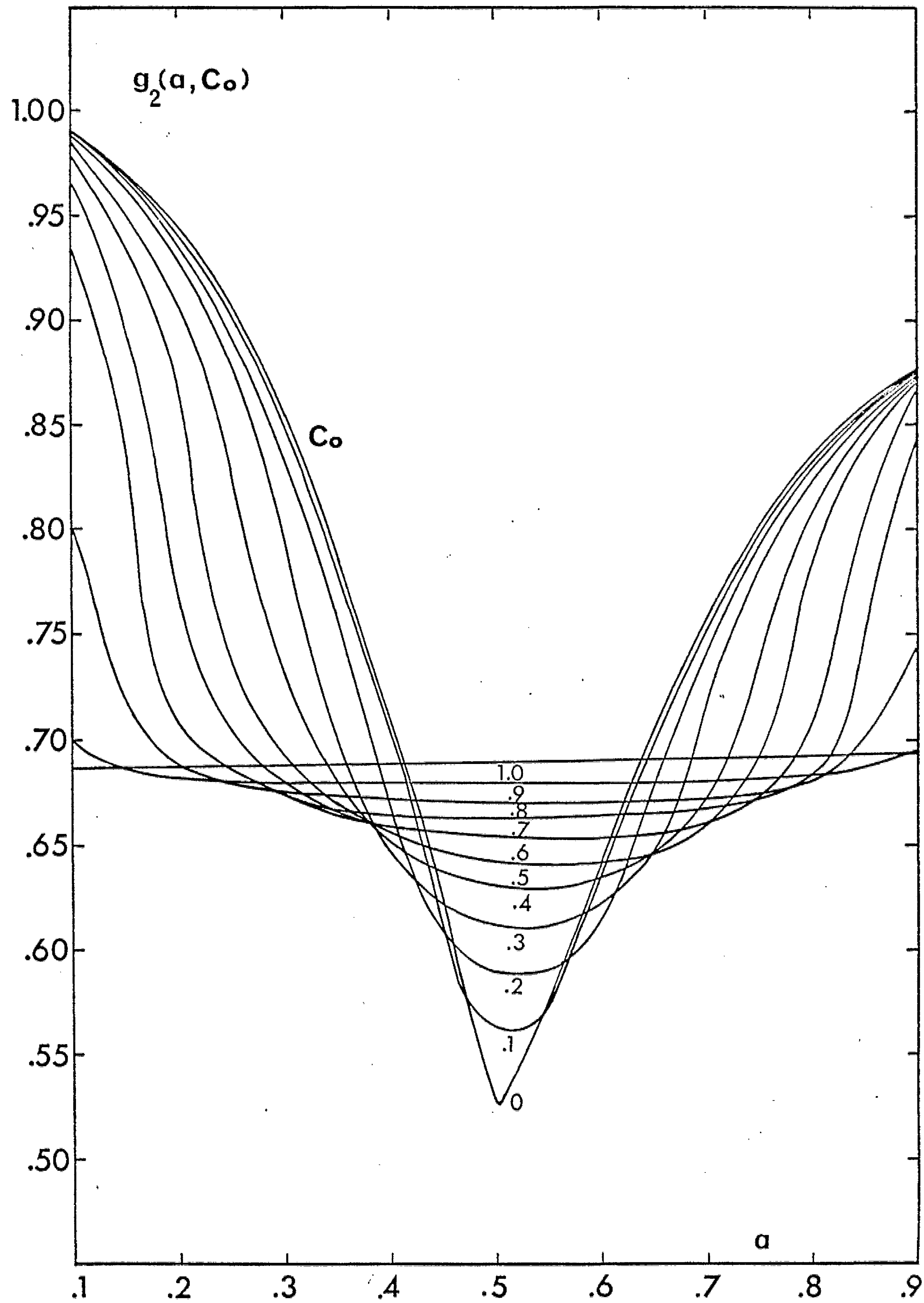

Figure 7. A plot of the function $g_2(a, c_o)$ versus a for fixed values of c_o.

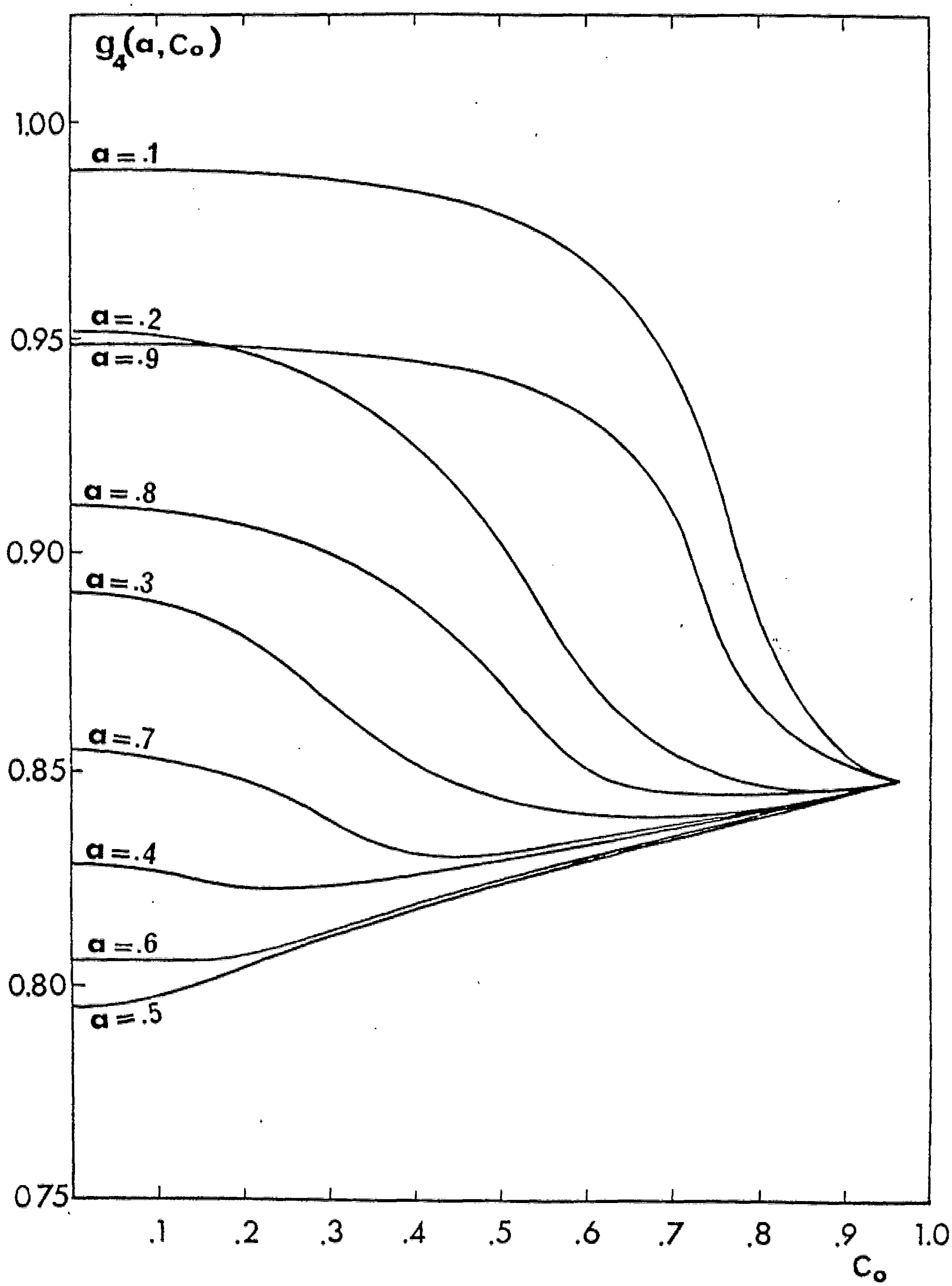

Figure 8. A plot of the function $g_4(a, c_o)$ versus c_o for fixed values of a .

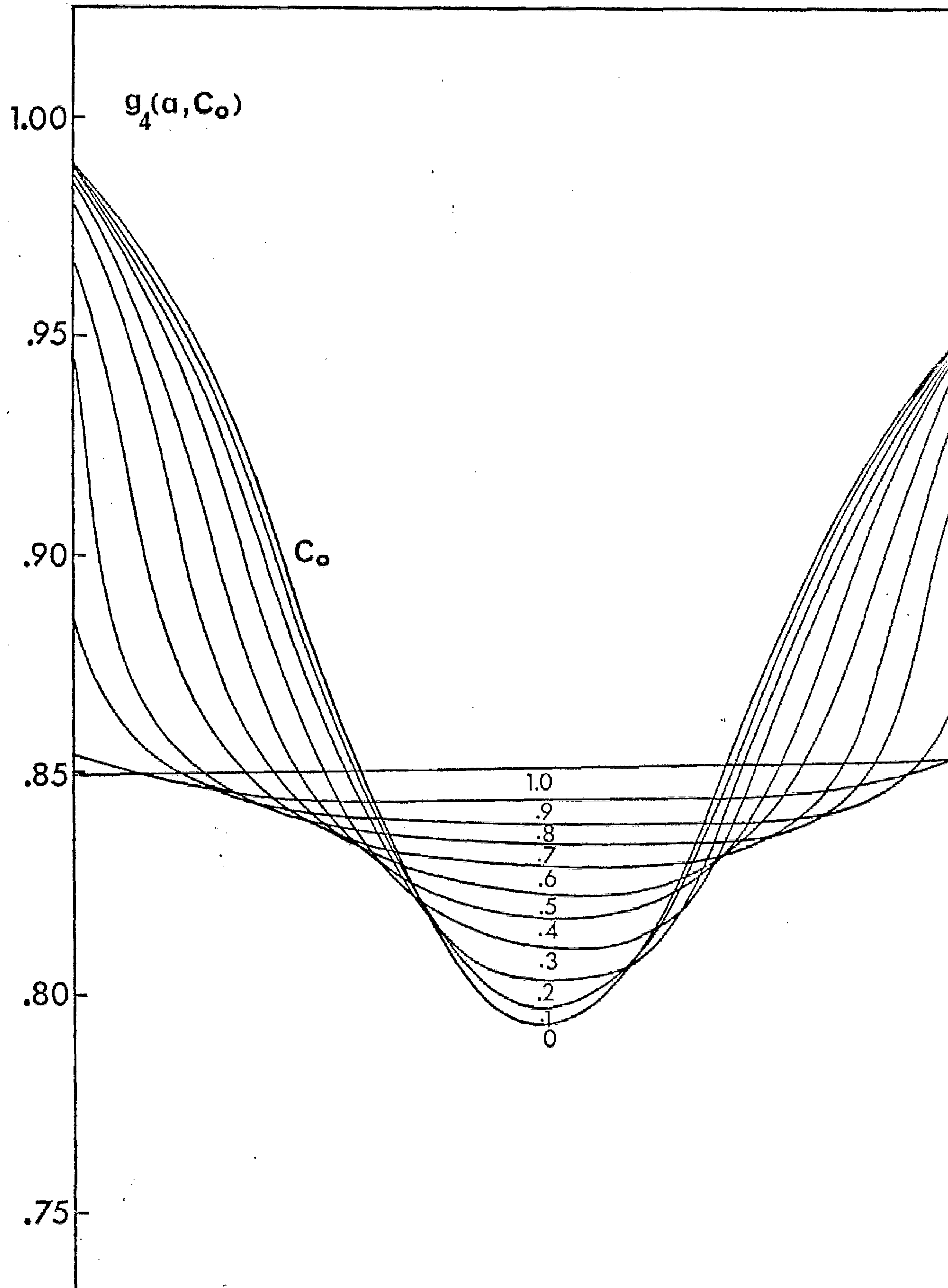

Figure 9. A plot of the function $g_4(a, c_o)$ versus a for fixed values of c_o.

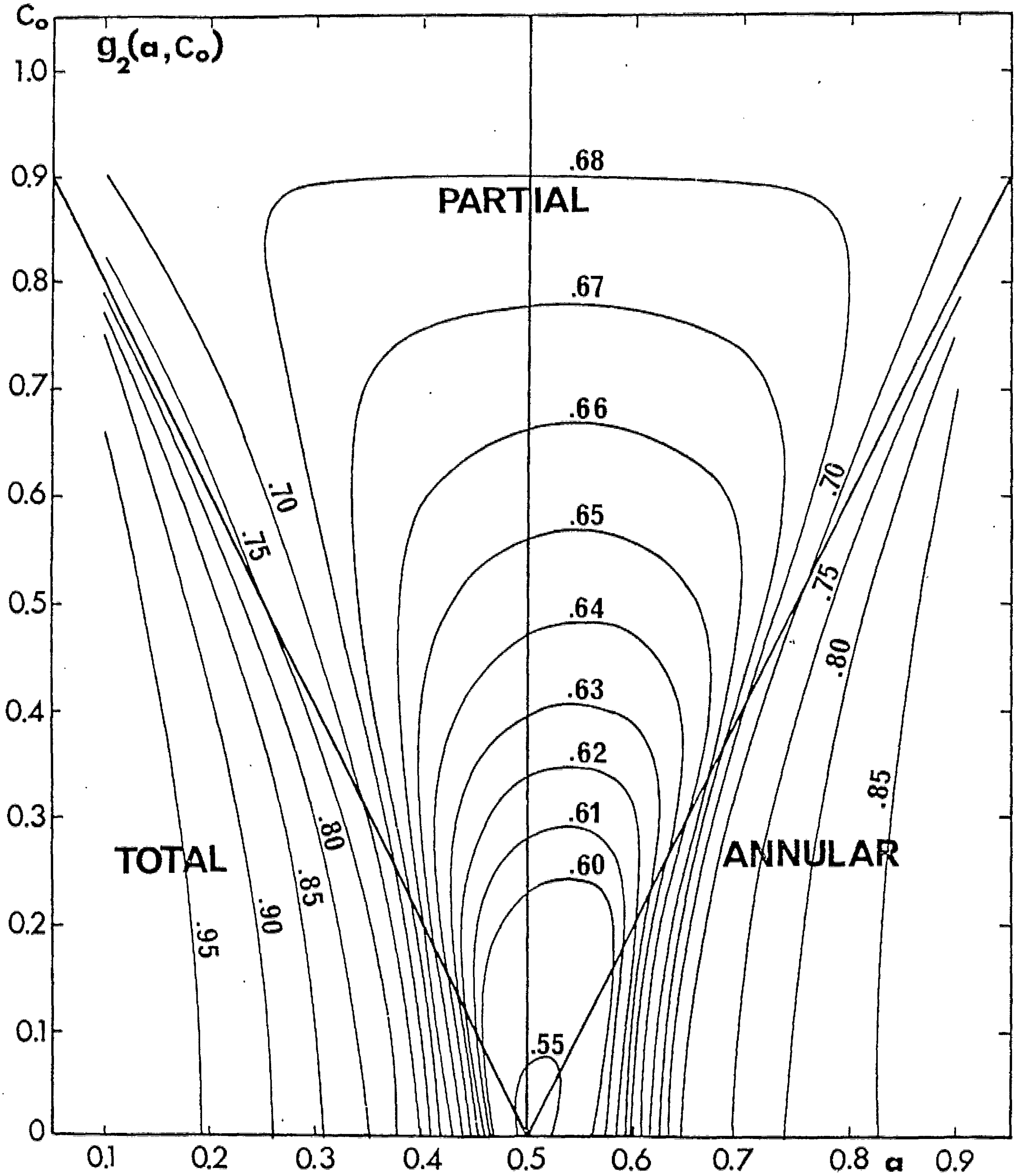

Figure 10. A diagrammatic representation of the function $g_2(a, c_o)$ as defined by Eq. (1.5), in the (a, c_o) plane, for every type of eclipse.

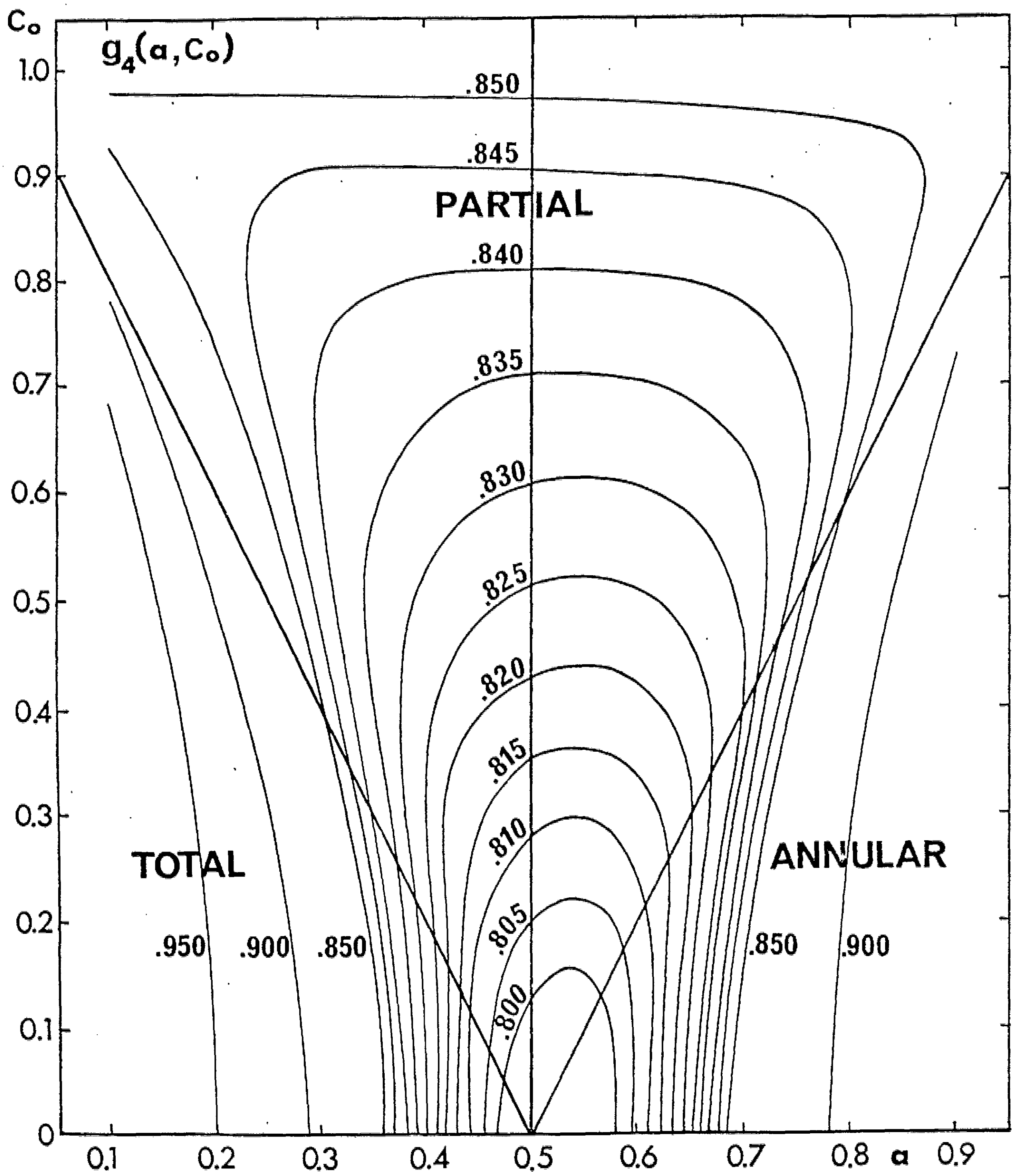

Figure 11. A diagrammatic representation of the function $g_4(a, c_o)$ as defined by Eq. (1.6), in the (a, c_o) plane, for every type of eclipse.

Table 8a. $a(g_2, g_4)$ - functions for the occultation-type eclipses.

$g_4 \backslash g_2$	0.60	0.61	0.62	0.63	0.64	0.65	0.66	0.67	0.68	0.69	0.70	0.75
0.800	0.47											
0.805	0.46	0.46	0.46	0.45								
0.810		0.46	0.44	0.44	0.44	0.44						
0.815			0.44	0.43	0.43	0.43	0.43					
0.820					0.42	0.42	0.42	0.42				
0.825						0.40	0.41	0.41	0.41			
0.830							0.38	0.38	0.40	0.40		
0.835								0.34	0.37	0.38	0.39	
0.840									0.31	0.34	0.35	0.38
0.845												
0.850												

Table 8b. $c_o(g_2, g_4)$ - functions for the occultation-type eclipses.

$g_4 \backslash g_2$	0.60	0.61	0.62	0.63	0.64	0.65	0.66	0.67	0.68	0.69	0.70	0.75
0.800	0	0										
0.805	0.06	0.04	0.03	0.01								
0.810		0.23	0.12	0.09	0.02	0						
0.815			0.20	0.13	0.10	0.06	0					
0.820					0.24	0.12	0.07	0				
0.825						0.28	0.15	0.10	0			
0.830							0.43	0.23	0.13	0.05		
0.835								0.51	0.29	0.22	0.14	
0.840									0.56	0.39	0.30	0.05
0.845												
0.850												

Table 9a. $a(g_2, g_4)$ - functions for the transit-type eclipses.

$g_4 \backslash g_2$	0.60	0.61	0.62	0.63	0.64	0.65	0.66	0.67	0.68	0.69	0.70	0.75
0.800	0.58	0.58										
0.805	0.58	0.59	0.59	0.60								
0.810			0.61	0.62	0.62	0.61						
0.815				0.63	0.64	0.64	0.63					
0.820					0.65	0.66	0.65	0.64	0.64			
0.825						0.67	0.68	0.67	0.66	0.65	0.65	
0.830							0.70	0.70	0.70	0.69	0.67	
0.835								0.72	0.72	0.71	0.69	
0.840									0.75	0.73	0.72	
0.845									0.80	0.78	0.75	
0.850										0.84	0.82	0.70

Table 9b. $c_o(g_2, g_4)$ - functions for the transit-type eclipses.

$g_4 \backslash g_2$	0.60	0.61	0.62	0.63	0.64	0.65	0.66	0.67	0.68	0.69	0.70	0.75
0.800	0.10	0.03										
0.805	0.20	0.17	0.12	0.05								
0.810			0.23	0.20	0.18	0.07						
0.815				0.30	0.25	0.22	0.08					
0.820					0.36	0.30	0.25	0.22	0.05			
0.825						0.39	0.33	0.30	0.24	0.12	0	
0.830							0.48	0.41	0.39	0.32	0.21	
0.835								0.49	0.45	0.39	0.31	
0.840									0.56	0.47	0.41	
0.845										0.62	0.52	
0.850											0.68	0.20

With the values of g_2 and g_4 ascertained from the observations we can enter these tables to establish the corresponding values of a and c_o. It is interesting to note that for many pairs of g_2 and g_4 there is no solution. If, however, the solution discloses that

$$0 < a < \tfrac{1}{2} \qquad (1.14)$$

the eclipse under analysis turns out to be an occultation (i.e., $r_1 < r_2$); while if

$$\tfrac{1}{2} < a < 1 \quad , \qquad (1.15)$$

it proves to be a transit ($r_2 < r_1$). Moreover, if the value of c_o happens to be such that

$$1 > c_o > |2a - 1| \quad , \qquad (1.16)$$

the eclipse is bound to be partial. If

$$c_o < 1 - 2a \quad , \qquad (1.17)$$

the eclipse becomes total; while if

$$c_o < 2a - 1 \quad , \qquad (1.18)$$

it happens to be annular.

To make the formulae (1.8) – (1.13) for the eclipse elements practicable, four-digit tables of $f_o'(a, c_o)$ and $f_o^* = f_2'/f_o'$, permitting linear interpolation for intermediate entries, have also been constructed and given in the Appendices 1 and 2. For the behaviour of $f_o'(a, c_o)$ – functions see Figures 12 – 14. The above tables should

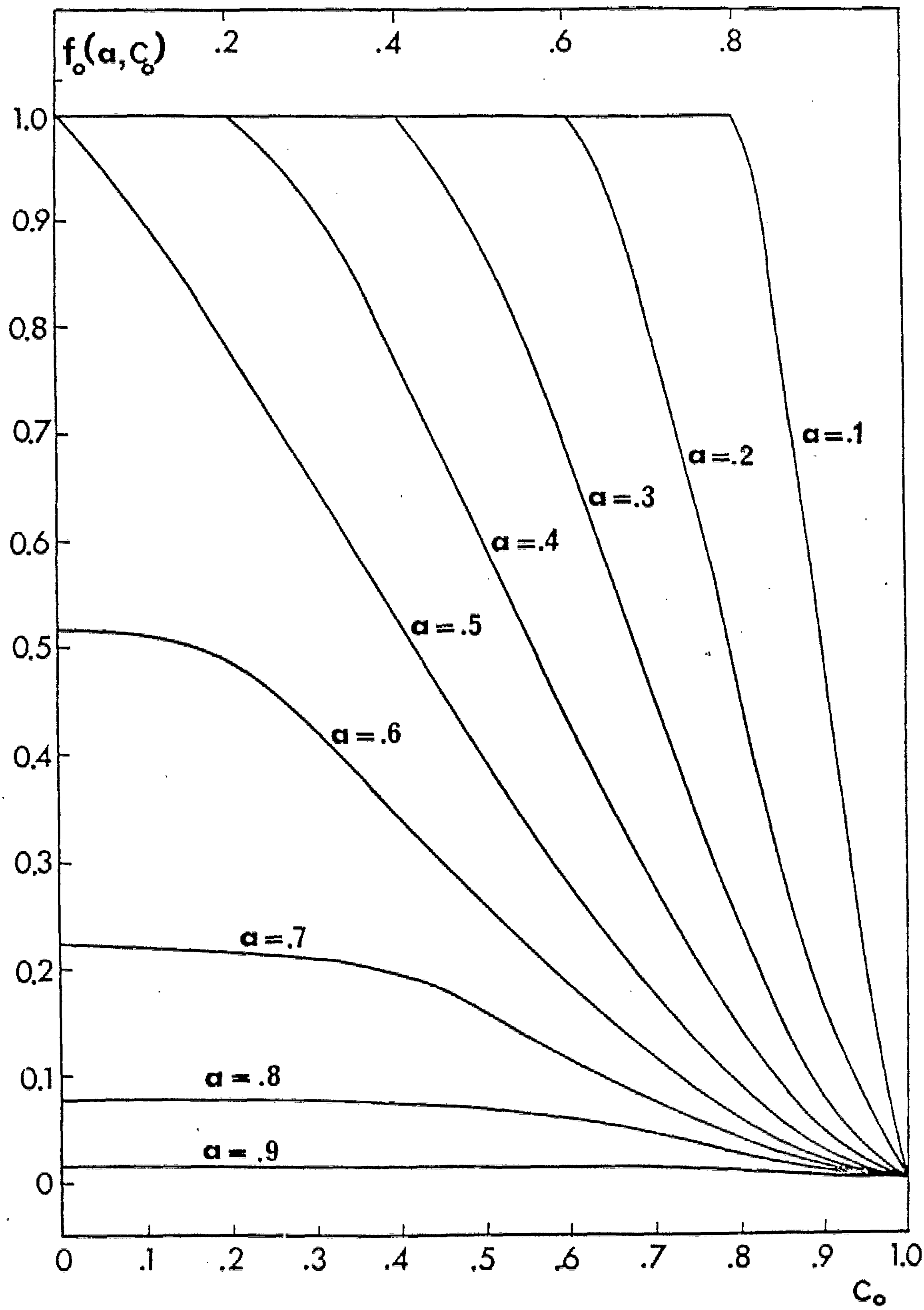

Figure 12. A plot of the function $f_o{}'(a, c_o)$ (see Eq. 4.9 in Chapter 3) versus c_o for fixed values of a.

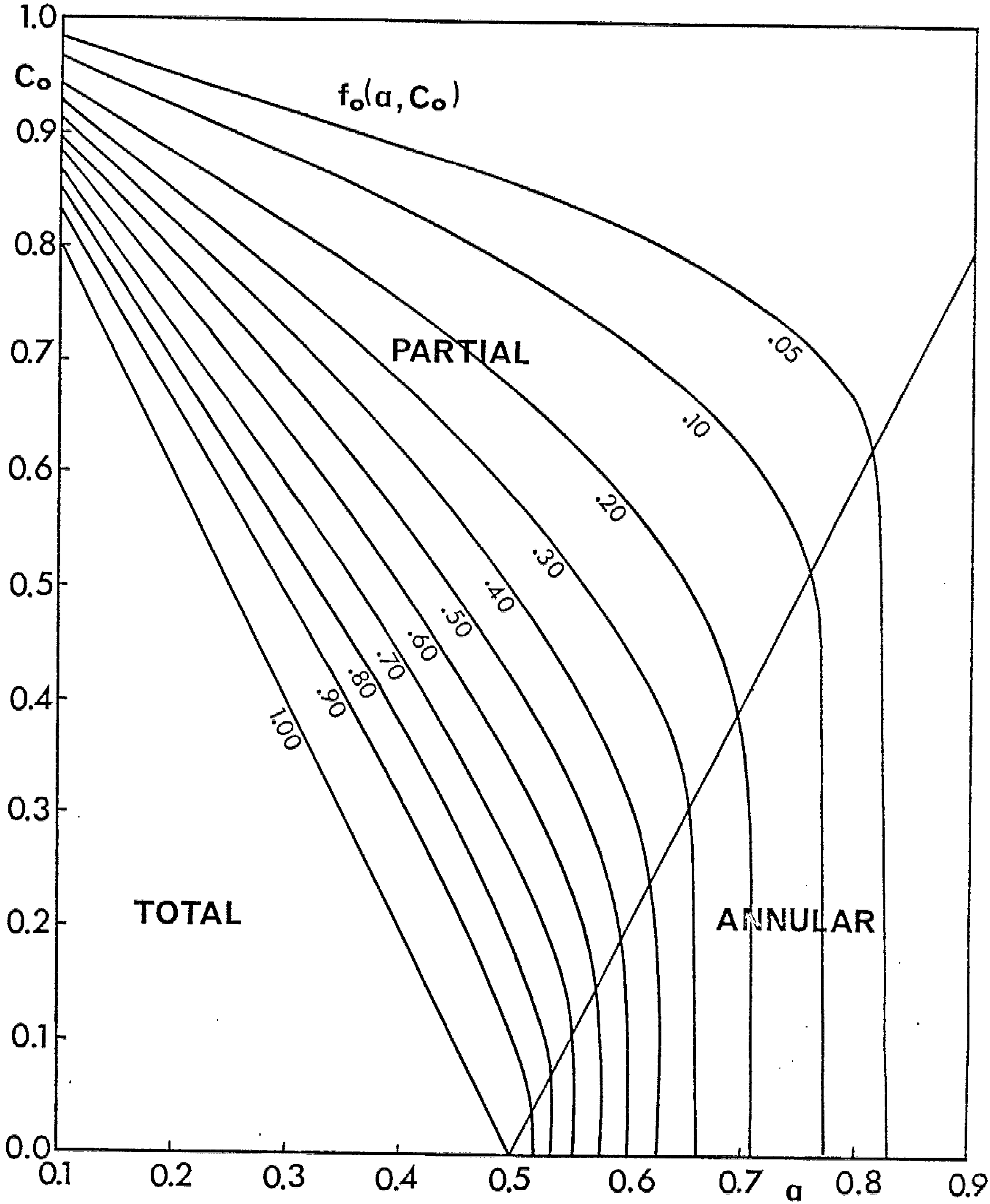

Figure 13. The functional behaviour of the function $f_o'(a, c_o)$ as defined by Eq.(4.9)of Chapter 3, in the (a, c_o) plane.

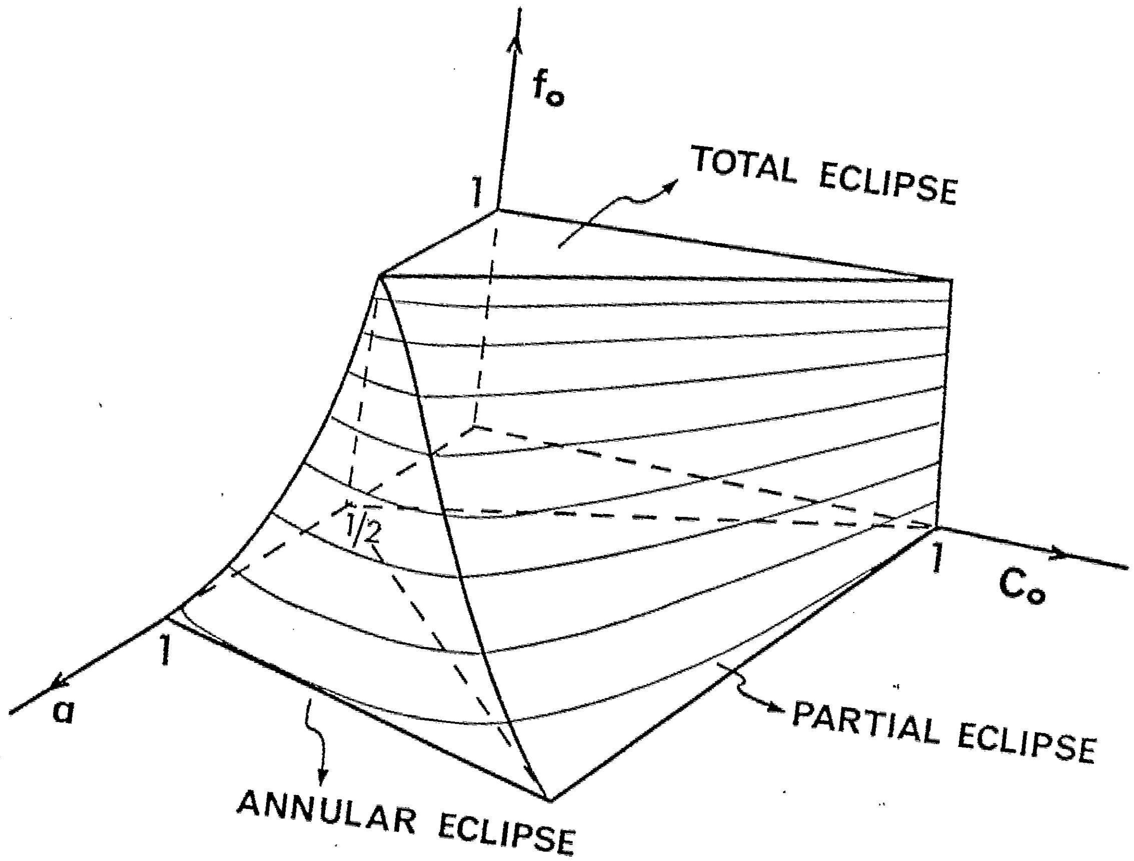

Figure 14. The functional behaviour of the function $f_o{}'(a, c_o)$ (see Eq. (4.9) of Chapter 3) in three dimensional space.

make a determination of r_1 from (1.9) a matter of simple algebra; and to obtain the other unknowns from Equations (1.10) - (1.13) is straightforward enough. It may be noted that if $a < \frac{1}{2}$ and $c_o < 1 - 2a$ - i.e., if the eclipse under investigation proves to be total - a solution for the elements becomes wholly algebraic, requiring no aid of any auxiliary tables, and can be carried out in a closed form by methods already investigated in Papers I and II.

The entire process described so far can be carried out on the basis of the light curve of any one minimum - say, the primary (deeper) one - alone, without any recourse to the secondary minimum; the only necessary prerequisite being a knowledge of the light level (i.e., the luminosity of the system outside eclipses) subtending the areas representing the moments A_{2m}. The process furnishes, moreover, a direct solution of our problem with the aid of necessary tables for a and c_o; r_1 (r_2, i); and L_1($L_2 = 1 - L_1$) in that order. Its feasibility depends, however, on the accuracy with which the two simultaneous equations like (1.6) and (1.7) based on the observations of the same minimum, can be solved for the parameters a and c_o (see Section 3 of the present Chapter); and this, in turn, depends on the numerical magnitude of the Jacobian

$$J = \frac{\partial(g^{(1)}, g^{(2)})}{\partial(a, c_o)} . \tag{1.19}$$

As is well known, the vanishing of this Jacobian would imply a functional relationship to exist between successive functions $g(a, c_o)$

which would render a simultaneous solution of Equations like (1.6) and (1.7) indeterminate. In general, this will not be the case. However, if the Jacobian J, while non vanishing, is numerically small - as is likely to be the case for shallow partial eclipses - a simultaneous solution of equations of the form (1.6) and (1.7) for a and c_o may be feasible only if the left-hand sides of these equations can be deduced from the observations with the requisite degree of accuracy.

Suppose, however, that the simultaneous solution of equations like (1.6) and (1.7) for a and c_o based upon a given light curve is weak, or borders on indeterminacy. If so, the determinacy of such a solution may be restored if we are in possession of the light curve of the alternate (secondary) minimum of the same system. In such a case, the roles of the fractional radii r_1 and r_2 are interchanged, and the unknown quantities a and c_o can be solved from the pair of equations like

$$\left(\frac{A_2^2}{A_0 A_4}\right)_{Pri.} = g_2(a, c_0) , \tag{1.20}$$

$$\left(\frac{A_2^2}{A_0 A_4}\right)_{Sec.} = g_2(b, c_0) , \tag{1.21}$$

based on the same moments A_{2m} of the alternate minima. Such a procedure is generally to be followed in an analysis of the light curves of systems (like WW Aur, for instance) whose minima are due to partial eclipses of comparable depth; and tables of $g_2(a, c_o)$ continue to be available to facilitate this task.

Suppose, however, that the depth of the secondary minimum is so shallow that the proportional errors ε_{2m} of the moments A_{2m} on the l.h.s. of Eq. (1.21) become too large to make this latter equation of any practical use - a situation frequently encountered in typical Algol systems with cool secondary components. In such cases, only Equation (1.20) remains significant, and additional independent relations between a and c_o must be sought.

In order to construct such a relation based on the depth of both minima alone, we may proceed as follows. Let $\lambda_{a,b}$ denote the remaining fractional light at maximum occultation or transit eclipses alternating in each system, and $\alpha(a, c_o)$, $\alpha(b, c_o)$ be the fractional losses of light of the components L_a, L_b. The subscript a will hereafter refer consistently to the smaller component of the two (an eclipse of which is, therefore, an occultation) and the subscript b to the larger star (an eclipse of which is a transit) - regardless of whether $L_a \gtrless L_b$; while, in the arguments of α, a continues to be equal to $r_1/(r_1 + r_2)$ and

$$b = \frac{r_2}{r_1+r_2} = 1 - a . \tag{1.22}$$

If so, then an occultation eclipse $(r_1 < r_2)$

$$\lambda_a = 1 - \alpha(a, c_o) L_a ; \tag{1.23}$$

while half a revolution later, during a transit $(r_1 > r_2)$,

$$\lambda_b = 1 - \alpha(b, c_0)\, L_b \ . \tag{1.24}$$

An elimination of $L_{a,b}$ between (1.23) and (1.24), taking advantage of (1.2), discloses that during an occultation eclipse,

$$\alpha(a, c_0) = 1 - \lambda_a + \frac{1 - \lambda_b}{\left(\frac{a}{b}\right)^2 Y(a, c_0)} \ ; \tag{1.25}$$

while during a transit,

$$\alpha(b, c_0) = 1 - \lambda_b + (1 - \lambda_a)\left(\frac{a}{b}\right)^2 Y(a, c_0) \ , \tag{1.26}$$

where we have abbreviated

$$Y(a, c_0) = \left(\frac{b}{a}\right)^2 \frac{\alpha(b, c_0)}{\alpha(a, c_0)} \ , \tag{1.27}$$

representing a slowly varying function of its parameters as well as of the coefficients of limb-darkening of both stars, whose theoretical value may be obtained by use of Eq. (1.1).

Equations (1.25) and (1.26) represent a second desired relation between the parameters a(or $b = 1-a$) and c_0 based on the depth of the two minima of light, and whichever of them should be adjoined to (1.20) to obtain the solution for a and c_0 depends on the nature of the eclipses giving rise to the observed minima. Should the primary minimum be due to an occultation eclipse (i.e., if $\lambda_1 \equiv \lambda_a$), Eq. (1.25) should be used; while if it is a transit ($\lambda_1 \equiv \lambda_b$), (1.26) should be adopted. If, moreover, the occultation happens to be total (i.e., if $\alpha(a, c_o) = 1$), Eq. (1.25) can be solved to yield

$$\left(\frac{a}{b}\right)^2 = \frac{1-\lambda_b}{\lambda_a \, Y(a, c_o)} \tag{1.28}$$

for the depth a/b (<1) of the respective system; and for an annular transit a similar equation holds good, provided that the fractional luminosity λ_b refers to the light intensity of the system at the commencement of the annular phase.

In general, only one of the alternatives $\lambda = \lambda_1$, or λ_b will lead to a real solution for a and c_o. Such a solution can be facilitated by a recourse to tables of $g_2(a, c_o)$, $\alpha(a, c_o) \equiv f_o'(a, c_o)$ and $Y(a, c_o)$. The tables of function $Y(a, c_o)$ have also been constructed for the same distribution of brightness over the apparent discs of the two stars, and given in the Appendix 3. For the behaviour of this $Y(a, c_o)$-function see Føgure 15. For machine computations, the fortran program F2M given in the previous chapter is sufficient alone to perform the respective tables for any adopted values of the coefficients of limb-darkening, and once these have been done and the parameters a and c_o inverted, the respective solutions can be obtained in the same way as before from the Equations (1.8) - (1.10) or alternatively, for the fractional luminosities $L_{a,b}$ of the two stars can be evaluated from the following equations:

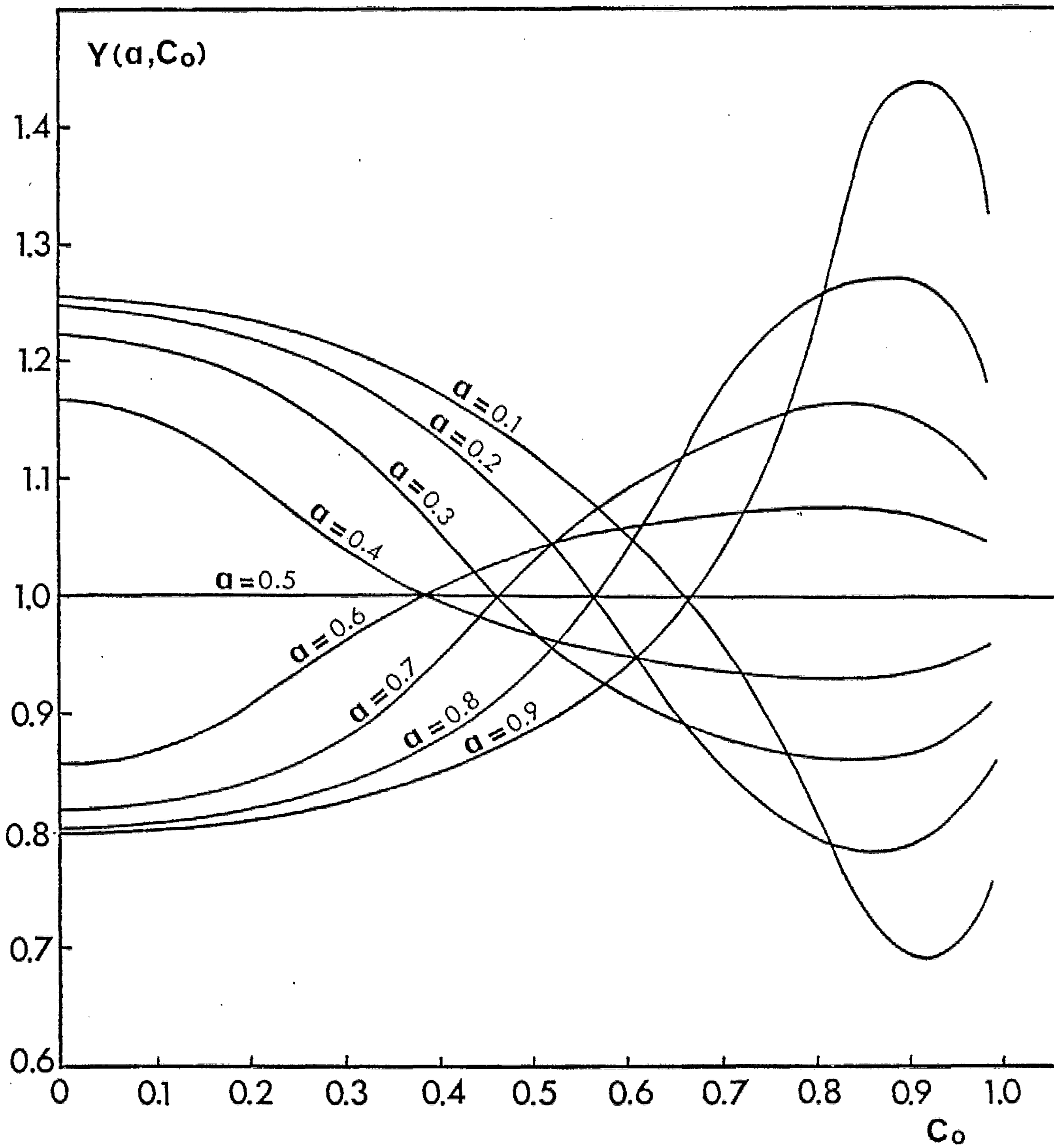

Figure 15. A plot of the function $Y(a, c_o)$ – as defined by Eq. (1.27) – versus c_o for fixed values of a.

$$\frac{L_a}{L_b} = \left(\frac{a}{b}\right)^2 \gamma(a, c_o) \frac{1-\lambda_a}{1-\lambda_b} \tag{1.29}$$

and

$$L_a + L_b = 1 \quad . \tag{1.30}$$

Eq. (1.29) can be derived with the aid of (1.23), (1.24) and (1.27).

Should, lastly, the depth $1 - \lambda_2$ of the secondary minimum be negligible - as it would be if the secondary component were effectively dark - this would imply that $L_1 = 1$ (or very close to it). In such a case, the second independent relation between a and c_o to be adjoined to (1.20) would follow from (1.1) in the form

$$\left.\begin{array}{l} \alpha(a, c_o) \\ \alpha(b, c_o) \end{array}\right\} = 1 - \lambda_1 \tag{1.31}$$

depending on whether the observed minimum of depth $1 - \lambda_1$, is due to an occultation or transit eclipse. Only one of these, together with (1.20), will yield a real solution for $a(b = 1 - a)$ and c_o, and once their values have been obtained, the remaining elements $r_{1,2}$ and i can proceed in the same way as before.

4.2 Systems Consisting of Distorted Stars.

In the preceding section the practicable procedures for an analysis of the light curves of any type of eclipsing system have been developed which are directly applicable to systems whose components can be regarded as spherical. However, as soon as photometric effects arising from their proximity cease to be negligible, such effects will produce a continuous light variation of the system throughout its orbital cycle, which will superpose upon the changes of light arising from possible eclipses. This fact will, in turn, necessitate certain modifications of the reduction procedures outlined in Section 1 to make them applicable also to an analysis of the light changes exhibited by close eclipsing systems; and the aim of the present section will be to outline the requisite modifications from Paper XIV.

In order to extend the methods of Section 1 to an analysis of the light curves of distorted eclipsing systems, we must first re-define in an appropriate manner the respective moments of the light curves. In the case of spherical systems treated in Section 1, the upper limit of integration on the r.h.s. of Eq. (0.1) of Chapter 2 was taken to be $\sin^{2m}\theta'$, where the value of the angle θ' of the first contact of the eclipse could be read off the observed light curve. For distorted eclipsing systems the variation of light $l(\theta)$ will continue beyond $\theta > \theta'$, and the position of the angle θ' on the light curve can no longer be ascertained in advance without ambiguity. In order to avoid this ambiguity, the moments of the light curves are redefined as

$$\tilde{A}_{2m} = \int_0^{\frac{\pi}{2}} \left[l\left(\tfrac{\pi}{2}\right) - l(\theta) \right] d\left(\sin^{2m}\theta\right) , \qquad (2.1)$$

where $l(\pi/2)$ denotes the maximum brightness of the system at the time of quadratures ($\theta = \frac{\pi}{2}, \frac{3\pi}{2}$). This new definition is indeed quite consistent with that represented by Eq. (0.1) of Chapter 2 – for in the case of spherical stars – an extension of the upper limit of integration beyond the moment of first contact would not affect the numerical values of A_{2m}, as their integrand $1 - l$ vanished outside eclipses It agrees, moreover, with Eq. (4.2) of Paper V. Since, however, the range of integration adopted on the r.h.s. of Eq. (2.1) extends now from the moments of conjunction to that of the quadrature, the empirical values of A_{2m} can again be ascertained directly from the observations – without any prior knowledge of the properties of the respective system. In particular, we do not need to know whether or not the system exhibits eclipses, or whether the observed light curves are due to the photometric proximity effects alone. We may also add that the definition (2.1) of A_{2m} – like (0.1) in Chapter 2 for A_{2m} – holds hood also for $m = 0$, in which case Eq. (1.1) is to be replaced by

$$\tilde{A}_0 = l\left(\frac{\pi}{2}\right) - l(0) , \tag{2.2}$$

and represents the total amplitude of the observed light changes between quadrature and conjunction of the respective minimum.

The straight theoretical moments A_{2m} of the spherical case as defined by Equations (3.17) – (3.20) of Chapter 3, in terms of the eclipse elements are then related exactly with the empirical moments A_{2m} by Eq. (2.11) of Paper IX, of the form

$$\tilde{A}_{2m} = -m! \sum_{j=1}^{n} \frac{\Gamma\left(\frac{j+2}{2}\right)}{\Gamma\left(\frac{j+2+2m}{2}\right)} c_j + A_{2m} + \beta_{2m} , \tag{2.3}$$

valid for any value of m (including zero), where the quantity on the l.h.s. of Eq. (2.3) represents the empirical)observed) moment obtainable by planimetry of the respective light curve, and the c_j's are the amplitudes of the proximity effects varying as $\cos^j\theta$. The physical significance of these constants has been established in Section 2 of Paper V (Kopal, 1975e; Eq. 2.48) to which the reader is referred for fuller details; hereafter we shall regard them as unknowns to be determined by subsequent analysis. Lastly, the β_{2m}'s stand for the 'photometric perturbations', within eclipses, arising from distortion both of the eclipsed portion of the disc and of the eclipsing limb.

The numerical magnitude of the β_{2m}'s is generally quite small, and for moderately distorted systems they constitute the smallest term of Eq. (2.3). This, however, is not true of the weighted sum

$$m! \sum_{j=1}^{n} \frac{\Gamma\left(\frac{j+2}{2}\right)}{\Gamma\left(\frac{j+2+2m}{2}\right)} \tag{2.4}$$

of the constants c_j . Indeed, for close eclipsing systems, those corresponding to even harmonics j (and in particular, c_2 factoring the second harmonic of tidal origin) may become relatively large, and their weighted sum (2.4) comparable in magnitude with A_{2m} on the left-hand side of Eq. (2.3). In such cases the numerical magnitudes of the c_j's must be established, from those parts of the light curves which are unaffected by eclipses, before an analysis for the elements of the eclipses (if any) can get under way.

As is well known (cf., e.g., Kopal, 1954 and subsequent publications), the range of possible eclipses in close binary systems is bounded even if their components are in actual contact, and the mass ratios not in excess of 10:1, their duration cannot exceed $\theta' = 60^\circ$. Moreover, if $i \ll 90^\circ$ and (or) if one (or both) components become detached from their respective Roche limits, $\theta' \ll 60^\circ$. If so, however, it was shown in Papers V and IX that the values of the individual constants c_j can be established by a suitable 'modulation' of the light curve between eclipses - i.e., by an evaluation (through quadratures or otherwise) of integrals of the form

$$c_j = \int_{-a}^{a} \left[l\left(\tfrac{\pi}{2}\right) - l(\theta) \right] \mathcal{P}_j^{(a,n)}(\cos\theta)\, d(\cos\theta) \tag{2.5}$$

where j denotes, as before, the degree of the respective harmonic factoring c_j, n is the total number of such harmonics included in a simultaneous solution, and $a = \cos\theta'$.* The explicit forms of the 'modulation polynomials' $\mathcal{P}_j^{(a,n)}(\cos\theta)$ for different ranges of eclipses and the first four partial harmonics included in the analysis ($n = 4$) have already been given in Paper IX for the following ranges assumed

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* Not to be confused with the a used before.

to be free from any effects of the eclipses:

(i) $60^\circ < \theta < 120^\circ$; $a = \frac{1}{2}$: Equations (5.18) - (5.21),

(ii) $45^\circ < \theta < 135^\circ$; $a = \sqrt{2}/2$: Equations (5.42) - (5.45),

(iii) $30^\circ < \theta < 150^\circ$; $a = \sqrt{3}/2$: Equations (5.65) - (5.68),

in which we have set $\cos\theta \equiv x$.

The actual range to adopt should be ascertained by an inspection of the respective light curve. If the light changes are continuous and exhibit no obvious indication of an onset of the eclipses, we can never go wrong by adopting range (i). Should, however, a piecewise discontinuity in the shape of the light curve indicate the phase angle θ' at which eclipses commence, then range (ii) or (iii) should be chosen within which $\theta > \theta'$. The more extended the range, the greater the accuracy with which the values of the c_j's can be determined from the light curve by a modulation represented by Eq. (2.5). On the other hand, should we allow θ' to lie within the adopted range, a determination of the c_j's may then be vitiated by eclipse effects.

It should also be noted that, inasmuch as these constants occur on the r.h.s. of Eq. (2.3) only through their sum (2.4), the value of this sum could also be obtained directly by an analogous modulation of the uneclipsed part of the light curve - by use of the polynomials as given by Equations (5.23) - (5.25), (5.46) - (5.48) or (5.69) - (5.71) of Paper IX, replacing the $\mathcal{P}_j^{(a,n)}(x)$'s on the r.h.s. of (2.5) without seeking to specify the c_j's individually. The latter possess, however, a distinct information content of their own, which can be utilized in due course (for instance, for a specification of the amount of gravity-darkening of the respective stars, or of their mass-ratio, with the aid of Eq. (2.48) of Paper V). Therefore, an individual determination of the c_j's and subsequent formation of the sum (2 4) from them should generally be preferable. Should the sum so obtained turn out to be sensibly equal to the corresponding empirical moments $\tilde{A}_{2m}$ (thereby implying the quantities A_{2m} as well as β_{2m} to be negligible), this would signify that the respective system does not eclipse, and that its observed light changes are due solely to the proximity effects.

Lastly, the leading terms of the 'photometric perturbations' β_{2m}, corresponding to the first three spherical-harmonics in rotational or tidal distortion, have been established for total eclipses in Section 6 of Paper V, and their investigation subsequently extended by Livaniou (1976, 1977, 1978) to every other type of eclipse (annular, partial; occultations or transits), and recently this quantity has been tabulated (Edalati and Budding, 1978; Paper XVII) by employing the automated procedures of Budding (1974) for the numerical evaluation (by simple trapezoidal quadratures) of the integrals

$$\beta_{2m} = m L_1 \csc^{2m} i \sum_{l=0}^{\Lambda} C^{(l)} \int_{\delta_0^2}^{\delta_1^2} (\delta^2 - \delta_0^2)^{m-1} \left[f_*^{(l)} + f_1^{(l)} + f_2^{(l)} \right] d\delta^2 \tag{2.6}$$

(cf. Paper V, Eq. 6.3), for the fixed mass ratio ($m_2/m_1 = 1$), limb-darkening ($u = 0.6$) and gravity-darkening ($\tau = 1$). The β_{2m}'s consist of two parts: i) a distortion of the eclipsing limb of the component in front, and ii) effects associated with the distortion of the eclipsed star. The former effects are purely geometrical and relatively simple; their evaluation discloses that, for total eclipses, and for m = 1, 2 and 3:

$$\beta_2 = L_1 C_1 \left[\left(\frac{r_2}{r_1}\right)^2 \left\{ \frac{1}{3} v_2^{(2)} - w_2^{(2)} + \frac{3}{4} w_2^{(4)} \right\} - \frac{3}{4} X_1 \left\{ \frac{1}{3} v_2^{(2)} - w_2^{(2)} \right\} \right], \tag{2.7}$$

$$\beta_4 = 2 C_3 \beta_2 + L_1 C_1^2 \left[\left(\frac{r_2}{r_1}\right)^2 X_1 \left\{ \frac{1}{3} v_2^{(2)} - w_2^{(2)} + \frac{15}{4} w_2^{(4)} \right\} + X_2 w_2^{(4)} \right], \tag{2.8}$$

$$\beta_6 = 3\left[(r_2^4 \csc^4 i - \cot^4 i)\beta_2 - \beta_4 \cot^2 i \right] +$$
$$+ 3 L_1 C_1^3 \left[\left(\frac{r_2}{r_1}\right)^2 X_1 \left\{ \frac{1}{2} v_2^{(2)} - 2 w_2^{(2)} + \frac{9}{2} w_2^{(4)} \right\} + \right.$$
$$\left. + X_2 \left\{ \frac{1}{18} v_2^{(2)} + \frac{1}{3} w_2^{(2)} - \frac{19}{4} w_2^{(4)} \right\} + \frac{3}{8} X_3 w_2^{(4)} \right], \tag{2.9}$$

where

$$X_j \equiv \sum_{l=0}^{\Lambda} \frac{(j+1)!\, C^{(l)}}{\nu(\nu+1)(\nu+2)\cdots(\nu+j)}, \qquad \nu = \frac{l+2}{2} \tag{2.10}$$

are constants which depend (through $C^{(l)}$) on the coefficients U_l of limb-darkening of the eclipsed star;

$$v_2^{(2)} = \left(1 + \frac{m_1}{m_2}\right) r_2^{\,3} \tag{2.11}$$

and

$$w_2^{(j)} = \frac{m_1}{m_2} r_2^{j+1} \tag{2.12}$$

are coefficients associated with the rotational and tidal distortion of the eclipsing component (where (cf. Equations 2.49 – 2.50 of Paper V) m_1/m_2 denotes the meas-ratio of the respective stars); and the constants $C_{1,3}$ continue to be related with the geometrical elements $r_{1,2}$ and i by Equations (3.26) – (3.28) of Paper I. The foregoing expressions (2.7) – (2.9) should be sufficient for an analysis of the primary minima, due to total eclipses, of systems with subgiant components in which the primary (early-type) star is essentially spherical – so that a distortion of the secondary's shadow cylinder represents the only perturbation which needs to be taken into account.

The explicit form of the terms arising from ii) depends not only on the geometrical distortion of the component undergoing eclipse, but also on the distribution of brightness (gravity-darkening) over the surface of the eclipsed star, and need not be reproduced in this place. For the explicit results the reader is referred to the sources already quoted. These contain, to be sure, only the β_{2m}'s corresponding to $m > 0$. However, for $m = 0$,

$$\beta_0 = L_1 \sum_{l=0}^{\Lambda} C^{(l)} \left[f_*^{(l)} + f_1^{(l)} + f_2^{(l)} \right] , \tag{2.13}$$

where the functions $f_*^{(l)}$ and $f_{1,2}^{(l)}$ are given by Equations (3.32), (3.33) and (3.34) of Paper V. If the eclipse is total, $f_1^{(l)} = f_2^{(l)} = 0$ identically at the moment of maximum eclipse ($\theta = 0$); and only $f_*^{(l)} \neq 0$ for the distorted primary star. If, however, the latter can be regarded as spherical, then also $f_*^{(l)} = 0$ and, in consequence, $\beta_o = 0$.

The actual method of computation of the elements of distorted eclipsing systems can now be summarized by the following scheme:

1) First, determine the requisite number of the empirical moments $\tilde{A}_{2m}$ of the observed light curves, as defined by Eq. (2.1) above.

2) Next, evaluate the requisite number n of the constants c_j by appropriate modulation of the 'uneclipsed' parts of the light curve, by use of Eq. (2.5), and form their weighted sum (2.4)

3) Transpose the sum (2.4) on the r.h.s. of Eq. (2.3) to the left, ignore β_{2m}, and evaluate the A_{2m}'s.

4) With the aid of 'rectified' moments A_{2m} of the light curve, evaluate the elements of the system by the method of Section 1.

5) By use of the elements $r_{1,2}$, i and $L_{1,2}$ so obtained, evaluate the corresponding 'photometric perturbations' β_{2m}, transpose them together with (2.4) to the l.h.s. of Eq. (2.3) to obtain an improved set of the A_{2m}'s : and from these improved elements $r_{1,2}$, i and $L_{1,2}$.

6) Should these improved elements differ significantly from their first version, repeat steps 3-5 until both sides of Eq. (2.3) can be satisfied by the same set of the values of $r_{1,2}$, i and $L_{1,2}$: and these constitute the final solution of our problem.

First, it may be noted that - unlike in the case of light changes exhibited by mutual eclipses of spherical stars treated in the preceding section of this chapter, which could be solved directly when we use the necessary tables - a solution for the elements of distorted systems can be obtained only by iterations even if we use the tables. The need to iterate arises solely from a presence of the photometric perturbations B_{2m} on the r.h.s. of Eq. (2.3) relating $\tilde{A}_{2m}$ with A_{2m}. Since, however, the numerical magnitudes of the B_{2m}'s will generally be small, iterative solutions should converge with sufficient rapidity to make more than one repetition of steps 3 - 5 of the foregoing cycle unnecessary. Should, however, this cycle fail to converge, our solution - in fact, any solution consistent with a physically sound model of the system - would then become indeterminate from the photometric evidence alone, and additional (e.g., spectroscopic) evidence may be required to alter this situation.

Second, it should be stressed that - unlike in the previous treatment of the subject in the time domain by more conventional methods - each step of our present analysis can be expressed in algebraic form (as closed formulae, or convergent series of satisfactory asymptotic properties) which is amenable to automation, and the entire solution can be obtained at high speed with the aid of electronic computers. The investigator without ready access to such computers, and working by hand or aided only by a desk-type (or pocket-type) computer, can perform his task expeditiously with the assistance of auxiliary tables accompanying this chapter; but their almost four-digit precision will also set the limits to the accuracy of his work. However, once the empirical values of the moments A_{2m} or $\tilde{A}_{2m}$ have been determined, automatic computers can be programmed to perform the rest of the solution internally. To do so, the only remaining difficulty is the solution of the parameters a and c_o from the simultaneous nonlinear equations like (1.6) and (1.7); in terms of the observed quantities $g(A_{2m})$. In the following section a method, for the solution of these key parameters a and c_o, will be given to complete the automation of the whole procedures outlined, for obtaining the elements of any eclipsing binary system.

4.3 A Method for the Solution of the Parameters a and c_o.

In the preceding two sections the procedures have been outlined for obtaining the elements of the eclipsing binary systems. The only remaining problem is the solution of the two simultaneous nonlinear equations like (1.6) and (1.7) of Section 1, for the two key parameters a and c_o of the eclipsing system concerned. In this section we shall develop an iterative method for the solution of the respective two simultaneous nonlinear equations of the form

$$B^{(1)} = g^{(1)}(a, c_o) \tag{3.1}$$

and

$$B^{(2)} = g^{(2)}(a, c_o) \tag{3.2}$$

(see Equations 1.2 and 1.3 in Section 1 of the present chapter) for the two different sets of parameters k_n, x_n, j_m and y_m. The l.h. sides in (3.1) and (3.2) can be established from the observations as ratios of the respective powers of the respective moments of the light curves. Thus, if we rewrite (3.1) and (3.2) as

$$F(a, c_o) = 0 \tag{3.3}$$

and

$$G(a, c_o) = 0 \tag{3.4}$$

and if a point (a_1, c_{o1}) close to a solution has been determined by graphical methods (for g_2 and g_4 pair this approximate solution may be taken directly from Tables 8 and 9) or otherwise, a closer solution (a, c_o) can usually be obtained as follows:

Let $a - a_1 = \delta a$, $c_o - c_{o1} = \delta c_o$. Expand $F(a, c_o)$ and $G(a, c_o)$ in Taylor's series to terms of the first degree, and assume that (a, c_o) is a solution, i.e., $F(a, c_o) = G(a, c_o) = 0$. Then, approximately,

$$F(a_1, c_{o1}) + \left(\frac{\partial F}{\partial a}\right)_{a_1} \delta a + \left(\frac{\partial F}{\partial c_o}\right)_{c_{o1}} \delta c_o = 0 \tag{3.5}$$

$$G(a_1, c_{o1}) + \left(\frac{\partial G}{\partial a}\right)_{a_1} \delta a + \left(\frac{\partial G}{\partial c_o}\right)_{c_{o1}} \delta c_o = 0 \tag{3.6}$$

These two linear simultaneous equations in two unknowns are solved for δa and δc_o :

$$\delta a = \frac{g^{(2)}\left(\frac{\partial g^{(1)}}{\partial c_o}\right)_{c_{o1}} - g^{(1)}\left(\frac{\partial g^{(2)}}{\partial c_o}\right)_{c_{o1}}}{\left(\frac{\partial g^{(1)}}{\partial a}\right)_{a_1}\left(\frac{\partial g^{(2)}}{\partial c_o}\right)_{c_{o1}} - \left(\frac{\partial g^{(1)}}{\partial c_o}\right)_{c_{o1}}\left(\frac{\partial g^{(2)}}{\partial a}\right)_{a_1}} \,, \tag{3.7}$$

$$\delta c_o = \frac{g^{(1)}\left(\frac{\partial g^{(2)}}{\partial a}\right)_{a_1} - g^{(2)}\left(\frac{\partial g^{(1)}}{\partial a}\right)_{a_1}}{\left(\frac{\partial g^{(1)}}{\partial a}\right)_{a_1}\left(\frac{\partial g^{(2)}}{\partial c_o}\right)_{c_{o1}} - \left(\frac{\partial g^{(1)}}{\partial c_o}\right)_{c_{o1}}\left(\frac{\partial g^{(2)}}{\partial a}\right)_{a_1}} \,, \tag{3.8}$$

and the new approximation to the solution is given by

$$a = a_1 + \delta a \,, \qquad c_o = c_{o1} + \delta c_o \,. \tag{3.9}$$

The process is repeated with the new values of a and c_o until the desired accuracy is secured. If it is found that the values of the partial derivatives of $g^{(1)}$ and $g^{(2)}$ with respect to a and c_o are not much changed in successive calculations, we need not recompute them at every step, but merely copy tham for the previous step.

For the evaluation of the partial derivatives occurring in (3.7) and (3.8), we may proceed as follows: Let us assume, for example, that g_2 given by (1.6) in Section 1 is one of the $g^{(i)}(a, c_o)$-functions concerned. Then, we can write that

$$\log g_2 = 2 \log f_2 - \log f_0 - \log f_4 \tag{3.10}$$

and by derivation of (3.10) with respect to a and c_o we have

$$\frac{\partial g_2}{\partial a} = g_2 \times \left[\frac{2}{f_2} \frac{\partial f_2}{\partial a} - \frac{1}{f_0} \frac{\partial f_0}{\partial a} - \frac{1}{f_4} \frac{\partial f_4}{\partial a} \right] \tag{3.11}$$

and

$$\frac{\partial g_2}{\partial c_0} = g_2 \times \left[\frac{2}{f_2} \frac{\partial f_2}{\partial c_0} - \frac{1}{f_0} \frac{\partial f_0}{\partial c_0} - \frac{1}{f_4} \frac{\partial f_4}{\partial c_0} \right] . \tag{3.12}$$

Finally, the partial derivatives $\dfrac{\partial f_{2m}}{\partial a}$ and $\dfrac{\partial f_{2m}}{\partial c_0}$ on the r.h. sides of (3.11) and (3.12) may be easily derived from one of the algebraic expressions (3.19), (3.20) and (3.21) given in Chapter 3. From (3.19), for example, we have

$$\frac{\partial f_{2m}}{\partial a} = -2\,\Gamma(m+1) \sum_{l=0}^{\Lambda} C^{(l)} \frac{(1-c_0^2)^{\nu}}{\nu\,\Gamma(\nu+2)} \sum_{n=0}^{\infty} \frac{n}{n+1} \frac{(\nu+n+2)(\nu+2n+2)\,\Gamma(\nu+n+1)\,\Gamma(\nu+n+2)}{(n!)^2\;\Gamma(m+\nu+2)} \times$$

$$\times\; {}_2F_1\left(\begin{matrix} -n,\, n+\nu+2 \\ \nu+1 \end{matrix} \,\middle|\, a\right) {}_2F_1\left(\begin{matrix} -n+1,\, n+\nu+3 \\ \nu+2 \end{matrix} \,\middle|\, a\right) {}_2F_1\left(\begin{matrix} -n,\, n+\nu+2 \\ m+\nu+2 \end{matrix} \,\middle|\, 1-c_0^2\right) . \tag{3.13}$$

and

$$\frac{\partial f_{2m}}{\partial c_0} = 2c_0\,\Gamma(m+1) \sum_{l=0}^{\Lambda} C^{(l)} \frac{(1-c_0^2)^{\nu}}{\nu\,\Gamma(\nu+1)} \sum_{n=0}^{\infty} \frac{1\;(\nu+2n+2)\,\Gamma(\nu+n+1)\,\Gamma(\nu+n+2)}{(n+1)!\,(n-1)!\;\Gamma(m+\nu+2)} \times$$

$$\times \left[{}_2F_1\left(\begin{matrix} -n,\, n+\nu+2 \\ \nu+1 \end{matrix} \,\middle|\, a\right) \right]^2 \cdot \cancel{{}_2F_1\left(\begin{matrix} -n+1,\, n+\nu+3 \\ m+\nu+3 \end{matrix} \,\middle|\, 1-c_0^2\right)} . \tag{3.14}$$

Let us now turn back to Equations (3.7) and (3.8), where if the

$$\left\{ \frac{n(n+\nu+2)}{(m+\nu+2)}\, {}_2F_1\left(\begin{matrix} -n+1,\, n+\nu+3 \\ m+\nu+3 \end{matrix} \,\middle|\, 1-c_0^2\right) - \frac{\nu}{1-c_0^2}\, {}_2F_1\left(\begin{matrix} -n,\, n+\nu+2 \\ m+\nu+2 \end{matrix} \,\middle|\, 1-c_0^2\right) \right\}$$

denominator vanishes at or near a supposed solution of $F(a, c_o) = 0$, $G(a, c_o) = 0$ we may ~~anticipate~~ expect difficulty with the method. The vanishing denominator may indicate i) the large difference between the approximate solution we start with and the true parameters a and c_o, ii) the existence of two or more solutions close together, or iii) no solution at all. In this case we may re-estimate the approximate solution (a_1, c_{o1}) as starting point for the iterations and may include the second partial derivatives in Equations (3.5) and (3.6).

4.4 Applications to YZ(21) Cassiopeiae and β Persei (Algol).

We shall illustrate in this section the numerical examples to the solutions of eclipse elements from one observed minimum alone by the applications to the light curves of YZ Cas and β Per, under the spherical model assumptions.

Our aim here is not only to solve the elements of a system, but also to try to establish how the procedures of the new methods work and acquire some understanding of the problems related with the determinacy of the solution. Having this in mind, it was decided to study the light curves of well known eclipsing binary systems YZ Cas and β Per.

(1) YZ(21) Cassiopeiae.

As is well known, YZ(21) Cassiopeiae has a simple model with detached spherical components and circular orbit. We have highly accurate (with the probable error ± 0.002 magnitude) observations of this system in $\lambda = 4500$ A and $\lambda = 6700$ A by Kron (1939 and 1942).

Almost no rectification is needed for the proximity effects. Thus, the solution may be expected to be free from any rectification error; and the accuracy of the solution may be directly related with the accuracy of the observational moments and the geometrical determinacy of the unknown parameters a and c_o .

The fundamental quantities from which we depart for the solution of eclipse elements of YZ Cas have been given in Table 10 for occultation (secondary minimum) and transit (primary minimum) eclipses in two colours. These data for the moments A_{2m} have been extracted from the work by Kurutaç (1976). The g_2's and g_4's in Table 10 have been derived from the moments by making use of the l.h. sides of Equations (1.6) and (1.7). There are conspicuous differences between the derived values of the g_2's and g_4's for the light curves in two colours which are noticeably greater than the probable error of observations. They may be caused mainly by errors arising from the numerical evaluation of moments A_{2m} by a simple trapezoidal quadrature. If we use the moments from Jurkevich (1976) obtained by a more sophisticated Kalman filter method for the transit eclipses of blue observations, it can be found that $g_2 = 0.7059$ and $g_4 = 0\ 8265$. As an inevitable result of these differences the eclipse elements will be different for the light curves observed in different wavelengths.

The uncertainties of moments in Table 10a,b have been obtained by using the formula (cf. Demircan, 1977a; Eq. 4.7)

$$\Delta A_{2m} \doteq \Delta U \sin^{2m}\theta' \quad , \tag{4.1}$$

Table 10a

The Observed Quantities of YZ Cas in λ = 4500 A

	Occultation min.		Transit min.	
A_0	0.0622	$\pm$0.0007	0.307	$\pm$0.006
A_2	0.001274	$\pm$0.000036	0.005243	$\pm$0.000028
A_4	0.00003434	$\pm$0.0000019	0.0001283	$\pm$0.0000013
A_6	0.000001099	$\pm$0.000000096	0.000003809	$\pm$0.000000062
g_2	0.7599	$\pm$0.0077	0.6979	$\pm$0.00098
g_4	0.8422	$\pm$0.0042	0.8243	$\pm$0.00111

Table 10b

The Observed Quantities of YZ Cas in λ = 6700 A

	Occultation min.		Transit min.	
A_0	0.1022	$\pm$0.001	0.267	$\pm$0.0006
A_2	0.002048	$\pm$0.000046	0.004895	$\pm$0.000028
A_4	0.00005357	$\pm$0.0000025	0.0001225	$\pm$0.0000013
A_6	0.000001701	$\pm$0.00000013	0.000003652	$\pm$0.000000063
g_2	0.7661	$\pm$0.0088	0.7328	$\pm$0.0010
g_4	0.8238	$\pm$0.0046	0.8394	$\pm$0015

where ΔU is the error in the unit of light U, while θ' stands for the phase angle of first contact point. In order to estimate the uncertainties in g_2 and g_4 we used the formulae

$$\Delta g_2 \doteq \frac{2A_0A_2A_4\Delta A_2 - A_2^2\left[A_4\Delta A_0 + A_0\Delta A_4\right]}{A_0^2A_4^2} \tag{4.2}$$

and

$$\Delta g_4 \doteq \frac{2A_2A_4A_6\Delta A_4 - A_4^2\left[A_6\Delta A_2 + A_2\Delta A_6\right]}{A_2^2A_6^2} \tag{4.3}$$

(see also Paper XV; Equations 5.2 and 5.3) which can be easily obtained by differentiation of Equations (1.6) and (1.7). The standard errors in the unit of light were taken from Kurutac (1976) as ± 0.0006 in blue and red colours for primary minimum and ± 0.007 in blue and ± 0.0010 in red colours for secondary minimum.

In principle, the intersection point $P(a, c_o)$ of g_2 and g_4 (see Figures 10 and 11) gives us two fundamental parameters a and c_o, but in practice the intersection point will not be precise as long as the observational values of g_2 and g_4 are subject to any uncertainty. This situation has been illustrated schematically in Figure 16. Every point in the batched area (solution domain) satisfies the nonlinear Equations (1.6) and (1.7) simultaneously within the error of the observational g_2 and g_4. Larger values of $\Delta\alpha$ and Δc_o for the solution domain will mean larger uncertainties in the final elements.

In order to estimate the solution points P_1, P_2 and M (see Figure 16) minimization routine given in the previous section has not been employed for the solution of the simultaneous nonlinear Equations (1.6) and (1.7), since it will give only any one of the points in the solution domain, and fails if there is no intersection for the adopted Δg_2 and Δg_4. The parameters a and c_o in the expected solution domain have been adjusted in small steps (0.001), and the $0 - C$ values for g_2 and g_4 have been computed for every step together with the final elements $r_{1,2}$ i and $L_{1,2}$ - Then by inspection of the 0–C values, the limit points $P_1(a_1, c_{o1})$ and $P_2(a_2, c_{o2})$, and the most probable point $M(a_m, c_{om})$ for the most accurate solution have been estimated graphically (see Table 11). For the computations, Equations (3.18) and (3.19) from Chapter 3 have been employed separately and noted that (3.18) is more practicable to work with. It is also noteworthy that whole computations have been performed in approximately 30 seconds on the CDC 7600. For occultation eclipses it was found that the functions g_2 and g_4 do not intersect each other (see Figure 17). Only under the assumption of large uncertainties could we estimate the points $M(a_m, c_{om})$ together with the most probable elements. The expected large errors in these final elements are due to the shallowness of secondary minimum and the larger dispersion of observations within these eclipses which increases the uncertainties in the fundamental observed quantities A_{2m}.

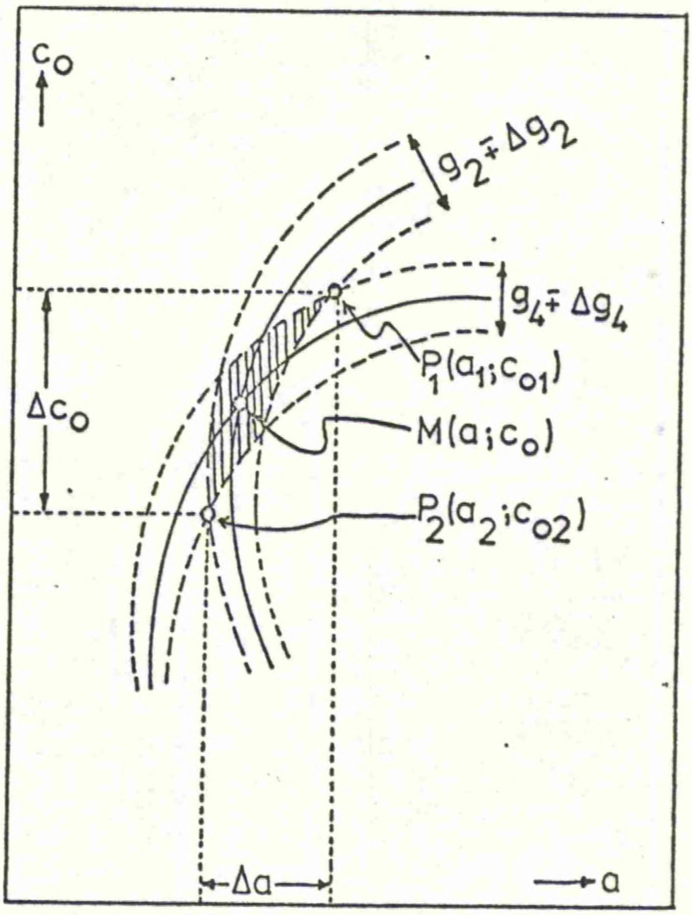

Figure 16. The geometrical illustration for the solution of the eclipse parameters a and c_o from the functions g_2 and g_4. Every point in the hatched area (solution domain) satisfies the observational g_2 and g_4 simultaneously within the errors Δg_2 and Δg_4 (after Demircan, 1978b, Paper XV).

Table 11a

The Elements of YZ Cas in λ = 4500 A

	Occultation min.	Transit min.		
	M(0.36, 0.11)	P_1(0.635,0.000)	P_2(0.66,0.23)	M(0.651,0.136)
r_g	0.145	0.140	0.148	0.143
r_s	0.082	0.080	0.076	0.077
i	88°.6	90°.0	87°.0	88°.3
L_1	0.062	0.810	1.039	0.938
U_1(adopted)	0.4	0.5	0.5	0.5

Table 11b

The Elements of YZ Cas in λ = 6700 A

	Occultation min.	Transit min.		
	M(0.37, 0.09)	P_1(0.645,0.000)	P_2(0.665,0.207)	M(0.655,0.140)
r_g	0.143	0.141	0.147	0.144
r_s	0.084	0.078	0.074	0.076
i	88°.8	90°.0	87°.4	88°.1
L_1	0.102	0.813	0.984	0.893
U_1 (adopted)	0.4	0.3	0.3	0.3

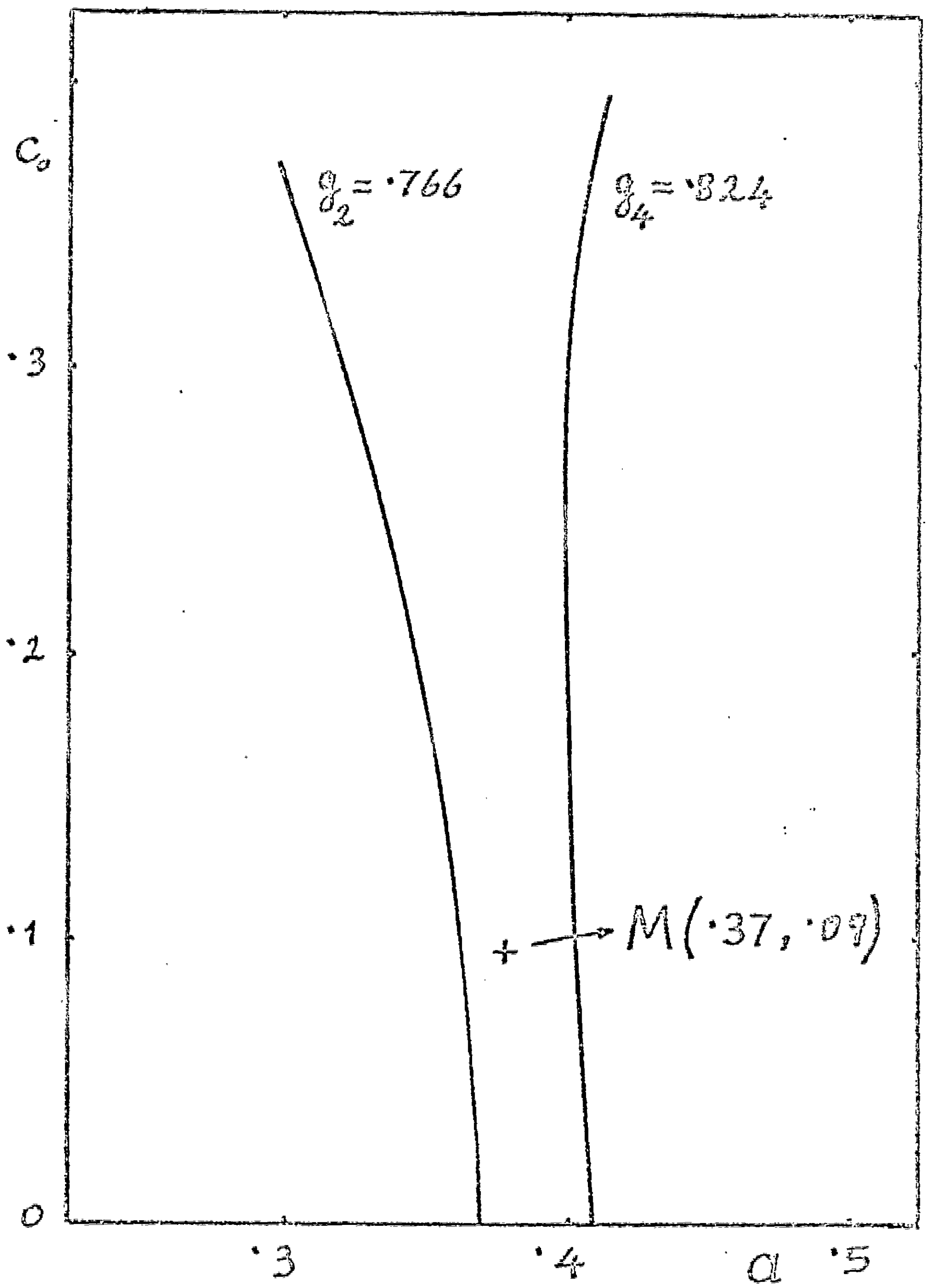

Figure 17. The functions g_2 and g_4 for the occultation eclipses of YZ Cas do not intersect each other. Only under the assumptions of large observational uncertainties Δg_2 and Δg_4 , intersection point may be estimated (see text).

(2) β Persei (Algol).

A study of the three light curves (in λ = 4350 Å narrow band, λ = 5500 A narrow band, and λ = 5500 A broad band observations of Wilson et al, 1972) of Algol has also been carried out by employing the same automated methods for obtaining the eclipse elements from one observed minimum alone, under the spherical model assumptions. Algol, as a semi-detached system, presents an extreme case of differing shapes for the components since the primary is nearly spherical, whereas the secondary almost certainly fills its Roche lobe and is thus very highly distorted. Although this is the main point against a spherical model analysis, we have found reaonable results in good agreement with one another and the other published results which have been derived using totally different approaches.

In the following Table 12 the observed quantities of Algol which have been extracted from Demircan (1977a) and in Table 13 the resulting elements from the present investigation of these data are presented.

The results of the certain applications to the light curves of YZ Cas and Algol as given in Tables 11a,b and 13 may show the success of the new methods. The same automated method - by the present author - for obtaining the elements of an eclipsing binary system from its one observed minimum alone (see Section 1 of the present chapter) has also been successfully employed by Kaskambas (1977), Al-Naimiy (1977), Güdür (1978) and Edalati (1978).

Table 12

The Observed Quantities of Algol for Occultation Minimum

	λ = 4350A narrow	λ = 5500A narrow	λ = 5500A broad	probable uncertainty
A_0	0.7025	0.6867	0.6832	$\pm$0.008
A_2	0.04335	0.04083	0.04114	$\pm$0.000387
A_4	0.00429	0.00391	0.00401	$\pm$0.000075
A_6	0.00052	0.00046	0.00048	$\pm$0.000014
g_2	0.624	0.621	0.618	$\pm$0.007
g_4	0.816	0.814	0.814	$\pm$0.001

Table 13

The Elements of Algol from Occultation Minimum

	λ = 4350A narrow	λ = 5500A narrow	λ = 5500A broad
r_g	0.250	0.245	0.247
r_s	0.216	0.212	0.214
i	81$^\circ_.$8	82$^\circ_.$0	81$^\circ_.$9
L_1	0.947	0.927	0.921
U_1 (adopted)	0.53	0.43	0.43

CHAPTER 5

CONCLUSIONS AND THE ACCURACY OF THE FOURIER TECHNIQUES

The practical procedures for the solutions of the elements of any eclipsing system in the frequency-domain have been outlined in the previous chapter (cf. also Paper XIV). The fundamental quantities from which we depart in quest of our solution are two g-functions defined by the moments A_{2m} (see Equations 1.3 - 1.5 in Chapter 4, and also Equations 2.13 - 2.16 in Paper XIV, or Equations 3.2 - 3.6 in Paper XV). If we establish the observational values for these functions, they constitute two independent relations between the unknown parameters a and c_o, and the observed quantities and can be solved numerically (see Section 3 of Chapter 4) with the aid of the general expressions for the respective moments.

The fundamental quantities

$$g_2(a, c_o) = \frac{A_2^2}{A_o A_4} = \frac{f_2^2}{f_o f_4} \tag{0.1}$$

and

$$g_4(a, c_o) = \frac{A_4^2}{A_2 A_6} = \frac{f_4^2}{f_2 f_6} \tag{0.2}$$

have been studied in detail for the solution of eclipse elements. The necessary functions for the analysis have also been tabulated for grey plane-parallel stellar atmospheres up to four significant digits at intervals permitting linear interpolation (see Appendices 1 - 5). The methods for obtaining the elements of wide binaries from their only one observed minimum have been automated and applied successfully on the light curves of YZ(21) Cassiopeiae and β Persei (Algol). From these practical applications it was noted that the errors in the final elements are caused by not only the observational uncertainties in the moments, but also by the geometrical behaviour of two employed g-functions. Determinacy of the parameters a and c_o which are the fundamental quantities for obtaining the final elements $r_{1,2}$, i and $L_{1,2}$ evidently depends on the intersection angle of the two g-functions employed. In particular, the g_2 and g_4 intersect each other, in most cases, at very low angles (see Figures 10 and 11 in Chapter 4). The larger intersection angles may be obtained by using different combinations of the moments A_{2m} for different positive values of real m; or in other ways.

The above idea has been worked out by the present author in Paper XVI, to gain a fuller understanding of the geometrical determinacy of the fundamental eclipse parameters a and c_o. In this final Chapter we have given the results of this work from Paper XVI. In Section 1, different combinations of the moments A_{2m} have been worked out as g-functions. For the index $2m$, the values between zero and six were applied. It has been noted that the behaviour of these functions vary but very little with applied different combinations of the moments. A choice of the most convenient moments to obtain a good determinacy for the eclipse elements were discussed. In this connection, in Section 2 the m dependence of the moments and the errors in their observational values have been considered. In Section 3, different practical procedures for the solution of eclipse elements were introduced, and finally in Section 4, a different type of moments was tested.

In the computations of f_{2m}-functions, Eq. (3.17) with (3.18) from Chapter 3 (hereafter (3.3.17) and (3.3.18)) have been employed; three terms for the first summation and eighty terms for the second summation have been used to construct 4D tables of the requisite g-functions. All the g-functions studied throughout this chapter have been tabulated for grey plane-parallel stellar atmospheres, and in terms of the values of $a = 0.1(0.01)0.9$ and $c_o = 0(0.02)0.98$ for every type of eclipse.

5.1 <u>On the g-Functions.</u>

These functions have been defined by Eqs. (1.3) - (1.5) of Chapter 4. In this section, first the following particular forms of g-functions have been studied to investigate the form and power dependence of these functions

$$\frac{A_2^3}{A_0^2 A_6}, \quad \frac{A_2^3 A_6}{A_0 A_4^3}, \quad \frac{A_2^5}{A_0^3 A_4 A_6}, \quad \frac{A_0 A_6}{A_2 A_4}, \quad \frac{A_0^2 A_6^3}{A_2 A_4^4}$$

$$\frac{A_0^3 A_6^4}{A_2^2 A_4^5}, \quad \frac{A_0^3 A_6^5}{A_2 A_4^7}, \quad \frac{A_0^5 A_6^6}{A_2^4 A_4^7}, \quad \frac{A_0^4 A_6^5}{A_2^3 A_4^6}, \quad \frac{A_0^5 A_6^8}{A_2^2 A_4^{11}},$$

$$\frac{A_4^5}{A_0 A_2 A_6^3}, \quad \frac{A_4^7}{A_0 A_2^2 A_6^4}, \quad \frac{A_4^8}{A_0^2 A_2 A_6^5}, \quad \text{and} \quad \frac{A_4^{12}}{A_0^2 A_2^3 A_6^7} \qquad (1.3)$$

for only integral powers and even moments. Note that every form - except first one - consist of the moments A_0, A_2, A_4 and A_6. The 0.5-th, 0.2-nd and 0.1-st powers and logarithmics of the foregoing g-functions have also been computed and their behaviour illustrated diagrammatically. The results are:

1. The intermediate curves between g_2 and g_4 (see Figures 10 and 11 in Chapter 4) have been obtained. This result enables us to say that the behaviour of g-functions depends but very little on different combinations and powers of the same moments.

2. Numerical values of g-functions increase and varying intervals shorten with decreasing positive real powers. So, for some particular powers the most useful short tables permitting linear interpolation can be constructed. These powers are approximately 0.2 and 0.5 for three-digit short tables of g_2 and g_4, respectively. These powers become 0.05 and 0.10 for four-digit short tables for the same g-functions, respectively. It should be noted that further decrease of these powers requires fifth significant digit in the numerical values and this cannot be achieved for the above forms of g-functions if we employ the series

expansion (3.17) with (3.18) for the constituent moments A_{2m}, from Chapter 3.

3. A logarithmic behaviour of g-functions is similar but the use of logarithms will increase considerably their numerical range; therefore in practice no advantage can be obtained by working with the logarithm of g-functions rather than with the functions themselves.

Next, it was decided to study the simplest form as given by

$$g(a, c_0) = \frac{A_x^2}{A_{y_1} A_{y_2}} = \frac{f_x^2}{f_{y_1} f_{y_2}}, \tag{1.4}$$

such that $x = \frac{1}{2}(y_1 + y_2)$, with fixed powers and gradually decreasing values of real x and $y_{1,2}$. The following particular g-functions in this form have been tabulated and their behaviours investigated:

$$\frac{A_3^2}{A_2 A_4}, \quad \frac{A_2^2}{A_1 A_3}, \quad \frac{A_1^2}{A_0 A_2}, \quad \frac{A_{\frac{1}{2}}^2}{A_0 A_1}, \quad \frac{A_{\frac{1}{4}}^2}{A_0 \cdot A_{\frac{1}{2}}},$$
$$\frac{A_{\frac{1}{8}}^2}{A_0 A_{\frac{1}{4}}}, \quad \frac{A_{\frac{1}{10}}^2}{A_0 A_{\frac{1}{5}}}, \quad \frac{A_{\frac{1}{100}}^2}{A_0 A_{\frac{1}{50}}}, \quad \text{and} \quad \frac{A_{\frac{1}{500}}^2}{A_0 A_{\frac{1}{250}}}. \tag{1.5}$$

An inspection of the numerical tables and diagrams of these g-functions leads to the following conclusions:

1. In general, the behaviour of g-functions depends very little on the specific moments employed.

2. Any change in the behaviour of the g-functions is caused by

changes of the orders x and $y_{1,2}$ (see the accompanying Figures 18–22). These orders control the curvature of g-functions. The greater the values of x and $y_{1,2}$, the lesser the curvatures in the respective g-functions.

3. Larger numerical values of any g-function are obtained if the difference between two radii r_1 and r_2 is large and the inclination i or orbital plane is near to 90° for the eclipsing system in question.

$$\lim_{\substack{a \to 0 \text{ or } 1 \\ c_0 \to 0}} g(a, c_0) = g_{max}. \tag{1.6}$$

The minimum values are always attained if the radii of the components are equal to each other and the inclination i of orbital plane is 90° ($a = \frac{1}{2}$ and $c_0 = 0$): i.e.,

$$\lim_{\substack{a \to \frac{1}{2} \\ c_0 \to 0}} g(a, c_0) = g_{min}. \tag{1.7}$$

Thus, $g(a, c_0)$ varies between these two values ($g_{min} \leqslant g(a, c_0) \leqslant g_{max}$). These limit values are listed in Table 14 for the g-functions given by (1.5).

4. In general, the scale for numerical values of g-functions can be altered by changing the differences between x and $y_{1,2}$. The smaller the differences, the larger are the numerical values for the respective g-function. Thus, we can note three factors which are effective on the scale of numerical values of g-functions.

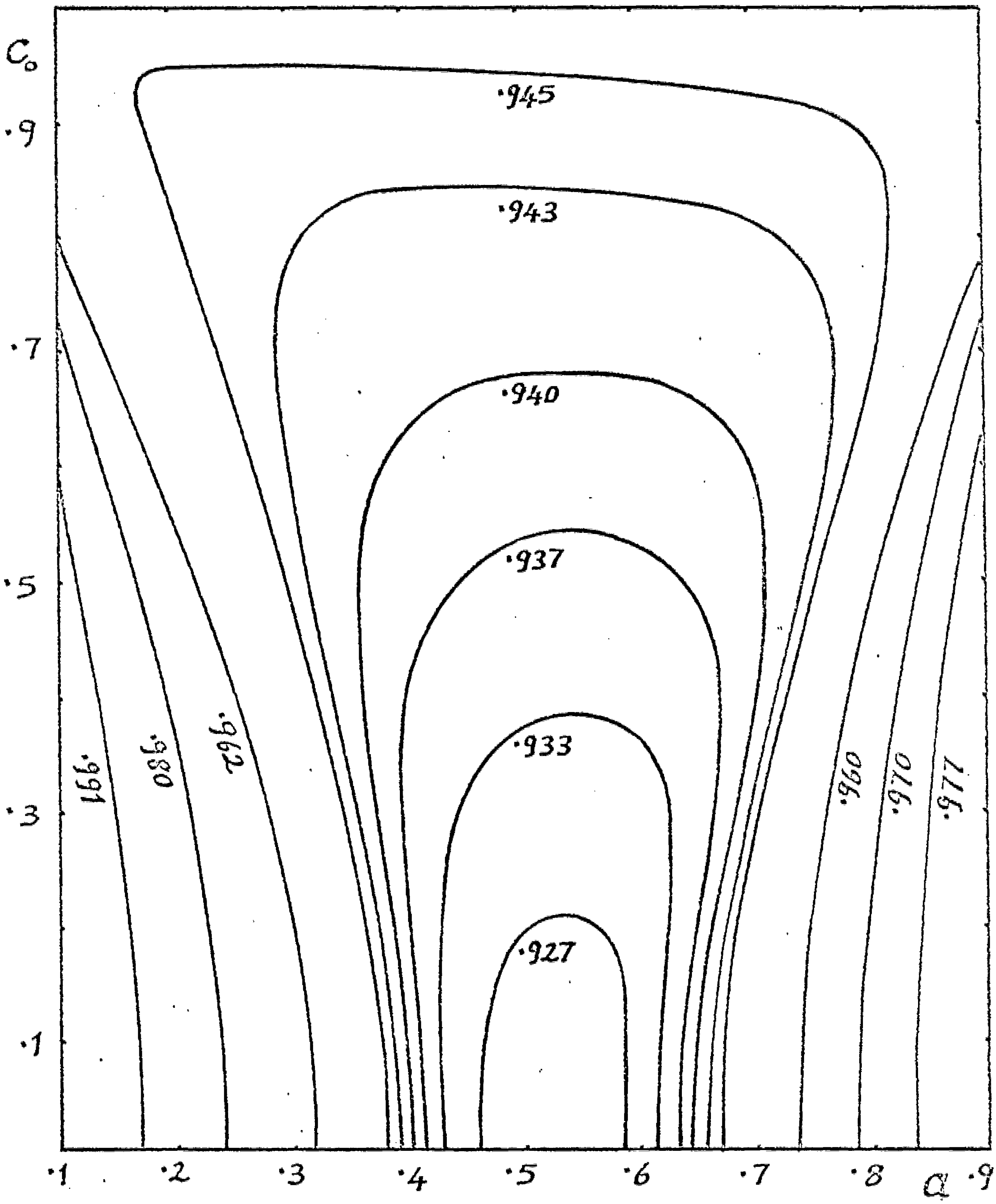

Figure 18. The functional behaviour of the g-function $\frac{A_3^2}{A_2A_4}$.

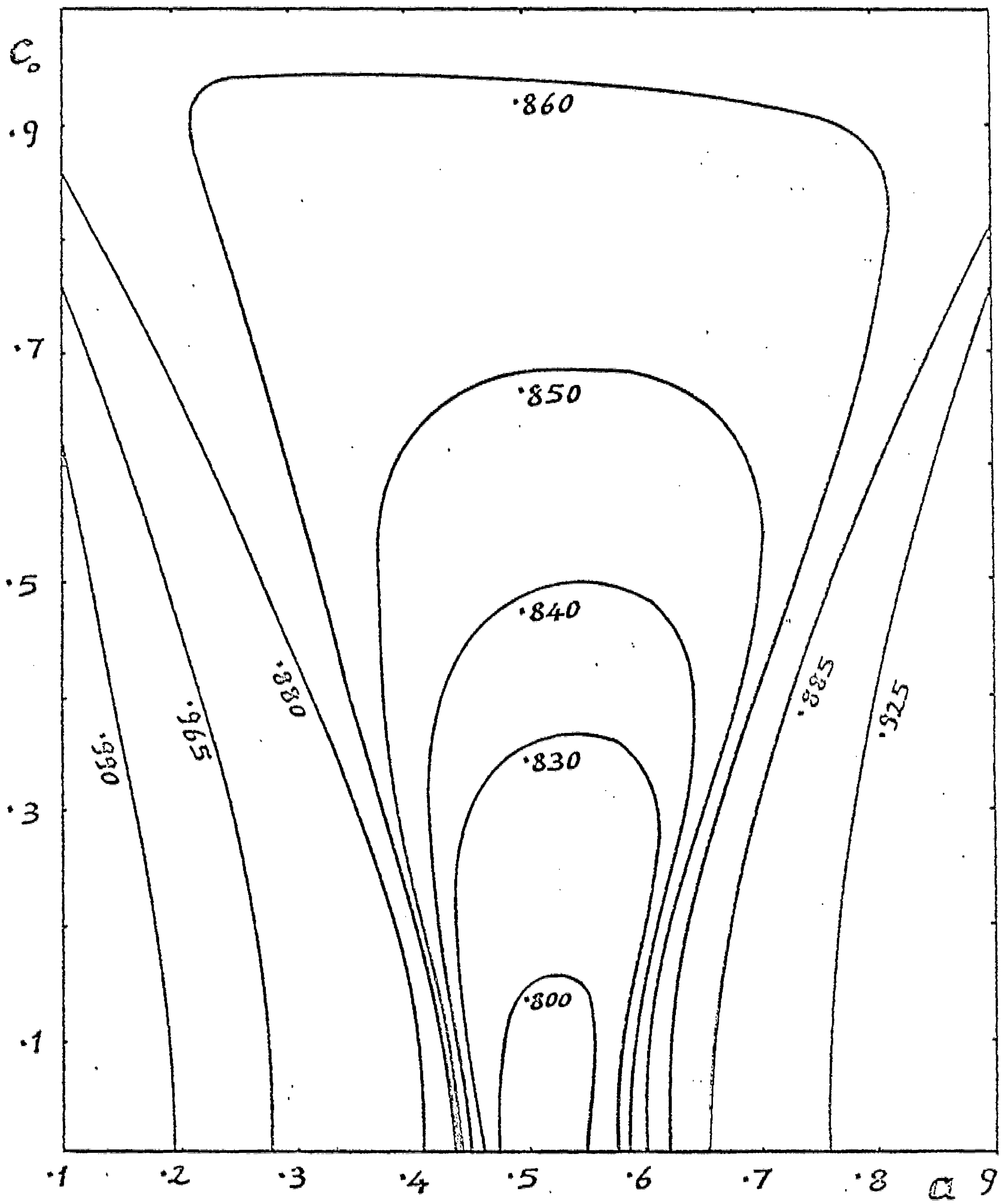

Figure 19. The functional behaviour of the g-function $\frac{A_1^2}{A_0A_2}$.

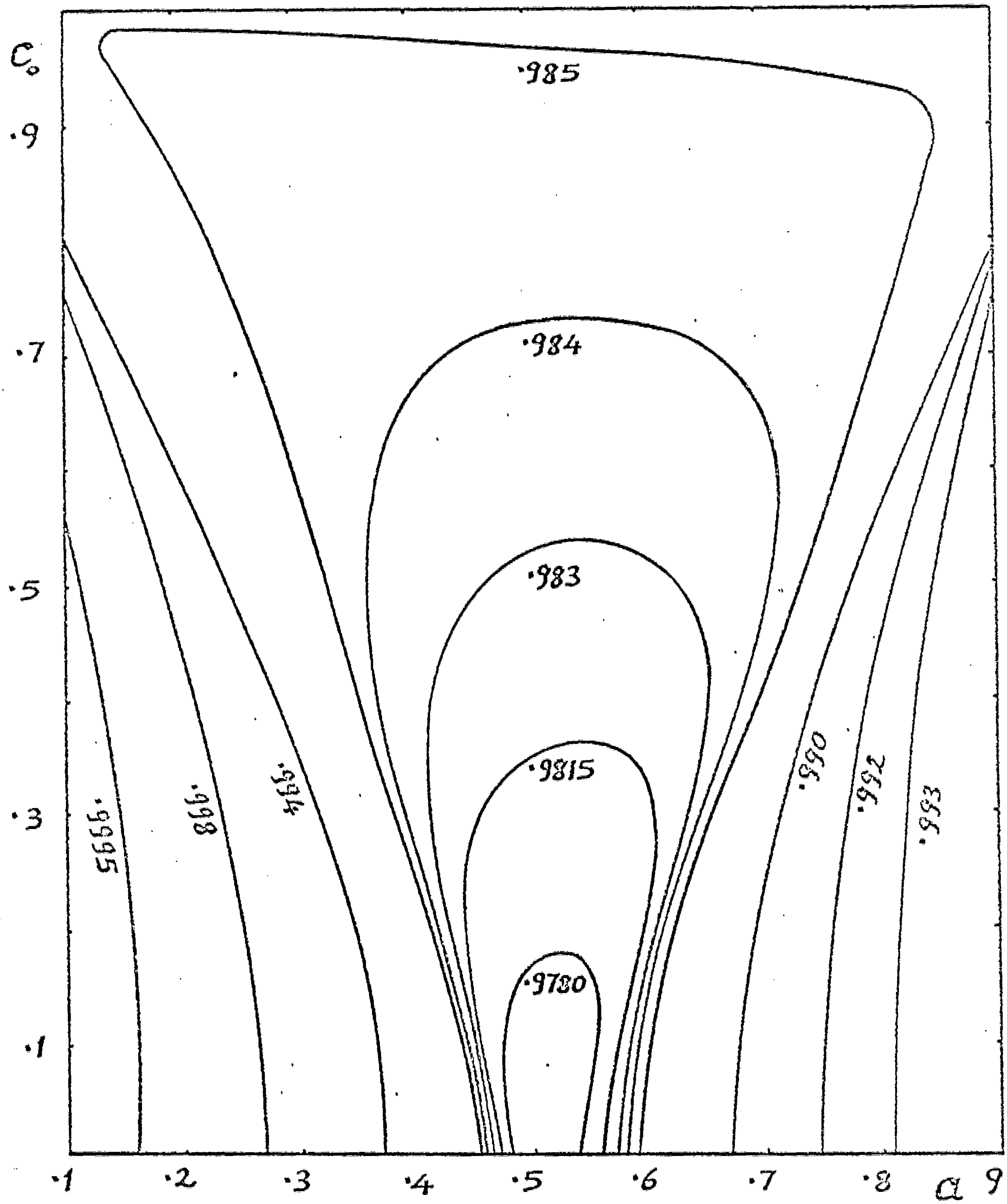

Figure 20. The functional behaviour of the g-function $\dfrac{A_{1/4}^2}{A_0 A_{1/2}}$.

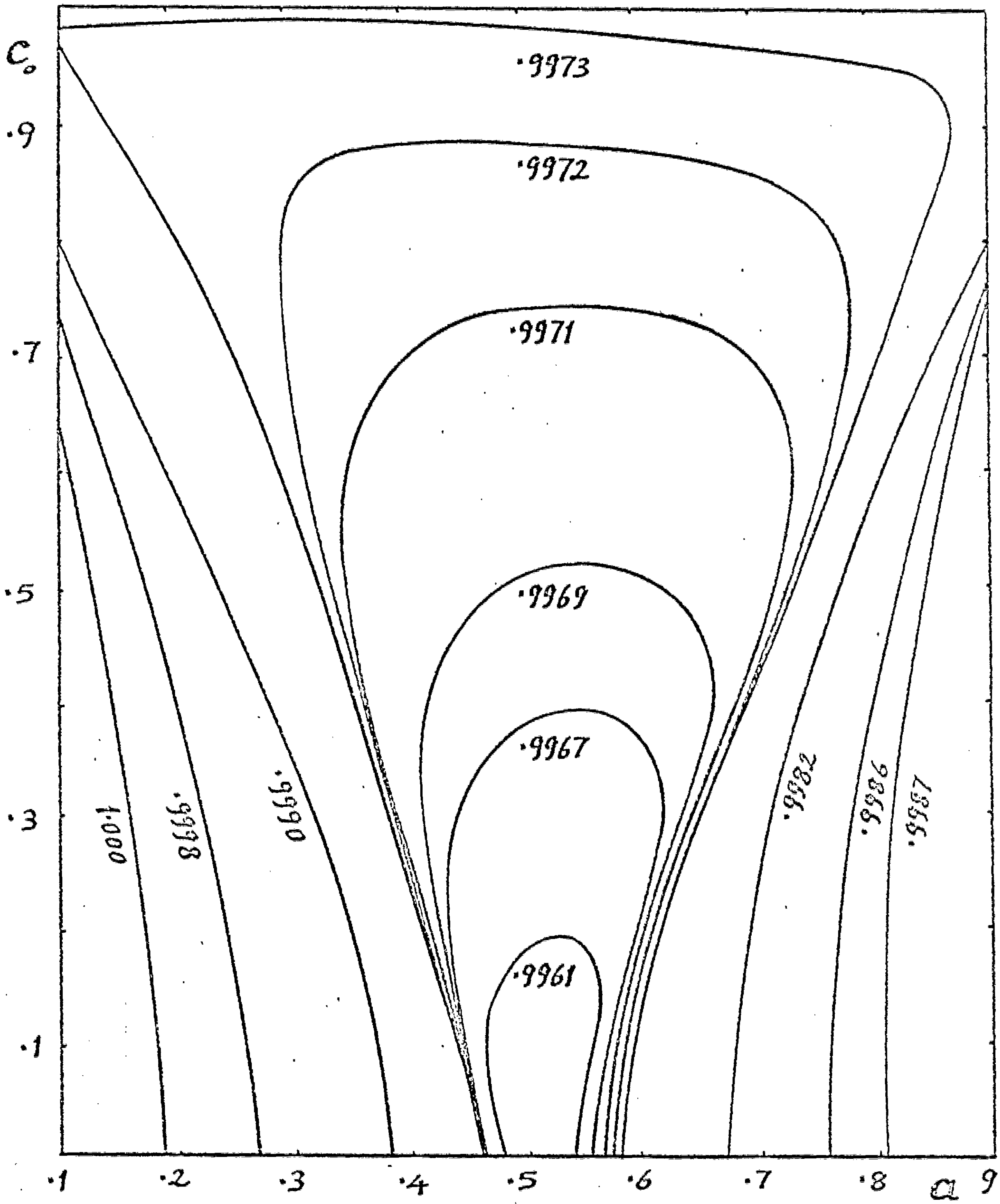

Figure 21. The functional behaviour of the g-function $\frac{A_{\frac{1}{10}}^2}{A_0 A_{\frac{1}{5}}}$.

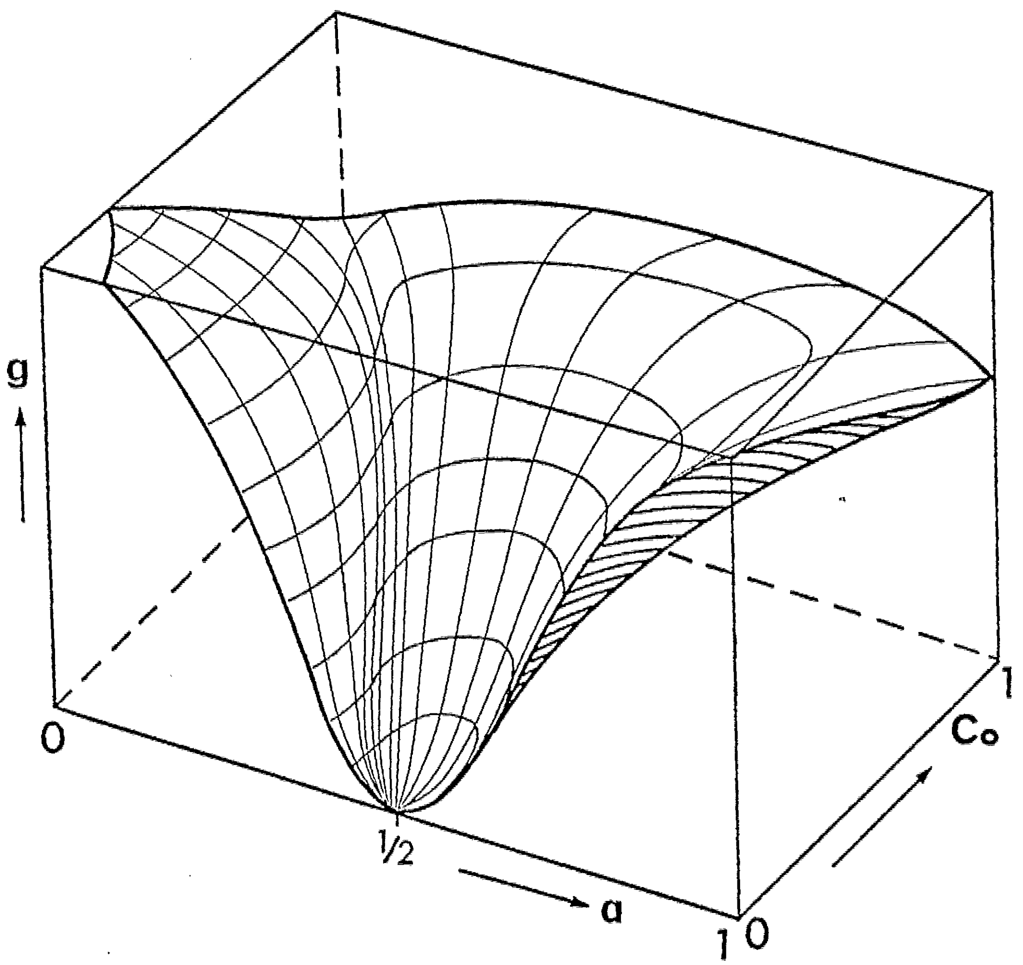

Figure 22. A schematic representation of the behaviour of g-functions (in general) in three dimensional space.

i) Powers of g-functions,

ii) different values of x and $y_{1,2}$, and

iii) different values of $|x-y_1|$, $|x-y_2|$ and $|y_1-y_2|$.

For the results 3 and 4 see Table 14.

In order to utilize the foregoing results for our purpose in choosing most useful two g-functions, we should look for a greater difference between their curvatures, which means better-defined intersections of these two functions. By resorting to second result mentioned above, if we recall Δ and Δ' as the mean values of the adopted orders x, y_1 and y_2 for two different g-functions, the most important conclusion is that

$$|\Delta - \Delta'| \propto \phi(a, c_0) \tag{1.8}$$

where ϕ stands for the angle of intersection between the two adopted g-functions. This means that when we use the first and last g-functions from Table 14, we get better intersections and, oncsequently, a better determinacy for the eclipse parameters a and c_0.

But, in practice, there are two restrictions to be considered:

i) Numerical values accurate to more than four digits are required for g-functions of very small orders. It is obvious that much higher accuracy can be achieved for empirical values of g-functions given by (1.4) if, and only if, the differences $|x-y_1|$, $|x-y_2|$ and $|y_1-y_2|$ between the adopted moments are sufficiently small. In this case, the errors of the respective moments become comparable and largely cancel in

Table 14

The Limit Values of Some g-Functions

g	g_{min}	g_{max}	g	g_{min}	g_{max}
$\frac{A_4^2}{A_2 A_6}$	0.794	0.989	$\frac{A_{\frac{1}{4}}^2}{A_0 A_{\frac{1}{2}}}$	0.963	1.000
$\frac{A_3^2}{A_2 A_4}$	0.921	0.997	$\frac{A_{\frac{1}{6}}^2}{A_0 A_{\frac{1}{3}}}$	0.980	1.000
$\frac{A_2^2}{A_1 A_3}$	0.872	0.997	$\frac{A_{\frac{1}{8}}^2}{A_0 A_{\frac{1}{4}}}$	0.986	1.000
$\frac{A_2^2}{A_0 A_4}$	0.528	0.989	$\frac{A_{\frac{1}{10}}^2}{A_0 A_{\frac{1}{5}}}$	0.990	1.000
$\frac{A_1^2}{A_0 A_2}$	0.752	0.997	$\frac{A_{\frac{1}{100}}^2}{A_0 A_{\frac{1}{50}}}$	0.996	1.000
$\frac{A_{\frac{1}{2}}^2}{A_0 A_1}$	0.897	1.000	$\frac{A_{\frac{1}{500}}^2}{A_0 A_{\frac{1}{250}}}$	1.000	1.000
$\frac{A_{\frac{1}{3}}^2}{A_0 A_{\frac{2}{3}}}$	0.943	1.000			

the formation of the ratio (1.4) for the g-functions. But even though this may be so, it will be difficult to deal with large numerical values if one wants to use the tables and diagrams for these functions in practice.

ii) The moments with the large orders which are used in constructing the first g-functions in Table 14 are subject to larger observational errors. Thus, these g-functions may be of little practical use.

The difficulty in dealing with large numerical values of the g-functions of very small orders may be removed by using the powers of respective g-functions (see result 4 i above). The better understanding of the second restriction will be the subject of the following section.

5.2 The m-Dependence of the Moments A_{2m} and Errors in their Observational Values.

In the study of different g-functions (section 1), required f-functions were also tabulated in terms of the values of $a = 0.1(0.01)0.9$ and $c_o = 0(0.02)0.98$ for every type of eclipse. For m fifteen different values between zero and three were applied. With the aid of these tables, theoretical values of the moments A_{2m} can be obtained for any set of elements $r_{1,2}$, i and L_1, and for any value of m between zero and three, by means of Equations (2.2.17) - (3.3.18). To illustrate the way in which the numerical values of thesemoments decrease with increasing values of m, their behaviour for $r_1 = 0.1$, 0.3 and $i = 90^o$ has been plotted on the accompanying Figures 23 and 24 for fixed values of a. Note that, if r_1 is fixed in 0.1, r_2 becomes 0.9 when $a = 0.1$ and diminishes to 0.025 when $a = 0.8$. If r_1 is 0.3, the minimum value of a will be 0.3 for $r_2 = 0.7$, and r_2 reduces to 0.375 for $a = 0.8$. For $a \leqslant 0.5$ (total eclipses) it is seen that $A_0 = L_1$, and when $a > 0.5$, then $A_0 < L_1$. It can be observed that the rate of decrease of the moments A_{2m} increases when a tends to ½. These results permit a conclusion that, in practice, more care should be exercised in dealing with higher moments – especially i) when the eclipse is of transit type and the luminosity of the eclipsed component constitutes a small portion of total luminosity – conditions which produce a shallow minimum in the light curve of the respective system. ii) When the radii of the components of the system are comparable to each other, which gives rise to two minima of shorter duration for the total phases.

Next, the proportional errors of the empirical moments as the functions of m and eclipse elements $r_{1,2}$, i and L_1 will be considered. The accompanying diagrammatic representations of these errors, as the functions of m and eclipse elements will aid us better to understand this important point in practice.

As has been shown in Demircan (1977, Eq. (4.7)), the probable errors in the empirical values of moments A_{2m} can be defined by Eq. (4.1) of Chapter 4 (hereafter 4.4.1). If we use Equations (3.3.17) and (4.4.1) to define the proportional errors of the empirical moments A_{2m}, we get

$$\varepsilon_{2m} \equiv \frac{\Delta A_{2m}}{A_{2m}} \doteq \frac{\Delta U}{L_1} \frac{\alpha^{2m}}{b^2(1-c_0^2)^{m+1}} \left(\frac{\sin\theta' \sin i}{r_1}\right)^{2m} \frac{1}{f_{2m}} . \qquad (2.1)$$

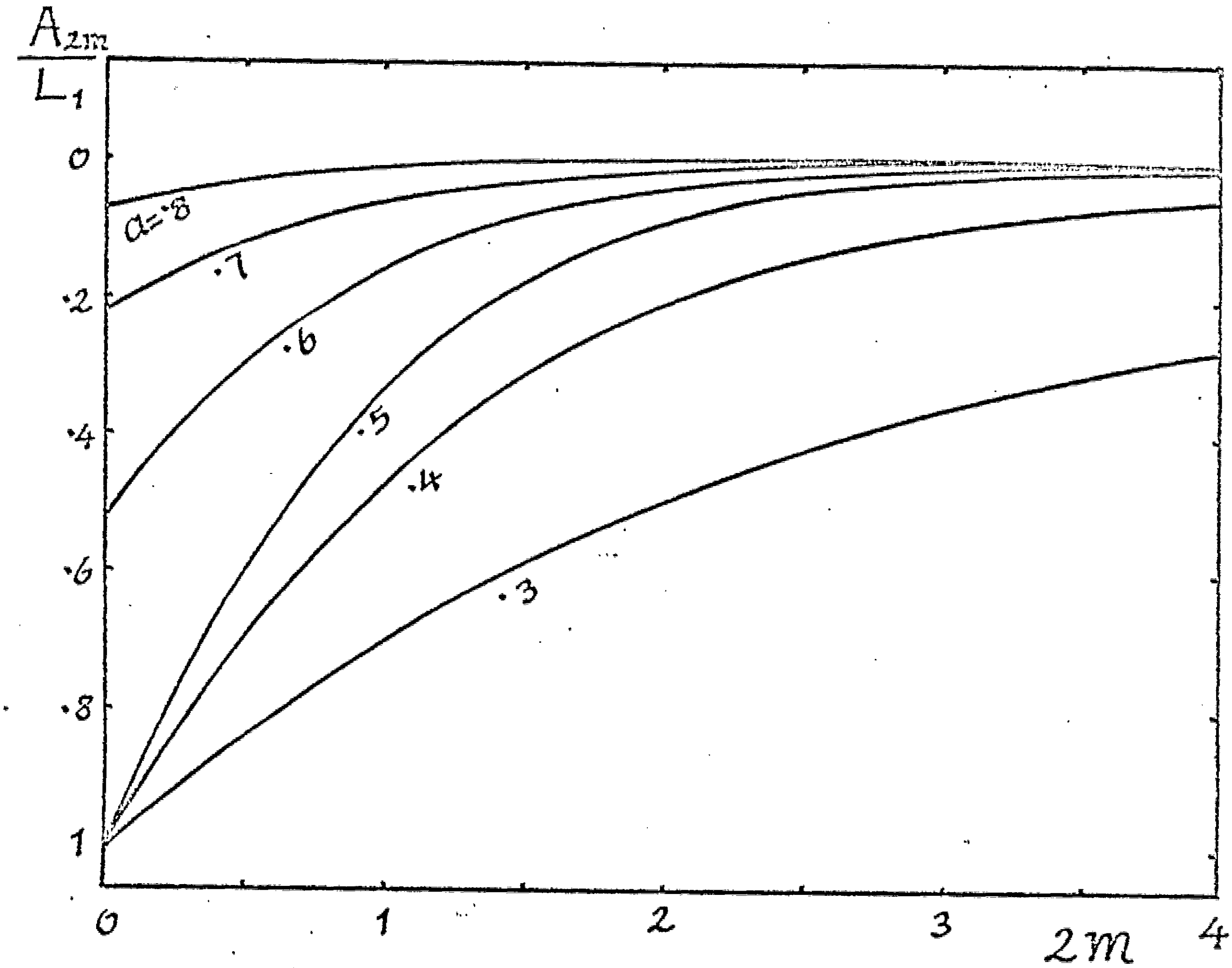

Figure 23. The m dependence of the moments A_{2m} for $r_1 = 0.3$, $i = 90^{\circ}$ and fixed values of a .

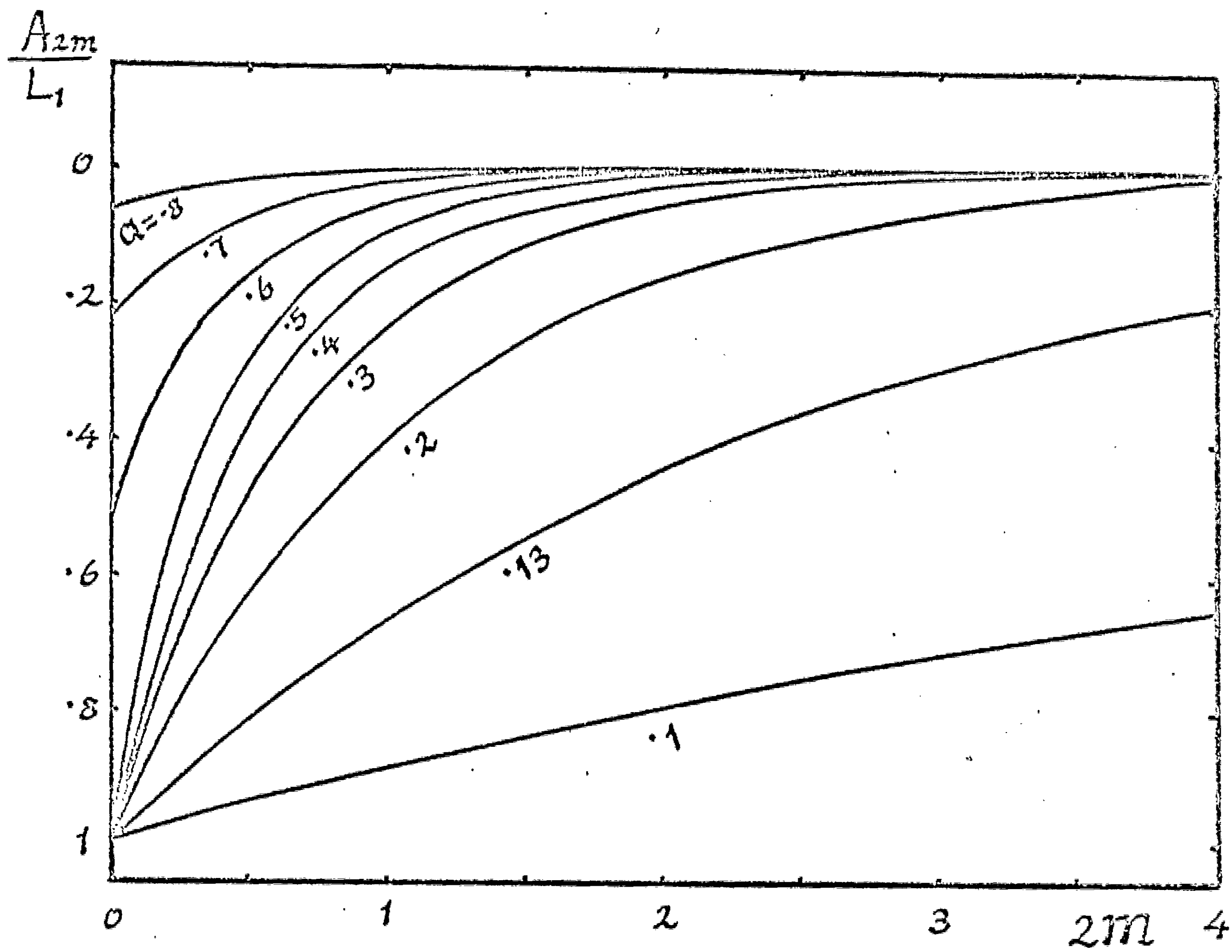

Figure 24. The m dependence of the moments A_{2m} for $r_1 = 0.1$, $i = 90^{\circ}$ and fixed values of a .

On the other hand, if we introduce the apparent separation of centres of the two stars, projected on the celestial sphere, as

$$\delta^2 = \sin^2\theta \sin^2 i + \cos^2 i \tag{2.2}$$

in terms of the phase angle θ, and the inclination i of the orbital plane of the system to the celestial sphere, this permits us to rewrite Eq. (2.1) as

$$\mathcal{E}_{2m} \doteq \frac{\Delta U}{L_1} \psi_{2m}(a, c_0) \qquad \text{with } \psi_{2m} = \left[b^2(1-c_0^2) f_{2m}\right]^{-1}, \tag{2.3}$$

where the error ΔU can be obtained from observations, so can L_1 if the eclipse it total. It is now obvious that the numerical values of the proportional errors $\mathcal{E}_{2m}$ in the empirical A_{2m}'s can be evaluated with the aid of availabl e tables of f_{2m} for any set of eclipse elements and for different values of real m between zero and three. So, the m dependence of the $\mathcal{E}_{2m}$'s and the effect of the inclination i can be worked out numerically. This has been done, and the results are diagrammatically shown in the accompanying Figures 25 and 26. If, for smaller values of a (occultation eclipses) L_1 has smaller values and for larger values of a (transit eclipses) if it becomes larger, then the effect of L_1 on the proportional error of A_{2m}'s becomes in the way to bring the curves of the Figures 25 and 26 nearer to one another. Another obvious fact can be observed from the Figures 25 and 26; namely, that <u>the proportional error of the moments A_{2m} increases</u>

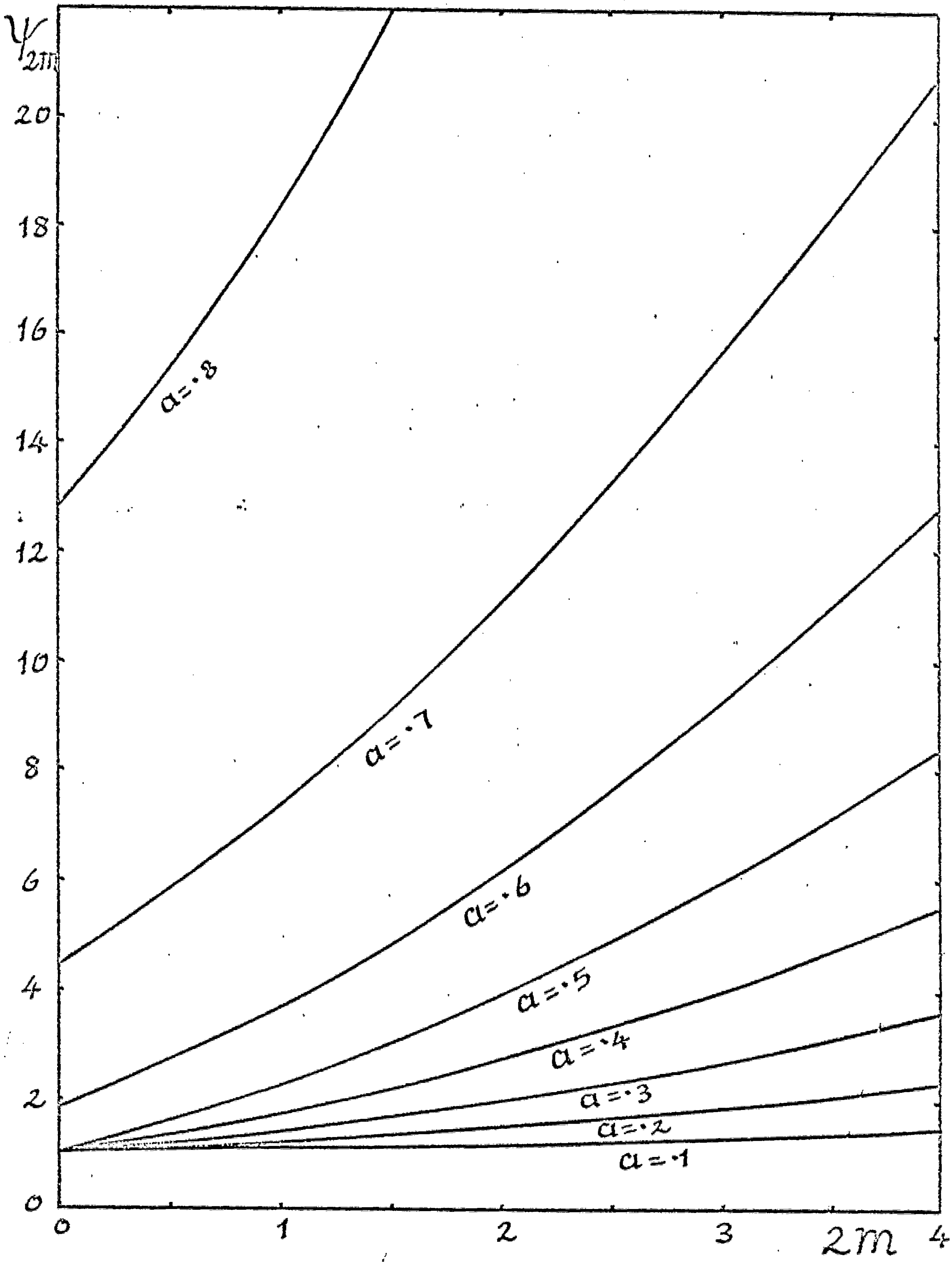

Figure 25. The m dependence of the Ψ_{2m}-functions.

$$\Psi_{2m} = \frac{L_1}{\Delta U} \cdot \frac{\Delta A_{2m}}{A_{2m}}$$ (see Eq. 2.3).

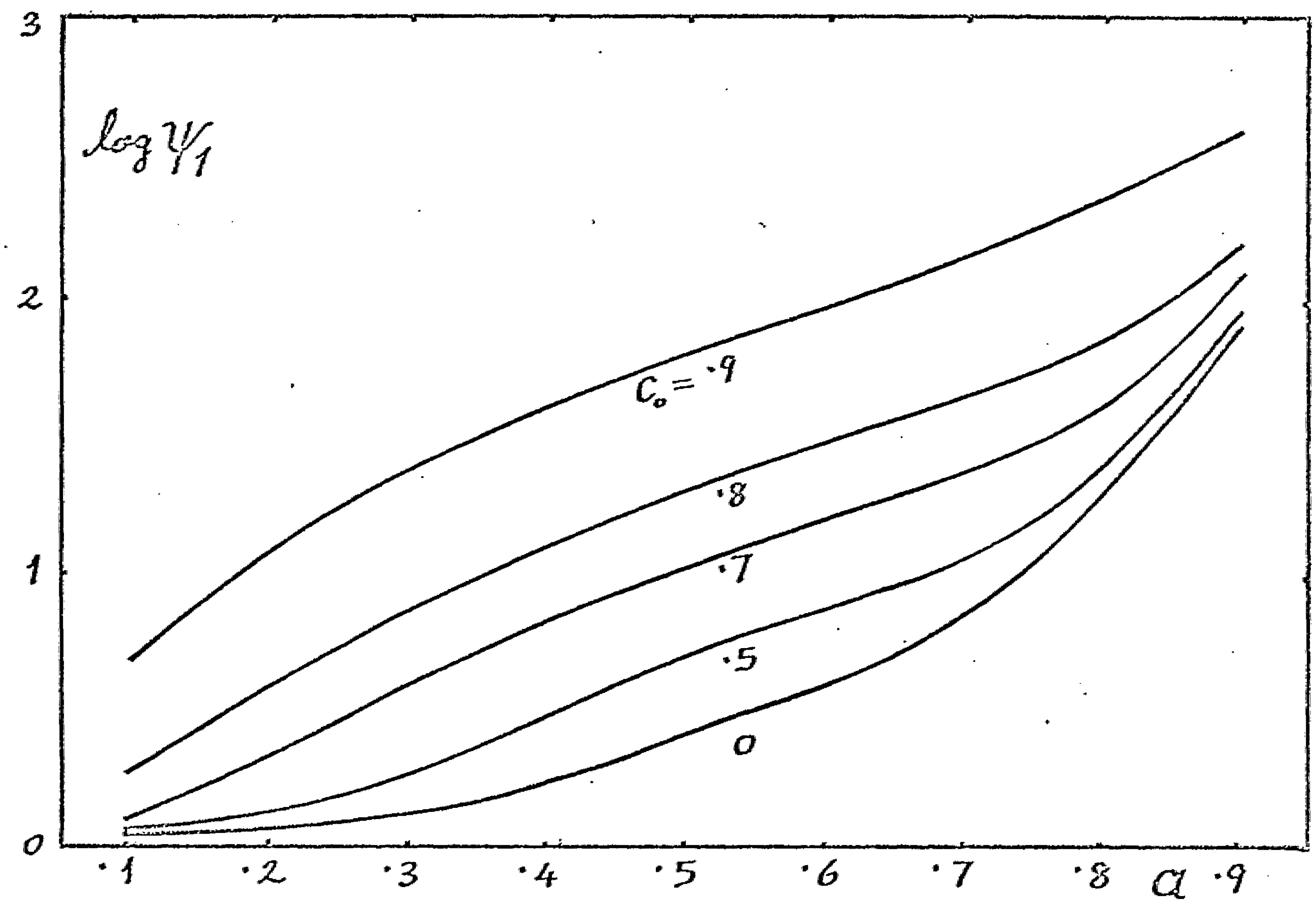

Figure 26. The functional behaviour of the ψ_{2m} for $m = 1$ and fixed values of c_o (for the ψ_{2m}–functions see Eq. 2.3).

with increasing values of real m and the parameter a . Figure 26 represents the effect of inclination i : the smalle r the value of inclination i (the smaller the eclipse) is, the larger the proportional error in the observational moments A_{2m}. For the comparable vaoues of r_1 and r_2 ($a \sim 0.5$) which produce a minimum with shorter duration of complete phases, note the relatively increasing values of $\mathcal{E}_{2m}$ in Figure 26. One more thing noteworthy is that for L_1 tends to zero (in the case of, for example, optical light curves for X-ray binaries), $\mathcal{E}_{2m}$ goes to infinity for $a = 0$ (see Eq. (2.3) and Figure 26) which can be given as

$$\lim_{\substack{L_1 \to 0 \\ a \to 0}} \mathcal{E}_{2m} = \infty \tag{2.4}$$

This situation coincides with the occultation minima of the optical light curves of X-ray binaries which are always absent in the observations. In the case of transit minimum of the optical light curve of any X-ray binary, L_1 becomes unity since L_1 of the X-ray component is zero in the optical domain, and the proportional error $\mathcal{E}_{2m}$ of the moments A_{2m} for this indistinguishably shallow minimum becomes approximately one (see Eq. (2.3) and Figure 26).

If the foregoing results given in the present and previous sections are utilized to choose the most useful two g-functions from Table 14 for a good determination of the eclipse parameters a and c_o in the geometrical sense, the following facts may be observed:

1. If we deal with a moderately deep occultation minimum with a moderately long duration of total phases, one can use higher moments in construction of the g-functions - i.e., one of the g-functions with higher orders x, y_1 and y_2 can be used from Table 14. For the second adopted g-function, the orders x, y_1 and y_2 should be smaller than those for the first g-function. Remember that the geometrical determinacy of a and c_o depends on the absolute value of the difference between the mean values of orders x, y_1 and y_2 for two different g-functions in question. In the applications to YZ Cassiopeiae (see Chapter 4) which shows moderately deep minima with total phases, the functions g_2 and g_4 (see Equations (0.1) and (0.2)) have been applied. Another g-function with smaller orders x, y_1 and y_2 could be applied instead of g_2 for better determination of the eclipse elements.

2. If the minimum we deal with is i) a transit type, ii) shallow, or iii) deep enough but of short duration of complete phases, we should restrict ourselves to the use of higher moments. In such a case, for example, the following two g-functions can be attempted:

$$\frac{A^2_{\frac{1}{2}}}{A_0 A_1} \quad \text{and} \quad \left(\frac{A^2_{\frac{1}{100}}}{A_0 A_{\frac{1}{50}}}\right)^{27}, \qquad (2.5)$$

where the twenty seventh power of the second g-function was taken to extend the very small numerical range (see Table 14) for this g-function, as well as that of the first g-function. This removes the difficulty in dealing with large numerical values of g-functions of very small orders.

Remember that the second g-function (without power) can be obtained from observations with high accuracy, since the proportional errors of the empirical A_{2m}'s become comparable for the smaller differences $|x-y_1|$, $|x-y_2|$, and $|y_1-y_2|$ (here these differences are $^1/100$, $^1/100$ and $^1/50$, respectively) and they largely cancel each other in the ratio for the respective g-function. But, as was noted before, it may be difficult to deal with large numerical values - especially if the tables and diagrams of these functions are used in practice.

It has been noted in Chapter 4 that the geometrical determinacy of the unknown parameters a and c_o depends on the numerical magnitude of the Jacobian

$$J \equiv \frac{\partial(g^{(1)}, g^{(2)})}{\partial(a, c_o)} \qquad (2.6)$$

for applied two g-functions $g^{(1)}$ and $g^{(2)}$. The vanishing of this Jacobian would imply a functional relationship to exist between successive g-functions, which would render the solution for a and c_o indeterminate. Therefore, the functional behaviour of this Jacobian for different pairs of g-functions would tell us definite results in connection with the geometrical determinacy of the parameters a and c_o, and the above results can be verified in this way. The explicit expressions for the above Jacobian have been developed by Kopal (1977b) and the promising numerical work was undertaken by Edalati (1978).

5.3 Different Procedures for the Solution of Eclipse Elements.

In the present section, two additional alternative ways will be outlined for obtaining the elements of eclipsing systems by the analysis of their observed photometric data in the frequency-domain. The same moments A_{2m} and the polynomial expansion (3.3.17) with (3.3.18) for the evaluation of their theoretical values will be utilized. In the first alternative way, methods can be applicable if the phase angle θ , in the first contact point when $\delta = \delta_1 \equiv r_1 + r_2$ is known from the observed light curves. In order to derive the requisite equations let us consider Equation (2.3) of the preceding section. If we write it out for two different moments A_{2m} for two different values of m and perform their ratio, we get

$$\frac{\varepsilon_{2\mu}}{\varepsilon_{2\mu'}} = \frac{\Delta A_{2\mu}}{\Delta A_{2\mu'}} \frac{A_{2\mu'}}{A_{2\mu}} = \frac{\psi_{2\mu}}{\psi_{2\mu'}} . \tag{3.1}$$

By making use of the expressions (4.4.1) and (2.3) for the probable errors ΔA_{2m}'s and for the functions ψ_{2m}'s , respectively, Equation (3.1) can be rewritten in the form

$$\frac{A_{2\mu'}}{A_{2\mu}} \left[\sin^2\theta'\right]^{\mu-\mu'} = \frac{f_{2\mu'}}{f_{2\mu}} \tag{3.2}$$

as a definition of the ratio of proportional errors $\varepsilon_{2\mu}$ and $\varepsilon_{2\mu'}$ of two respective moments $A_{2\mu}$ and $A_{2\mu'}$. It can be proved by use of the general expansion (3.3.17) for the moments together with Eq. (2.2) that Eq. (3.2) is exact. To illustrate this, let us consider Eq. (2.2) for $\delta = \delta_1$. It can be shown that

$$\left(\frac{r_1}{\sin i}\right)^2 = \frac{a^2 \sin^2\theta'}{(1-c_0^2)} . \tag{3.3}$$

On insertion of (3.3) in (3.3.17) it follows that

$$A_{2m}\left[\sin\theta'\right]^{-2m} = L_1 b^2 (1-c_0^2) f_{2m}(a, c_0) . \tag{3.4}$$

Now, if we write (3.4) for two different values of m and perform their ratio we get Eq. (3.2). This proof also shows that the expressions (4.4.1) and (2.3) for the errors of the moments A_{2m} are correct.

If the values of the left-hand side of the preceding equation (3.2) can be established for two different values of real μ and μ' from the observations, these two simultaneous equations constitute two independent relations between the unknown constants a and c_o ; and can be solved numerically for them. Therefore, these functions can be utilized just as the g-functions for obtaining the eclipse elements of an eclipsing system, but if the phase angle θ of first contact point can be established from the observations in addition to the respective moments A_{2m}. These functions given by (3.2) were also tabulated in the same manner as in the g-functions, for the values of m between zero and three, It may be noted that they behave just like the g-functions. The functional behaviour of $A_1 \sin^{-3/4}\theta_1 / A_{1/4}$ has been presented in Figure 27, for the sake of an example.

As another way to deduce the eclipse elements, the use of a product of a g-function and an $F(a, c_o)$-function can be considered. But, the new function $F(a, c_o)$ should be set in such a way that the intersection points of the g-functions and the $g \times F$-products become well determined in respect to the intersection points of two g-functions. If, in practice, theadopted suitable F-function cannot be established from observations as g's, it may be approached iteratively just like the use of $Y(a, c_o)$-functions (see Eq. (1.27) in Chapter 4). However, the construction of these F-functions suitably for our purpose remains yet to be studied. In this work, the behaviour of $g \times F$-product was examined for only the $F(a)$ being a parabola given by

$$F(a) = -3.75\,a^2 + 3.75\,a + 0.3625 \tag{3.5}$$

as a polynomial of second degree in a . The $g_2(a, c_o)$ (see Eq. D.1) has been applied as g-function. It is found that the behaviour of $g \times F$ in this case becomes very different (see Figure 28) from those of g-functions so that the intersection points $P(a, c_o)$ of a g-function and $g_2(a, c_o) \times F(a)$-product (where $F(a)$ being given by (3.5)) are well determined.

In this way, the strategy for obtaining the eclipse elements of

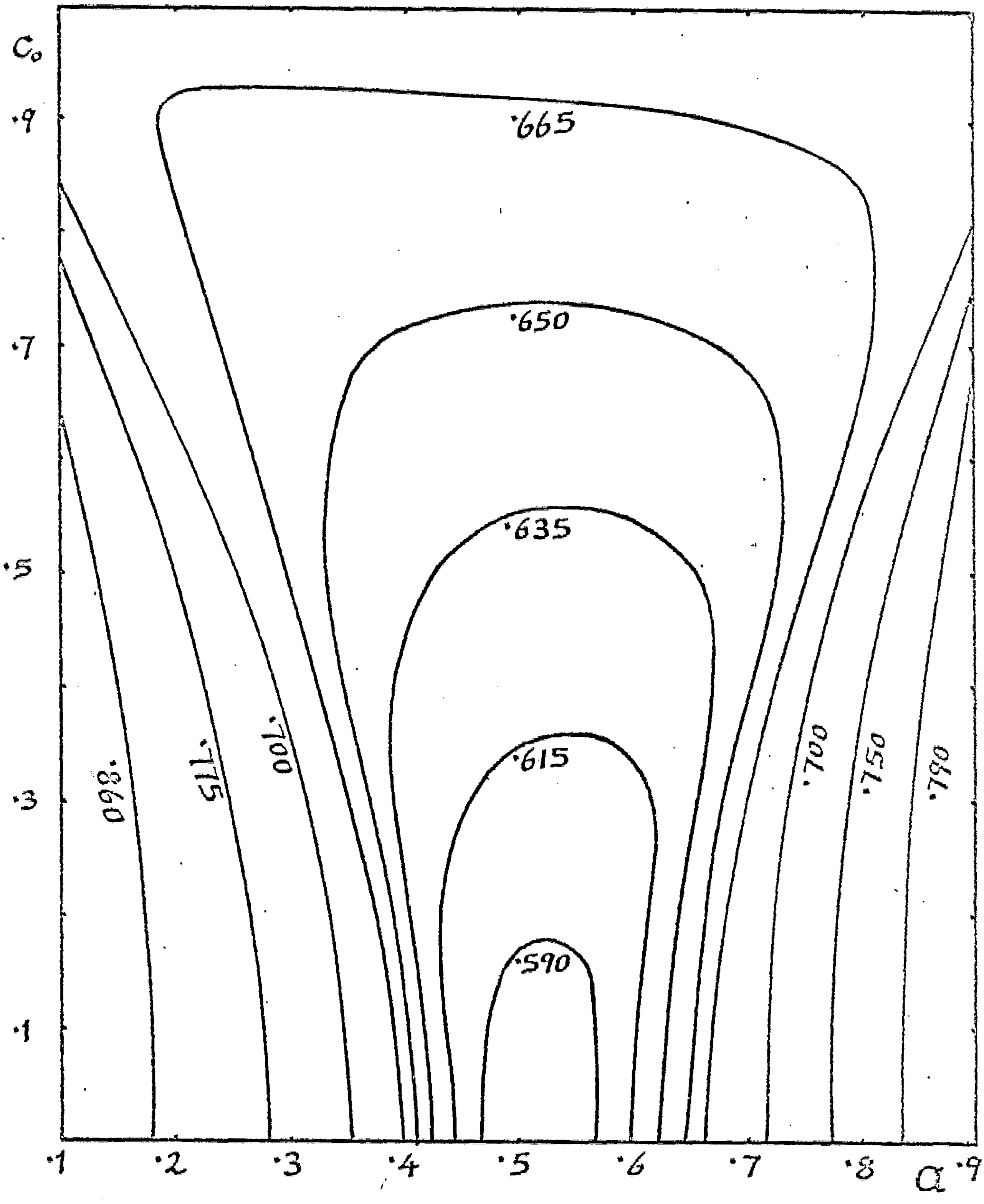

Figure 27. The functional behaviour of $\frac{A_1}{A_{1/4}} \sin^{-3/4}\theta'$.

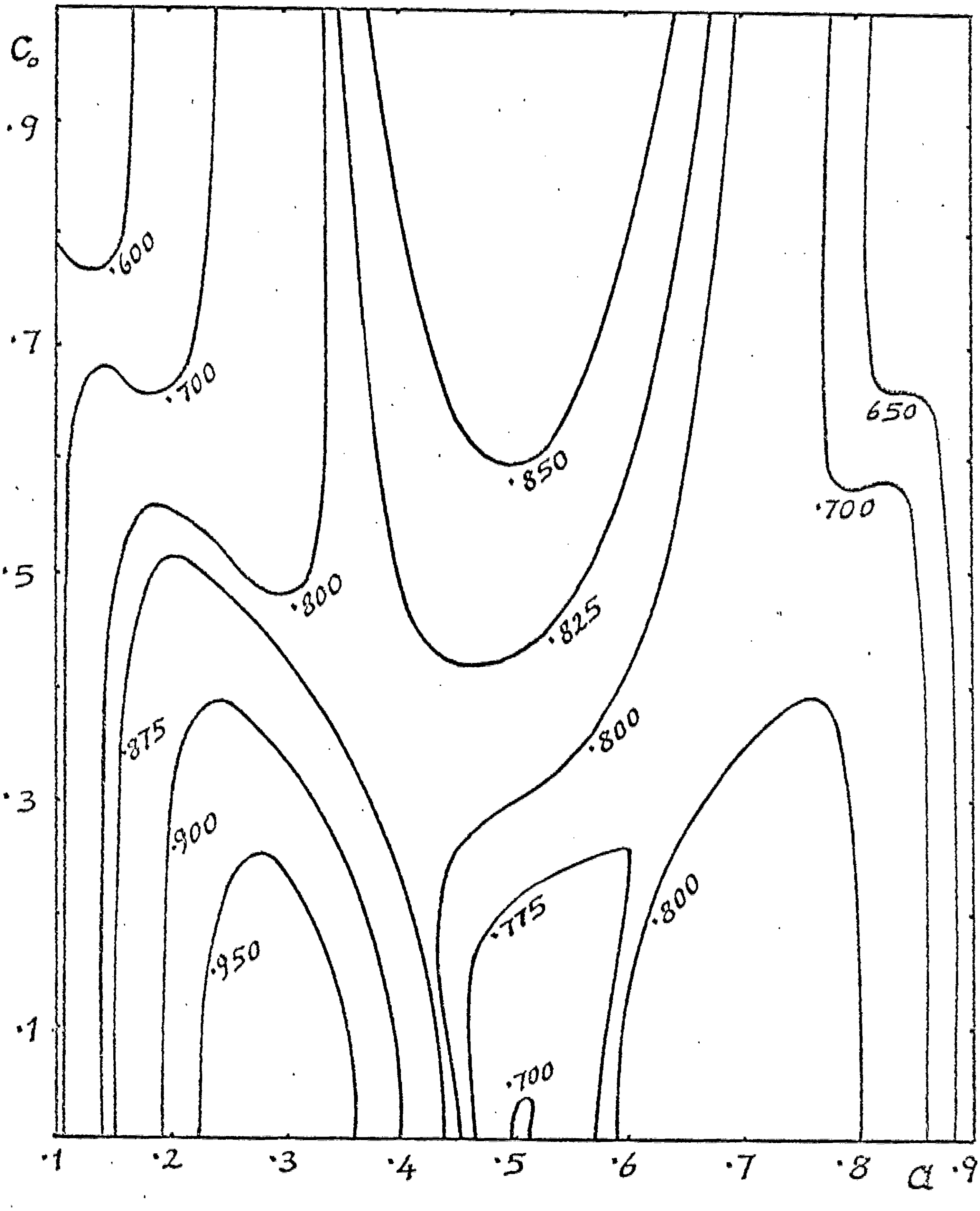

Figure 28. The functional behaviour of $g_2 \times F$ (for g_2 and F see Equations 1.5 and 3.5).

any eclipsing system can be outlines as follows:

1. First, determine the requisite number of empirical moments A_{2m} of the observed light curves.

2. Free these moments from the proximity effects, ignore the "photometric perturbations" (see Chapter 4), and evaluate the respective empirical g-function with the aid of so-called "rectified" moments.

3. Adopt a trial value for a – if we do not know what kind of eclipse we deal with, we may depart from an assumption of $a = ½$, and set $F = F(½, 0)$ from the adopted F-functions for $a = ½$ and $c_o = 0$.

4. From the product $g(a, c_o) \times F(a, c_o)$ establish the corresponding value of c_o for known a and F.

5. From the g-function establish new value of a for known value of c_o, and determine the new value of $F(a, c_o)$ for known a and c_o.

6. Repeat the operations 4 and 5 until the differences between the newly derived and the previously adopted values for a and c_o become tolerable.

7. By using the resulting values of a and c_o, determine the eclipse elements $r_{1,2}$, i and L_1 of the system as described in Section 1 of Chapter 4, and by these elements perform the reductions for the "photometric perturbations" to obtain an improved set of the A_{2m}'s and from these improved set of elements.

8. Repeat the operations 3–8 until the differences between the

improved elements and their previously adopted values become insignificant,

5.4 A Different Type Moments and the φ -Functions.

A different type moments were introduced by Kopal (1977b) for obtaining the eclipse elements of close binary systems in the frequency-domain, which can be described as follows: Let us first consider our conventional moments with their probable errors, as

$$A_{2m} = A_{2m} \mp \Delta U \sin^{2m}\theta' . \tag{4.1}$$

If we write

$$A_{2m'} \sin^{k}\theta' - A_{2m} \equiv B_{2m} \tag{4.2}$$

as the definition of new moments, provided that $2m' + k = 2m$, $(m \neq m')$, probable uncertainties of the moments A_{2m} and $A_{2m'}$ cancel. Therefore, the moments B_{2m} should be free from any observational error except that inherent in the observed value of θ'. Let us note that these moments are applicable only for the wide binaries if the phase angle θ' can be determined from the observations.

In this section, the explicit expressions for these new type moments will be developed and they will be tested numerically to obtain the elements of eclipsing binaries.

It can be shown, by using the expression (3.3.17) with (3.3.18) for the moments A_{2m} that the moments B_{2m} can be given by a similar expansion to that of A_{2m}'s as

$$B_{2m} = L_1 \frac{b^2}{a^{2m}} \left(\frac{r_1}{\sin i}\right)^{2m} (1-c_0^2)^{m+1} h_{2m}(a, c_0) , \tag{4.3}$$

where we have abbreviated

$$h_{2m}(a, c_0) = \Gamma(m) \sum_{l=0}^{\Lambda} C^{(l)} \Gamma(v) (1-c_0^2)^{v} \cdot$$

$$\cdot \sum_{n=0}^{\infty} \frac{n!(v+2n+2)}{(n+1)\Gamma(v+n+1)} \left[R_n^{(l,v)}(a)\right]^2 Q_n^{(m,v+2)}(1-c_0^2) , \quad \mathrm{Re}\, m > 0 \tag{4.4}$$

with the polynomials

$$Q_n^{(m,\alpha)}(x) = \sum_{j=0}^{n} \frac{(-1)^j \Gamma(n+\alpha+j)(v+j+1)}{\Gamma(m+\alpha+j)\, j!\,(n-j)!} x^j . \tag{4.5}$$

These polynomials satisfy a three term recursion relation of the form

$$Q_{n+1}(x) = (A_n x + B_n) Q_n(x) - C_n Q_{n-1}(x) \tag{4.6}$$

with the coefficients

$$A_n = - \frac{(2n+\alpha)(2n+\alpha+1)(n+\alpha)}{(n+1)(m+n+\alpha)(n+\alpha-1)} , \tag{4.7}$$

$$B_n = (2n+\alpha) + \frac{n(m+n+\alpha-1)(n+\nu)}{(2n+\alpha-1)(n+\alpha-1)} A_n \qquad (4.8)$$

and

$$C_n = \frac{(n+\alpha-1)}{n} B_n - \frac{(n+\alpha)(n+\alpha-1)}{n(n+1)}, \qquad (4.9)$$

where $X \equiv 1 - c_o^2$ and $\alpha \equiv \nu + 2$. In the foregoing expressions for the moments B_{2m}, the notations are consistent with those used in the preceding papers of this series. This expansion (4.3) for the moments B_m is also valid for every type of eclipse for any positive real value of $m > 0$ and for any arbitrary degree l of the adopted law of limb darkening. It is readily seen from Equations (4.2) and (4.3) that, with the aid of expansion (3.3.17) for A_{2m}'s, the functions $h_{2m}(a, c_o)$ can be given in terms of the functions $f_{2m}(a, c_o)$, as

$$h_{2m}(\alpha, c_0) = f_{2m-k}(\alpha, c_0) - f_{2m}(\alpha, c_0), \qquad (4.10)$$

$$Re\ m > 0 \quad \text{and} \quad Re\ k > 0 .$$

Thus Eq. (4.3) can be rewritten by use of (4.10), as

$$B_{2m} = L_1 \frac{b^2}{a^{2m}} \left(\frac{r_1}{\sin i}\right)^{2m} (1-c_0^2)^{m+1} \left[f_{2m-k}(\alpha, c_0) - f_{2m}(\alpha, c_0) \right], \qquad (4.11)$$

$$Re\ m > 0 \quad \text{and} \quad Re\ k > 0 ,$$

in terms of f-functions given by (3.3.18). The ratios

$$\frac{B_{2m}}{A_{2m}}, \qquad \text{for } \operatorname{Re} m > 0 \tag{4.12}$$

will depend on only the parameters a and c_o through the h and f-functions, as

$$\frac{B_{2m}}{A_{2m}} = \frac{h_{2m}(a, c_o)}{f_{2m}(a, c_o)} \equiv \varphi_{2m}(a, c_o) \tag{4.13}$$

which can be rwrittten, with the aid of Eq. (4.10), in terms of f-functions, as

$$\frac{B_{2m}}{A_{2m}} = \frac{f_{2m-k}(a, c_o) - f_{2m}(a, c_o)}{f_{2m}(a, c_o)} \equiv \varphi_{2m}(a, c_o), \tag{4.14}$$

$$\operatorname{Re} m > 0 \text{ and } \operatorname{Re} k > 0 .$$

The φ -functions given by (4.13) and (4.14) can serve - just like the g-functions - to evaluate the elements of an eclipsing system. Their theoretical values have been computed for m = 1 and k = 2 and their similar behaviour as g-functions was presented diagrammatically in Figures 29-31. When we use the φ -functions for obtaining the elements of an eclipsing system a difficulty arises in finding the phase angle θ' which occurs in the left-hand side of Eq. (4.14) from the observed light curve of the system. It can be estimated within about 1^{o} error

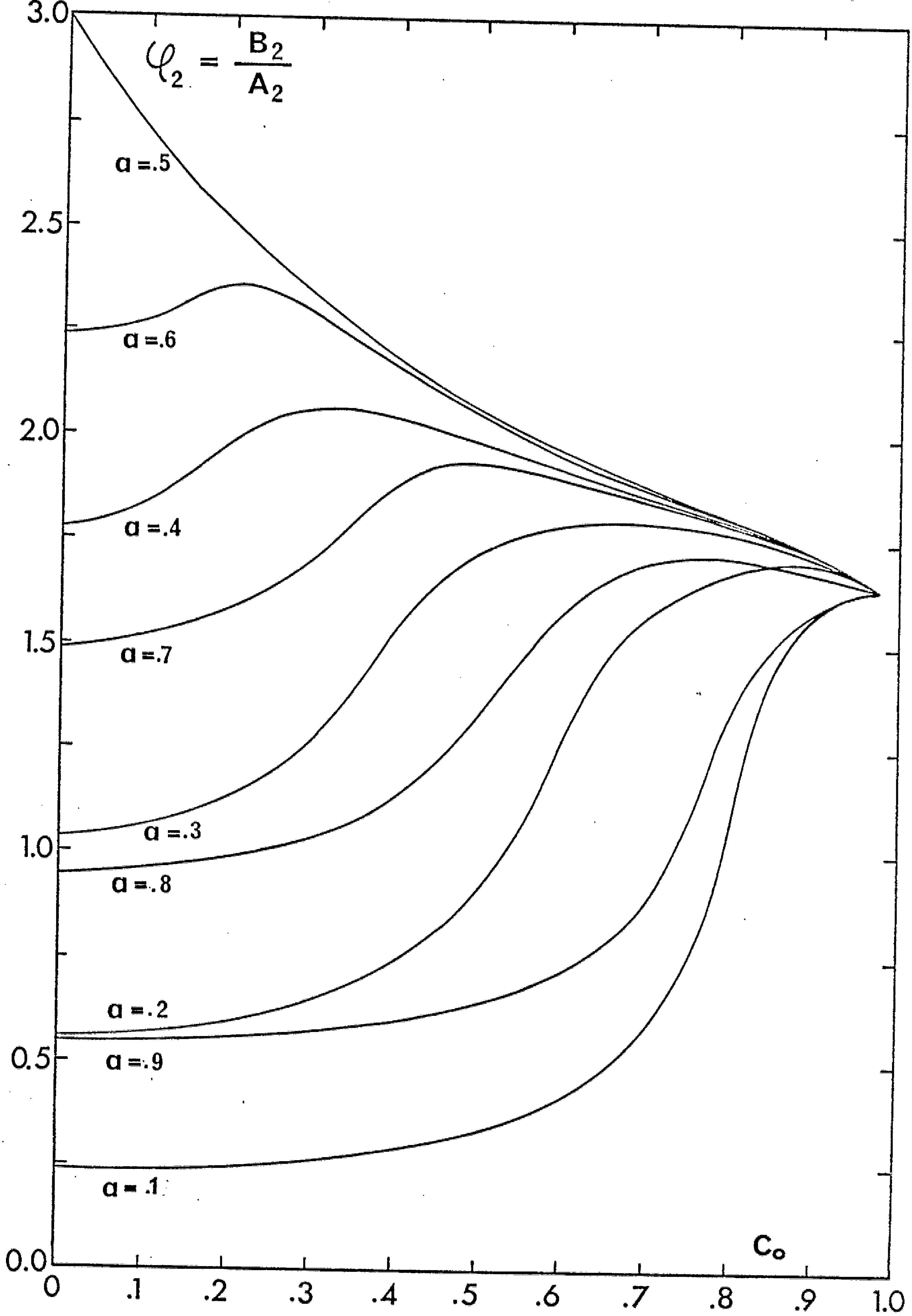

Figure 29. A plot of the function $\varphi_2 = \frac{B_2}{A_2}$ versus c_o for fixed values of a .

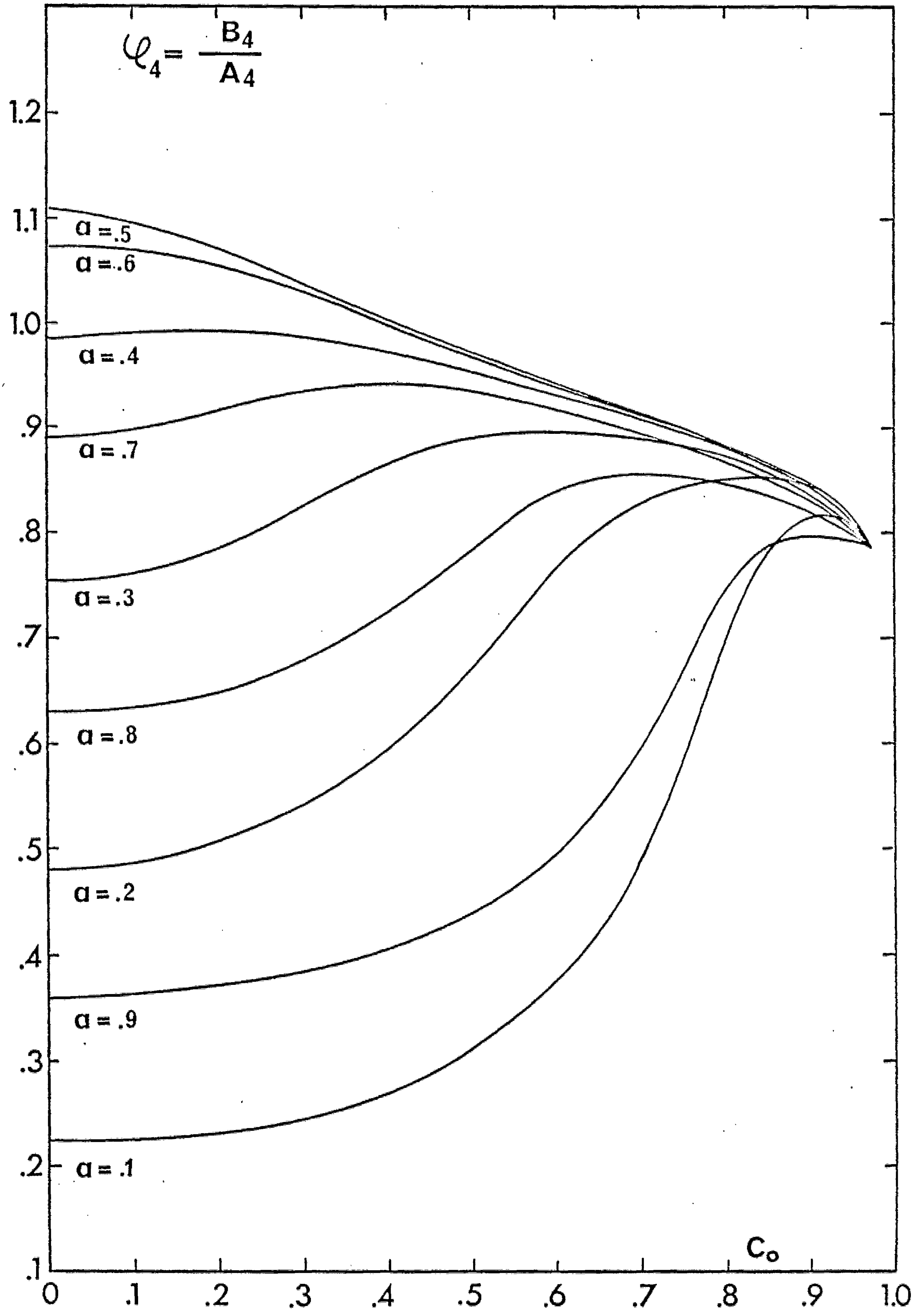

Figure 30. A plot of the function $\varphi_4 = \frac{B_4}{A_4}$ versus c_o for fixed values of a .

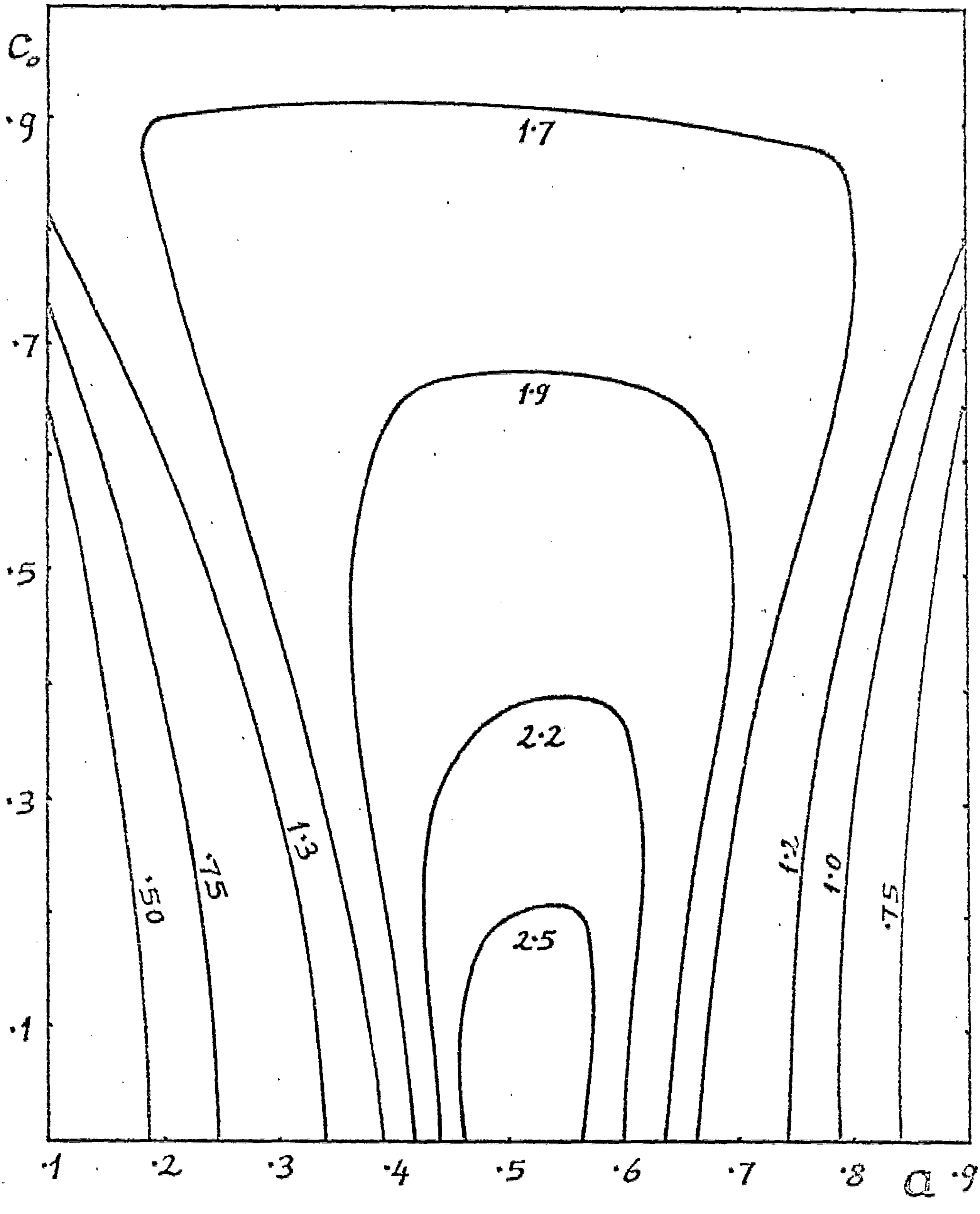

Figure 31. The functional behaviour of $\varphi_{2m} = \frac{B_{2m}}{A_{2m}}$ -functions for m = 1 (see Eq. 4.14).

from the photoelectric observations of only well-behaved wide binaries. In this connection, if the θ' is not treated as an additional unknown in the solution, the uncertainties encountered by its estimated empirical values should be considered. By differentiation from Eq. (4.2) we get

$$\Delta B_{2m} \doteq \mp \Delta\theta' \times A_{2m-k} \times k \times \sin^{k-1}\theta' \cdot \cos\theta' . \qquad (4.15)$$

In what follows, we shall demonstrate the comparison of the probable errors in the empirical moments A_{2m} and B_{2m}, on an actual example. To do so, let us consider a system consisting of two components of fractional radii $r_1 = r_2 = 0.2$ and the inclination i of the orbital plane is 90°. The corresponding values of a, c_o and θ' then are $a = 0.5$, $c_o = 0$ and $\theta' = 23^\circ\!.58$, respectively. Let us, moreover, assume that $\Delta U = \pm 0.001$, $\Delta\theta' = \pm 1^\circ$, and the limb darkening coefficient U_1 for the undergoing star is $U_1 = 0.6$. Under these conditions, the even moments A_{2m} and B_{2m} of the light changes arising during eclipses can be evaluated together with the probable errors in their empirical values with the aid of Equations (1.3), (2.1), (4.2) and (4.15) of the present chapter for $m = 0(1)3$ and $k = 2$; and the results for the linear law of limb darkening are presented in Table 15 for $L_1 = 0.20$ and 0.80. It can be seen from Table 15 that the numerical values of the B-moments diminish more rapidly than those of conventional A-moments. The uncertainties in the observational B-moments are caused by the error in determining the phase angle θ' from the observations, while the uncertainties in the observational A-moments are caused by the error in determining the unit of light U. On the basis of an inspection the numerical values in Table 15 and the equations from which these numerical values are established, we can conclude that the probable errors encountered by empirical determination of θ' are larger than those caused by the empirical determination of U, in other words, the probable errors of the empirical B-moments appear to be larger than those of the empirical A-moments even if the phase angle θ' is well determined (this is possible for only wide binaries) from the observations. Therefore, the usefulness of the B-moments is likely to be limited, and the conventional A-moments preferred for practical work.

Table 15a

$L_1 = 0.20$		
A_0	0.20	$\pm$ 0.001
A_2	0.008	$\pm$ 0.000167
A_4	0.000608	$\pm$ 0.0000278
A_6	0.0000582	$\pm$ 0.00000464
B_2	0.024	$\pm$ 0.00256
B_4	0.000672	$\pm$ 0.000102
B_6	0.0000391	$\pm$ 0.00000778

Table 15b

$L_1 = 0.80$		
A_0	0.80	$\pm$ 0.001
A_2	0.032	$\pm$ 0.000167
A_4	0.00243	$\pm$ 0.0000278
A_6	0.000233	$\pm$ 0.00000464
B_2	0.096	$\pm$ 0.0102
B_4	0.00269	$\pm$ 0.000410
B_6	0.000156	$\pm$ 0.0000311

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APPENDICES

APPENDIX 1. Tables of the $f_0'(\alpha, c_0)$ - functions.

c_0 \ α	.10	.11	.12	.13	.14	.15	.16	.17	.18	.19
0.00	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.02	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.04	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.06	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.08	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.10	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.12	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.14	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.16	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.18	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.20	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.22	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.24	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.26	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.28	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.30	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.32	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.34	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.36	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.38	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.40	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.42	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.44	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.46	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.48	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.50	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.52	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.54	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.56	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.58	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.60	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.62	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.64	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	.9845
.66	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	.9997	.9832	.9521
.68	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	.9817	.9481	.9077
.70	1.0000	1.0000	1.0000	1.0000	1.0000	.9997	.9799	.9435	.9000	.8542
.72	1.0000	1.0000	1.0000	1.0000	1.0000	.9779	.9381	.8912	.8422	.7937
.74	1.0000	1.0000	1.0000	.9997	.9754	.9317	.8809	.8285	.7770	.7277
.76	1.0000	1.0000	1.0000	.9724	.9241	.8688	.8125	.7579	.7062	.6580
.78	1.0000	.9996	.9587	.9149	.8545	.7938	.7359	.6817	.6318	.5860
.80	.9995	.9642	.9037	.8372	.7717	.7101	.6534	.6019	.5552	.5131
.82	.9584	.8898	.8153	.7454	.6800	.6208	.5578	.5204	.4781	.4404
.84	.8722	.7904	.7137	.6444	.5829	.5287	.4810	.4390	.4019	.3691
.86	.7581	.6750	.6019	.5386	.4839	.4364	.3953	.3595	.3281	.3006
.88	.6272	.5509	.4854	.4320	.3858	.3464	.3126	.2834	.2580	.2358
.90	.4889	.4249	.3722	.3285	.2920	.2612	.2349	.2124	.1930	.1760
.92	.3522	.3035	.2642	.2320	.2054	.1831	.1543	.1482	.1344	.1225
.94	.2255	.1931	.1672	.1463	.1291	.1149	.1028	.0926	.0839	.0763
.96	.1175	.1001	.0864	.0754	.0664	.0590	.0527	.0475	.0430	.0391
.98	.0376	.0320	.0276	.0241	.0212	.0188	.0168	.0151	.0137	.0125

C_o \ α	.20	.21	.22	.23	.24	.25	.26	.27	.28	.29
0.00	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.02	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.04	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.06	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.08	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.10	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.12	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.14	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.16	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.18	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.20	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.22	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.24	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.26	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.28	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.30	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.32	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.34	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.36	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.38	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.40	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.42	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.44	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	.9998	.9912
.46	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	.9909	.9730
.48	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	.9998	.9905	.9719	.9484
.50	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	.9900	.9705	.9450	.9185
.52	1.0000	1.0000	1.0000	1.0000	.9998	.9894	.9591	.9435	.9148	.8843
.54	1.0000	1.0000	1.0000	1.0000	.9889	.9674	.9406	.9107	.8790	.8456
.56	1.0000	1.0000	.9998	.9882	.9555	.9374	.9061	.8732	.8396	.8061
.58	1.0000	1.0000	.9874	.9635	.9338	.9011	.8567	.8319	.7972	.7532
.60	.9998	.9866	.9512	.9298	.8954	.8595	.8233	.7875	.7524	.7185
.62	.9856	.9585	.9252	.8890	.8514	.8138	.7767	.7406	.7058	.6725
.64	.9555	.9201	.8819	.8425	.8032	.7647	.7275	.6919	.6579	.6255
.66	.9143	.8738	.8324	.7914	.7515	.7131	.6765	.6418	.6090	.5780
.68	.8646	.8211	.7781	.7368	.6972	.6597	.6243	.5909	.5597	.5303
.70	.8082	.7633	.7203	.6795	.6411	.6050	.5712	.5397	.5102	.4828
.72	.7466	.7019	.6599	.6205	.5838	.5497	.5179	.4885	.4611	.4357
.74	.6813	.6380	.5977	.5605	.5261	.4943	.4649	.4377	.4125	.3894
.76	.6135	.5725	.5348	.5002	.4685	.4393	.4125	.3879	.3652	.3442
.78	.5444	.5064	.4718	.4403	.4115	.3853	.3613	.3393	.3190	.3004
.80	.4781	.4407	.4097	.3816	.3560	.3328	.3116	.2923	.2746	.2583
.82	.4066	.3763	.3491	.3245	.3024	.2823	.2640	.2473	.2321	.2182
.84	.3400	.3140	.2907	.2699	.2511	.2341	.2187	.2047	.1920	.1803
.86	.2762	.2546	.2354	.2182	.2028	.1889	.1763	.1649	.1545	.1450
.88	.2163	.1991	.1838	.1702	.1580	.1470	.1371	.1281	.1200	.1125
.90	.1612	.1482	.1367	.1254	.1172	.1090	.1016	.0949	.0888	.0833
.92	.1120	.1028	.0947	.0876	.0812	.0754	.0703	.0656	.0614	.0575
.94	.0698	.0640	.0589	.0544	.0504	.0468	.0436	.0407	.0381	.0357
.96	.0357	.0327	.0301	.0278	.0258	.0239	.0223	.0208	.0195	.0183
.98	.0114	.0105	.0096	.0089	.0083	.0077	.0072	.0067	.0063	.0059

C_0 \ a	.30	.31	.32	.33	.34	.35	.36	.37	.38	.39
0.00	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.02	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.04	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.06	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.08	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.10	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.12	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.14	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.16	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.18	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.20	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.22	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	.9998
.24	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	.9932
.26	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	.9998	.9931	.9794
.28	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	.9930	.9791	.9612
.30	1.0000	1.0000	1.0000	1.0000	1.0000	.9998	.9929	.9788	.9606	.9395
.32	1.0000	1.0000	1.0000	1.0000	1.0000	.9928	.9784	.9599	.9384	.9150
.34	1.0000	1.0000	1.0000	.9998	.9925	.9779	.9590	.9372	.9133	.8881
.36	1.0000	1.0000	1.0000	.9924	.9773	.9580	.9357	.9114	.8858	.8593
.38	1.0000	1.0000	.9921	.9766	.9568	.9340	.9092	.8831	.8563	.8290
.40	.9998	.9919	.9759	.9555	.9321	.9067	.8801	.8527	.8251	.7974
.42	.9915	.9750	.9540	.9299	.9039	.8767	.8488	.8207	.7925	.7647
.44	.9741	.9523	.9275	.9007	.8729	.8443	.8157	.7871	.7589	.7312
.46	.9504	.9248	.8972	.8686	.8394	.8102	.7811	.7525	.7244	.6970
.48	.9218	.8933	.8539	.8340	.8041	.7745	.7454	.7170	.6893	.6624
.50	.8890	.8587	.8280	.7974	.7672	.7376	.7087	.6807	.6536	.6274
.52	.8530	.8214	.7900	.7592	.7290	.6997	.6714	.6440	.6177	.5923
.54	.8141	.7819	.7504	.7195	.6899	.6612	.6335	.6070	.5815	.5572
.56	.7730	.7407	.7094	.6791	.6500	.6221	.5954	.5698	.5454	.5221
.58	.7301	.6982	.6574	.6379	.6097	.5828	.5572	.5327	.5094	.4873
.60	.6859	.6546	.6248	.5963	.5692	.5435	.5190	.4958	.4737	.4528
.62	.6407	.6105	.5917	.5545	.5287	.5042	.4811	.4591	.4384	.4187
.64	.5949	.5660	.5386	.5127	.4883	.4653	.4435	.4230	.4036	.3852
.66	.5488	.5214	.4955	.4712	.4483	.4268	.4065	.3874	.3694	.3524
.68	.5028	.4770	.4529	.4302	.4089	.3890	.3702	.3526	.3360	.3203
.70	.4571	.4332	.4108	.3899	.3703	.3519	.3347	.3186	.3034	.2891
.72	.4120	.3900	.3595	.3504	.3325	.3158	.3002	.2855	.2718	.2589
.74	.3678	.3478	.3293	.3120	.2959	.2808	.2668	.2536	.2413	.2297
.76	.3248	.3069	.2903	.2748	.2604	.2471	.2346	.2229	.2120	.2017
.78	.2832	.2674	.2527	.2391	.2255	.2147	.2037	.1935	.1840	.1750
.80	.2433	.2295	.2158	.2050	.1941	.1839	.1744	.1656	.1574	.1497
.82	.2054	.1936	.1827	.1727	.1634	.1548	.1468	.1393	.1323	.1258
.84	.1696	.1598	.1508	.1424	.1347	.1275	.1209	.1147	.1089	.1036
.86	.1363	.1284	.1210	.1143	.1081	.1023	.0969	.0919	.0873	.0830
.88	.1058	.0995	.0938	.0886	.0837	.0792	.0751	.0712	.0676	.0642
.90	.0782	.0736	.0594	.0655	.0619	.0585	.0554	.0526	.0499	.0474
.92	.0540	.0508	.0479	.0452	.0427	.0404	.0383	.0363	.0344	.0327
.94	.0335	.0315	.0297	.0280	.0265	.0251	.0237	.0225	.0214	.0203
.96	.0171	.0161	.0152	.0143	.0136	.0128	.0122	.0115	.0110	.0104
.98	.0055	.0052	.0049	.0046	.0044	.0041	.0039	.0037	.0035	.0034

C_0 \ α	.40	.41	.42	.43	.44	.45	.46	.47	.48	.49
0.00	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.02	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	.9995
.04	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	.9997	.9893
.06	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	.9909	.9721
.08	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	.9998	.9918	.9751	.9529
.10	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	.9923	.9768	.9564	.9319
.12	1.0000	1.0000	1.0000	1.0000	.9998	.9927	.9780	.9587	.9356	.9097
.14	1.0000	1.0000	1.0000	1.0000	.9929	.9787	.9501	.9380	.9133	.8865
.16	1.0000	1.0000	.9998	.9931	.9792	.9611	.9397	.9158	.8899	.8625
.18	1.0000	1.0000	.9932	.9795	.9617	.9407	.9173	.8921	.8655	.8378
.20	.9998	.9932	.9796	.9620	.9413	.9182	.8935	.8674	.8404	.8126
.22	.9932	.9797	.9621	.9414	.9185	.8941	.8684	.8418	.8145	.7859
.24	.9795	.9620	.9413	.9185	.8942	.8686	.8423	.8154	.7882	.7608
.26	.9617	.9410	.9181	.8938	.8683	.8421	.8154	.7885	.7614	.7344
.28	.9403	.9174	.8929	.8674	.8413	.8147	.7879	.7610	.7343	.7078
.30	.9163	.8917	.8661	.8398	.8132	.7865	.7598	.7333	.7070	.6810
.32	.8901	.8642	.8379	.8112	.7845	.7578	.7313	.7052	.6794	.6540
.34	.8620	.8354	.8086	.7817	.7550	.7286	.7025	.6769	.6517	.6271
.36	.8325	.8054	.7784	.7516	.7251	.6990	.6735	.6484	.6240	.6001
.38	.8017	.7744	.7475	.7209	.6948	.6692	.6443	.6199	.5962	.5731
.40	.7699	.7427	.7160	.6898	.6642	.6393	.6150	.5914	.5685	.5462
.42	.7373	.7104	.6841	.6584	.6335	.6092	.5857	.5629	.5409	.5195
.44	.7041	.6776	.6519	.6269	.6025	.5792	.5565	.5346	.5134	.4930
.46	.6704	.6445	.6195	.5952	.5718	.5492	.5274	.5064	.4862	.4666
.48	.6364	.6113	.5870	.5636	.5411	.5194	.4986	.4785	.4592	.4406
.50	.6022	.5779	.5546	.5321	.5105	.4898	.4699	.4508	.4324	.4148
.52	.5680	.5446	.5222	.5008	.4802	.4605	.4416	.4235	.4061	.3894
.54	.5338	.5115	.4901	.4697	.4502	.4315	.4136	.3965	.3801	.3644
.56	.4998	.4786	.4583	.4390	.4205	.4029	.3861	.3699	.3546	.3398
.58	.4662	.4461	.4269	.4087	.3913	.3748	.3590	.3439	.3295	.3157
.60	.4329	.4140	.3960	.3789	.3626	.3472	.3324	.3184	.3050	.2922
.62	.4001	.3824	.3656	.3497	.3345	.3201	.3064	.2934	.2810	.2691
.64	.3678	.3514	.3359	.3211	.3070	.2937	.2811	.2691	.2576	.2467
.66	.3363	.3211	.3068	.2932	.2803	.2680	.2564	.2454	.2349	.2249
.68	.3055	.2916	.2785	.2660	.2543	.2431	.2325	.2225	.2129	.2038
.70	.2757	.2630	.2510	.2398	.2291	.2190	.2094	.2003	.1917	.1835
.72	.2468	.2353	.2243	.2144	.2048	.1957	.1871	.1789	.1712	.1639
.74	.2189	.2087	.1991	.1900	.1814	.1733	.1657	.1585	.1516	.1451
.76	.1921	.1831	.1746	.1666	.1591	.1520	.1453	.1389	.1328	.1271
.78	.1667	.1588	.1514	.1444	.1379	.1317	.1258	.1203	.1151	.1101
.80	.1425	.1357	.1294	.1234	.1179	.1125	.1075	.1027	.0982	.0940
.82	.1197	.1140	.1087	.1036	.0989	.0944	.0902	.0852	.0825	.0789
.84	.0985	.0938	.0894	.0852	.0813	.0775	.0742	.0709	.0678	.0649
.86	.0789	.0751	.0716	.0683	.0651	.0622	.0594	.0568	.0543	.0519
.88	.0611	.0581	.0554	.0528	.0504	.0481	.0459	.0439	.0420	.0402
.90	.0451	.0429	.0409	.0390	.0372	.0355	.0339	.0324	.0310	.0297
.92	.0311	.0296	.0282	.0269	.0257	.0245	.0234	.0224	.0214	.0205
.94	.0193	.0184	.0175	.0167	.0159	.0152	.0146	.0139	.0133	.0127
.96	.0099	.0094	.0090	.0086	.0082	.0078	.0075	.0071	.0068	.0066
.98	.0032	.0031	.0029	.0028	.0027	.0025	.0024	.0023	.0022	.0021

C_o \ a	.50	.51	.52	.53	.54	.55	.56	.57	.58	.59
0,00	1,0000	.9529	.8999	.8467	.7942	.7428	.6937	.6463	.6015	.5585
.02	,9851	.9496	.8986	.8457	.7933	.7423	.6930	.6460	.6009	.5583
.04	,9665	.9356	.8940	.8427	.7911	.7404	.6915	.6446	.6000	.5573
.06	,9471	.9161	.8802	.8371	.7870	.7374	.6890	.6426	.5981	.5559
.08	,9261	.8955	.8613	.8237	.7808	.7327	.6854	.6395	.5956	.5537
.10	,9042	.8737	.8408	.8055	.7679	.7262	.6804	.6355	.5922	.5509
.12	,8814	.8512	.8192	.7857	.7505	.7137	.6737	.6303	.5880	.5473
.14	,8580	.8280	.7969	.7647	.7315	.6972	.6617	.6236	.5828	.5430
.16	,8339	.8043	.7740	.7429	.7113	.6790	.6460	.6122	.5762	.5378
.18	,8093	.7802	.7506	.7206	.6903	.6597	.6287	.5973	.5652	.5313
.20	,7844	.7557	.7269	.6979	.6688	.6396	.6103	.5809	.5511	.5209
.22	,7590	.7310	.7029	.6749	.6469	.6190	.5912	.5634	.5356	.5075
.24	,7334	.7060	.6787	.6516	.6247	.5980	.5715	.5452	.5190	.4928
.26	,7075	.6808	.6544	.6282	.6023	.5768	.5515	.5265	.5017	.4772
.28	,6815	.6555	.6299	.6047	.5798	.5553	.5312	.5074	.4840	.4608
.30	,6554	.6302	.6054	.5811	.5572	.5338	.5107	.4881	.4659	.4440
.32	,6292	.6048	.5809	.5575	.5345	.5121	.4902	.4687	.4476	.4269
.34	,6030	.5794	.5564	.5339	.5120	.4905	.4696	.4492	.4292	.4095
.36	,5768	.5541	.5319	.5104	.4894	.4690	.4490	.4296	.4107	.3921
.38	,5507	.5288	.5076	.4870	.4670	.4475	.4285	.4101	.3921	.3745
.40	,5247	.5037	.4834	.4637	.4445	.4261	.4081	.3906	.3736	.3571
.42	,4988	.4788	.4594	.4407	.4225	.4049	.3878	.3713	.3552	.3395
.44	,4732	.4541	.4355	.4178	.4005	.3839	.3677	.3521	.3369	.3222
.46	,4478	.4296	.4121	.3952	.3788	.3631	.3478	.3331	.3188	.3050
.48	,4227	.4054	.3888	.3728	.3574	.3425	.3282	.3143	.3009	.2879
.50	,3979	.3816	.3659	.3508	.3363	.3223	.3088	.2957	.2832	.2710
.52	,3734	.3580	.3433	.3291	.3155	.3023	.2897	.2775	.2657	.2544
.54	,3494	.3349	.3211	.3078	.2950	.2827	.2709	.2595	.2486	.2380
.56	,3257	.3122	.2993	.2869	.2749	.2635	.2525	.2419	.2317	.2219
.58	,3026	.2900	.2779	.2664	.2553	.2447	.2345	.2247	.2152	.2061
.60	,2799	.2683	.2571	.2464	.2361	.2263	.2168	.2078	.1991	.1907
.62	,2578	.2470	.2357	.2269	.2174	.2084	.1997	.1913	.1833	.1756
.64	,2363	.2264	.2159	.2079	.1992	.1909	.1830	.1753	.1680	.1610
.66	,2154	.2064	.1977	.1895	.1815	.1740	.1668	.1598	.1532	.1458
.68	,1952	.1870	.1791	.1716	.1645	.1576	.1511	.1448	.1388	.1330
.70	,1757	.1683	.1612	.1544	.1480	.1418	.1359	.1303	.1249	.1197
.72	,1569	.1502	.1439	.1379	.1321	.1266	.1214	.1164	.1115	.1069
.74	,1389	.1330	.1274	.1220	.1169	.1121	.1074	.1030	.0987	.0947
.76	,1217	.1165	.1115	.1069	.1025	.0982	.0941	.0903	.0865	.0830
.78	,1054	.1009	.0966	.0926	.0887	.0850	.0815	.0782	.0750	.0719
.80	,0900	.0861	.0825	.0790	.0758	.0726	.0696	.0668	.0640	.0614
.82	,0755	.0723	.0693	.0664	.0636	.0610	.0584	.0561	.0538	.0516
.84	,0621	.0594	.0569	.0546	.0523	.0501	.0481	.0461	.0442	.0424
.86	,0497	.0476	.0456	.0437	.0419	.0402	.0385	.0369	.0354	.0340
.88	,0385	.0368	.0353	.0338	.0324	.0311	.0298	.0286	.0274	.0263
.90	,0284	.0272	.0261	.0250	.0240	.0230	.0220	.0211	.0203	.0195
.92	,0196	.0188	.0180	.0173	.0165	.0159	.0152	.0146	.0140	.0135
.94	,0122	.0117	.0112	.0107	.0103	.0099	.0095	.0091	.0087	.0084
.96	,0063	.0060	.0058	.0055	.0053	.0051	.0049	.0047	.0045	.0043
.98	,0020	.0020	.0019	.0018	.0017	.0017	.0016	.0015	.0015	.0014

C_o \ α	.60	.61	.62	.63	.64	.65	.56	.67	.68	.59
0.00	.5182	.4798	.4438	.4098	.3780	.3479	.3199	.2936	.2689	.2460
.02	.5178	.4796	.4435	.4097	.3777	.3478	.3197	.2935	.2689	.2459
.04	.5171	.4789	.4430	.4091	.3773	.3474	.3194	.2932	.2686	.2457
.06	.5157	.4778	.4420	.4083	.3765	.3468	.3189	.2927	.2682	.2453
.08	.5139	.4762	.4407	.4071	.3755	.3459	.3181	.2920	.2676	.2448
.10	.5115	.4742	.4389	.4056	.3742	.3448	.3171	.2911	.2669	.2441
.12	.5086	.4716	.4367	.4037	.3726	.3433	.3159	.2901	.2659	.2433
.14	.5049	.4686	.4341	.4015	.3706	.3417	.3144	.2888	.2648	.2423
.16	.5006	.4649	.4310	.3988	.3684	.3397	.3127	.2873	.2635	.2412
.18	.4954	.4606	.4273	.3957	.3657	.3374	.3107	.2856	.2620	.2399
.20	.4891	.4556	.4231	.3921	.3627	.3347	.3084	.2836	.2603	.2384
.22	.4792	.4495	.4182	.3880	.3591	.3318	.3058	.2814	.2584	.2368
.24	.4666	.4401	.4124	.3833	.3552	.3284	.3030	.2789	.2562	.2349
.26	.4527	.4282	.4035	.3777	.3507	.3246	.2997	.2752	.2539	.2329
.28	.4379	.4151	.3923	.3693	.3454	.3203	.2961	.2731	.2512	.2306
.30	.4224	.4011	.3799	.3587	.3374	.3152	.2920	.2696	.2483	.2281
.32	.4066	.3865	.3667	.3471	.3275	.3078	.2372	.2657	.2450	.2253
.34	.3904	.3716	.3530	.3347	.3155	.2984	.2302	.2012	.2413	.2222
.36	.3740	.3563	.3389	.3217	.3049	.2881	.2714	.2546	.2371	.2187
.38	.3575	.3408	.3245	.3085	.2927	.2771	.2517	.2463	.2309	.2147
.40	.3410	.3253	.3099	.2949	.2802	.2657	.2514	.2373	.2231	.2089
.42	.3244	.3097	.2953	.2812	.2674	.2540	.2407	.2276	.2145	.2017
.44	.3080	.2941	.2806	.2674	.2546	.2420	.2297	.2175	.2056	.1937
.46	.2915	.2786	.2659	.2535	.2415	.2299	.2184	.2072	.1961	.1852
.48	.2753	.2631	.2513	.2398	.2285	.2177	.2071	.1967	.1865	.1754
.50	.2593	.2479	.2358	.2261	.2157	.2055	.1957	.1860	.1766	.1673
.52	.2434	.2328	.2225	.2125	.2029	.1935	.1843	.1754	.1667	.1581
.54	.2278	.2179	.2083	.1991	.1901	.1814	.1730	.1647	.1567	.1488
.56	.2124	.2033	.1944	.1859	.1775	.1695	.1517	.1541	.1467	.1395
.58	.1973	.1889	.1807	.1728	.1652	.1578	.1506	.1436	.1369	.1302
.60	.1926	.1748	.1673	.1601	.1531	.1463	.1397	.1333	.1271	.1210
.62	.1682	.1611	.1542	.1475	.1412	.1349	.1289	.1231	.1174	.1120
.64	.1542	.1477	.1414	.1354	.1295	.1239	.1184	.1131	.1080	.1030
.66	.1406	.1347	.1290	.1235	.1182	.1131	.1082	.1034	.0987	.0942
.68	.1274	.1221	.1170	.1120	.1073	.1026	.0982	.0939	.0897	.0857
.70	.1147	.1099	.1053	.1009	.0966	.0925	.0885	.0847	.0810	.0774
.72	.1025	.0982	.0941	.0902	.0864	.0827	.0792	.0758	.0725	.0693
.74	.0908	.0870	.0834	.0799	.0766	.0734	.0703	.0672	.0643	.0615
.76	.0796	.0763	.0731	.0701	.0672	.0644	.0617	.0591	.0565	.0541
.78	.0589	.0661	.0634	.0608	.0583	.0558	.0535	.0513	.0491	.0470
.80	.0589	.0565	.0542	.0519	.0498	.0478	.0458	.0439	.0420	.0402
.82	.0495	.0474	.0455	.0437	.0419	.0402	.0385	.0369	.0354	.0339
.84	.0407	.0390	.0375	.0359	.0345	.0331	.0317	.0304	.0292	.0279
.86	.0326	.0313	.0300	.0288	.0277	.0265	.0255	.0244	.0234	.0225
.88	.0253	.0243	.0233	.0223	.0215	.0206	.0198	.0190	.0182	.0174
.90	.0187	.0179	.0172	.0165	.0159	.0152	.0146	.0141	.0135	.0129
.92	.0129	.0124	.0119	.0115	.0110	.0106	.0101	.0097	.0094	.0090
.94	.0081	.0077	.0074	.0072	.0069	.0066	.0063	.0061	.0058	.0056
.96	.0042	.0040	.0038	.0037	.0036	.0034	.0033	.0032	.0030	.0029
.98	.0014	.0013	.0013	.0012	.0012	.0011	.0011	.0010	.0010	.0010

C_o \ α	.70	.71	.72	.73	.74	.75	.76	.77	.78	.79
0.00	.2545	.2045	.1858	.1685	.1524	.1374	.1236	.1107	.0989	.0880
.02	.2544	.2044	.1858	.1685	.1523	.1374	.1235	.1137	.0989	.0879
.04	.2542	.2043	.1856	.1683	.1522	.1373	.1235	.1106	.0988	.0879
.06	.2539	.2040	.1854	.1681	.1521	.1371	.1233	.1105	.0987	.0878
.08	.2535	.2036	.1851	.1678	.1518	.1369	.1231	.1104	.0986	.0877
.10	.2529	.2031	.1846	.1674	.1515	.1366	.1229	.1101	.0984	.0875
.12	.2522	.2025	.1841	.1670	.1511	.1363	.1225	.1099	.0981	.0873
.14	.2513	.2017	.1834	.1664	.1506	.1358	.1222	.1096	.0979	.0871
.16	.2503	.2009	.1827	.1658	.1500	.1353	.1218	.1092	.0975	.0868
.18	.2492	.1999	.1818	.1650	.1493	.1348	.1213	.1088	.0972	.0865
.20	.2479	.1988	.1809	.1642	.1485	.1341	.1207	.1083	.0968	.0861
.22	.2465	.1975	.1798	.1632	.1478	.1334	.1201	.1078	.0963	.0857
.24	.2449	.1961	.1785	.1622	.1469	.1327	.1194	.1072	.0958	.0853
.26	.2431	.1946	.1772	.1610	.1459	.1318	.1187	.1055	.0952	.0848
.28	.2411	.1929	.1758	.1597	.1448	.1308	.1179	.1058	.0946	.0843
.30	.2090	.1910	.1741	.1583	.1436	.1298	.1170	.1050	.0940	.0837
.32	.2065	.1890	.1724	.1568	.1423	.1287	.1160	.1042	.0933	.0831
.34	.2040	.1867	.1705	.1552	.1408	.1274	.1149	.1033	.0925	.0825
.36	.2011	.1842	.1684	.1534	.1393	.1261	.1138	.1023	.0916	.0817
.38	.1978	.1815	.1660	.1514	.1375	.1247	.1126	.1012	.0907	.0810
.40	.1941	.1785	.1635	.1492	.1358	.1231	.1112	.1001	.0897	.0801
.42	.1885	.1750	.1607	.1469	.1338	.1214	.1098	.0989	.0887	.0792
.44	.1819	.1699	.1574	.1442	.1315	.1195	.1082	.0975	.0875	.0783
.46	.1744	.1636	.1526	.1412	.1291	.1175	.1065	.0961	.0863	.0772
.48	.1665	.1566	.1457	.1368	.1263	.1152	.1045	.0945	.0850	.0761
.50	.1582	.1492	.1402	.1313	.1222	.1126	.1025	.0928	.0835	.0749
.52	.1497	.1415	.1333	.1252	.1171	.1088	.1001	.0908	.0820	.0735
.54	.1411	.1336	.1251	.1187	.1114	.1040	.0966	.0886	.0802	.0721
.56	.1325	.1255	.1187	.1120	.1054	.0988	.0921	.0854	.0782	.0705
.58	.1238	.1175	.1113	.1052	.0992	.0932	.0872	.0813	.0752	.0587
.60	.1152	.1094	.1038	.0982	.0928	.0874	.0821	.0757	.0714	.0659
.62	.1066	.1014	.0953	.0912	.0863	.0815	.0767	.0719	.0672	.0524
.64	.0982	.0934	.0888	.0843	.0799	.0755	.0712	.0670	.0627	.0585
.66	.0899	.0856	.0814	.0774	.0734	.0695	.0557	.0619	.0581	.0544
.68	.0818	.0779	.0742	.0705	.0670	.0636	.0502	.0558	.0535	.0502
.70	.0739	.0705	.0672	.0639	.0608	.0577	.0547	.0517	.0488	.0459
.72	.0662	.0632	.0603	.0574	.0546	.0519	.0493	.0457	.0441	.0416
.74	.0588	.0562	.0536	.0511	.0487	.0463	.0440	.0417	.0395	.0373
.76	.0517	.0494	.0472	.0450	.0429	.0409	.0389	.0369	.0350	.0331
.78	.0449	.0430	.0411	.0392	.0374	.0356	.0339	.0322	.0306	.0290
.80	.0385	.0368	.0352	.0336	.0321	.0305	.0292	.0278	.0264	.0250
.82	.0324	.0311	.0297	.0284	.0271	.0259	.0247	.0235	.0223	.0212
.84	.0268	.0256	.0245	.0235	.0224	.0214	.0204	.0195	.0185	.0176
.86	.0215	.0206	.0197	.0189	.0181	.0173	.0165	.0157	.0150	.0143
.88	.0167	.0160	.0154	.0147	.0141	.0135	.0129	.0123	.0117	.0111
.90	.0124	.0119	.0114	.0109	.0105	.0100	.0096	.0091	.0087	.0083
.92	.0086	.0083	.0079	.0076	.0073	.0070	.0067	.0054	.0061	.0058
.94	.0054	.0052	.0050	.0048	.0046	.0044	.0042	.0040	.0038	.0037
.96	.0028	.0027	.0026	.0025	.0024	.0023	.0022	.0021	.0020	.0019
.98	.0009	.0009	.0008	.0008	.0008	.0008	.0007	.0007	.0007	.0006

C_0 \ a	.80	.81	.82	.83	.84	.85	.86	.87	.88	.89
0.00	.0779	.0687	.0602	.0525	.0454	.0390	.0332	.0280	.0233	.0192
.02	.0779	.0686	.0602	.0524	.0454	.0390	.0332	.0280	.0233	.0192
.04	.0778	.0686	.0602	.0524	.0454	.0390	.0332	.0280	.0233	.0192
.06	.0778	.0685	.0601	.0524	.0453	.0389	.0332	.0280	.0233	.0192
.08	.0777	.0685	.0600	.0523	.0453	.0389	.0331	.0279	.0233	.0191
.10	.0775	.0683	.0599	.0522	.0452	.0388	.0331	.0279	.0232	.0191
.12	.0773	.0682	.0598	.0521	.0451	.0388	.0330	.0278	.0232	.0191
.14	.0771	.0680	.0596	.0520	.0450	.0387	.0329	.0278	.0232	.0190
.16	.0769	.0678	.0595	.0518	.0449	.0385	.0329	.0277	.0231	.0190
.18	.0766	.0676	.0593	.0517	.0447	.0385	.0328	.0276	.0230	.0189
.20	.0763	.0673	.0591	.0515	.0446	.0383	.0327	.0275	.0230	.0189
.22	.0760	.0670	.0588	.0513	.0444	.0382	.0325	.0274	.0229	.0188
.24	.0756	.0667	.0585	.0511	.0442	.0380	.0324	.0273	.0228	.0187
.26	.0752	.0664	.0582	.0508	.0440	.0378	.0323	.0272	.0227	.0187
.28	.0748	.0660	.0579	.0505	.0438	.0377	.0321	.0271	.0226	.0186
.30	.0743	.0656	.0576	.0502	.0435	.0374	.0319	.0269	.0225	.0185
.32	.0738	.0651	.0572	.0499	.0433	.0372	.0317	.0268	.0224	.0184
.34	.0732	.0646	.0568	.0495	.0430	.0370	.0315	.0266	.0222	.0183
.36	.0726	.0641	.0563	.0492	.0427	.0367	.0313	.0265	.0221	.0182
.38	.0719	.0636	.0559	.0488	.0423	.0364	.0311	.0263	.0219	.0181
.40	.0712	.0629	.0553	.0484	.0420	.0361	.0308	.0261	.0218	.0179
.42	.0704	.0623	.0548	.0479	.0416	.0358	.0306	.0258	.0216	.0178
.44	.0696	.0616	.0542	.0474	.0412	.0355	.0303	.0256	.0214	.0176
.46	.0687	.0609	.0536	.0469	.0407	.0351	.0300	.0254	.0212	.0175
.48	.0678	.0601	.0529	.0463	.0403	.0347	.0297	.0251	.0210	.0173
.50	.0668	.0592	.0522	.0457	.0398	.0343	.0293	.0248	.0208	.0171
.52	.0657	.0583	.0514	.0451	.0392	.0339	.0290	.0245	.0205	.0169
.54	.0645	.0573	.0506	.0444	.0387	.0334	.0286	.0242	.0203	.0167
.56	.0631	.0562	.0497	.0436	.0381	.0329	.0282	.0239	.0200	.0165
.58	.0617	.0550	.0487	.0428	.0374	.0324	.0277	.0235	.0197	.0163
.60	.0600	.0537	.0477	.0420	.0367	.0318	.0273	.0232	.0194	.0161
.62	.0574	.0522	.0455	.0410	.0359	.0311	.0268	.0227	.0191	.0158
.64	.0542	.0498	.0451	.0399	.0351	.0305	.0262	.0223	.0187	.0155
.66	.0507	.0469	.0429	.0387	.0341	.0297	.0255	.0218	.0184	.0152
.68	.0469	.0436	.0402	.0368	.0330	.0289	.0250	.0213	.0180	.0149
.70	.0430	.0401	.0372	.0343	.0312	.0279	.0242	.0207	.0175	.0146
.72	.0391	.0366	.0341	.0315	.0290	.0263	.0234	.0201	.0170	.0142
.74	.0351	.0330	.0308	.0287	.0255	.0242	.0219	.0194	.0165	.0138
.76	.0312	.0294	.0275	.0257	.0238	.0220	.0201	.0181	.0158	.0133
.78	.0274	.0258	.0243	.0227	.0212	.0196	.0180	.0164	.0147	.0128
.80	.0237	.0224	.0211	.0198	.0185	.0172	.0159	.0146	.0132	.0117
.82	.0201	.0190	.0179	.0169	.0158	.0148	.0137	.0126	.0115	.0104
.84	.0167	.0158	.0150	.0141	.0133	.0124	.0116	.0107	.0099	.0090
.86	.0135	.0128	.0122	.0115	.0108	.0101	.0095	.0088	.0082	.0075
.88	.0106	.0101	.0095	.0090	.0085	.0080	.0075	.0070	.0065	.0060
.90	.0079	.0075	.0071	.0068	.0064	.0060	.0057	.0053	.0049	.0046
.92	.0055	.0053	.0050	.0048	.0045	.0042	.0040	.0038	.0035	.0033
.94	.0035	.0033	.0032	.0030	.0028	.0027	.0025	.0024	.0022	.0021
.96	.0018	.0017	.0017	.0016	.0015	.0014	.0013	.0013	.0012	.0011
.98	.0005	.0006	.0005	.0005	.0005	.0005	.0004	.0004	.0004	.0004

APPENDIX 2. Tables of the $f_0^* = f_2' / f_0'$ - functions.

C_0 \ a	,10	,11	,12	.13	,14	,15	,16	,17	,18	,19
0,00	81,00	65,46	53,78	44.79	37,73	32,11	27,56	23.84	20,75	18,17
-,02	80,96	65,43	53,75	44.76	37,71	32,09	27,55	23.82	20,74	18,16
,04	80,84	65,33	53,67	44.69	37,65	32,04	27,50	23.78	20,70	18,13
,06	80,64	65,17	53,53	44.57	37,55	31,95	27,42	23.71	20,64	18,07
,08	80,36	64,93	53,33	44.41	37,41	31,83	27,31	23.52	20,56	18,00
,10	80,00	64,64	53,08	44.20	37,22	31,67	27,17	23.49	20,44	17,90
,12	79,56	64,27	52,78	43.93	37,00	31,47	27,00	23.34	20,31	17,78
,14	79,04	63,84	52,42	43.63	36,73	31,24	26,80	23.16	20,15	17,63
,16	78,44	63,35	52,00	43.27	36,43	30,97	26,56	22.95	19,96	17,47
,18	77,76	62,79	51,53	42.87	36,08	30,67	26,30	22.72	19,75	17,28
,20	77,00	62,16	51,00	42.42	35,69	30,33	26,00	22.45	19,52	17,07
,22	76,16	61,46	50,42	41.92	35,27	29,96	25,67	22.16	19,26	16,83
,24	75,24	60,70	49,78	41.38	34,80	29,55	25,31	21.84	18,98	16,58
,26	74,24	59,88	49,08	40.79	34,29	29,11	24,92	21.50	18,67	16,30
,28	73,16	58,98	48,33	40.15	33,73	28,63	24,50	21.12	18,33	16,00
,30	72,00	58,02	47,53	39.46	33,14	28,11	24,05	20.72	17,98	15,68
,32	70,76	57,00	46,67	38.73	32,51	27,56	23,55	20.29	17,59	15,34
,34	69,44	55,91	45,75	37.95	31,84	26,97	23,05	19.84	17,19	14,97
,36	68,04	54,75	44,78	37.12	31,12	26,35	22,50	19.35	16,75	14,58
,38	66,56	53,53	43,75	36.24	30,37	25,69	21,92	18.84	16,30	14,17
,40	65,00	52,24	42,67	35.32	29,57	25,00	21,31	18.30	15,81	13,74
,42	63,36	50,88	41,53	34.35	28,73	24,27	20,67	17.73	15,31	13,29
,44	61,64	49,46	40,33	33.33	27,85	23,51	20,00	17.14	14,78	12,81
,46	59,84	47,98	39,08	32.27	26,94	22,71	19,30	16.52	14,22	12,31
,48	57,96	46,42	37,78	31.15	25,98	21,87	18,56	15.87	13,64	11,79
,50	56,00	44,80	36,42	29.99	24,98	21,00	17,80	15.19	13,04	11,25
,52	53,96	43,12	35,00	28.79	23,94	20,09	17,00	14.48	12,41	10,68
,54	51,84	41,36	33,53	27.53	22,85	19,15	16,17	13.75	11,75	10,10
,56	49,64	39,55	32,00	26.23	21,73	18,17	15,31	12.99	11,07	9,49
,58	47,35	37,66	30,42	24.88	20,57	17,16	14,42	12.20	10,37	8,85
,60	45,00	35,71	28,78	23.49	19,37	16,11	13,50	11.38	9,64	8,20
,62	42,56	33,69	27,08	22.04	18,12	15,03	12,55	10.54	8,89	7,53
,64	40,04	31,61	25,33	20.55	16,84	13,91	11,56	9.56	8,11	6,94
,66	37,44	29,46	23,53	19.01	15,51	12,75	10,55	8.77	7,44	6,44
,68	34,76	27,25	21,67	17.43	14,14	11,56	9,50	7.99	6,87	6,00
,70	32,00	24,97	19,75	15.79	12,73	10,34	8,60	7.34	6,36	5,58
,72	29,16	22,62	17,78	14.11	11,29	9,29	7,85	6.76	5,89	5,19
,74	26,24	20,21	15,75	12.39	10,06	8,42	7,18	6.22	5,45	4,82
,76	23,24	17,73	13,67	10.93	9,04	7,64	6,56	5.71	5,02	4,45
,78	20,16	15,19	11,93	9.73	8,14	6,93	5,98	5.22	4,61	4,09
,80	17,01	13,07	10,51	8.68	7,31	6,26	5,42	4.75	4,20	3,74
,82	14,41	11,37	9,26	7.71	6,53	5,61	4,88	4.29	3,79	3,38
,84	12,34	9,88	8,11	6.80	5,79	4,99	4,35	3.83	3,39	3,03
,86	10,54	8,51	7,04	5.92	5,05	4,37	3,82	3.37	2,99	2,67
,88	8,89	7,23	6,01	5.07	4,35	3,75	3,29	2.91	2,58	2,31
,90	7,34	6,00	5,00	4.24	3,64	3,16	2,77	2.44	2,18	1,95
,92	5,85	4,80	4,01	3.41	2,93	2,55	2,23	1.98	1,76	1,58
,94	4,39	3,61	3,03	2.58	2,22	1,93	1,69	1.50	1,34	1,20
,96	2,94	2,43	2,04	1.74	1,50	1,30	1,15	1.02	,91	,81
,98	1,49	1,23	1,04	.89	,76	,67	,59	.52	,46	,42

C_0 \ a	.20	.21	.22	.23	.24	.25	.26	.27	.28	.29
0,00	15.000	14,152	12.570	11.208	10.028	9,000	8.101	7,310	6.612	5,994
,02	15.990	14,143	12.562	11.200	10.021	8,994	8.095	7,305	6.607	5,989
,04	15.960	14,116	12.537	11.178	10.000	8,974	8.077	7,288	6.592	5,975
,06	15.910	14,070	12.496	11.140	9.965	8,942	8.047	7,261	6.566	5,951
,08	15.840	14,007	12.438	11.087	9.917	8,898	8.006	7,222	6.531	5,918
,10	15.750	13,925	12.364	11.019	9.854	8,840	7.953	7,173	6.485	5,875
,12	15.640	13,825	12.273	10.936	9.778	8,770	7.888	7,112	6.429	5,823
,14	15.510	13,707	12.155	10.837	9.687	8,686	7.811	7,041	6.362	5,761
,16	15.360	13,571	12.041	10.724	9.583	8,590	7.722	6,959	6.286	5,690
,18	15.190	13,417	11.901	10.595	9.465	8,482	7.621	6,866	6.199	5,609
,20	15.000	13,245	11.744	10.452	9.333	8,360	7.509	6,761	6.102	5,518
,22	14.790	13,054	11.570	10.293	9.187	8,225	7.385	6,646	5.995	5,419
,24	14.560	12,846	11.380	10.119	9.028	8,078	7.249	6,520	5.878	5,309
,26	14.310	12,619	11.174	9.930	8.854	7,918	7.101	6,383	5.750	5,190
,28	14.040	12,374	10.950	9.726	8.667	7,746	6.941	6,235	5.612	5,062
,30	13.750	12,111	10.711	9.507	8.465	7,560	6.769	6,075	5.464	4,924
,32	13.440	11,830	10.455	9.272	8.250	7,362	6.586	5,905	5.306	4,776
,34	13.110	11,531	10.182	9.023	8.021	7,150	6.391	5,724	5.138	4,620
,36	12.760	11,213	9.893	8.758	7.778	6,926	6.183	5,532	4.959	4,453
,38	12.390	10,878	9.587	8.478	7.521	5,690	5.964	5,329	4.770	4,277
,40	12.000	10,524	9.264	8.183	7.250	5,440	5.734	5,115	4.571	4,092
,42	11.590	10,152	8.926	7.873	6.965	5,178	5.491	4,890	4.362	3,897
,44	11.160	9,762	8.570	7.548	6.667	5,902	5.237	4,654	4.144	3,725
,46	10.710	9,354	8.198	7.208	6.354	5,614	4.970	4,407	3.950	3,579
,48	10.240	8,927	7.810	6.853	6.028	5,314	4.593	4,191	3.785	3,446
,50	9.750	8,483	7.405	6.482	5.688	5,000	4.448	4,005	3.635	3,321
,52	9.240	8,020	6.983	6.096	5.334	4,725	4.239	3,836	3.494	3,202
,54	8.710	7,540	6.545	5.696	5.023	4,489	4.049	3,677	3.361	3,086
,56	8.160	7,041	6.092	5.345	4.757	4,275	3.871	3,527	3.231	2,974
,58	7.590	6,524	5.694	5.045	4.516	4,075	3.702	3,382	3.104	2,852
,60	7.002	6,073	5.355	4.774	4.291	3,885	3.539	3,240	2.979	2,750
,62	6.487	5,689	5.048	4.520	4.078	3,702	3.379	3,099	2.854	2,639
,64	6.051	5,343	4.763	4.280	3.872	3,523	3.222	2,960	2.729	2,526
,66	5.658	5,020	4.493	4.049	3.672	3,347	3.066	2,820	2.603	2,412
,68	5.294	4,715	4.233	3.824	3.474	3,173	2.910	2,679	2.476	2,295
,70	4.949	4,422	3.979	3.602	3.278	2,998	2.753	2,537	2.347	2,178
,72	4.617	4,137	3.730	3.383	3.083	2,822	2.594	2,393	2.216	2,057
,74	4.295	3,856	3.483	3.163	2.887	2,645	2.434	2,247	2.082	1,934
,76	3.978	3,578	3.237	2.943	2.689	2,466	2.271	2,098	1.945	1,808
,78	3.665	3,301	2.990	2.722	2.489	2,285	2.105	1,947	1.805	1,679
,80	3.352	3,024	2.742	2.498	2.286	2,100	1.937	1,791	1.662	1,547
,82	3.039	2,744	2.491	2.272	2.080	1,912	1.764	1,633	1.516	1,411
,84	2.724	2,462	2.237	2.041	1.871	1,721	1.588	1,470	1.366	1,272
,86	2.405	2,176	1.979	1.807	1.657	1,525	1.408	1,304	1.212	1,129
,88	2.084	1,886	1.716	1.568	1.438	1,324	1.223	1,134	1.053	,982
,90	1.756	1,591	1.448	1.324	1.215	1,119	1.034	,959	.891	,831
,92	1.423	1,290	1.175	1.074	.987	,909	.840	,779	.724	,675
,94	1.083	,982	.895	.819	.752	,693	.641	,594	.553	,515
,96	.734	,666	.507	.556	.511	,471	.435	,404	.376	,350
,98	.376	,341	.311	.285	.261	,241	.223	,207	.192	,179

Co \ a	.30	.31	.32	.33	.34	.35	.36	.37	.38	.39
0,00	5.444	4,954	4.516	4.122	3.768	3,449	3,160	2,899	2.662	2,446
.02	5.440	4,950	4.512	4.118	3.765	3,446	3,157	2,896	2,659	2,444
.04	5.427	4,938	4.500	4.107	3.754	3,436	3,148	2,888	2,651	2,435
.06	5.404	4,917	4.480	4.089	3.737	3,420	3,133	2,873	2,637	2,423
.08	5.373	4,888	4.453	4.063	3.713	3,397	3.111	2,852	2,618	2,404
.10	5.333	4,850	4.418	4.030	3.682	3,367	3,083	2,826	2.593	2,381
.12	5.284	4,804	4.375	3.990	3.644	3,331	3,049	2,794	2.562	2,352
.14	5.227	4,750	4.324	3.942	3.599	3,289	3,009	2,756	2.526	2,318
.16	5.160	4,688	4.256	3.887	3.547	3,240	2,963	2,712	2,485	2,278
.18	5.084	4,617	4.199	3.825	3.488	3,184	2.910	2,663	2.438	2,233
.20	5.000	4,538	4.125	3.755	3.422	3,122	2,852	2,607	2.385	2,183
.22	4.907	4,451	4.043	3.678	3.349	3,054	2,787	2,546	2.327	2,129
.24	4.804	4,355	3.953	3.593	3.270	2,979	2,716	2,478	2,263	2,082
.26	4.693	4,251	3.855	3.501	3.183	2,897	2,539	2,406	2,209	2,045
.28	4.573	4,138	3.750	3.402	3.090	2,809	2,556	2,343	2,166	2,012
.30	4.444	4,018	3.637	3.296	2.990	2,715	2,484	2,292	2.125	1,932
.32	4.307	3,889	3.516	3.182	2.882	2,632	2.424	2,245	2,090	1,952
.34	4.160	3,751	3.387	3.061	2.789	2,564	2.370	2,202	2,054	1,923
.36	4.004	3,606	3.250	2.955	2.711	2,502	2.320	2,151	2,020	1,894
.38	3.840	3,452	3.131	2.866	2.539	2,443	2.272	2,120	1.985	1,854
.40	3.667	3,317	3.029	2.784	2.573	2,388	2.225	2,080	1.950	1,833
.42	3.515	3,202	2.937	2.708	2.509	2,333	2.178	2,039	1.914	1,801
.44	3.385	3,097	2.850	2.635	2.445	2,280	2.131	1,997	1.877	1,768
.46	3.267	2,999	2.757	2.564	2.385	2,225	2.083	1,955	1.839	1,734
.48	3.156	2,906	2.587	2.494	2.324	2,171	2.034	1,911	1,799	1,598
.50	3.050	2,815	2.608	2.425	2.262	2,116	1.985	1,856	1.758	1,560
.52	2.948	2,725	2.529	2.355	2.199	2,059	1,933	1,819	1.715	1,520
.54	2.847	2,637	2.450	2.284	2.135	2,001	1.880	1,770	1.670	1,579
.56	2.748	2,548	2.371	2.212	2.070	1,942	1.825	1,720	1.623	1,535
.58	2.548	2,459	2.290	2.139	2.003	1,880	1.769	1,657	1.575	1,490
.60	2.548	2,369	2.208	2.064	1.934	1,817	1.710	1,613	1.524	1,442
.62	2.447	2,277	2.125	1.988	1.864	1,751	1.549	1,556	1.471	1,393
.64	2.345	2,184	2.039	1.909	1.791	1,684	1.586	1,497	1.416	1,341
.66	2.241	2,088	1.951	1.828	1.715	1,614	1.521	1,436	1.359	1,287
.68	2.135	1,991	1.851	1.744	1.638	1,541	1.453	1,373	1.299	1,231
.70	2.026	1,891	1.759	1.658	1.558	1,467	1.383	1,307	1.237	1,173
.72	1.915	1,788	1.673	1.570	1.475	1,389	1.311	1,239	1.173	1,112
.74	1.802	1,683	1.576	1.478	1.390	1,310	1.235	1,158	1.106	1,049
.76	1.685	1,575	1.475	1.384	1.302	1,227	1.158	1,095	1.037	,984
.78	1.566	1,463	1.371	1.287	1.211	1,142	1.078	1,020	.956	,916
.80	1.443	1,349	1.254	1.188	1.119	1,054	.995	,941	.892	,846
.82	1.317	1,232	1.155	1.085	1.021	,963	.910	,851	.815	,774
.84	1.187	1,111	1.042	.979	.921	,869	.821	,777	.737	,599
.86	1.054	,986	.925	.870	.819	,773	.730	,691	.655	,622
.88	.917	,858	.805	.757	.713	,673	.536	,602	.571	,542
.90	.776	,727	.682	.641	.604	,570	.539	,510	.484	,459
.92	.631	,591	.555	.522	.491	,464	.439	,415	.394	,374
.94	.482	,451	.424	.398	.375	,354	.335	,317	.301	,286
.96	.328	,307	.288	.271	.255	,241	.228	,216	.205	,194
.98	.168	,157	.148	.139	.131	,124	.117	,111	.105	,100

Co \ a	.40	.41	.42	.43	.44	.45	.46	.47	.48	.49
0,00	2,2500	2,0708	1,9070	1,7572	1,6198	1,4938	1,3781	1,2716	1,1736	1,0833
,02	2,2475	2,0684	1,9048	1,7550	1,6178	1,4919	1,3762	1,2698	1,1719	1,0822
,04	2,2400	2,0613	1,8980	1,7485	1,6116	1,4859	1,3705	1,2644	1,1670	1,0884
,06	2,2275	2,0494	1,8866	1,7377	1,6012	1,4760	1,3511	1,2553	1,1687	1,0992
,08	2,2100	2,0327	1,8707	1,7226	1,5868	1,4622	1,3481	1,2530	1,1754	1,1095
,10	2,1875	2,0113	1,8503	1,7031	1,5682	1,4444	1,3412	1,2558	1,1826	1,1194
,12	2,1600	1,9851	1,8254	1,6793	1,5457	1,4332	1,3399	1,2595	1,1895	1,1282
,14	2,1275	1,9542	1,7959	1,6512	1,5295	1,4279	1,3400	1,2633	1,1957	1,1357
,16	2,0900	1,9185	1,7622	1,6301	1,5197	1,4241	1,3405	1,2666	1,2008	1,1420
,18	2,0475	1,8780	1,7353	1,6157	1,5119	1,4211	1,3407	1,2690	1,2048	1,1469
,20	2,0003	1,8455	1,7159	1,6035	1,5051	1,4179	1,3401	1,2703	1,2073	1,1504
,22	1,9609	1,8207	1,6991	1,5927	1,4983	1,4141	1,3385	1,2703	1,2086	1,1524
,24	1,9304	1,7989	1,6839	1,5820	1,4911	1,4095	1,3358	1,2691	1,2083	1,1530
,26	1,9032	1,7790	1,6690	1,5710	1,4831	1,4037	1,3318	1,2654	1,2067	1,1520
,28	1,8781	1,7596	1,6541	1,5594	1,4741	1,3968	1,3264	1,2623	1,2035	1,1496
,30	1,8539	1,7403	1,6385	1,5469	1,4639	1,3884	1,3196	1,2566	1,1988	1,1455
,32	1,8299	1,7206	1,6222	1,5333	1,4524	1,3787	1,3113	1,2494	1,1925	1,1400
,34	1,8055	1,7001	1,6048	1,5183	1,4395	1,3675	1,3014	1,2406	1,1847	1,1329
,36	1,7807	1,6787	1,5852	1,5020	1,4251	1,3547	1,2900	1,2303	1,1752	1,1243
,38	1,7548	1,6560	1,5562	1,4842	1,4092	1,3403	1,2769	1,2184	1,1642	1,1141
,40	1,7278	1,6320	1,5447	1,4649	1,3917	1,3243	1,2522	1,2048	1,1517	1,1023
,42	1,6994	1,6065	1,5215	1,4439	1,3724	1,3066	1,2459	1,1896	1,1375	1,0890
,44	1,6695	1,5794	1,4959	1,4212	1,3516	1,2873	1,2279	1,1728	1,1217	1,0741
,46	1,6381	1,5507	1,4705	1,3968	1,3289	1,2662	1,2082	1,1543	1,1043	1,0577
,48	1,6050	1,5202	1,4423	1,3707	1,3045	1,2434	1,1868	1,1342	1,0852	1,0396
,50	1,5701	1,4879	1,4124	1,3428	1,2784	1,2189	1,1537	1,1124	1,0645	1,0201
,52	1,5335	1,4538	1,3805	1,3130	1,2505	1,1926	1,1389	1,0889	1,0423	,9989
,54	1,4949	1,4178	1,3459	1,2814	1,2208	1,1646	1,1124	1,0637	1,0184	,9761
,56	1,4543	1,3799	1,3113	1,2480	1,1892	1,1348	1,0841	1,0369	,9929	,9518
,58	1,4118	1,3401	1,2738	1,2126	1,1558	1,1031	1,0541	1,0084	,9657	,9259
,60	1,3673	1,2982	1,2344	1,1754	1,1205	1,0697	1,0223	,9782	,9369	,8983
,62	1,3207	1,2544	1,1930	1,1362	1,0835	1,0345	,9888	,9463	,9065	,8693
,64	1,2720	1,2084	1,1495	1,0951	1,0445	,9974	,9536	,9126	,8744	,8386
,66	1,2213	1,1605	1,1042	1,0521	1,0037	,9586	,9166	,8773	,8406	,8063
,68	1,1683	1,1104	1,0568	1,0071	,9609	,9179	,8778	,8403	,8052	,7724
,70	1,1132	1,0583	1,0074	,9601	,9162	,8753	,8372	,8015	,7681	,7369
,72	1,0559	1,0040	,9559	,9112	,8696	,8309	,7948	,7610	,7294	,6998
,74	,9964	,9476	,9023	,8603	,8211	,7847	,7506	,7188	,6890	,6611
,76	,9345	,8890	,8456	,8073	,7707	,7365	,7046	,6748	,6469	,6207
,78	,8705	,8281	,7888	,7523	,7182	,6865	,6568	,6291	,6031	,5787
,80	,8042	,7651	,7289	,6952	,6638	,6345	,6072	,5816	,5576	,5351
,82	,7355	,6999	,6668	,6360	,6074	,5806	,5556	,5323	,5103	,4898
,84	,6644	,6323	,6025	,5748	,5489	,5248	,5023	,4811	,4613	,4428
,86	,5910	,5625	,5360	,5114	,4884	,4670	,4469	,4282	,4106	,3941
,88	,5150	,4902	,4672	,4458	,4258	,4071	,3897	,3734	,3580	,3436
,90	,4366	,4156	,3961	,3779	,3610	,3452	,3305	,3166	,3036	,2915
,92	,3555	,3384	,3226	,3078	,2941	,2812	,2692	,2579	,2474	,2375
,94	,2716	,2586	,2465	,2353	,2248	,2149	,2058	,1972	,1891	,1815
,96	,1849	,1760	,1678	,1601	,1530	,1463	,1401	,1342	,1287	,1236
,98	,0948	,0902	,0860	,0821	,0784	,0750	,0718	,0688	,0660	,0633

C_o \ a	.50	.51	.52	.53	.54	.55	.56	.57	.58	.59
0.00	1.0000	.9688	.9459	.9288	.9137	.9012	.8899	.8805	.8719	.8547
.02	1.0135	.9705	.9467	.9284	.9134	.9005	.8895	.8798	.8714	.8538
.04	1.0281	.9805	.9472	.9275	.9118	.8988	.8875	.8779	.8693	.8518
.06	1.0412	.9935	.9546	.9265	.9095	.8959	.8845	.8745	.8660	.8534
.08	1.0535	1.0055	.9651	.9316	.9072	.8923	.8802	.8701	.8613	.8537
.10	1.0644	1.0166	.9751	.9396	.9101	.8884	.8752	.8645	.8555	.8477
.12	1.0741	1.0264	.9843	.9475	.9158	.8892	.8697	.8580	.8484	.8405
.14	1.0824	1.0348	.9924	.9548	.9215	.8929	.8688	.8511	.8406	.8321
.16	1.0893	1.0419	.9993	.9611	.9269	.8957	.8705	.8484	.8323	.8230
.18	1.0948	1.0475	1.0049	.9662	.9314	.9002	.8721	.8483	.8280	.8133
.20	1.0987	1.0518	1.0090	.9702	.9349	.9029	.8742	.8486	.8263	.8075
.22	1.1013	1.0546	1.0119	.9729	.9373	.9048	.8753	.8487	.8250	.8043
.24	1.1023	1.0559	1.0133	.9743	.9384	.9056	.8755	.8482	.8235	.8015
.26	1.1019	1.0558	1.0134	.9743	.9384	.9053	.8749	.8470	.8215	.7985
.28	1.0999	1.0542	1.0120	.9731	.9371	.9039	.8732	.8450	.8190	.7954
.30	1.0965	1.0511	1.0093	.9704	.9345	.9012	.8705	.8419	.8157	.7915
.32	1.0915	1.0466	1.0050	.9655	.9307	.8974	.8665	.8379	.8114	.7859
.34	1.0850	1.0406	.9994	.9611	.9255	.8924	.8615	.8329	.8052	.7815
.36	1.0770	1.0332	.9924	.9544	.9191	.8861	.8554	.8257	.8000	.7752
.38	1.0675	1.0242	.9839	.9453	.9113	.8786	.8480	.8195	.7928	.7679
.40	1.0565	1.0138	.9740	.9359	.9022	.8698	.8395	.8111	.7845	.7597
.42	1.0439	1.0019	.9627	.9251	.8918	.8597	.8297	.8016	.7752	.7505
.44	1.0299	.9885	.9500	.9138	.8801	.8484	.8187	.7909	.7647	.7402
.46	1.0142	.9737	.9358	.9003	.8670	.8358	.8065	.7791	.7532	.7289
.48	.9971	.9573	.9202	.8853	.8525	.8219	.7931	.7660	.7405	.7166
.50	.9785	.9396	.9031	.8690	.8369	.8068	.7785	.7519	.7268	.7032
.52	.9583	.9203	.8847	.8513	.8199	.7904	.7626	.7365	.7119	.6887
.54	.9366	.8995	.8648	.8322	.8015	.7727	.7456	.7200	.6959	.6732
.56	.9133	.8773	.8434	.8117	.7818	.7537	.7272	.7023	.6787	.6565
.58	.8885	.8535	.8207	.7898	.7608	.7335	.7077	.6834	.6604	.6388
.60	.8622	.8283	.7965	.7666	.7384	.7119	.6869	.6633	.6410	.6199
.62	.8344	.8016	.7709	.7420	.7147	.6891	.6649	.6420	.6204	.6000
.64	.8050	.7735	.7433	.7150	.6897	.6650	.6415	.6195	.5987	.5789
.66	.7741	.7438	.7154	.6886	.6634	.6395	.6171	.5959	.5758	.5568
.68	.7415	.7126	.6854	.6598	.6356	.6128	.5913	.5710	.5517	.5335
.70	.7075	.6800	.6540	.6296	.6066	.5848	.5643	.5449	.5265	.5091
.72	.6720	.6458	.6212	.5980	.5752	.5555	.5360	.5176	.5001	.4835
.74	.6348	.6101	.5869	.5650	.5444	.5249	.5065	.4891	.4725	.4569
.76	.5961	.5730	.5512	.5307	.5113	.4930	.4757	.4593	.4438	.4291
.78	.5558	.5343	.5140	.4948	.4768	.4597	.4436	.4283	.4139	.4001
.80	.5139	.4940	.4753	.4576	.4409	.4251	.4102	.3951	.3827	.3700
.82	.4704	.4522	.4351	.4189	.4035	.3892	.3755	.3626	.3504	.3387
.84	.4253	.4089	.3934	.3788	.3650	.3519	.3395	.3279	.3168	.3053
.86	.3786	.3639	.3502	.3372	.3249	.3133	.3023	.2919	.2820	.2727
.88	.3301	.3174	.3054	.2940	.2833	.2732	.2636	.2546	.2460	.2378
.90	.2800	.2692	.2590	.2494	.2403	.2318	.2235	.2159	.2086	.2017
.92	.2281	.2193	.2110	.2032	.1958	.1888	.1822	.1759	.1700	.1643
.94	.1744	.1676	.1613	.1553	.1497	.1443	.1393	.1345	.1299	.1256
.96	.1187	.1141	.1098	.1057	.1019	.0983	.0948	.0915	.0884	.0855
.98	.0608	.0585	.0563	.0542	.0522	.0503	.0486	.0459	.0453	.0438

Co \ a	.60	.61	.62	.63	.64	.65	.66	.67	.68	.69
0.00	.8577	.8520	.8464	.8417	.8372	.8333	.8295	.8263	.8234	.8205
.02	.8572	.8512	.8459	.8410	.8367	.8327	.8291	.8258	.8228	.8201
.04	.8551	.8492	.8438	.8390	.8345	.8307	.8271	.8239	.8210	.8182
.06	.8518	.8457	.8403	.8357	.8314	.8275	.8240	.8208	.8179	.8153
.08	.8469	.8411	.8357	.8310	.8268	.8230	.8195	.8164	.8137	.8111
.10	.8410	.8350	.8293	.8250	.8209	.8172	.8139	.8109	.8082	.8058
.12	.8335	.8277	.8224	.8178	.8137	.8101	.8069	.8040	.8015	.7992
.14	.8252	.8190	.8139	.8093	.8054	.8018	.7988	.7960	.7936	.7915
.16	.8155	.8094	.8040	.7996	.7955	.7923	.7893	.7868	.7845	.7825
.18	.8050	.7985	.7932	.7886	.7848	.7815	.7788	.7763	.7743	.7725
.20	.7941	.7868	.7811	.7766	.7727	.7695	.7669	.7647	.7628	.7613
.22	.7870	.7747	.7683	.7635	.7597	.7565	.7540	.7519	.7503	.7489
.24	.7823	.7663	.7551	.7496	.7455	.7424	.7399	.7380	.7365	.7354
.26	.7781	.7603	.7454	.7352	.7305	.7272	.7249	.7230	.7217	.7208
.28	.7739	.7548	.7382	.7245	.7152	.7113	.7087	.7070	.7058	.7061
.30	.7694	.7494	.7315	.7161	.7034	.6950	.6919	.6900	.6890	.6884
.32	.7643	.7437	.7250	.7084	.6947	.6822	.6747	.6723	.6711	.6707
.34	.7585	.7375	.7182	.7008	.6852	.6719	.6610	.6543	.6525	.6521
.36	.7521	.7307	.7110	.6930	.6765	.6622	.6497	.6397	.6367	.6328
.38	.7447	.7232	.7032	.6847	.6679	.6525	.6391	.6275	.6183	.6131
.40	.7365	.7149	.6947	.6760	.6588	.6430	.6287	.6152	.6054	.5970
.42	.7273	.7057	.6855	.6666	.6492	.6330	.6183	.6050	.5933	.5833
.44	.7172	.6956	.6754	.6566	.6390	.6227	.6076	.5938	.5814	.5705
.46	.7061	.6847	.6645	.6458	.6281	.6117	.5964	.5823	.5695	.5579
.48	.6940	.6728	.6529	.6341	.6155	.6001	.5848	.5705	.5574	.5454
.50	.6810	.6600	.6403	.6217	.6042	.5879	.5725	.5582	.5449	.5325
.52	.6668	.6462	.6268	.6084	.5911	.5749	.5596	.5453	.5320	.5196
.54	.6517	.6314	.6123	.5943	.5772	.5612	.5460	.5318	.5185	.5061
.56	.6355	.6157	.5959	.5792	.5625	.5467	.5318	.5177	.5045	.4921
.58	.6183	.5989	.5806	.5633	.5469	.5314	.5167	.5029	.4898	.4775
.60	.6000	.5811	.5633	.5464	.5304	.5152	.5009	.4873	.4745	.4624
.62	.5807	.5624	.5450	.5286	.5130	.4983	.4843	.4711	.4585	.4457
.64	.5602	.5425	.5258	.5099	.4948	.4805	.4669	.4540	.4418	.4303
.66	.5388	.5217	.5055	.4902	.4756	.4618	.4487	.4362	.4244	.4132
.68	.5162	.4998	.4843	.4696	.4555	.4422	.4295	.4176	.4062	.3953
.70	.4925	.4769	.4621	.4480	.4345	.4218	.4097	.3982	.3872	.3768
.72	.4679	.4530	.4388	.4254	.4125	.4005	.3889	.3779	.3675	.3575
.74	.4421	.4280	.4146	.4019	.3898	.3783	.3673	.3569	.3469	.3375
.76	.4151	.4019	.3893	.3773	.3660	.3551	.3448	.3350	.3255	.3167
.78	.3871	.3747	.3630	.3518	.3412	.3310	.3214	.3122	.3034	.2951
.80	.3580	.3465	.3355	.3253	.3154	.3060	.2971	.2886	.2804	.2727
.82	.3277	.3172	.3072	.2978	.2887	.2801	.2719	.2641	.2566	.2495
.84	.2963	.2868	.2778	.2692	.2610	.2532	.2458	.2387	.2319	.2254
.86	.2638	.2553	.2473	.2396	.2323	.2253	.2187	.2124	.2063	.2005
.88	.2300	.2226	.2156	.2089	.2025	.1965	.1907	.1852	.1799	.1748
.90	.1951	.1888	.1829	.1772	.1718	.1666	.1617	.1570	.1525	.1482
.92	.1589	.1538	.1490	.1443	.1399	.1357	.1317	.1279	.1242	.1207
.94	.1215	.1176	.1139	.1103	.1069	.1037	.1006	.0977	.0949	.0922
.96	.0827	.0800	.0775	.0751	.0729	.0706	.0685	.0665	.0645	.0627
.98	.0424	.0410	.0397	.0384	.0373	.0361	.0351	.0340	.0330	.0321

C_o \ α	.70	.71	.72	.73	.74	.75	.76	.77	.78	.79
0.00	.8182	.8157	.8138	.8118	.8100	.8085	.8068	.8057	.8044	.8031
.02	.8175	.8154	.8132	.8113	.8095	.8080	.8066	.8052	.8040	.8029
.04	.8158	.8135	.8115	.8097	.8079	.8064	.8049	.8037	.8025	.8014
.06	.8129	.8108	.8087	.8070	.8053	.8038	.8025	.8012	.8001	.7991
.08	.8088	.8067	.8048	.8031	.8015	.8001	.7988	.7977	.7966	.7957
.10	.8035	.8016	.7998	.7982	.7968	.7954	.7943	.7932	.7923	.7914
.12	.7972	.7953	.7937	.7922	.7908	.7897	.7886	.7877	.7869	.7861
.14	.7896	.7879	.7864	.7851	.7839	.7829	.7820	.7812	.7805	.7800
.16	.7809	.7793	.7780	.7769	.7759	.7751	.7743	.7737	.7732	.7728
.18	.7710	.7697	.7686	.7676	.7669	.7662	.7657	.7652	.7649	.7647
.20	.7600	.7589	.7580	.7573	.7567	.7563	.7560	.7558	.7556	.7556
.22	.7478	.7470	.7463	.7459	.7455	.7453	.7453	.7453	.7454	.7456
.24	.7345	.7340	.7336	.7333	.7333	.7334	.7335	.7338	.7342	.7345
.26	.7202	.7199	.7197	.7198	.7200	.7204	.7208	.7214	.7220	.7227
.28	.7047	.7047	.7048	.7052	.7057	.7063	.7071	.7079	.7088	.7098
.30	.6883	.6884	.6889	.6895	.6903	.6913	.6923	.6935	.6947	.6960
.32	.6707	.6712	.6719	.6728	.6739	.6752	.6765	.6781	.6796	.6812
.34	.6523	.6529	.6539	.6551	.6565	.6581	.6598	.6617	.6635	.6655
.36	.6329	.6337	.6348	.6364	.6381	.6400	.6421	.6443	.6465	.6488
.38	.6128	.6136	.6150	.6167	.6188	.6210	.6234	.6259	.6285	.6312
.40	.5925	.5928	.5942	.5962	.5984	.6010	.6038	.6066	.6096	.6127
.42	.5756	.5718	.5728	.5747	.5773	.5801	.5832	.5864	.5897	.5932
.44	.5513	.5543	.5511	.5527	.5552	.5583	.5517	.5652	.5690	.5728
.46	.5478	.5393	.5320	.5303	.5325	.5357	.5394	.5432	.5473	.5515
.48	.5345	.5252	.5174	.5117	.5097	.5125	.5162	.5203	.5247	.5292
.50	.5515	.5114	.5027	.4956	.4904	.4890	.4924	.4956	.5013	.5052
.52	.5081	.4977	.4884	.4804	.4738	.4693	.4584	.4724	.4771	.4823
.54	.4945	.4839	.4742	.4656	.4582	.4522	.4482	.4478	.4524	.4577
.56	.4805	.4698	.4599	.4509	.4430	.4362	.4307	.4272	.4274	.4325
.58	.4661	.4553	.4453	.4362	.4279	.4205	.4142	.4093	.4063	.4069
.60	.4511	.4404	.4304	.4212	.4127	.4050	.3982	.3925	.3881	.3856
.62	.4355	.4250	.4151	.4059	.3973	.3895	.3824	.3751	.3709	.3670
.64	.4193	.4090	.3993	.3902	.3815	.3737	.3565	.3600	.3543	.3495
.66	.4025	.3925	.3830	.3740	.3656	.3577	.3505	.3438	.3378	.3326
.68	.3851	.3753	.3661	.3573	.3491	.3414	.3342	.3275	.3214	.3159
.70	.3669	.3575	.3486	.3401	.3321	.3246	.3175	.3109	.3048	.2992
.72	.3480	.3390	.3305	.3223	.3145	.3073	.3004	.2940	.2879	.2824
.74	.3285	.3199	.3117	.3039	.2965	.2896	.2829	.2767	.2708	.2653
.76	.3082	.3000	.2923	.2849	.2779	.2713	.2649	.2589	.2533	.2480
.78	.2871	.2795	.2722	.2653	.2587	.2524	.2464	.2407	.2354	.2303
.80	.2652	.2582	.2514	.2450	.2388	.2329	.2273	.2220	.2169	.2121
.82	.2425	.2361	.2299	.2240	.2183	.2129	.2077	.2027	.1980	.1936
.84	.2192	.2133	.2077	.2023	.1971	.1922	.1874	.1829	.1786	.1745
.86	.1950	.1897	.1847	.1798	.1752	.1708	.1666	.1625	.1586	.1549
.88	.1700	.1654	.1609	.1567	.1526	.1487	.1450	.1415	.1380	.1348
.90	.1441	.1401	.1364	.1328	.1293	.1260	.1228	.1198	.1169	.1141
.92	.1173	.1141	.1110	.1081	.1052	.1025	.0999	.0974	.0950	.0927
.94	.0896	.0872	.0848	.0825	.0804	.0783	.0763	.0744	.0725	.0707
.96	.0610	.0593	.0577	.0561	.0546	.0532	.0518	.0505	.0493	.0481
.98	.0312	.0303	.0295	.0287	.0279	.0272	.0265	.0258	.0252	.0245

C_0 \ a	.80	.81	.82	.83	.84	.85	.86	.87	.88	.89
0.00	.8024	.8012	.8003	.7998	.7989	.7981	.7977	.7974	.7967	.7960
.02	.8013	.8010	.8001	.7993	.7987	.7981	.7975	.7969	.7965	.7962
.04	.8004	.7995	.7987	.7980	.7973	.7967	.7962	.7958	.7953	.7948
.06	.7981	.7973	.7965	.7958	.7952	.7947	.7942	.7937	.7934	.7931
.08	.7948	.7940	.7933	.7928	.7922	.7917	.7913	.7910	.7906	.7903
.10	.7905	.7900	.7894	.7888	.7884	.7880	.7877	.7873	.7871	.7870
.12	.7855	.7849	.7844	.7841	.7837	.7833	.7831	.7830	.7828	.7826
.14	.7794	.7790	.7787	.7784	.7782	.7780	.7779	.7778	.7778	.7778
.16	.7725	.7722	.7710	.7718	.7717	.7717	.7717	.7718	.7719	.7720
.18	.7645	.7644	.7644	.7644	.7645	.7647	.7648	.7650	.7653	.7656
.20	.7557	.7557	.7559	.7561	.7564	.7567	.7570	.7574	.7578	.7582
.22	.7458	.7462	.7465	.7470	.7474	.7479	.7484	.7490	.7496	.7502
.24	.7351	.7357	.7363	.7369	.7375	.7383	.7391	.7398	.7406	.7414
.26	.7234	.7243	.7251	.7260	.7269	.7279	.7288	.7298	.7308	.7318
.28	.7109	.7119	.7131	.7142	.7154	.7166	.7178	.7190	.7202	.7215
.30	.6973	.6987	.7001	.7015	.7030	.7044	.7059	.7074	.7089	.7103
.32	.6829	.6846	.6863	.6880	.6897	.6915	.6932	.6950	.6967	.6985
.34	.6675	.6695	.6715	.6735	.6755	.6775	.6797	.6818	.6838	.6858
.36	.6511	.6535	.6559	.6583	.6607	.6631	.6654	.6677	.6701	.6724
.38	.6339	.6366	.6393	.6421	.6448	.6475	.6502	.6529	.6555	.6581
.40	.6157	.6188	.6217	.6251	.6281	.6312	.6342	.6372	.6402	.6432
.42	.5965	.6001	.6035	.6071	.6105	.6140	.6174	.6208	.6241	.6274
.44	.5765	.5805	.5844	.5883	.5922	.5960	.5997	.6035	.6072	.6108
.46	.5557	.5600	.5643	.5686	.5729	.5771	.5813	.5854	.5895	.5935
.48	.5339	.5386	.5433	.5480	.5527	.5573	.5619	.5665	.5709	.5753
.50	.5112	.5164	.5215	.5266	.5317	.5368	.5418	.5467	.5516	.5565
.52	.4877	.4932	.4987	.5043	.5098	.5153	.5208	.5262	.5315	.5367
.54	.4634	.4692	.4751	.4811	.4871	.4931	.4990	.5048	.5106	.5162
.56	.4382	.4444	.4505	.4571	.4635	.4700	.4763	.4826	.4888	.4949
.58	.4125	.4189	.4255	.4323	.4392	.4460	.4528	.4596	.4662	.4728
.60	.3865	.3928	.3996	.4067	.4140	.4212	.4285	.4357	.4429	.4499
.62	.3649	.3664	.3731	.3803	.3879	.3957	.4034	.4110	.4186	.4262
.64	.3460	.3443	.3453	.3534	.3612	.3692	.3774	.3855	.3935	.4016
.66	.3283	.3252	.3239	.3252	.3330	.3422	.3506	.3592	.3678	.3763
.68	.3111	.3073	.3045	.3036	.3063	.3145	.3232	.3321	.3411	.3501
.70	.2942	.2899	.2854	.2840	.2831	.2865	.2951	.3043	.3137	.3231
.72	.2773	.2727	.2588	.2557	.2537	.2634	.2568	.2759	.2855	.2954
.74	.2602	.2556	.2515	.2480	.2452	.2435	.2435	.2472	.2568	.2559
.76	.2430	.2384	.2342	.2305	.2274	.2249	.2235	.2238	.2277	.2378
.78	.2255	.2210	.2159	.2131	.2098	.2070	.2048	.2037	.2041	.2083
.80	.2075	.2033	.1993	.1955	.1923	.1893	.1868	.1850	.1840	.1847
.82	.1893	.1853	.1815	.1780	.1747	.1717	.1691	.1669	.1653	.1646
.84	.1705	.1669	.1634	.1600	.1569	.1541	.1515	.1491	.1472	.1458
.86	.1514	.1480	.1449	.1418	.1389	.1362	.1337	.1315	.1295	.1278
.88	.1317	.1287	.1258	.1231	.1205	.1181	.1158	.1137	.1118	.1101
.90	.1114	.1088	.1064	.1040	.1018	.0997	.0976	.0957	.0940	.0924
.92	.0905	.0884	.0864	.0844	.0825	.0808	.0791	.0775	.0760	.0746
.94	.0690	.0674	.0658	.0643	.0629	.0615	.0602	.0589	.0577	.0566
.96	.0469	.0458	.0447	.0436	.0427	.0417	.0408	.0399	.0390	.0382
.98	.0240	.0234	.0228	.0223	.0218	.0213	.0208	.0203	.0199	.0194

APPENDIX 3. Tables of the $Y(a, c_0)$ - functions.

C_0 \ a	.10	.11	.12	.13	.14	.15	.16	.17	.18	.19
0.00	1.2570	1.2562	1.2552	1.2541	1.2536	1.2530	1.2517	1.2503	1.2495	1.2481
.02	1.2556	1.2551	1.2546	1.2540	1.2531	1.2521	1.2512	1.2502	1.2489	1.2476
.04	1.2555	1.2550	1.2542	1.2534	1.2526	1.2518	1.2507	1.2495	1.2484	1.2470
.06	1.2542	1.2536	1.2531	1.2524	1.2515	1.2505	1.2495	1.2484	1.2470	1.2457
.08	1.2533	1.2526	1.2518	1.2509	1.2501	1.2492	1.2480	1.2468	1.2455	1.2441
.10	1.2512	1.2505	1.2499	1.2491	1.2481	1.2470	1.2460	1.2448	1.2433	1.2418
.12	1.2494	1.2487	1.2478	1.2468	1.2459	1.2448	1.2436	1.2422	1.2408	1.2393
.14	1.2466	1.2459	1.2451	1.2442	1.2431	1.2418	1.2406	1.2393	1.2377	1.2360
.16	1.2440	1.2431	1.2421	1.2410	1.2399	1.2387	1.2373	1.2358	1.2342	1.2324
.18	1.2405	1.2396	1.2385	1.2375	1.2363	1.2349	1.2334	1.2319	1.2301	1.2282
.20	1.2370	1.2360	1.2348	1.2335	1.2322	1.2308	1.2292	1.2274	1.2256	1.2235
.22	1.2328	1.2316	1.2304	1.2291	1.2276	1.2260	1.2243	1.2225	1.2204	1.2183
.24	1.2283	1.2271	1.2257	1.2242	1.2225	1.2209	1.2191	1.2170	1.2149	1.2125
.26	1.2234	1.2220	1.2205	1.2189	1.2172	1.2153	1.2132	1.2111	1.2087	1.2061
.28	1.2180	1.2165	1.2148	1.2131	1.2112	1.2091	1.2069	1.2045	1.2020	1.1992
.30	1.2122	1.2105	1.2087	1.2068	1.2047	1.2025	1.2000	1.1975	1.1947	1.1917
.32	1.2058	1.2040	1.2021	1.2000	1.1977	1.1952	1.1926	1.1898	1.1868	1.1835
.34	1.1992	1.1972	1.1950	1.1926	1.1902	1.1875	1.1846	1.1815	1.1783	1.1747
.36	1.1918	1.1896	1.1873	1.1848	1.1820	1.1791	1.1760	1.1727	1.1691	1.1652
.38	1.1842	1.1818	1.1791	1.1763	1.1734	1.1702	1.1568	1.1631	1.1593	1.1551
.40	1.1758	1.1731	1.1703	1.1673	1.1641	1.1606	1.1569	1.1529	1.1487	1.1441
.42	1.1671	1.1642	1.1610	1.1577	1.1542	1.1504	1.1464	1.1420	1.1373	1.1324
.44	1.1577	1.1545	1.1511	1.1475	1.1436	1.1395	1.1350	1.1303	1.1252	1.1197
.46	1.1476	1.1442	1.1405	1.1366	1.1323	1.1278	1.1230	1.1178	1.1122	1.1062
.48	1.1372	1.1333	1.1292	1.1249	1.1203	1.1154	1.1101	1.1044	1.0983	1.0917
.50	1.1258	1.1216	1.1172	1.1126	1.1075	1.1020	1.0963	1.0901	1.0833	1.0760
.52	1.1139	1.1094	1.1045	1.0993	1.0938	1.0879	1.0816	1.0747	1.0673	1.0593
.54	1.1012	1.0968	1.0909	1.0853	1.0793	1.0728	1.0657	1.0582	1.0500	1.0410
.56	1.0877	1.0823	1.0765	1.0704	1.0637	1.0565	1.0488	1.0405	1.0313	1.0213
.58	1.0735	1.0675	1.0612	1.0543	1.0470	1.0392	1.0306	1.0212	1.0111	.9998
.60	1.0582	1.0516	1.0447	1.0372	1.0292	1.0204	1.0109	1.0005	.9889	.9757
.62	1.0419	1.0348	1.0272	1.0189	1.0099	1.0001	.9895	.9777	.9642	.9483
.64	1.0247	1.0168	1.0083	.9991	.9891	.9783	.9561	.9522	.9359	.9198
.66	1.0061	.9978	.9879	.9777	.9666	.9541	.9399	.9234	.9065	.8946
.68	.9861	.9764	.9660	.9545	.9417	.9271	.9098	.8927	.8807	.8727
.70	.9647	.9539	.9421	.9288	.9139	.8963	.8784	.8663	.8584	.8537
.72	.9414	.9292	.9156	.9002	.8817	.8635	.8513	.8437	.8393	.8373
.74	.9158	.9018	.8859	.8671	.8479	.8357	.8283	.8244	.8228	.8231
.76	.8876	.8711	.8512	.8317	.8194	.8124	.8090	.8080	.8088	.8109
.78	.8559	.8353	.8147	.8024	.7959	.7931	.7927	.7942	.7969	.8006
.80	.8183	.7968	.7847	.7787	.7766	.7771	.7793	.7827	.7871	.7920
.82	.7779	.7661	.7609	.7597	.7611	.7642	.7584	.7735	.7791	.7852
.84	.7465	.7423	.7423	.7448	.7489	.7541	.7600	.7665	.7732	.7801
.86	.7231	.7244	.7253	.7337	.7400	.7469	.7542	.7617	.7693	.7770
.88	.7064	.7119	.7187	.7263	.7343	.7426	.7509	.7593	.7676	.7759
.90	.6959	.7045	.7136	.7229	.7323	.7416	.7507	.7598	.7687	.7774
.92	.6919	.7028	.7135	.7242	.7345	.7446	.7544	.7639	.7731	.7821
.94	.6958	.7083	.7203	.7318	.7428	.7533	.7534	.7731	.7824	.7915
.96	.7116	.7250	.7376	.7494	.7605	.7710	.7810	.7906	.7997	.8084
.98	.7518	.7849	.7769	.7880	.7984	.8080	.8171	.8257	.8338	.8415

Co \ α	.20	.21	.22	.23	.24	.25	.26	.27	.28	.29
0.00	1.2464	1.2451	1.2432	1.2412	1.2394	1.2368	1.2346	1.2319	1.2288	1.2259
.02	1.2462	1.2445	1.2428	1.2409	1.2388	1.2366	1.2341	1.2314	1.2285	1.2253
.04	1.2454	1.2439	1.2421	1.2401	1.2381	1.2357	1.2333	1.2306	1.2276	1.2244
.06	1.2442	1.2425	1.2407	1.2388	1.2366	1.2343	1.2317	1.2289	1.2260	1.2225
.08	1.2425	1.2408	1.2389	1.2369	1.2347	1.2322	1.2297	1.2258	1.2237	1.2204
.10	1.2403	1.2384	1.2365	1.2344	1.2321	1.2296	1.2269	1.2240	1.2208	1.2173
.12	1.2375	1.2357	1.2336	1.2314	1.2291	1.2264	1.2235	1.2205	1.2172	1.2135
.14	1.2343	1.2322	1.2301	1.2278	1.2253	1.2226	1.2195	1.2154	1.2129	1.2091
.16	1.2305	1.2284	1.2262	1.2237	1.2211	1.2181	1.2150	1.2117	1.2079	1.2040
.18	1.2262	1.2239	1.2215	1.2190	1.2161	1.2131	1.2097	1.2051	1.2023	1.1980
.20	1.2213	1.2190	1.2164	1.2136	1.2105	1.2073	1.2038	1.2000	1.1958	1.1914
.22	1.2159	1.2134	1.2106	1.2077	1.2044	1.2009	1.1972	1.1931	1.1887	1.1838
.24	1.2100	1.2072	1.2043	1.2010	1.1976	1.1939	1.1898	1.1854	1.1807	1.1755
.26	1.2034	1.2005	1.1972	1.1938	1.1901	1.1860	1.1817	1.1770	1.1719	1.1663
.28	1.1963	1.1930	1.1896	1.1859	1.1818	1.1775	1.1728	1.1677	1.1621	1.1551
.30	1.1885	1.1850	1.1813	1.1772	1.1729	1.1681	1.1631	1.1575	1.1515	1.1450
.32	1.1800	1.1762	1.1722	1.1678	1.1631	1.1580	1.1524	1.1454	1.1399	1.1326
.34	1.1709	1.1669	1.1624	1.1577	1.1525	1.1469	1.1409	1.1343	1.1271	1.1193
.36	1.1611	1.1566	1.1518	1.1467	1.1410	1.1350	1.1283	1.1211	1.1132	1.1044
.38	1.1505	1.1457	1.1404	1.1347	1.1285	1.1219	1.1147	1.1057	1.0978	1.0882
.40	1.1392	1.1338	1.1281	1.1219	1.1151	1.1078	1.0998	1.0908	1.0811	1.0698
.42	1.1269	1.1211	1.1149	1.1080	1.1005	1.0925	1.0835	1.0736	1.0623	1.0490
.44	1.1138	1.1075	1.1005	1.0930	1.0848	1.0757	1.0657	1.0542	1.0410	1.0274
.46	1.0997	1.0927	1.0851	1.0767	1.0675	1.0574	1.0458	1.0322	1.0187	1.0077
.48	1.0845	1.0768	1.0683	1.0590	1.0487	1.0369	1.0234	1.0094	.9984	.9898
.50	1.0682	1.0595	1.0501	1.0397	1.0277	1.0137	.9998	.9887	.9801	.9735
.52	1.0505	1.0409	1.0302	1.0181	1.0040	.9897	.9785	.9700	.9635	.9590
.54	1.0313	1.0205	1.0080	.9935	.9791	.9679	.9594	.9531	.9487	.9457
.56	1.0103	.9976	.9831	.9681	.9568	.9483	.9422	.9380	.9352	.9336
.58	.9869	.9718	.9557	.9453	.9369	.9309	.9268	.9243	.9229	.9225
.60	.9605	.9449	.9333	.9250	.9191	.9153	.9130	.9119	.9118	.9125
.62	.9325	.9209	.9126	.9069	.9033	.9012	.9005	.9007	.9017	.9035
.64	.9080	.8998	.8943	.8909	.8892	.8887	.8892	.8906	.8926	.8952
.66	.8865	.8812	.8781	.8767	.8765	.8774	.8791	.8814	.8843	.8877
.68	.8677	.8649	.8638	.8640	.8653	.8673	.8700	.8732	.8769	.8809
.70	.8513	.8506	.8512	.8528	.8553	.8583	.8619	.8659	.8702	.8749
.72	.8370	.8381	.8401	.8430	.8454	.8504	.8547	.8594	.8643	.8695
.74	.8246	.8271	.8304	.8343	.8387	.8434	.8484	.8537	.8592	.8648
.76	.8139	.8177	.8221	.8269	.8320	.8374	.8430	.8488	.8548	.8608
.78	.8049	.8098	.8150	.8206	.8264	.8324	.8385	.8448	.8511	.8575
.80	.7974	.8032	.8092	.8155	.8219	.8283	.8349	.8415	.8482	.8549
.82	.7915	.7981	.8048	.8116	.8184	.8253	.8323	.8392	.8462	.8531
.84	.7872	.7944	.8017	.8089	.8162	.8234	.8307	.8379	.8451	.8522
.86	.7847	.7924	.8001	.8077	.8153	.8228	.8303	.8377	.8450	.8523
.88	.7841	.7922	.8002	.8081	.8160	.8237	.8313	.8388	.8462	.8536
.90	.7859	.7943	.8026	.8106	.8185	.8263	.8340	.8416	.8490	.8563
.92	.7909	.7994	.8077	.8158	.8237	.8315	.8391	.8456	.8539	.8611
.94	.8002	.8086	.8169	.8248	.8325	.8402	.8475	.8548	.8619	.8688
.96	.8168	.8249	.8327	.8402	.8475	.8546	.8616	.8683	.8749	.8813
.98	.8489	.8559	.8627	.8691	.8754	.8814	.8873	.8930	.8985	.9033

C_o \ a	.30	.31	.32	.33	.34	.35	.36	.37	.38	.39
0.00	1.2222	1.2187	1.2145	1.2102	1.2054	1.2000	1.1945	1.1880	1.1815	1.1737
.02	1.2220	1.2181	1.2141	1.2097	1.2049	1.1996	1.1938	1.1877	1.1807	1.1734
.04	1.2209	1.2172	1.2130	1.2085	1.2037	1.1983	1.1926	1.1862	1.1793	1.1717
.06	1.2192	1.2152	1.2111	1.2065	1.2015	1.1962	1.1902	1.1838	1.1767	1.1690
.08	1.2167	1.2128	1.2084	1.2038	1.1987	1.1931	1.1870	1.1803	1.1732	1.1651
.10	1.2135	1.2094	1.2050	1.2001	1.1949	1.1891	1.1828	1.1750	1.1683	1.1601
.12	1.2095	1.2054	1.2008	1.1957	1.1902	1.1842	1.1777	1.1704	1.1626	1.1538
.14	1.2051	1.2005	1.1957	1.1904	1.1845	1.1784	1.1714	1.1639	1.1555	1.1463
.16	1.1995	1.1950	1.1898	1.1842	1.1782	1.1715	1.1642	1.1561	1.1473	1.1373
.18	1.1935	1.1885	1.1831	1.1771	1.1706	1.1636	1.1557	1.1472	1.1375	1.1269
.20	1.1865	1.1812	1.1754	1.1690	1.1622	1.1545	1.1462	1.1358	1.1264	1.1145
.22	1.1785	1.1729	1.1667	1.1600	1.1525	1.1443	1.1351	1.1250	1.1133	1.0998
.24	1.1699	1.1638	1.1571	1.1497	1.1417	1.1325	1.1227	1.1112	1.0978	1.0840
.26	1.1602	1.1536	1.1463	1.1384	1.1294	1.1196	1.1083	1.0953	1.0815	1.0696
.28	1.1495	1.1423	1.1344	1.1256	1.1158	1.1047	1.0915	1.0782	1.0665	1.0564
.30	1.1377	1.1299	1.1211	1.1114	1.1003	1.0875	1.0740	1.0626	1.0528	1.0445
.32	1.1248	1.1160	1.1064	1.0953	1.0823	1.0692	1.0579	1.0483	1.0403	1.0335
.34	1.1104	1.1008	1.0897	1.0769	1.0637	1.0525	1.0431	1.0354	1.0289	1.0235
.36	1.0947	1.0836	1.0706	1.0575	1.0464	1.0372	1.0297	1.0235	1.0184	1.0143
.38	1.0770	1.0639	1.0508	1.0398	1.0307	1.0234	1.0174	1.0126	1.0088	1.0058
.40	1.0569	1.0435	1.0325	1.0236	1.0164	1.0107	1.0062	1.0026	.9999	.9979
.42	1.0357	1.0248	1.0159	1.0089	1.0034	.9991	.9958	.9934	.9917	.9907
.44	1.0165	1.0077	1.0009	.9955	.9914	.9884	.9863	.9849	.9841	.9839
.46	.9990	.9923	.9871	.9833	.9805	.9785	.9775	.9770	.9771	.9777
.48	.9832	.9782	.9746	.9720	.9704	.9696	.9694	.9698	.9706	.9719
.50	.9688	.9654	.9631	.9617	.9612	.9612	.9619	.9630	.9645	.9665
.52	.9558	.9537	.9526	.9523	.9526	.9535	.9549	.9568	.9589	.9614
.54	.9439	.9430	.9429	.9436	.9448	.9464	.9485	.9510	.9537	.9568
.56	.9330	.9332	.9341	.9355	.9375	.9399	.9425	.9456	.9489	.9524
.58	.9231	.9242	.9260	.9282	.9308	.9338	.9371	.9406	.9444	.9484
.60	.9140	.9160	.9185	.9214	.9247	.9282	.9320	.9360	.9403	.9447
.62	.9058	.9085	.9117	.9152	.9190	.9231	.9273	.9318	.9365	.9412
.64	.8982	.9017	.9055	.9095	.9138	.9184	.9231	.9279	.9329	.9381
.66	.8914	.8955	.8998	.9044	.9091	.9141	.9192	.9244	.9297	.9352
.68	.8853	.8899	.8947	.8997	.9049	.9102	.9155	.9212	.9269	.9325
.70	.8798	.8849	.8901	.8955	.9011	.9067	.9125	.9183	.9243	.9303
.72	.8749	.8804	.8860	.8918	.8977	.9036	.9097	.9158	.9220	.9282
.74	.8705	.8765	.8825	.8886	.8948	.9010	.9073	.9136	.9200	.9264
.76	.8670	.8732	.8795	.8859	.8923	.8987	.9052	.9118	.9183	.9249
.78	.8640	.8705	.8771	.8837	.8903	.8969	.9036	.9103	.9170	.9238
.80	.8617	.8684	.8752	.8820	.8888	.8955	.9024	.9092	.9161	.9229
.82	.8601	.8670	.8740	.8809	.8879	.8948	.9017	.9086	.9156	.9225
.84	.8593	.8664	.8735	.8805	.8876	.8945	.9016	.9085	.9155	.9225
.86	.8595	.8667	.8739	.8810	.8880	.8950	.9021	.9090	.9160	.9229
.88	.8609	.8681	.8752	.8823	.8894	.8964	.9033	.9102	.9171	.9240
.90	.8636	.8708	.8778	.8849	.8918	.8987	.9055	.9124	.9191	.9259
.92	.8682	.8752	.8822	.8890	.8958	.9025	.9091	.9157	.9223	.9288
.94	.8755	.8823	.8890	.8955	.9019	.9083	.9145	.9208	.9270	.9332
.96	.8875	.8938	.8998	.9058	.9117	.9175	.9232	.9289	.9345	.9400
.98	.9091	.9142	.9192	.9241	.9289	.9337	.9384	.9430	.9475	.9520

Co \ α	.40	.41	.42	.43	.44	.45	.46	.47	.48	.49
0,00	1,1659	1,1565	1,1470	1,1356	1,1237	1,1097	1,0944	1,0756	1,0561	1,0323
,02	1,1650	1,1561	1,1450	1,1351	1,1226	1,1089	1,0932	1,0754	1,0546	1,0292
,04	1,1634	1,1542	1,1441	1,1328	1,1203	1,1061	1,0902	1,0716	1,0495	1,0245
,06	1,1604	1,1511	1,1406	1,1291	1,1160	1,1015	1,0346	1,0644	1,0425	1,0209
,08	1,1584	1,1465	1,1359	1,1237	1,1102	1,0946	1,0762	1,0561	1,0367	1,0181
,10	1,1509	1,1408	1,1293	1,1167	1,1021	1,0848	1,0563	1,0486	1,0318	1,0157
,12	1,1442	1,1334	1,1214	1,1076	1,0915	1,0740	1,0576	1,0422	1,0276	1,0136
,14	1,1360	1,1248	1,1113	1,0958	1,0795	1,0642	1,0499	1,0356	1,0240	1,0118
,16	1,1263	1,1136	1,0989	1,0831	1,0687	1,0554	1,0431	1,0316	1,0207	1,0102
,18	1,1146	1,1002	1,0852	1,0715	1,0590	1,0476	1,0370	1,0272	1,0178	1,0088
,20	1,1007	1,0860	1,0728	1,0610	1,0503	1,0406	1,0316	1,0232	1,0152	1,0075
,22	1,0855	1,0729	1,0516	1,0515	1,0425	1,0342	1,0266	1,0195	1,0128	1,0063
,24	1,0717	1,0609	1,0514	1,0429	1,0353	1,0284	1,0221	1,0162	1,0106	1,0052
,26	1,0592	1,0501	1,0421	1,0351	1,0288	1,0231	1,0179	1,0131	1,0086	1,0042
,28	1,0478	1,0402	1,0337	1,0279	1,0228	1,0183	1,0141	1,0103	1,0067	1,0033
,30	1,0373	1,0312	1,0259	1,0213	1,0173	1,0138	1,0105	1,0077	1,0050	1,0025
,32	1,0278	1,0229	1,0188	1,0153	1,0122	1,0096	1,0073	1,0053	1,0034	1,0017
,34	1,0190	1,0153	1,0122	1,0096	1,0075	1,0058	1,0043	1,0030	1,0019	1,0009
,36	1,0109	1,0082	1,0061	1,0044	1,0031	1,0022	1,0014	1,0009	1,0005	1,0002
,38	1,0035	1,0017	1,0004	,9996	,9991	,9988	,9988	,9989	,9992	,9995
,40	,9965	,9956	,9952	,9951	,9953	,9957	,9963	,9971	,9980	,9990
,42	,9901	,9900	,9903	,9909	,9917	,9928	,9940	,9954	,9969	,9984
,44	,9842	,9848	,9857	,9869	,9884	,9900	,9918	,9938	,9958	,9979
,46	,9785	,9799	,9815	,9833	,9853	,9875	,9898	,9923	,9948	,9974
,48	,9735	,9753	,9775	,9799	,9824	,9851	,9879	,9908	,9939	,9969
,50	,9686	,9711	,9738	,9767	,9797	,9829	,9861	,9895	,9930	,9965
,52	,9642	,9671	,9703	,9737	,9772	,9808	,9845	,9883	,9921	,9961
,54	,9600	,9635	,9671	,9709	,9748	,9788	,9829	,9871	,9914	,9957
,56	,9561	,9600	,9641	,9683	,9726	,9770	,9815	,9860	,9906	,9953
,58	,9525	,9569	,9613	,9659	,9705	,9753	,9801	,9850	,9900	,9950
,60	,9492	,9539	,9587	,9636	,9685	,9737	,9788	,9841	,9893	,9946
,62	,9461	,9512	,9553	,9615	,9669	,9722	,9777	,9832	,9888	,9944
,64	,9433	,9487	,9541	,9596	,9652	,9709	,9766	,9824	,9882	,9941
,66	,9407	,9464	,9521	,9579	,9637	,9697	,9756	,9817	,9877	,9938
,68	,9384	,9443	,9503	,9563	,9624	,9685	,9747	,9810	,9873	,9936
,70	,9363	,9425	,9487	,9549	,9612	,9675	,9739	,9804	,9869	,9934
,72	,9345	,9408	,9472	,9537	,9601	,9667	,9732	,9799	,9865	,9932
,74	,9329	,9394	,9460	,9526	,9592	,9659	,9726	,9794	,9862	,9931
,76	,9316	,9383	,9450	,9517	,9585	,9653	,9722	,9790	,9860	,9930
,78	,9305	,9373	,9442	,9510	,9579	,9648	,9718	,9788	,9858	,9929
,80	,9298	,9367	,9436	,9505	,9575	,9645	,9715	,9786	,9857	,9928
,82	,9294	,9364	,9433	,9503	,9573	,9643	,9714	,9785	,9855	,9928
,84	,9294	,9364	,9434	,9504	,9574	,9644	,9714	,9785	,9856	,9928
,86	,9299	,9368	,9438	,9507	,9577	,9647	,9717	,9787	,9857	,9929
,88	,9309	,9378	,9445	,9515	,9583	,9652	,9721	,9790	,9860	,9930
,90	,9326	,9393	,9460	,9527	,9594	,9661	,9728	,9796	,9864	,9932
,92	,9353	,9417	,9482	,9546	,9611	,9675	,9740	,9804	,9869	,9934
,94	,9393	,9454	,9514	,9575	,9635	,9696	,9756	,9817	,9878	,9939
,96	,9456	,9511	,9565	,9620	,9674	,9728	,9782	,9836	,9891	,9945
,98	,9565	,9609	,9653	,9697	,9740	,9784	,9827	,9870	,9913	,9957

C_0 \ α	.50	.51	.52	.53	.54	.55	.56	.57	.58	.59
0.00	1.0000	.9687	.9459	.9288	.9137	.9012	.8899	.8806	.8718	.8547
.02	1.0000	.9716	.9482	.9298	.9148	.9018	.8908	.8810	.8725	.8550
.04	1.0000	.9761	.9531	.9332	.9173	.9041	.8925	.8828	.8740	.8564
.06	1.0000	.9795	.9592	.9395	.9220	.9079	.8960	.8856	.8767	.8587
.08	1.0000	.9823	.9545	.9459	.9292	.9136	.9007	.8899	.8804	.8722
.10	1.0000	.9846	.9592	.9536	.9378	.9217	.9074	.8955	.8855	.8756
.12	1.0000	.9866	.9731	.9595	.9455	.9311	.9164	.9029	.8917	.8823
.14	1.0000	.9883	.9756	.9547	.9525	.9397	.9264	.9126	.8998	.8893
.16	1.0000	.9899	.9797	.9694	.9587	.9475	.9357	.9232	.9100	.8979
.18	1.0000	.9913	.9825	.9735	.9543	.9546	.9443	.9333	.9215	.9089
.20	1.0000	.9926	.9850	.9774	.9694	.9610	.9521	.9425	.9321	.9208
.22	1.0000	.9937	.9874	.9809	.9741	.9669	.9593	.9510	.9420	.9321
.24	1.0000	.9948	.9895	.9841	.9784	.9724	.9559	.9588	.9511	.9426
.26	1.0000	.9958	.9915	.9870	.9824	.9774	.9720	.9651	.9596	.9523
.28	1.0000	.9967	.9933	.9898	.9861	.9821	.9777	.9728	.9674	.9513
.30	1.0000	.9975	.9950	.9924	.9895	.9864	.9830	.9791	.9747	.9697
.32	1.0000	.9983	.9966	.9948	.9927	.9905	.9879	.9850	.9816	.9776
.34	1.0000	.9991	.9981	.9970	.9957	.9943	.9925	.9905	.9880	.9849
.36	1.0000	.9998	.9995	.9991	.9986	.9978	.9969	.9956	.9939	.9918
.38	1.0000	1.0004	1.0008	1.0011	1.0012	1.0012	1.0009	1.0004	.9996	.9983
.40	1.0000	1.0010	1.0020	1.0029	1.0037	1.0043	1.0048	1.0050	1.0049	1.0044
.42	1.0000	1.0016	1.0031	1.0046	1.0060	1.0073	1.0084	1.0092	1.0098	1.0101
.44	1.0000	1.0021	1.0042	1.0063	1.0082	1.0101	1.0117	1.0132	1.0145	1.0155
.46	1.0000	1.0026	1.0052	1.0078	1.0103	1.0127	1.0149	1.0170	1.0189	1.0205
.48	1.0000	1.0031	1.0062	1.0092	1.0122	1.0151	1.0179	1.0206	1.0230	1.0253
.50	1.0000	1.0035	1.0071	1.0105	1.0141	1.0174	1.0207	1.0239	1.0269	1.0298
.52	1.0000	1.0040	1.0079	1.0119	1.0158	1.0196	1.0234	1.0271	1.0306	1.0340
.54	1.0000	1.0044	1.0087	1.0131	1.0174	1.0217	1.0259	1.0300	1.0340	1.0379
.56	1.0000	1.0047	1.0094	1.0142	1.0189	1.0235	1.0282	1.0328	1.0372	1.0415
.58	1.0000	1.0051	1.0101	1.0152	1.0203	1.0254	1.0304	1.0354	1.0403	1.0451
.60	1.0000	1.0054	1.0108	1.0162	1.0215	1.0270	1.0324	1.0378	1.0431	1.0483
.62	1.0000	1.0057	1.0114	1.0171	1.0229	1.0286	1.0343	1.0400	1.0457	1.0513
.64	1.0000	1.0059	1.0119	1.0179	1.0239	1.0300	1.0360	1.0421	1.0481	1.0541
.66	1.0000	1.0062	1.0124	1.0187	1.0250	1.0313	1.0376	1.0440	1.0503	1.0556
.68	1.0000	1.0064	1.0129	1.0194	1.0259	1.0325	1.0391	1.0457	1.0523	1.0590
.70	1.0000	1.0066	1.0133	1.0200	1.0268	1.0335	1.0404	1.0472	1.0541	1.0511
.72	1.0000	1.0068	1.0137	1.0205	1.0275	1.0345	1.0415	1.0486	1.0557	1.0529
.74	1.0000	1.0070	1.0140	1.0210	1.0281	1.0353	1.0425	1.0498	1.0571	1.0545
.76	1.0000	1.0071	1.0142	1.0214	1.0285	1.0360	1.0433	1.0507	1.0582	1.0558
.78	1.0000	1.0072	1.0144	1.0217	1.0290	1.0365	1.0439	1.0515	1.0591	1.0569
.80	1.0000	1.0072	1.0145	1.0219	1.0293	1.0368	1.0444	1.0520	1.0598	1.0575
.82	1.0000	1.0073	1.0146	1.0220	1.0295	1.0370	1.0446	1.0523	1.0601	1.0580
.84	1.0000	1.0073	1.0145	1.0220	1.0294	1.0369	1.0145	1.0522	1.0600	1.0579
.86	1.0000	1.0072	1.0145	1.0218	1.0292	1.0366	1.0442	1.0518	1.0595	1.0574
.88	1.0000	1.0071	1.0142	1.0214	1.0287	1.0360	1.0435	1.0510	1.0586	1.0564
.90	1.0000	1.0069	1.0138	1.0208	1.0279	1.0351	1.0423	1.0496	1.0571	1.0546
.92	1.0000	1.0066	1.0133	1.0200	1.0257	1.0336	1.0405	1.0475	1.0546	1.0519
.94	1.0000	1.0062	1.0124	1.0186	1.0250	1.0314	1.0378	1.0444	1.0510	1.0578
.96	1.0000	1.0055	1.0110	1.0166	1.0223	1.0280	1.0337	1.0395	1.0455	1.0515
.98	1.0000	1.0044	1.0088	1.0132	1.0175	1.0221	1.0267	1.0313	1.0350	1.0407

C_o \ α	.60	.61	.62	.63	.64	.65	.66	.67	.68	.69
0,00	.8577	.8520	.8453	.8417	.8372	.8333	.8296	.8263	.8234	.8205
.02	.8583	.8522	.8469	.8420	.8376	.8336	.8300	.8267	.8236	.8209
.04	.8595	.8535	.8479	.8430	.8385	.8345	.8308	.8274	.8244	.8216
.06	.8618	.8554	.8499	.8447	.8402	.8360	.8323	.8288	.8257	.8229
.08	.8648	.8583	.8524	.8472	.8424	.8382	.8343	.8307	.8275	.8245
.10	.8689	.8620	.8559	.8504	.8455	.8410	.8369	.8332	.8299	.8268
.12	.8739	.8667	.8601	.8544	.8491	.8444	.8402	.8363	.8328	.8296
.14	.8803	.8723	.8654	.8592	.8537	.8486	.8441	.8400	.8363	.8330
.16	.8878	.8793	.8716	.8650	.8590	.8536	.8488	.8444	.8405	.8368
.18	.8972	.8874	.8791	.8717	.8653	.8594	.8542	.8495	.8453	.8414
.20	.9087	.8973	.8878	.8797	.8725	.8662	.8605	.8554	.8508	.8466
.22	.9212	.9092	.8982	.8889	.8810	.8739	.8677	.8621	.8571	.8526
.24	.9331	.9225	.9108	.8999	.8907	.8829	.8759	.8698	.8642	.8593
.26	.9441	.9349	.9246	.9132	.9023	.8932	.8854	.8785	.8724	.8668
.28	.9544	.9466	.9376	.9275	.9161	.9052	.8962	.8884	.8815	.8754
.30	.9640	.9574	.9499	.9411	.9311	.9196	.9088	.8997	.8920	.8850
.32	.9730	.9676	.9613	.9539	.9453	.9353	.9239	.9130	.9038	.8960
.34	.9814	.9770	.9719	.9658	.9587	.9501	.9402	.9287	.9175	.9084
.36	.9892	.9859	.9819	.9770	.9711	.9641	.9556	.9456	.9341	.9228
.38	.9965	.9943	.9913	.9875	.9829	.9771	.9702	.9617	.9517	.9398
.40	1.0035	1.0021	1.0001	.9974	.9939	.9894	.9838	.9769	.9685	.9583
.42	1.0100	1.0094	1.0083	1.0066	1.0042	1.0009	.9966	.9912	.9843	.9758
.44	1.0161	1.0163	1.0151	1.0153	1.0139	1.0117	1.0086	1.0045	.9991	.9924
.46	1.0219	1.0228	1.0234	1.0235	1.0230	1.0218	1.0199	1.0170	1.0131	1.0078
.48	1.0273	1.0290	1.0303	1.0312	1.0316	1.0314	1.0305	1.0288	1.0261	1.0223
.50	1.0324	1.0347	1.0367	1.0384	1.0396	1.0403	1.0404	1.0398	1.0383	1.0358
.52	1.0372	1.0401	1.0428	1.0452	1.0472	1.0487	1.0497	1.0501	1.0498	1.0485
.54	1.0417	1.0452	1.0485	1.0516	1.0543	1.0566	1.0585	1.0598	1.0605	1.0605
.56	1.0459	1.0500	1.0539	1.0575	1.0609	1.0640	1.0667	1.0689	1.0706	1.0716
.58	1.0498	1.0544	1.0589	1.0631	1.0671	1.0709	1.0743	1.0774	1.0800	1.0820
.60	1.0535	1.0586	1.0635	1.0683	1.0730	1.0773	1.0815	1.0853	1.0887	1.0917
.62	1.0569	1.0624	1.0679	1.0732	1.0783	1.0833	1.0881	1.0926	1.0969	1.1007
.64	1.0601	1.0660	1.0719	1.0777	1.0833	1.0889	1.0943	1.0995	1.1044	1.1090
.66	1.0630	1.0693	1.0756	1.0818	1.0879	1.0940	1.0999	1.1057	1.1113	1.1167
.68	1.0656	1.0723	1.0789	1.0855	1.0921	1.0987	1.1051	1.1115	1.1177	1.1238
.70	1.0680	1.0750	1.0819	1.0889	1.0959	1.1029	1.1098	1.1167	1.1235	1.1301
.72	1.0701	1.0774	1.0846	1.0919	1.0993	1.1066	1.1140	1.1213	1.1286	1.1359
.74	1.0719	1.0794	1.0870	1.0946	1.1022	1.1099	1.1176	1.1254	1.1331	1.1409
.76	1.0734	1.0812	1.0889	1.0968	1.1047	1.1127	1.1207	1.1288	1.1370	1.1452
.78	1.0746	1.0825	1.0905	1.0985	1.1067	1.1149	1.1232	1.1317	1.1402	1.1488
.80	1.0755	1.0835	1.0916	1.0998	1.1081	1.1166	1.1251	1.1338	1.1426	1.1515
.82	1.0759	1.0840	1.0922	1.1005	1.1090	1.1176	1.1263	1.1352	1.1442	1.1533
.84	1.0759	1.0840	1.0923	1.1007	1.1092	1.1178	1.1267	1.1357	1.1448	1.1542
.86	1.0754	1.0835	1.0917	1.1001	1.1086	1.1173	1.1261	1.1351	1.1444	1.1538
.88	1.0742	1.0822	1.0903	1.0986	1.1070	1.1156	1.1244	1.1334	1.1426	1.1520
.90	1.0723	1.0800	1.0880	1.0961	1.1043	1.1127	1.1213	1.1301	1.1392	1.1484
.92	1.0692	1.0767	1.0843	1.0920	1.1000	1.1081	1.1164	1.1249	1.1336	1.1425
.94	1.0646	1.0716	1.0787	1.0860	1.0934	1.1010	1.1087	1.1157	1.1249	1.1333
.96	1.0576	1.0638	1.0701	1.0766	1.0832	1.0899	1.0969	1.1040	1.1113	1.1189
.98	1.0455	1.0504	1.0554	1.0605	1.0657	1.0710	1.0765	1.0821	1.0879	1.0939

C_o \ α	.70	.71	.72	.73	.74	.75	.76	.77	.78	.79
0.00	.8182	.8157	.8138	.8118	.8100	.8085	.8068	.8057	.8044	.8031
.02	.8184	.8161	.8140	.8121	.8103	.8087	.8073	.8058	.8046	.8035
.04	.8191	.8167	.8146	.8126	.8109	.8093	.8077	.8054	.8051	.8039
.06	.8202	.8179	.8157	.8137	.8119	.8102	.8087	.8072	.8060	.8048
.08	.8219	.8194	.8172	.8151	.8132	.8115	.8099	.8085	.8072	.8059
.10	.8240	.8215	.8191	.8170	.8151	.8132	.8116	.8101	.8087	.8075
.12	.8267	.8240	.8215	.8193	.8173	.8154	.8136	.8121	.8106	.8093
.14	.8298	.8271	.8244	.8221	.8199	.8179	.8161	.8144	.8129	.8115
.16	.8336	.8306	.8278	.8253	.8230	.8209	.8190	.8172	.8156	.8140
.18	.8379	.8347	.8317	.8291	.8265	.8244	.8223	.8204	.8186	.8170
.20	.8428	.8394	.8362	.8333	.8307	.8283	.8260	.8240	.8221	.8204
.22	.8484	.8447	.8413	.8382	.8353	.8327	.8303	.8280	.8260	.8242
.24	.8548	.8507	.8470	.8436	.8405	.8376	.8350	.8326	.8304	.8283
.26	.8619	.8574	.8533	.8496	.8462	.8431	.8403	.8377	.8353	.8330
.28	.8699	.8650	.8605	.8564	.8527	.8492	.8461	.8433	.8406	.8382
.30	.8789	.8734	.8684	.8639	.8598	.8561	.8526	.8495	.8466	.8439
.32	.8890	.8829	.8773	.8723	.8677	.8636	.8598	.8563	.8531	.8502
.34	.9005	.8935	.8872	.8815	.8765	.8719	.8676	.8638	.8603	.8570
.36	.9135	.9055	.8983	.8920	.8863	.8811	.8764	.8721	.8682	.8646
.38	.9285	.9190	.9109	.9036	.8971	.8913	.8860	.8813	.8769	.8728
.40	.9464	.9347	.9250	.9167	.9093	.9027	.8968	.8913	.8865	.8820
.42	.9655	.9533	.9414	.9314	.9230	.9154	.9086	.9025	.8970	.8920
.44	.9838	.9733	.9606	.9486	.9383	.9297	.9218	.9149	.9087	.9030
.46	1.0010	.9924	.9817	.9688	.9562	.9457	.9368	.9287	.9216	.9152
.48	1.0171	1.0103	1.0016	.9907	.9774	.9644	.9535	.9443	.9361	.9287
.50	1.0322	1.0271	1.0203	1.0115	1.0002	.9865	.9731	.9618	.9523	.9438
.52	1.0463	1.0428	1.0378	1.0310	1.0220	1.0105	.9960	.9823	.9706	.9608
.54	1.0595	1.0574	1.0541	1.0492	1.0423	1.0332	1.0213	1.0065	.9920	.9800
.56	1.0718	1.0711	1.0693	1.0661	1.0613	1.0545	1.0452	1.0329	1.0175	1.0024
.58	1.0833	1.0839	1.0835	1.0819	1.0789	1.0742	1.0674	1.0579	1.0452	1.0291
.60	1.0941	1.0958	1.0967	1.0966	1.0953	1.0926	1.0880	1.0811	1.0715	1.0584
.62	1.1040	1.1068	1.1090	1.1103	1.1105	1.1096	1.1070	1.1026	1.0958	1.0859
.64	1.1133	1.1171	1.1203	1.1229	1.1245	1.1253	1.1247	1.1224	1.1182	1.1114
.66	1.1218	1.1265	1.1308	1.1345	1.1375	1.1397	1.1409	1.1407	1.1388	1.1348
.68	1.1296	1.1352	1.1404	1.1452	1.1494	1.1530	1.1557	1.1574	1.1576	1.1562
.70	1.1367	1.1430	1.1491	1.1549	1.1602	1.1651	1.1592	1.1726	1.1748	1.1756
.72	1.1430	1.1501	1.1569	1.1636	1.1700	1.1760	1.1815	1.1853	1.1903	1.1932
.74	1.1486	1.1563	1.1639	1.1714	1.1787	1.1857	1.1924	1.1986	1.2042	1.2090
.76	1.1534	1.1617	1.1699	1.1781	1.1862	1.1942	1.2019	1.2094	1.2164	1.2229
.78	1.1574	1.1662	1.1750	1.1838	1.1926	1.2014	1.2101	1.2186	1.2270	1.2349
.80	1.1605	1.1697	1.1789	1.1883	1.1977	1.2072	1.2168	1.2263	1.2357	1.2450
.82	1.1627	1.1721	1.1818	1.1916	1.2015	1.2116	1.2219	1.2322	1.2426	1.2530
.84	1.1637	1.1734	1.1833	1.1935	1.2039	1.2144	1.2252	1.2352	1.2474	1.2588
.86	1.1634	1.1733	1.1834	1.1938	1.2044	1.2154	1.2266	1.2381	1.2499	1.2620
.88	1.1616	1.1715	1.1817	1.1922	1.2030	1.2141	1.2256	1.2374	1.2496	1.2623
.90	1.1580	1.1677	1.1778	1.1883	1.1990	1.2101	1.2217	1.2336	1.2460	1.2589
.92	1.1518	1.1613	1.1711	1.1812	1.1918	1.2027	1.2140	1.2258	1.2381	1.2510
.94	1.1420	1.1510	1.1603	1.1699	1.1799	1.1902	1.2011	1.2124	1.2242	1.2366
.96	1.1266	1.1347	1.1430	1.1517	1.1607	1.1701	1.1799	1.1901	1.2009	1.2123
.98	1.1000	1.1064	1.1130	1.1199	1.1271	1.1345	1.1424	1.1506	1.1592	1.1684

C_o \ a	.80	.81	.82	.83	.84	.85	.86	.87	.88	.89
0.00	.8023	.8012	.8003	.7998	.7989	.7981	.7977	.7974	.7967	.7960
.02	.8024	.8016	.8007	.7999	.7992	.7985	.7980	.7974	.7970	.7967
.04	.8029	.8019	.8011	.8003	.7995	.7989	.7983	.7979	.7973	.7968
.06	.8037	.8028	.8019	.8010	.8003	.7997	.7991	.7985	.7980	.7977
.08	.8049	.8038	.8029	.8021	.8013	.8005	.7999	.7994	.7989	.7983
.10	.8063	.8053	.8043	.8034	.8026	.8019	.8012	.8006	.8001	.7997
.12	.8081	.8069	.8059	.8050	.8041	.8033	.8027	.8020	.8014	.8009
.14	.8102	.8090	.8080	.8069	.8061	.8053	.8045	.8038	.8032	.8027
.16	.8127	.8114	.8102	.8092	.8082	.8073	.8065	.8058	.8051	.8044
.18	.8155	.8142	.8130	.8118	.8108	.8098	.8089	.8081	.8074	.8067
.20	.8188	.8173	.8159	.8147	.8135	.8125	.8115	.8107	.8099	.8091
.22	.8224	.8208	.8194	.8180	.8168	.8155	.8145	.8136	.8127	.8119
.24	.8265	.8247	.8231	.8217	.8203	.8190	.8179	.8158	.8158	.8149
.26	.8310	.8291	.8273	.8257	.8242	.8229	.8215	.8204	.8193	.8183
.28	.8359	.8339	.8320	.8302	.8285	.8271	.8257	.8244	.8232	.8221
.30	.8414	.8391	.8370	.8351	.8333	.8315	.8301	.8287	.8273	.8251
.32	.8474	.8449	.8425	.8405	.8385	.8367	.8350	.8334	.8319	.8306
.34	.8540	.8512	.8487	.8463	.8441	.8421	.8402	.8385	.8368	.8353
.36	.8612	.8582	.8554	.8527	.8503	.8481	.8460	.8440	.8423	.8406
.38	.8692	.8657	.8625	.8597	.8570	.8545	.8522	.8501	.8481	.8462
.40	.8778	.8741	.8705	.8673	.8544	.8615	.8590	.8567	.8545	.8524
.42	.8874	.8831	.8792	.8757	.8723	.8693	.8564	.8638	.8613	.8590
.44	.8978	.8931	.8887	.8847	.8810	.8776	.8744	.8715	.8688	.8662
.46	.9093	.9040	.8991	.8946	.8905	.8867	.8832	.8799	.8768	.8740
.48	.9221	.9160	.9105	.9055	.9008	.8965	.8926	.8890	.8855	.8823
.50	.9361	.9293	.9231	.9174	.9122	.9074	.9029	.8988	.8951	.8916
.52	.9520	.9440	.9359	.9305	.9245	.9192	.9142	.9096	.9054	.9014
.54	.9697	.9606	.9523	.9450	.9383	.9322	.9265	.9214	.9156	.9122
.56	.9898	.9791	.9597	.9611	.9534	.9465	.9401	.9343	.9289	.9240
.58	1.0133	1.0002	.9890	.9792	.9703	.9623	.9551	.9485	.9424	.9367
.60	1.0414	1.0249	1.0112	.9995	.9893	.9800	.9716	.9641	.9572	.9509
.62	1.0723	1.0545	1.0372	1.0228	1.0105	.9999	.9902	.9815	.9736	.9664
.64	1.1013	1.0872	1.0584	1.0502	1.0351	1.0222	1.0110	1.0009	.9918	.9835
.66	1.1281	1.1178	1.1031	1.0830	1.0640	1.0481	1.0345	1.0228	1.0122	1.0027
.68	1.1525	1.1459	1.1355	1.1202	1.0991	1.0787	1.0520	1.0476	1.0352	1.0241
.70	1.1747	1.1713	1.1549	1.1544	1.1384	1.1160	1.0942	1.0756	1.0615	1.0483
.72	1.1947	1.1943	1.1915	1.1853	1.1747	1.1581	1.1342	1.1109	1.0922	1.0762
.74	1.2127	1.2149	1.2153	1.2130	1.2073	1.1967	1.1793	1.1533	1.1288	1.1089
.76	1.2285	1.2332	1.2354	1.2376	1.2362	1.2309	1.2204	1.2024	1.1748	1.1479
.78	1.2424	1.2491	1.2548	1.2592	1.2515	1.2610	1.2565	1.2452	1.2275	1.1977
.80	1.2540	1.2626	1.2705	1.2776	1.2832	1.2869	1.2875	1.2842	1.2744	1.2551
.82	1.2634	1.2736	1.2835	1.2928	1.3013	1.3086	1.3139	1.3153	1.3143	1.3054
.84	1.2703	1.2818	1.2933	1.3047	1.3157	1.3261	1.3353	1.3427	1.3472	1.3471
.86	1.2744	1.2870	1.2999	1.3129	1.3250	1.3389	1.3514	1.3630	1.3731	1.3804
.88	1.2753	1.2888	1.3027	1.3170	1.3317	1.3467	1.3518	1.3769	1.3914	1.4048
.90	1.2724	1.2863	1.3009	1.3161	1.3320	1.3485	1.3556	1.3833	1.4014	1.4195
.92	1.2645	1.2786	1.2934	1.3091	1.3255	1.3430	1.3514	1.3809	1.4014	1.4229
.94	1.2497	1.2635	1.2781	1.2935	1.3099	1.3275	1.3463	1.3665	1.3882	1.4117
.96	1.2243	1.2370	1.2505	1.2649	1.2803	1.2970	1.3149	1.3345	1.3558	1.3793
.98	1.1780	1.1883	1.1993	1.2111	1.2238	1.2376	1.2526	1.2690	1.2871	1.3074

APPENDIX 4. Tables of the $g_2(a, c_0)$ - functions.

c_0 \ a	.10	.11	.12	.13	.14	.15	.16	.17	.18	.19
0,00	.9891	.9865	.9836	.9804	.9768	.9729	.9585	.9638	.9586	.9530
.02	.9891	.9865	.9836	.9804	.9768	.9728	.9585	.9637	.9586	.9530
.04	.9890	.9864	.9836	.9803	.9767	.9727	.9584	.9636	.9584	.9528
.06	.9890	.9864	.9835	.9802	.9766	.9726	.9582	.9634	.9582	.9525
.08	.9889	.9863	.9833	.9801	.9764	.9724	.9580	.9631	.9579	.9521
.10	.9888	.9862	.9832	.9799	.9762	.9721	.9576	.9628	.9574	.9516
.12	.9887	.9860	.9830	.9796	.9759	.9718	.9572	.9623	.9569	.9510
.14	.9885	.9858	.9828	.9794	.9756	.9714	.9568	.9617	.9562	.9502
.16	.9883	.9856	.9825	.9790	.9752	.9709	.9562	.9610	.9554	.9493
.18	.9881	.9853	.9822	.9786	.9747	.9703	.9555	.9603	.9545	.9483
.20	.9879	.9851	.9818	.9782	.9742	.9697	.9548	.9594	.9535	.9471
.22	.9876	.9847	.9814	.9777	.9735	.9690	.9539	.9583	.9523	.9457
.24	.9873	.9843	.9809	.9771	.9728	.9681	.9529	.9572	.9509	.9441
.26	.9870	.9839	.9804	.9765	.9721	.9672	.9518	.9558	.9494	.9423
.28	.9866	.9834	.9798	.9757	.9712	.9661	.9505	.9543	.9476	.9402
.30	.9862	.9829	.9791	.9749	.9701	.9649	.9591	.9526	.9456	.9379
.32	.9857	.9823	.9784	.9740	.9690	.9635	.9574	.9507	.9433	.9353
.34	.9852	.9816	.9775	.9729	.9677	.9620	.9556	.9485	.9408	.9323
.36	.9846	.9808	.9765	.9717	.9663	.9602	.9535	.9461	.9379	.9289
.38	.9839	.9799	.9755	.9704	.9646	.9583	.9511	.9433	.9346	.9251
.40	.9831	.9790	.9742	.9688	.9628	.9560	.9485	.9401	.9308	.9206
.42	.9822	.9779	.9728	.9671	.9607	.9534	.9454	.9364	.9265	.9156
.44	.9813	.9766	.9712	.9652	.9583	.9505	.9419	.9322	.9216	.9098
.46	.9801	.9752	.9694	.9629	.9555	.9472	.9378	.9274	.9158	.9030
.48	.9789	.9735	.9674	.9603	.9523	.9433	.9331	.9218	.9092	.8952
.50	.9774	.9716	.9650	.9573	.9486	.9388	.9277	.9153	.9014	.8860
.52	.9757	.9694	.9622	.9538	.9443	.9335	.9213	.9076	.8923	.8752
.54	.9737	.9669	.9589	.9497	.9392	.9273	.9137	.8985	.8814	.8623
.56	.9714	.9639	.9551	.9449	.9332	.9199	.9047	.8876	.8684	.8469
.58	.9687	.9603	.9505	.9391	.9260	.9110	.8939	.8745	.8526	.8281
.60	.9654	.9560	.9450	.9322	.9173	.9002	.8807	.8585	.8334	.8052
.62	.9615	.9509	.9384	.9237	.9068	.8870	.8544	.8387	.8095	.7768
.64	.9567	.9446	.9302	.9133	.8935	.8705	.8441	.8139	.7797	.7538
.66	.9508	.9367	.9200	.9001	.8768	.8497	.8184	.7827	.7557	.7379
.68	.9434	.9268	.9069	.8833	.8555	.8229	.7852	.7572	.7390	.7259
.70	.9338	.9140	.8901	.8614	.8275	.7880	.7585	.7398	.7266	.7165
.72	.9214	.8972	.8878	.8323	.7903	.7596	.7404	.7270	.7169	.7090
.74	.9047	.8744	.8374	.7929	.7605	.7407	.7271	.7170	.7092	.7030
.76	.8816	.8428	.7950	.7610	.7407	.7269	.7168	.7091	.7031	.6982
.78	.8485	.7975	.7612	.7403	.7263	.7163	.7088	.7029	.6982	.6944
.80	.7996	.7609	.7395	.7254	.7155	.7081	.7024	.6979	.6943	.6913
.82	.7603	.7382	.7241	.7143	.7072	.7017	.6974	.6940	.6911	.6888
.84	.7364	.7224	.7128	.7059	.7008	.6967	.6934	.6908	.6887	.6868
.86	.7201	.7109	.7043	.6995	.6957	.6928	.6904	.6884	.6868	.6854
.88	.7085	.7024	.6979	.6945	.6919	.6897	.6880	.6866	.6854	.6843
.90	.7001	.6961	.6931	.6908	.6889	.6875	.6863	.6853	.6844	.6837
.92	.6940	.6915	.6896	.6881	.6869	.6860	.6852	.6846	.6840	.6836
.94	.6897	.6882	.6871	.6863	.6857	.6851	.6847	.6844	.6841	.6838
.96	.6872	.6866	.6861	.6858	.6855	.6853	.6852	.6851	.6850	.6850
.98	.6869	.6868	.6868	.6869	.6870	.6870	.6871	.6872	.6873	.6874

Co \ α	.20	.21	.22	.23	.24	.25	.26	.27	.28	.29
0.00	.9470	.9405	.9335	.9260	.9180	.9095	.9004	.8908	.8807	.8700
.02	.9469	.9404	.9334	.9259	.9179	.9094	.9003	.8907	.8805	.8698
.04	.9467	.9402	.9332	.9256	.9176	.9090	.8999	.8903	.8801	.8693
.06	.9464	.9398	.9327	.9252	.9171	.9084	.8993	.8895	.8792	.8684
.08	.9460	.9393	.9322	.9245	.9163	.9076	.8983	.8885	.8781	.8671
.10	.9454	.9386	.9314	.9235	.9153	.9065	.8971	.8871	.8765	.8654
.12	.9447	.9378	.9304	.9226	.9141	.9051	.8956	.8854	.8747	.8633
.14	.9438	.9368	.9293	.9213	.9127	.9035	.8937	.8833	.8724	.8608
.16	.9427	.9356	.9279	.9197	.9109	.9015	.8915	.8809	.8696	.8577
.18	.9415	.9342	.9264	.9179	.9089	.8992	.8890	.8780	.8665	.8542
.20	.9401	.9326	.9245	.9158	.9065	.8966	.8860	.8747	.8628	.8501
.22	.9385	.9308	.9224	.9134	.9038	.8935	.8826	.8709	.8585	.8454
.24	.9367	.9287	.9200	.9107	.9007	.8901	.8787	.8665	.8537	.8400
.26	.9346	.9263	.9173	.9076	.8972	.8861	.8742	.8615	.8481	.8338
.28	.9322	.9235	.9142	.9041	.8932	.8816	.8691	.8558	.8417	.8268
.30	.9295	.9205	.9106	.9000	.8885	.8764	.8633	.8493	.8345	.8187
.32	.9265	.9169	.9056	.8954	.8834	.8705	.8567	.8419	.8252	.8095
.34	.9230	.9130	.9020	.8902	.8775	.8638	.8491	.8335	.8158	.7990
.36	.9191	.9084	.8958	.8843	.8707	.8561	.8405	.8238	.8059	.7869
.38	.9145	.9032	.8909	.8775	.8630	.8474	.8306	.8127	.7935	.7731
.40	.9095	.8973	.8840	.8696	.8541	.8373	.8192	.7999	.7792	.7572
.42	.9036	.8905	.8752	.8606	.8438	.8256	.8060	.7851	.7626	.7388
.44	.8968	.8826	.8671	.8502	.8319	.8121	.7908	.7679	.7436	.7241
.46	.8889	.8735	.8555	.8381	.8180	.7963	.7730	.7479	.7281	.7137
.48	.8798	.8628	.8442	.8239	.8018	.7779	.7523	.7318	.7172	.7056
.50	.8690	.8502	.8296	.8072	.7827	.7562	.7354	.7205	.7088	.6993
.52	.8563	.8354	.8124	.7873	.7602	.7387	.7235	.7116	.7021	.6943
.54	.8411	.8177	.7919	.7637	.7417	.7262	.7143	.7046	.6968	.6902
.56	.8229	.7964	.7574	.7446	.7288	.7167	.7069	.6991	.6925	.6870
.58	.8008	.7705	.7472	.7311	.7188	.7090	.7011	.6946	.6890	.6844
.60	.7739	.7496	.7331	.7207	.7108	.7029	.6964	.6909	.6863	.6823
.62	.7519	.7350	.7224	.7124	.7045	.6980	.6926	.6880	.6840	.6806
.64	.7365	.7238	.7138	.7059	.6994	.6940	.6895	.6856	.6822	.6793
.66	.7250	.7149	.7070	.7006	.6952	.6907	.6869	.6837	.6808	.6783
.68	.7158	.7079	.7015	.6962	.6918	.6881	.6849	.6822	.6797	.6776
.70	.7085	.7022	.6971	.6927	.6891	.6860	.6833	.6810	.6789	.6771
.72	.7027	.6977	.6934	.6899	.6869	.6843	.6820	.6800	.6783	.6768
.74	.6981	.6939	.6905	.6876	.6851	.6829	.6811	.6794	.6780	.6767
.76	.6943	.6909	.6882	.6858	.6837	.6819	.6803	.6790	.6777	.6767
.78	.6912	.6885	.6863	.6843	.6826	.6811	.6798	.6787	.6777	.6768
.80	.6887	.6866	.6848	.6832	.6818	.6806	.6796	.6786	.6778	.6771
.82	.6868	.6851	.6835	.6824	.6813	.6803	.6795	.6787	.6780	.6775
.84	.6853	.6840	.6828	.6819	.6810	.6802	.6796	.6790	.6784	.6780
.86	.6842	.6832	.6823	.6816	.6809	.6803	.6798	.6794	.6790	.6786
.88	.6835	.6828	.6821	.6816	.6811	.6806	.6803	.6799	.6796	.6794
.90	.6831	.6826	.6822	.6818	.6815	.6812	.6809	.6807	.6805	.6803
.92	.6832	.6829	.6826	.6824	.6822	.6820	.6819	.6818	.6816	.6816
.94	.6837	.6838	.6834	.6833	.6832	.6831	.6831	.6830	.6831	.6830
.96	.6849	.6849	.6849	.6849	.6849	.6850	.6850	.6851	.6851	.6851
.98	.6875	.6876	.6877	.6878	.6879	.6880	.6881	.6881	.6883	.6883

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0.00	.8587	.8469	.8345	.8215	.8080	.7938	.7792	.7640	.7482	.7320
.02	.8585	.8467	.8342	.8212	.8077	.7935	.7788	.7636	.7479	.7316
.04	.8579	.8460	.8335	.8205	.8068	.7925	.7778	.7625	.7467	.7303
.06	.8569	.8449	.8323	.8191	.8054	.7910	.7761	.7607	.7447	.7282
.08	.8555	.8433	.8306	.8173	.8033	.7888	.7737	.7581	.7419	.7251
.10	.8537	.8413	.8284	.8148	.8005	.7859	.7706	.7547	.7382	.7212
.12	.8513	.8388	.8256	.8117	.7973	.7823	.7666	.7504	.7336	.7162
.14	.8485	.8357	.8222	.8080	.7933	.7779	.7618	.7452	.7280	.7102
.16	.8452	.8320	.8181	.8036	.7885	.7726	.7562	.7391	.7214	.7031
.18	.8413	.8277	.8134	.7985	.7828	.7665	.7495	.7319	.7136	.6948
.20	.8368	.8227	.8080	.7925	.7763	.7594	.7418	.7236	.7045	.6851
.22	.8315	.8170	.8016	.7856	.7687	.7512	.7329	.7139	.6943	.6741
.24	.8256	.8104	.7944	.7776	.7601	.7418	.7227	.7029	.6824	.6658
.26	.8187	.8028	.7851	.7685	.7502	.7310	.7110	.6903	.6735	.6608
.28	.8109	.7942	.7756	.7582	.7388	.7186	.6975	.6807	.6679	.6574
.30	.8020	.7844	.7658	.7453	.7259	.7047	.6874	.6745	.6639	.6552
.32	.7918	.7731	.7534	.7328	.7111	.6937	.6806	.6700	.6612	.6538
.34	.7802	.7603	.7393	.7174	.6996	.6864	.6755	.6657	.6593	.6529
.36	.7668	.7455	.7231	.7051	.6917	.6809	.6719	.6644	.6580	.6526
.38	.7515	.7286	.7103	.6967	.6858	.6768	.6692	.6628	.6573	.6525
.40	.7340	.7152	.7014	.6904	.6813	.6737	.6672	.6617	.6569	.6528
.42	.7198	.7058	.6945	.6854	.6778	.6713	.6658	.6610	.6568	.6532
.44	.7099	.6986	.6893	.6817	.6751	.6696	.6648	.6606	.6570	.6538
.46	.7022	.6929	.6852	.6787	.6731	.6683	.6642	.6605	.6573	.6545
.48	.6962	.6885	.6819	.6764	.6716	.6674	.6638	.6606	.6578	.6553
.50	.6915	.6849	.6794	.6746	.6705	.6668	.6637	.6609	.6584	.6562
.52	.6877	.6821	.6774	.6733	.6697	.6665	.6637	.6613	.6591	.6572
.54	.6847	.6799	.6758	.6723	.6691	.6664	.6640	.6618	.6599	.6582
.56	.6823	.6782	.6747	.6716	.6689	.6664	.6643	.6624	.6607	.6592
.58	.6803	.6768	.6738	.6711	.6687	.6666	.6648	.6631	.6616	.6603
.60	.6788	.6758	.6732	.6709	.6688	.6670	.6653	.6639	.6625	.6614
.62	.6777	.6751	.6728	.6708	.6690	.6674	.6659	.6646	.6635	.6625
.64	.6768	.6746	.6725	.6708	.6693	.6679	.6666	.6655	.6645	.6636
.66	.6762	.6743	.6725	.6710	.6697	.6685	.6674	.6664	.6655	.6647
.68	.6758	.6741	.6727	.6713	.6702	.6691	.6681	.6673	.6665	.6658
.70	.6755	.6741	.6728	.6717	.6707	.6698	.6690	.6682	.6676	.6669
.72	.6754	.6742	.6732	.6722	.6713	.6705	.6698	.6692	.6685	.6681
.74	.6755	.6745	.6735	.6728	.6720	.6713	.6707	.6702	.6697	.6693
.76	.6757	.6748	.6741	.6734	.6728	.6722	.6717	.6712	.6708	.6704
.78	.6760	.6753	.6746	.6740	.6735	.6731	.6725	.6723	.6719	.6716
.80	.6764	.6758	.6753	.6748	.6744	.6740	.6737	.6733	.6731	.6728
.82	.6769	.6764	.6760	.6756	.6753	.6750	.6747	.6745	.6742	.6740
.84	.6775	.6772	.6768	.6765	.6763	.6760	.6758	.6756	.6755	.6753
.86	.6783	.6780	.6778	.6775	.6773	.6772	.6770	.6769	.6767	.6766
.88	.6792	.6790	.6788	.6785	.6785	.6784	.6783	.6782	.6781	.6780
.90	.6802	.6801	.6800	.6799	.6798	.6797	.6797	.6796	.6795	.6796
.92	.6815	.6814	.6814	.6813	.6813	.6813	.6813	.6813	.6813	.6813
.94	.6830	.6830	.6830	.6830	.6830	.6831	.6831	.6831	.6831	.6832
.96	.6852	.6852	.6853	.6853	.6854	.6854	.6855	.6855	.6856	.6857
.98	.6884	.6885	.6886	.6887	.6887	.6889	.6889	.6890	.6891	.6892

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0.00	.7153	.6981	.6804	.6624	.6439	.6252	.6061	.5867	.5672	.5474
.02	.7148	.6976	.6799	.6618	.6434	.6245	.6054	.5860	.5664	.5469
.04	.7134	.6961	.6783	.6602	.6416	.6227	.6034	.5840	.5644	.5503
.06	.7112	.6937	.6757	.6574	.6386	.6195	.6001	.5805	.5658	.5562
.08	.7079	.6902	.6720	.6534	.6344	.6151	.5955	.5803	.5698	.5622
.10	.7037	.6856	.6672	.6482	.6289	.6093	.5939	.5828	.5744	.5682
.12	.6984	.6800	.6611	.6418	.6222	.6065	.5951	.5861	.5792	.5739
.14	.6919	.6731	.6538	.6340	.6183	.6066	.5972	.5899	.5840	.5795
.16	.6843	.6649	.6452	.6292	.6173	.6077	.6000	.5937	.5887	.5848
.18	.6754	.6554	.6393	.6272	.6174	.6095	.6030	.5976	.5933	.5898
.20	.6651	.6488	.6363	.6266	.6184	.6117	.6061	.6015	.5977	.5947
.22	.6576	.6452	.6351	.6258	.6199	.6141	.6093	.6052	.6019	.5992
.24	.6533	.6431	.6347	.6275	.6217	.6167	.6124	.6089	.6059	.6036
.26	.6505	.6420	.6348	.6288	.6235	.6192	.6155	.6124	.6098	.6077
.28	.6488	.6416	.6354	.6302	.6257	.6219	.6186	.6158	.6135	.6115
.30	.6479	.6417	.6363	.6318	.6278	.6244	.6216	.6191	.6170	.6153
.32	.6475	.6421	.6375	.6335	.6300	.6270	.6244	.6223	.6204	.6189
.34	.6475	.6428	.6388	.6352	.6322	.6295	.6272	.6253	.6235	.6222
.36	.6478	.6437	.6402	.6371	.6343	.6320	.6299	.6282	.6267	.6254
.38	.6484	.6448	.6416	.6389	.6365	.6344	.6325	.6310	.6296	.6285
.40	.6491	.6460	.6432	.6407	.6385	.6367	.6351	.6336	.6324	.6314
.42	.6500	.6472	.6447	.6425	.6405	.6390	.6375	.6362	.6351	.6342
.44	.6510	.6485	.6463	.6443	.6425	.6411	.6398	.6387	.6377	.6369
.46	.6520	.6498	.6479	.6461	.6445	.6433	.6421	.6411	.6402	.6394
.48	.6531	.6512	.6494	.6479	.6465	.6453	.6443	.6433	.6425	.6419
.50	.6543	.6525	.6510	.6496	.6484	.6473	.6464	.6455	.6448	.6442
.52	.6554	.6539	.6525	.6513	.6502	.6492	.6484	.6477	.6470	.6465
.54	.6566	.6553	.6540	.6530	.6520	.6511	.6504	.6497	.6491	.6486
.56	.6579	.6566	.6555	.6546	.6537	.6530	.6523	.6517	.6512	.6507
.58	.6591	.6580	.6570	.6562	.6554	.6547	.6541	.6536	.6531	.6527
.60	.6603	.6593	.6585	.6577	.6570	.6564	.6559	.6554	.6550	.6547
.62	.6615	.6607	.6599	.6593	.6585	.6581	.6576	.6572	.6569	.6566
.64	.6627	.6620	.6613	.6607	.6602	.6597	.6593	.6589	.6586	.6584
.66	.6640	.6633	.6627	.6622	.6617	.6613	.6610	.6606	.6604	.6601
.68	.6652	.6646	.6641	.6637	.6633	.6629	.6626	.6623	.6621	.6619
.70	.6664	.6659	.6655	.6651	.6647	.6644	.6641	.6639	.6637	.6635
.72	.6675	.6672	.6668	.6665	.6662	.6659	.6657	.6655	.6653	.6652
.74	.6689	.6685	.6682	.6679	.6676	.6674	.6672	.6670	.6669	.6668
.76	.6701	.6698	.6695	.6693	.6691	.6689	.6687	.6686	.6685	.6684
.78	.6713	.6711	.6709	.6707	.6705	.6703	.6702	.6701	.6700	.6699
.80	.6725	.6724	.6722	.6720	.6719	.6718	.6717	.6716	.6715	.6715
.82	.6739	.6737	.6736	.6734	.6733	.6733	.6732	.6731	.6731	.6730
.84	.6752	.6751	.6750	.6749	.6748	.6747	.6747	.6747	.6746	.6746
.86	.6765	.6765	.6764	.6763	.6763	.6763	.6763	.6762	.6762	.6762
.88	.6780	.6779	.6779	.6779	.6779	.6779	.6779	.6779	.6779	.6779
.90	.6795	.6795	.6795	.6795	.6795	.6795	.6796	.6796	.6796	.6796
.92	.6813	.6813	.6813	.6814	.6814	.6814	.6814	.6815	.6815	.6816
.94	.6832	.6833	.6833	.6833	.6834	.6834	.6835	.6835	.6836	.6836
.96	.6857	.6858	.6858	.6859	.6860	.6860	.6861	.6862	.6862	.6863
.98	.6892	.6893	.6894	.6894	.6895	.6896	.6897	.6897	.6898	.6898

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0.00	.5275	.5326	.5419	.5522	.5636	.5758	.5882	.6012	.6140	.6272
.02	.5347	.5337	.5418	.5520	.5635	.5755	.5881	.6008	.6138	.6268
.04	.5426	.5393	.5422	.5517	.5628	.5748	.5872	.6000	.6129	.6260
.06	.5499	.5469	.5468	.5517	.5620	.5736	.5859	.5986	.6115	.6245
.08	.5570	.5541	.5534	.5553	.5612	.5721	.5841	.5967	.6095	.6226
.10	.5637	.5611	.5601	.5610	.5640	.5707	.5820	.5942	.6071	.6200
.12	.5701	.5677	.5666	.5668	.5688	.5726	.5799	.5915	.6040	.6170
.14	.5762	.5739	.5728	.5728	.5739	.5766	.5810	.5888	.6007	.6133
.16	.5818	.5798	.5787	.5785	.5792	.5811	.5843	.5891	.5972	.6094
.18	.5872	.5854	.5843	.5840	.5844	.5858	.5881	.5917	.5970	.6053
.20	.5923	.5907	.5895	.5892	.5895	.5904	.5922	.5950	.5989	.6044
.22	.5971	.5956	.5945	.5942	.5943	.5950	.5964	.5985	.6016	.6058
.24	.6017	.6003	.5994	.5989	.5989	.5995	.6005	.6023	.6047	.6079
.26	.6060	.6047	.6039	.6034	.6034	.6037	.6045	.6060	.6079	.6105
.28	.6101	.6089	.6081	.6077	.6075	.6079	.6085	.6096	.6112	.6134
.30	.6139	.6129	.6122	.6117	.6116	.6118	.6123	.6132	.6145	.6163
.32	.6175	.6167	.6160	.6156	.6154	.6155	.6160	.6167	.6178	.6193
.34	.6211	.6202	.6195	.6192	.6190	.6191	.6195	.6201	.6210	.6222
.36	.6244	.6236	.6230	.6227	.6225	.6225	.6228	.6233	.6241	.6252
.38	.6275	.6268	.6263	.6260	.6258	.6258	.6260	.6265	.6271	.6280
.40	.6305	.6299	.6294	.6291	.6289	.6289	.6291	.6295	.6300	.6308
.42	.6334	.6329	.6324	.6321	.6319	.6319	.6321	.6324	.6328	.6335
.44	.6362	.6356	.6352	.6349	.6348	.6348	.6349	.6351	.6355	.6361
.46	.6388	.6383	.6379	.6377	.6375	.6375	.6375	.6378	.6382	.6387
.48	.6413	.6409	.6405	.6403	.6401	.6401	.6402	.6404	.6407	.6411
.50	.6437	.6433	.6430	.6428	.6426	.6425	.6427	.6428	.6431	.6435
.52	.6460	.6456	.6454	.6452	.6450	.6450	.6451	.6452	.6454	.6458
.54	.6482	.6479	.6476	.6475	.6474	.6473	.6474	.6475	.6477	.6480
.56	.6504	.6501	.6498	.6497	.6495	.6496	.6496	.6497	.6499	.6501
.58	.6524	.6521	.6519	.6518	.6517	.6517	.6517	.6518	.6520	.6522
.60	.6544	.6541	.6540	.6538	.6538	.6537	.6538	.6539	.6540	.6542
.62	.6563	.6561	.6559	.6558	.6558	.6558	.6558	.6559	.6560	.6562
.64	.6581	.6580	.6578	.6577	.6577	.6577	.6577	.6578	.6579	.6581
.66	.6599	.6598	.6597	.6596	.6595	.6595	.6596	.6596	.6598	.6599
.68	.6617	.6616	.6615	.6614	.6614	.6614	.6614	.6615	.6616	.6617
.70	.6634	.6633	.6632	.6631	.6631	.6631	.6632	.6632	.6633	.6635
.72	.6651	.6650	.6649	.6649	.6648	.6649	.6649	.6650	.6651	.6652
.74	.6667	.6666	.6666	.6665	.6665	.6666	.6666	.6667	.6668	.6669
.76	.6683	.6682	.6682	.6682	.6682	.6682	.6683	.6683	.6684	.6685
.78	.6699	.6698	.6698	.6698	.6698	.6699	.6699	.6700	.6701	.6702
.80	.6714	.6714	.6714	.6714	.6714	.6715	.6715	.6716	.6717	.6718
.82	.6730	.6730	.6730	.6730	.6731	.6731	.6732	.6732	.6733	.6734
.84	.6746	.6746	.6747	.6747	.6747	.6748	.6748	.6749	.6750	.6751
.86	.6763	.6763	.6763	.6763	.6764	.6764	.6765	.6766	.6766	.6767
.88	.6779	.6779	.6780	.6780	.6781	.6781	.6782	.6783	.6784	.6784
.90	.6797	.6797	.6798	.6798	.6799	.6799	.6800	.6801	.6801	.6802
.92	.6816	.6817	.6817	.6818	.6818	.6819	.6820	.6821	.6821	.6822
.94	.6837	.6838	.6839	.6839	.6840	.6840	.6841	.6842	.6843	.6843
.96	.6863	.6864	.6865	.6866	.6866	.6867	.6867	.6868	.6869	.6870
.98	.6899	.6899	.6900	.6901	.6902	.6902	.6903	.6903	.6904	.6904

C_o \ a	.60	.61	.62	.63	.64	.65	.66	.67	.68	.69
0.00	.6400	.6532	.6657	.6785	.6907	.7029	.7146	.7260	.7371	.7476
.02	.6399	.6528	.6656	.6782	.6906	.7026	.7144	.7258	.7369	.7475
.04	.6390	.6520	.6648	.6775	.6898	.7020	.7138	.7252	.7364	.7470
.06	.6377	.6506	.6635	.6762	.6887	.7008	.7127	.7243	.7354	.7463
.08	.6355	.6487	.6616	.6744	.6870	.6993	.7112	.7229	.7342	.7450
.10	.6332	.6462	.6593	.6721	.6848	.6972	.7093	.7211	.7325	.7435
.12	.6300	.6432	.6563	.6693	.6821	.6947	.7069	.7189	.7304	.7416
.14	.6264	.6396	.6528	.6659	.6789	.6916	.7041	.7162	.7279	.7393
.16	.6222	.6355	.6487	.6620	.6751	.6880	.7007	.7130	.7250	.7366
.18	.6177	.6307	.6441	.6574	.6708	.6839	.6968	.7093	.7216	.7334
.20	.6131	.6256	.6388	.6523	.6657	.6791	.6922	.7051	.7176	.7297
.22	.6116	.6204	.6332	.6466	.6602	.6737	.6872	.7003	.7131	.7256
.24	.6124	.6184	.6274	.6404	.6540	.6678	.6814	.6949	.7080	.7208
.26	.6140	.6187	.6248	.6341	.6473	.6611	.6750	.6887	.7023	.7155
.28	.6162	.6199	.6247	.6310	.6404	.6539	.6679	.6820	.6958	.7094
.30	.6186	.6216	.6254	.6304	.6368	.6465	.6602	.6744	.6887	.7026
.32	.6212	.6237	.6268	.6307	.6358	.6424	.6522	.6662	.6807	.6951
.34	.6239	.6259	.6285	.6317	.6358	.6409	.6477	.6577	.6720	.6867
.36	.6265	.6283	.6304	.6331	.6364	.6406	.6458	.6527	.6629	.6776
.38	.6292	.6307	.6325	.6347	.6375	.6409	.6451	.6505	.6575	.6679
.40	.6318	.6331	.6346	.6365	.6389	.6417	.6452	.6495	.6550	.6621
.42	.6344	.6354	.6368	.6384	.6404	.6428	.6457	.6492	.6536	.6592
.44	.6369	.6378	.6390	.6404	.6421	.6441	.6465	.6495	.6531	.6576
.46	.6393	.6401	.6411	.6423	.6438	.6455	.6476	.6501	.6531	.6568
.48	.6417	.6424	.6433	.6443	.6456	.6471	.6489	.6510	.6535	.6566
.50	.6440	.6446	.6454	.6463	.6474	.6487	.6502	.6520	.6542	.6568
.52	.6462	.6468	.6474	.6482	.6492	.6503	.6517	.6532	.6551	.6572
.54	.6484	.6489	.6494	.6502	.6510	.6520	.6531	.6545	.6561	.6580
.56	.6505	.6509	.6514	.6521	.6528	.6537	.6547	.6558	.6572	.6588
.58	.6525	.6529	.6534	.6539	.6545	.6553	.6562	.6572	.6584	.6598
.60	.6545	.6548	.6552	.6557	.6563	.6570	.6577	.6587	.6597	.6609
.62	.6564	.6567	.6571	.6575	.6580	.6586	.6593	.6601	.6610	.6620
.64	.6583	.6586	.6589	.6593	.6597	.6602	.6608	.6615	.6623	.6632
.66	.6601	.6604	.6606	.6610	.6614	.6619	.6624	.6630	.6637	.6645
.68	.6619	.6621	.6624	.6627	.6630	.6634	.6639	.6645	.6651	.6658
.70	.6636	.6638	.6641	.6643	.6647	.6650	.6654	.6659	.6665	.6671
.72	.6653	.6655	.6657	.6660	.6662	.6666	.6669	.6674	.6678	.6684
.74	.6670	.6672	.6674	.6676	.6678	.6681	.6685	.6688	.6693	.6697
.76	.6687	.6688	.6690	.6692	.6694	.6697	.6700	.6703	.6707	.6711
.78	.6703	.6704	.6706	.6708	.6710	.6712	.6715	.6718	.6721	.6724
.80	.6719	.6720	.6722	.6723	.6725	.6727	.6730	.6732	.6735	.6738
.82	.6735	.6736	.6738	.6739	.6741	.6743	.6745	.6747	.6750	.6752
.84	.6752	.6753	.6754	.6755	.6757	.6759	.6760	.6762	.6765	.6767
.86	.6768	.6769	.6770	.6772	.6773	.6775	.6776	.6778	.6780	.6782
.88	.6785	.6786	.6787	.6789	.6790	.6791	.6793	.6794	.6796	.6798
.90	.6803	.6804	.6805	.6806	.6807	.6808	.6810	.6811	.6813	.6814
.92	.6823	.6824	.6825	.6826	.6827	.6828	.6829	.6830	.6831	.6833
.94	.6844	.6845	.6846	.6847	.6847	.6849	.6849	.6851	.6851	.6853
.96	.6870	.6871	.6872	.6872	.6873	.6874	.6875	.6876	.6877	.6877
.98	.6905	.6906	.6906	.6907	.6907	.6908	.6908	.6910	.6910	.6911

Co \ a	.70	.71	.72	.73	.74	.75	.76	.77	.78	.79
0.00	.7581	.7677	.7773	.7862	.7948	.8031	.8105	.8181	.8248	.8311
.02	.7578	.7678	.7772	.7862	.7948	.8029	.8107	.8180	.8248	.8313
.04	.7574	.7673	.7768	.7858	.7945	.8027	.8104	.8178	.8246	.8310
.06	.7566	.7667	.7762	.7853	.7940	.8022	.8101	.8174	.8243	.8309
.08	.7556	.7656	.7753	.7845	.7932	.8016	.8094	.8169	.8239	.8304
.10	.7541	.7644	.7741	.7834	.7923	.8007	.8087	.8162	.8233	.8300
.12	.7524	.7627	.7727	.7821	.7911	.7997	.8077	.8154	.8225	.8293
.14	.7503	.7608	.7709	.7805	.7897	.7984	.8066	.8144	.8217	.8286
.16	.7478	.7585	.7688	.7786	.7880	.7969	.8052	.8132	.8205	.8276
.18	.7448	.7559	.7664	.7764	.7860	.7951	.8037	.8118	.8194	.8266
.20	.7415	.7527	.7636	.7739	.7837	.7930	.8018	.8102	.8180	.8253
.22	.7376	.7492	.7603	.7710	.7811	.7907	.7998	.8083	.8164	.8239
.24	.7332	.7452	.7567	.7676	.7781	.7880	.7974	.8062	.8145	.8223
.26	.7283	.7406	.7525	.7639	.7747	.7850	.7947	.8038	.8124	.8205
.28	.7227	.7355	.7478	.7596	.7708	.7815	.7916	.8011	.8100	.8184
.30	.7164	.7296	.7424	.7547	.7664	.7776	.7881	.7980	.8073	.8160
.32	.7093	.7231	.7364	.7492	.7615	.7731	.7841	.7945	.8042	.8134
.34	.7014	.7157	.7295	.7430	.7558	.7680	.7796	.7905	.8007	.8103
.36	.6925	.7075	.7219	.7360	.7495	.7623	.7745	.7859	.7967	.8068
.38	.6830	.6982	.7134	.7281	.7422	.7558	.7686	.7808	.7922	.8028
.40	.6728	.6881	.7037	.7192	.7341	.7484	.7620	.7749	.7870	.7983
.42	.6664	.6774	.6932	.7091	.7249	.7400	.7544	.7682	.7810	.7931
.44	.6632	.6706	.6818	.6980	.7144	.7304	.7458	.7604	.7742	.7871
.46	.6613	.6670	.6746	.6860	.7028	.7195	.7359	.7516	.7664	.7803
.48	.6603	.6649	.6707	.6784	.6901	.7074	.7245	.7414	.7573	.7723
.50	.6599	.6636	.6683	.6741	.6820	.6941	.7119	.7296	.7468	.7630
.52	.6598	.6630	.6667	.6715	.6774	.6854	.6979	.7164	.7346	.7522
.54	.6601	.6628	.6659	.6697	.6745	.6806	.6887	.7015	.7208	.7396
.56	.6607	.6629	.6655	.6687	.6723	.6774	.6836	.6919	.7051	.7251
.58	.6614	.6633	.6655	.6682	.6714	.6752	.6801	.6854	.6949	.7086
.60	.6623	.6639	.6658	.6680	.6707	.6739	.6778	.6827	.6891	.6978
.62	.6633	.6646	.6663	.6682	.6704	.6730	.6762	.6802	.6852	.6916
.64	.6643	.6655	.6669	.6685	.6704	.6725	.6753	.6785	.6824	.6875
.66	.6654	.6664	.6677	.6690	.6706	.6725	.6747	.6774	.6806	.6846
.68	.6666	.6675	.6685	.6697	.6710	.6727	.6745	.6767	.6793	.6825
.70	.6678	.6685	.6694	.6705	.6715	.6730	.6745	.6764	.6786	.6812
.72	.6690	.6697	.6704	.6713	.6723	.6735	.6748	.6763	.6781	.6803
.74	.6703	.6708	.6715	.6723	.6731	.6741	.6752	.6765	.6780	.6798
.76	.6715	.6721	.6726	.6733	.6740	.6749	.6758	.6769	.6782	.6796
.78	.6728	.6733	.6738	.6744	.6750	.6757	.6765	.6774	.6785	.6797
.80	.6742	.6746	.6750	.6755	.6760	.6766	.6773	.6781	.6790	.6800
.82	.6755	.6759	.6763	.6767	.6771	.6776	.6782	.6789	.6796	.6805
.84	.6770	.6773	.6776	.6779	.6783	.6788	.6792	.6798	.6804	.6811
.86	.6784	.6787	.6790	.6793	.6795	.6799	.6804	.6808	.6813	.6819
.88	.6800	.6802	.6804	.6807	.6809	.6812	.6816	.6819	.6824	.6828
.90	.6815	.6818	.6820	.6822	.6824	.6827	.6829	.6832	.6836	.6839
.92	.6834	.6836	.6837	.6839	.6841	.6843	.6845	.6847	.6850	.6853
.94	.6854	.6855	.6856	.6857	.6859	.6861	.6862	.6864	.6866	.6868
.96	.6878	.6879	.6880	.6882	.6882	.6884	.6885	.6886	.6888	.6889
.98	.6911	.6912	.6913	.6913	.6914	.6915	.6915	.6917	.6917	.6918

Co \ a	.80	.81	.82	.83	.84	.85	.86	.87	.88	.89
0.00	.8374	.8428	.8479	.8530	.8573	.8611	.8651	.8686	.8714	.8738
.02	.8373	.8429	.8482	.8529	.8574	.8616	.8652	.8685	.8717	.8745
.04	.8371	.8427	.8479	.8528	.8573	.8613	.8651	.8685	.8715	.8742
.06	.8369	.8426	.8479	.8527	.8572	.8614	.8651	.8684	.8716	.8744
.08	.8365	.8422	.8475	.8525	.8570	.8611	.8649	.8684	.8715	.8742
.10	.8361	.8419	.8473	.8522	.8568	.8611	.8648	.8683	.8715	.8743
.12	.8355	.8414	.8468	.8519	.8565	.8607	.8646	.8682	.8713	.8741
.14	.8349	.8409	.8464	.8515	.8562	.8606	.8645	.8680	.8713	.8742
.16	.8341	.8402	.8458	.8511	.8558	.8602	.8642	.8679	.8711	.8740
.18	.8332	.8395	.8452	.8505	.8554	.8599	.8640	.8676	.8710	.8740
.20	.8322	.8385	.8444	.8499	.8549	.8594	.8635	.8674	.8708	.8738
.22	.8310	.8375	.8435	.8491	.8543	.8590	.8632	.8671	.8706	.8737
.24	.8295	.8363	.8425	.8483	.8535	.8584	.8628	.8668	.8704	.8735
.26	.8280	.8350	.8414	.8474	.8528	.8578	.8623	.8664	.8701	.8734
.28	.8262	.8334	.8401	.8463	.8519	.8571	.8618	.8660	.8698	.8732
.30	.8242	.8317	.8385	.8451	.8509	.8563	.8611	.8655	.8695	.8729
.32	.8218	.8297	.8370	.8437	.8498	.8554	.8604	.8650	.8691	.8727
.34	.8192	.8274	.8350	.8421	.8485	.8543	.8595	.8643	.8686	.8723
.36	.8162	.8249	.8329	.8403	.8470	.8531	.8587	.8636	.8681	.8721
.38	.8128	.8219	.8304	.8382	.8453	.8517	.8575	.8628	.8674	.8715
.40	.8088	.8186	.8275	.8358	.8434	.8502	.8563	.8618	.8668	.8711
.42	.8044	.8147	.8243	.8331	.8411	.8483	.8549	.8607	.8659	.8705
.44	.7992	.8103	.8205	.8300	.8385	.8462	.8532	.8594	.8650	.8698
.46	.7932	.8052	.8153	.8263	.8355	.8438	.8513	.8579	.8638	.8691
.48	.7863	.7992	.8112	.8221	.8320	.8409	.8490	.8562	.8625	.8680
.50	.7782	.7923	.8053	.8171	.8279	.8377	.8464	.8541	.8610	.8670
.52	.7688	.7841	.7983	.8113	.8231	.8337	.8432	.8517	.8591	.8656
.54	.7577	.7746	.7901	.8044	.8174	.8291	.8395	.8488	.8569	.8640
.56	.7445	.7632	.7805	.7962	.8105	.8235	.8351	.8453	.8542	.8621
.58	.7294	.7496	.7687	.7864	.8024	.8168	.8297	.8411	.8510	.8596
.60	.7120	.7338	.7543	.7744	.7925	.8087	.8231	.8359	.8471	.8568
.62	.7075	.7153	.7381	.7500	.7803	.7987	.8151	.8296	.8422	.8532
.64	.6940	.7032	.7185	.7425	.7653	.7862	.8051	.8217	.8351	.8486
.66	.6895	.6963	.7057	.7215	.7470	.7709	.7925	.8117	.8285	.8429
.68	.6865	.6917	.6984	.7031	.7247	.7516	.7765	.7990	.8186	.8355
.70	.6843	.6884	.6935	.7004	.7104	.7273	.7564	.7826	.8058	.8258
.72	.6829	.6861	.6901	.6952	.7022	.7125	.7308	.7613	.7890	.8130
.74	.6820	.6845	.6877	.6917	.6958	.7039	.7145	.7338	.7666	.7959
.76	.6814	.6835	.6850	.6891	.6931	.6982	.7055	.7164	.7368	.7722
.78	.6811	.6828	.6843	.6873	.6904	.6943	.6995	.7058	.7181	.7398
.80	.6812	.6826	.6842	.6851	.6885	.6916	.6954	.7006	.7080	.7196
.82	.6814	.6825	.6839	.6854	.6873	.6895	.6925	.6963	.7015	.7090
.84	.6819	.6828	.6839	.6851	.6865	.6884	.6905	.6934	.6971	.7022
.86	.6825	.6832	.6841	.6851	.6862	.6875	.6893	.6914	.6941	.6977
.88	.6833	.6839	.6845	.6854	.6863	.6873	.6885	.6902	.6921	.6947
.90	.6843	.6848	.6853	.6859	.6865	.6874	.6883	.6895	.6909	.6927
.92	.6855	.6860	.6853	.6868	.6873	.6879	.6886	.6894	.6905	.6917
.94	.6871	.6873	.6875	.6879	.6883	.6887	.6892	.6898	.6904	.6912
.96	.6891	.6893	.6894	.6897	.6899	.6902	.6905	.6908	.6913	.6917
.98	.6919	.6920	.6921	.6923	.6923	.6925	.6927	.6928	.6930	.6932

APPENDIX 5. Tables of the $g_4(a, c_0)$ - functions.

c_0 \ a	.10	.11	.12	.13	.14	.15	.16	.17	.18	.19
0.00	.9893	.9869	.9841	.9811	.9778	.9743	.9704	.9662	.9618	.9571
.02	.9893	.9868	.9841	.9811	.9778	.9742	.9703	.9662	.9617	.9570
.04	.9893	.9868	.9841	.9811	.9777	.9741	.9703	.9661	.9616	.9569
.06	.9892	.9867	.9840	.9810	.9776	.9740	.9701	.9659	.9614	.9567
.08	.9891	.9866	.9839	.9808	.9775	.9738	.9699	.9657	.9612	.9564
.10	.9890	.9865	.9837	.9807	.9773	.9736	.9696	.9653	.9608	.9559
.12	.9889	.9864	.9836	.9804	.9770	.9733	.9693	.9649	.9603	.9554
.14	.9888	.9862	.9833	.9802	.9767	.9729	.9689	.9645	.9598	.9548
.16	.9886	.9860	.9831	.9799	.9764	.9725	.9684	.9639	.9592	.9541
.18	.9884	.9858	.9828	.9795	.9759	.9720	.9678	.9633	.9584	.9533
.20	.9882	.9855	.9825	.9791	.9755	.9715	.9671	.9625	.9576	.9523
.22	.9879	.9852	.9821	.9787	.9749	.9708	.9664	.9617	.9566	.9512
.24	.9877	.9848	.9817	.9781	.9743	.9701	.9656	.9607	.9555	.9500
.26	.9873	.9844	.9812	.9776	.9736	.9693	.9646	.9596	.9543	.9486
.28	.9870	.9840	.9806	.9769	.9728	.9684	.9636	.9584	.9529	.9471
.30	.9866	.9835	.9800	.9762	.9719	.9673	.9624	.9570	.9513	.9453
.32	.9861	.9829	.9793	.9753	.9709	.9662	.9610	.9555	.9496	.9434
.34	.9856	.9823	.9785	.9744	.9698	.9649	.9595	.9538	.9477	.9412
.36	.9851	.9816	.9777	.9733	.9686	.9634	.9578	.9519	.9455	.9388
.38	.9844	.9808	.9767	.9722	.9672	.9618	.9560	.9497	.9431	.9361
.40	.9837	.9799	.9756	.9709	.9656	.9600	.9539	.9473	.9404	.9331
.42	.9829	.9789	.9744	.9694	.9639	.9579	.9515	.9446	.9374	.9297
.44	.9820	.9778	.9730	.9677	.9619	.9556	.9488	.9416	.9340	.9260
.46	.9810	.9765	.9714	.9658	.9597	.9530	.9458	.9382	.9302	.9218
.48	.9798	.9750	.9697	.9637	.9571	.9501	.9425	.9344	.9260	.9172
.50	.9785	.9734	.9676	.9612	.9543	.9467	.9386	.9301	.9212	.9121
.52	.9770	.9715	.9653	.9585	.9510	.9429	.9343	.9253	.9160	.9065
.54	.9753	.9693	.9626	.9553	.9472	.9386	.9294	.9199	.9101	.9003
.56	.9732	.9668	.9596	.9516	.9429	.9337	.9239	.9138	.9037	.8937
.58	.9709	.9639	.9560	.9473	.9380	.9280	.9177	.9071	.8967	.8867
.60	.9682	.9604	.9518	.9424	.9323	.9216	.9107	.8997	.8893	.8797
.62	.9649	.9564	.9470	.9367	.9257	.9143	.9029	.8919	.8818	.8733
.64	.9611	.9516	.9412	.9300	.9182	.9062	.8945	.8838	.8749	.8682
.66	.9565	.9459	.9344	.9222	.9096	.8973	.8859	.8754	.8694	.8640
.68	.9509	.9391	.9255	.9133	.9002	.8880	.8779	.8705	.8649	.8605
.70	.9441	.9310	.9171	.9032	.8901	.8793	.8716	.8658	.8612	.8575
.72	.9358	.9213	.9054	.8923	.8807	.8725	.8665	.8618	.8581	.8550
.74	.9257	.9098	.8946	.8820	.8734	.8671	.8623	.8585	.8555	.8529
.76	.9136	.8970	.8833	.8741	.8676	.8627	.8588	.8558	.8533	.8512
.78	.8997	.8846	.8747	.8679	.8629	.8590	.8559	.8535	.8514	.8497
.80	.8858	.8752	.8681	.8629	.8590	.8560	.8536	.8516	.8499	.8485
.82	.8755	.8680	.8628	.8589	.8559	.8535	.8516	.8500	.8487	.8476
.84	.8678	.8624	.8585	.8556	.8533	.8515	.8500	.8487	.8477	.8468
.86	.8618	.8580	.8551	.8529	.8512	.8498	.8487	.8477	.8470	.8463
.88	.8572	.8545	.8524	.8508	.8495	.8486	.8477	.8470	.8464	.8459
.90	.8535	.8518	.8503	.8492	.8483	.8476	.8470	.8466	.8461	.8458
.92	.8509	.8497	.8488	.8481	.8475	.8470	.8467	.8464	.8461	.8459
.94	.8490	.8483	.8478	.8474	.8471	.8469	.8467	.8465	.8464	.8463
.96	.8480	.8477	.8475	.8474	.8473	.8472	.8472	.8472	.8471	.8471
.98	.8482	.8482	.8483	.8484	.8485	.8485	.8486	.8487	.8488	.8488

Co \ a	.20	.21	.22	.23	.24	.25	.26	.27	.28	.29
0.00	.9521	.9469	.9414	.9357	.9298	.9236	.9174	.9110	.9044	.8978
.02	.9520	.9468	.9413	.9356	.9297	.9235	.9173	.9109	.9043	.8977
.04	.9519	.9466	.9411	.9354	.9295	.9233	.9170	.9106	.9041	.8974
.06	.9516	.9464	.9408	.9351	.9291	.9229	.9166	.9102	.9036	.8969
.08	.9513	.9460	.9404	.9346	.9285	.9224	.9160	.9095	.9029	.8963
.10	.9508	.9455	.9398	.9340	.9279	.9217	.9153	.9087	.9021	.8954
.12	.9503	.9448	.9391	.9332	.9271	.9208	.9143	.9077	.9010	.8943
.14	.9495	.9441	.9383	.9323	.9261	.9197	.9132	.9065	.8998	.8930
.16	.9488	.9432	.9373	.9313	.9250	.9185	.9119	.9051	.8983	.8914
.18	.9478	.9422	.9362	.9300	.9236	.9171	.9104	.9035	.8966	.8897
.20	.9468	.9410	.9349	.9286	.9221	.9154	.9085	.9017	.8947	.8877
.22	.9455	.9396	.9334	.9270	.9204	.9136	.9067	.8996	.8926	.8855
.24	.9442	.9381	.9318	.9252	.9184	.9115	.9045	.8974	.8902	.8831
.26	.9427	.9364	.9299	.9232	.9163	.9092	.9020	.8948	.8876	.8804
.28	.9409	.9345	.9278	.9209	.9139	.9066	.8993	.8920	.8847	.8775
.30	.9390	.9324	.9255	.9184	.9112	.9038	.8964	.8890	.8816	.8744
.32	.9368	.9300	.9229	.9157	.9082	.9007	.8932	.8856	.8782	.8710
.34	.9344	.9274	.9201	.9126	.9050	.8973	.8895	.8821	.8747	.8676
.36	.9318	.9244	.9169	.9092	.9014	.8935	.8859	.8783	.8709	.8640
.38	.9288	.9212	.9134	.9055	.8975	.8896	.8818	.8743	.8671	.8604
.40	.9255	.9176	.9096	.9015	.8933	.8853	.8776	.8701	.8632	.8570
.42	.9218	.9137	.9054	.8971	.8888	.8808	.8731	.8660	.8596	.8540
.44	.9177	.9093	.9008	.8923	.8840	.8761	.8687	.8620	.8563	.8515
.46	.9132	.9045	.8958	.8873	.8790	.8713	.8644	.8584	.8536	.8495
.48	.9083	.8993	.8905	.8820	.8739	.8667	.8605	.8555	.8513	.8478
.50	.9029	.8937	.8849	.8765	.8690	.8626	.8574	.8531	.8495	.8464
.52	.8970	.8878	.8791	.8712	.8645	.8592	.8548	.8511	.8479	.8452
.54	.8907	.8816	.8734	.8664	.8609	.8564	.8526	.8493	.8466	.8442
.56	.8841	.8755	.8682	.8625	.8579	.8540	.8507	.8479	.8455	.8433
.58	.8775	.8700	.8640	.8593	.8553	.8520	.8491	.8467	.8445	.8427
.60	.8717	.8655	.8606	.8565	.8531	.8503	.8478	.8456	.8438	.8421
.62	.8669	.8618	.8577	.8542	.8513	.8488	.8467	.8448	.8432	.8417
.64	.8629	.8587	.8552	.8522	.8497	.8476	.8457	.8441	.8427	.8414
.66	.8596	.8561	.8531	.8506	.8484	.8466	.8450	.8435	.8423	.8412
.68	.8569	.8539	.8513	.8492	.8473	.8457	.8443	.8431	.8420	.8410
.70	.8545	.8520	.8498	.8480	.8464	.8450	.8438	.8428	.8418	.8410
.72	.8525	.8504	.8486	.8470	.8457	.8445	.8435	.8426	.8417	.8410
.74	.8508	.8491	.8475	.8462	.8451	.8441	.8432	.8424	.8417	.8411
.76	.8494	.8480	.8467	.8456	.8446	.8438	.8430	.8424	.8418	.8413
.78	.8483	.8471	.8460	.8451	.8443	.8436	.8430	.8424	.8419	.8415
.80	.8474	.8464	.8455	.8447	.8441	.8435	.8430	.8425	.8421	.8418
.82	.8466	.8458	.8451	.8445	.8440	.8435	.8431	.8427	.8424	.8421
.84	.8461	.8454	.8449	.8444	.8440	.8436	.8433	.8430	.8427	.8425
.86	.8457	.8452	.8448	.8444	.8441	.8438	.8436	.8433	.8432	.8430
.88	.8455	.8452	.8448	.8446	.8443	.8441	.8440	.8438	.8437	.8435
.90	.8455	.8453	.8451	.8449	.8447	.8446	.8445	.8444	.8443	.8442
.92	.8457	.8456	.8455	.8454	.8453	.8452	.8452	.8451	.8451	.8451
.94	.8462	.8462	.8461	.8461	.8461	.8461	.8461	.8461	.8461	.8461
.96	.8472	.8472	.8472	.8472	.8473	.8473	.8473	.8474	.8474	.8475
.98	.8489	.8490	.8491	.8492	.8492	.8493	.8494	.8494	.8495	.8496

Co \ α	.30	.31	.32	.33	.34	.35	.36	.37	.38	.39
0.00	.8911	.8845	.8778	.8711	.8545	.8581	.8517	.8456	.8395	.8339
.02	.8910	.8843	.8777	.8710	.8644	.8579	.8515	.8455	.8395	.8335
.04	.8907	.8840	.8773	.8707	.8641	.8576	.8513	.8452	.8392	.8335
.06	.8902	.8835	.8768	.8701	.8636	.8571	.8508	.8447	.8388	.8331
.08	.8895	.8828	.8760	.8694	.8628	.8563	.8501	.8440	.8381	.8325
.10	.8886	.8818	.8751	.8684	.8618	.8554	.8491	.8431	.8373	.8318
.12	.8875	.8807	.8739	.8672	.8607	.8542	.8480	.8421	.8364	.8310
.14	.8861	.8793	.8725	.8658	.8593	.8529	.8468	.8409	.8353	.8301
.16	.8845	.8777	.8709	.8642	.8577	.8514	.8453	.8396	.8341	.8291
.18	.8827	.8759	.8691	.8624	.8559	.8497	.8438	.8381	.8329	.8282
.20	.8807	.8738	.8670	.8604	.8540	.8479	.8421	.8357	.8317	.8273
.22	.8785	.8716	.8648	.8582	.8519	.8459	.8404	.8352	.8305	.8256
.24	.8760	.8691	.8624	.8559	.8497	.8440	.8387	.8339	.8297	.8252
.26	.8733	.8664	.8598	.8534	.8475	.8420	.8371	.8328	.8291	.8259
.28	.8704	.8636	.8571	.8510	.8453	.8402	.8357	.8320	.8287	.8258
.30	.8674	.8606	.8543	.8485	.8432	.8385	.8347	.8314	.8284	.8259
.32	.8641	.8576	.8516	.8461	.8414	.8374	.8340	.8310	.8283	.8250
.34	.8608	.8546	.8490	.8441	.8400	.8365	.8334	.8307	.8283	.8252
.36	.8575	.8517	.8467	.8425	.8389	.8357	.8330	.8305	.8284	.8255
.38	.8544	.8492	.8449	.8412	.8380	.8352	.8327	.8305	.8285	.8255
.40	.8517	.8472	.8434	.8401	.8373	.8348	.8325	.8306	.8288	.8272
.42	.8494	.8455	.8422	.8393	.8367	.8345	.8325	.8307	.8291	.8277
.44	.8476	.8441	.8412	.8385	.8363	.8343	.8325	.8309	.8295	.8282
.46	.8460	.8430	.8404	.8381	.8360	.8342	.8325	.8311	.8299	.8287
.48	.8447	.8421	.8397	.8377	.8358	.8342	.8327	.8314	.8303	.8292
.50	.8437	.8413	.8392	.8374	.8357	.8343	.8330	.8318	.8307	.8298
.52	.8428	.8407	.8388	.8372	.8357	.8344	.8332	.8322	.8312	.8304
.54	.8420	.8402	.8385	.8371	.8358	.8346	.8335	.8326	.8317	.8310
.56	.8415	.8398	.8384	.8370	.8359	.8348	.8339	.8330	.8323	.8316
.58	.8410	.8396	.8382	.8371	.8350	.8351	.8343	.8335	.8328	.8322
.60	.8407	.8394	.8382	.8372	.8363	.8354	.8347	.8340	.8334	.8328
.62	.8404	.8393	.8383	.8373	.8365	.8358	.8351	.8345	.8340	.8335
.64	.8403	.8393	.8384	.8375	.8368	.8362	.8355	.8351	.8346	.8341
.66	.8402	.8393	.8385	.8378	.8372	.8366	.8361	.8356	.8352	.8348
.68	.8402	.8394	.8387	.8381	.8375	.8370	.8366	.8362	.8358	.8355
.70	.8402	.8396	.8390	.8384	.8379	.8375	.8371	.8368	.8364	.8351
.72	.8404	.8398	.8393	.8388	.8384	.8380	.8377	.8374	.8371	.8353
.74	.8406	.8401	.8396	.8392	.8389	.8385	.8382	.8380	.8377	.8375
.76	.8408	.8404	.8400	.8397	.8393	.8391	.8388	.8386	.8384	.8382
.78	.8411	.8407	.8404	.8401	.8399	.8397	.8394	.8393	.8391	.8389
.80	.8414	.8411	.8409	.8407	.8404	.8403	.8401	.8399	.8398	.8397
.82	.8418	.8416	.8414	.8412	.8410	.8409	.8408	.8407	.8405	.8405
.84	.8423	.8421	.8420	.8418	.8417	.8416	.8415	.8414	.8413	.8413
.86	.8428	.8427	.8426	.8425	.8424	.8423	.8423	.8422	.8422	.8421
.88	.8435	.8434	.8433	.8432	.8432	.8431	.8431	.8431	.8430	.8430
.90	.8442	.8441	.8441	.8441	.8440	.8440	.8440	.8440	.8440	.8440
.92	.8450	.8450	.8450	.8450	.8450	.8450	.8450	.8451	.8451	.8451
.94	.8461	.8461	.8451	.8462	.8462	.8462	.8462	.8453	.8463	.8463
.96	.8475	.8476	.8475	.8477	.8477	.8477	.8478	.8478	.8479	.8479
.98	.8496	.8497	.8497	.8498	.8499	.8499	.8500	.8500	.8501	.8501

Co \ α	.40	.41	.42	.43	.44	.45	.46	.47	.48	.49
0.00	.8285	.8233	.8185	.8141	.8100	.8063	.8031	.8003	.7979	.7960
.02	.8284	.8233	.8185	.8140	.8100	.8063	.8031	.8003	.7980	.7961
.04	.8282	.8231	.8183	.8139	.8100	.8064	.8032	.8005	.7983	.7966
.06	.8278	.8228	.8181	.8138	.8099	.8065	.8035	.8009	.7989	.7972
.08	.8273	.8224	.8178	.8137	.8099	.8067	.8039	.8015	.7996	.7981
.10	.8267	.8219	.8175	.8135	.8100	.8070	.8044	.8023	.8005	.7991
.12	.8260	.8214	.8172	.8135	.8102	.8075	.8052	.8032	.8015	.8002
.14	.8253	.8209	.8169	.8135	.8106	.8081	.8060	.8042	.8025	.8014
.16	.8245	.8204	.8168	.8138	.8111	.8089	.8069	.8052	.8038	.8027
.18	.8239	.8201	.8169	.8142	.8119	.8097	.8079	.8054	.8051	.8040
.20	.8234	.8201	.8172	.8147	.8125	.8106	.8089	.8075	.8063	.8053
.22	.8232	.8202	.8176	.8153	.8133	.8115	.8100	.8087	.8076	.8067
.24	.8231	.8204	.8180	.8159	.8141	.8125	.8111	.8099	.8089	.8080
.26	.8231	.8207	.8186	.8167	.8150	.8135	.8123	.8111	.8102	.8094
.28	.8233	.8211	.8191	.8174	.8159	.8146	.8134	.8124	.8115	.8108
.30	.8236	.8216	.8198	.8182	.8168	.8156	.8145	.8136	.8128	.8121
.32	.8239	.8221	.8205	.8190	.8178	.8167	.8157	.8148	.8141	.8135
.34	.8243	.8227	.8212	.8199	.8187	.8177	.8168	.8160	.8153	.8148
.36	.8248	.8233	.8219	.8208	.8197	.8188	.8179	.8172	.8166	.8161
.38	.8253	.8239	.8227	.8216	.8207	.8198	.8190	.8184	.8178	.8173
.40	.8259	.8246	.8235	.8225	.8216	.8208	.8201	.8195	.8190	.8186
.42	.8264	.8253	.8243	.8234	.8225	.8219	.8212	.8207	.8202	.8198
.44	.8270	.8260	.8251	.8243	.8235	.8229	.8223	.8218	.8214	.8210
.46	.8277	.8267	.8259	.8251	.8245	.8239	.8234	.8229	.8225	.8222
.48	.8283	.8274	.8267	.8260	.8254	.8249	.8244	.8240	.8236	.8233
.50	.8289	.8282	.8275	.8269	.8253	.8258	.8254	.8250	.8247	.8244
.52	.8295	.8289	.8283	.8277	.8272	.8268	.8264	.8251	.8258	.8255
.54	.8303	.8297	.8291	.8286	.8282	.8278	.8274	.8271	.8268	.8256
.56	.8310	.8304	.8299	.8295	.8291	.8287	.8284	.8281	.8279	.8277
.58	.8317	.8312	.8307	.8303	.8299	.8296	.8293	.8291	.8289	.8287
.60	.8323	.8319	.8315	.8311	.8308	.8305	.8303	.8301	.8299	.8297
.62	.8330	.8326	.8323	.8320	.8317	.8314	.8312	.8310	.8308	.8307
.64	.8337	.8334	.8331	.8328	.8325	.8323	.8321	.8319	.8318	.8317
.66	.8344	.8341	.8339	.8335	.8334	.8332	.8330	.8329	.8327	.8326
.68	.8352	.8349	.8346	.8344	.8342	.8341	.8339	.8338	.8337	.8336
.70	.8359	.8356	.8354	.8352	.8351	.8349	.8348	.8347	.8346	.8345
.72	.8366	.8364	.8352	.8360	.8359	.8358	.8357	.8356	.8355	.8354
.74	.8373	.8372	.8370	.8369	.8367	.8366	.8366	.8355	.8364	.8354
.76	.8381	.8379	.8378	.8377	.8376	.8375	.8374	.8374	.8373	.8373
.78	.8388	.8387	.8386	.8385	.8384	.8384	.8383	.8383	.8382	.8382
.80	.8396	.8395	.8394	.8394	.8393	.8392	.8392	.8392	.8392	.8391
.82	.8404	.8403	.8403	.8402	.8402	.8401	.8401	.8401	.8401	.8401
.84	.8412	.8412	.8411	.8411	.8411	.8411	.8410	.8410	.8410	.8411
.86	.8421	.8421	.8420	.8420	.8420	.8420	.8420	.8420	.8420	.8420
.88	.8430	.8430	.8430	.8430	.8430	.8430	.8430	.8430	.8431	.8431
.90	.8440	.8440	.8440	.8440	.8441	.8441	.8441	.8441	.8442	.8442
.92	.8451	.8451	.8452	.8452	.8452	.8452	.8453	.8453	.8454	.8454
.94	.8464	.8464	.8465	.8465	.8455	.8466	.8466	.8457	.8467	.8457
.96	.8480	.8480	.8481	.8481	.8482	.8482	.8483	.8483	.8483	.8484
.98	.8502	.8502	.8503	.8503	.8504	.8504	.8505	.8505	.8506	.8506

Co \ α	.50	.51	.52	.53	.54	.55	.56	.57	.58	.59
0.00	.7945	.7936	.7931	.7932	.7937	.7946	.7961	.7980	.8003	.8031
.02	.7947	.7938	.7933	.7933	.7938	.7947	.7961	.7980	.8003	.8031
.04	.7952	.7943	.7937	.7937	.7941	.7950	.7963	.7981	.8004	.8031
.06	.7959	.7950	.7945	.7943	.7945	.7954	.7965	.7983	.8005	.8031
.08	.7969	.7960	.7954	.7952	.7954	.7960	.7971	.7987	.8007	.8032
.10	.7979	.7971	.7965	.7963	.7964	.7969	.7978	.7991	.8010	.8033
.12	.7991	.7983	.7978	.7975	.7975	.7979	.7986	.7998	.8014	.8035
.14	.8004	.7996	.7991	.7989	.7989	.7991	.7997	.8006	.8020	.8038
.16	.8017	.8010	.8005	.8003	.8002	.8004	.8009	.8017	.8028	.8043
.18	.8031	.8024	.8020	.8017	.8016	.8018	.8022	.8028	.8038	.8050
.20	.8045	.8039	.8034	.8032	.8031	.8032	.8035	.8041	.8049	.8060
.22	.8059	.8053	.8049	.8047	.8046	.8047	.8049	.8054	.8061	.8070
.24	.8073	.8068	.8064	.8062	.8061	.8061	.8064	.8068	.8073	.8081
.26	.8088	.8083	.8079	.8077	.8076	.8076	.8078	.8081	.8086	.8093
.28	.8102	.8097	.8094	.8091	.8090	.8091	.8092	.8095	.8100	.8106
.30	.8115	.8111	.8108	.8106	.8105	.8105	.8106	.8109	.8113	.8118
.32	.8129	.8125	.8122	.8120	.8119	.8119	.8121	.8123	.8126	.8131
.34	.8143	.8139	.8136	.8135	.8134	.8134	.8135	.8136	.8140	.8144
.36	.8155	.8153	.8150	.8148	.8148	.8147	.8148	.8150	.8153	.8156
.38	.8169	.8166	.8164	.8162	.8161	.8161	.8162	.8163	.8166	.8169
.40	.8182	.8179	.8177	.8175	.8175	.8174	.8175	.8176	.8179	.8182
.42	.8195	.8192	.8190	.8188	.8188	.8188	.8188	.8189	.8191	.8194
.44	.8207	.8204	.8203	.8201	.8201	.8200	.8201	.8202	.8204	.8206
.46	.8219	.8217	.8215	.8214	.8213	.8213	.8213	.8214	.8215	.8218
.48	.8231	.8229	.8227	.8226	.8225	.8225	.8226	.8226	.8228	.8230
.50	.8242	.8240	.8239	.8238	.8237	.8237	.8237	.8238	.8240	.8241
.52	.8253	.8252	.8250	.8249	.8249	.8249	.8249	.8250	.8251	.8253
.54	.8264	.8263	.8262	.8261	.8260	.8260	.8261	.8261	.8262	.8264
.56	.8275	.8274	.8273	.8272	.8271	.8271	.8272	.8272	.8273	.8275
.58	.8285	.8284	.8283	.8283	.8282	.8282	.8283	.8283	.8284	.8285
.60	.8296	.8295	.8294	.8293	.8293	.8293	.8293	.8294	.8295	.8296
.62	.8305	.8305	.8304	.8304	.8304	.8304	.8304	.8305	.8305	.8306
.64	.8316	.8315	.8314	.8314	.8314	.8314	.8314	.8315	.8316	.8317
.66	.8325	.8325	.8324	.8324	.8324	.8324	.8324	.8325	.8325	.8327
.68	.8335	.8335	.8334	.8334	.8334	.8334	.8334	.8335	.8336	.8337
.70	.8345	.8344	.8344	.8344	.8344	.8344	.8344	.8345	.8345	.8346
.72	.8354	.8354	.8354	.8353	.8354	.8354	.8354	.8355	.8355	.8356
.74	.8363	.8363	.8363	.8363	.8363	.8363	.8364	.8364	.8365	.8366
.76	.8373	.8373	.8373	.8373	.8373	.8373	.8373	.8374	.8375	.8375
.78	.8382	.8382	.8382	.8382	.8382	.8383	.8383	.8384	.8384	.8385
.80	.8391	.8391	.8392	.8392	.8392	.8392	.8393	.8393	.8394	.8395
.82	.8401	.8401	.8401	.8401	.8402	.8402	.8403	.8403	.8404	.8404
.84	.8411	.8411	.8411	.8411	.8412	.8412	.8413	.8413	.8414	.8414
.86	.8421	.8421	.8421	.8422	.8422	.8422	.8423	.8423	.8424	.8425
.88	.8431	.8431	.8432	.8432	.8433	.8433	.8433	.8434	.8435	.8435
.90	.8442	.8443	.8443	.8443	.8444	.8444	.8445	.8445	.8445	.8446
.92	.8454	.8455	.8455	.8456	.8456	.8457	.8457	.8458	.8458	.8459
.94	.8469	.8468	.8469	.8469	.8470	.8470	.8471	.8471	.8472	.8472
.96	.8484	.8485	.8485	.8486	.8486	.8487	.8487	.8488	.8488	.8488
.98	.8506	.8507	.8507	.8508	.8508	.8508	.8509	.8509	.8510	.8510

Co \ α	.60	.61	.62	.63	.64	.65	.66	.67	.68	.69
0,00	.8063	.8098	.8137	.8180	.8226	.8274	.8325	.8378	.8433	.8490
.02	.8062	.8098	.8137	.8180	.8225	.8274	.8325	.8378	.8433	.8489
.04	.8062	.8097	.8136	.8178	.8224	.8272	.8323	.8376	.8430	.8487
.06	.8062	.8096	.8134	.8176	.8221	.8269	.8319	.8372	.8427	.8483
.08	.8061	.8094	.8132	.8173	.8217	.8264	.8315	.8367	.8422	.8478
.10	.8061	.8093	.8129	.8169	.8212	.8259	.8309	.8361	.8415	.8471
.12	.8061	.8091	.8126	.8165	.8207	.8253	.8302	.8353	.8407	.8463
.14	.8062	.8090	.8123	.8160	.8201	.8246	.8293	.8344	.8398	.8453
.16	.8064	.8090	.8120	.8155	.8194	.8238	.8284	.8334	.8387	.8442
.18	.8068	.8090	.8118	.8151	.8188	.8229	.8275	.8323	.8375	.8430
.20	.8074	.8093	.8117	.8147	.8182	.8221	.8264	.8312	.8362	.8415
.22	.8082	.8098	.8118	.8144	.8175	.8213	.8254	.8299	.8349	.8401
.24	.8092	.8105	.8122	.8144	.8172	.8205	.8244	.8287	.8334	.8385
.26	.8102	.8114	.8128	.8146	.8170	.8199	.8234	.8275	.8320	.8369
.28	.8114	.8124	.8135	.8151	.8171	.8195	.8226	.8263	.8305	.8353
.30	.8125	.8134	.8145	.8158	.8174	.8195	.8221	.8253	.8292	.8336
.32	.8137	.8145	.8154	.8166	.8180	.8197	.8218	.8246	.8280	.8320
.34	.8149	.8156	.8164	.8175	.8187	.8202	.8220	.8242	.8270	.8305
.36	.8161	.8167	.8175	.8184	.8195	.8208	.8223	.8242	.8265	.8295
.38	.8173	.8179	.8185	.8193	.8203	.8214	.8228	.8244	.8253	.8287
.40	.8185	.8190	.8196	.8203	.8212	.8222	.8234	.8248	.8254	.8285
.42	.8197	.8202	.8207	.8213	.8221	.8230	.8240	.8252	.8257	.8284
.44	.8209	.8213	.8218	.8223	.8230	.8238	.8247	.8258	.8271	.8286
.46	.8221	.8224	.8229	.8234	.8240	.8247	.8255	.8264	.8275	.8289
.48	.8232	.8235	.8239	.8244	.8249	.8255	.8263	.8271	.8281	.8293
.50	.8244	.8246	.8250	.8254	.8259	.8264	.8271	.8279	.8287	.8298
.52	.8255	.8257	.8260	.8264	.8268	.8274	.8279	.8286	.8294	.8303
.54	.8265	.8268	.8271	.8274	.8278	.8283	.8288	.8294	.8301	.8309
.56	.8275	.8279	.8281	.8284	.8288	.8292	.8297	.8302	.8308	.8315
.58	.8287	.8289	.8291	.8294	.8297	.8301	.8305	.8310	.8315	.8322
.60	.8297	.8299	.8301	.8304	.8307	.8310	.8314	.8318	.8323	.8329
.62	.8308	.8309	.8311	.8314	.8316	.8319	.8323	.8327	.8331	.8335
.64	.8318	.8319	.8321	.8323	.8325	.8328	.8331	.8335	.8339	.8343
.66	.8328	.8329	.8331	.8333	.8335	.8337	.8340	.8343	.8347	.8351
.68	.8338	.8339	.8340	.8342	.8344	.8346	.8349	.8352	.8355	.8358
.70	.8347	.8349	.8350	.8351	.8353	.8355	.8358	.8360	.8363	.8366
.72	.8357	.8358	.8359	.8361	.8362	.8364	.8365	.8369	.8371	.8374
.74	.8367	.8368	.8369	.8370	.8372	.8373	.8375	.8377	.8380	.8382
.76	.8375	.8377	.8378	.8379	.8381	.8382	.8384	.8386	.8388	.8390
.78	.8385	.8387	.8388	.8389	.8390	.8391	.8393	.8395	.8396	.8399
.80	.8395	.8396	.8397	.8398	.8399	.8401	.8402	.8403	.8405	.8407
.82	.8405	.8406	.8407	.8408	.8409	.8410	.8411	.8412	.8414	.8415
.84	.8415	.8416	.8417	.8417	.8418	.8419	.8421	.8422	.8423	.8425
.86	.8425	.8426	.8427	.8427	.8428	.8429	.8430	.8431	.8433	.8434
.88	.8435	.8436	.8437	.8438	.8439	.8440	.8440	.8441	.8443	.8444
.90	.8447	.8448	.8448	.8449	.8450	.8450	.8451	.8452	.8453	.8454
.92	.8459	.8460	.8460	.8461	.8462	.8462	.8463	.8464	.8455	.8455
.94	.8473	.8473	.8474	.8474	.8475	.8476	.8476	.8477	.8478	.8478
.96	.8489	.8489	.8490	.8490	.8491	.8491	.8492	.8493	.8493	.8494
.98	.8510	.8511	.8511	.8512	.8512	.8512	.8513	.8513	.8513	.8514

C_0 \ a	.70	.71	.72	.73	.74	.75	.76	.77	.78	.79
0.00	.8547	.8606	.8564	.8723	.8782	.8840	.8897	.8953	.9008	.9061
.02	.8547	.8605	.8564	.8723	.8781	.8839	.8897	.8953	.9007	.9060
.04	.8544	.8603	.8552	.8720	.8779	.8837	.8895	.8951	.9005	.9059
.06	.8541	.8599	.8558	.8717	.8776	.8834	.8892	.8948	.9003	.9057
.08	.8535	.8594	.8553	.8712	.8771	.8830	.8887	.8944	.9000	.9053
.10	.8528	.8587	.8546	.8706	.8765	.8824	.8882	.8939	.8995	.9049
.12	.8520	.8579	.8538	.8698	.8757	.8817	.8875	.8933	.8989	.9043
.14	.8510	.8569	.8528	.8688	.8748	.8808	.8867	.8925	.8982	.9037
.16	.8499	.8558	.8517	.8677	.8738	.8798	.8857	.8916	.8973	.9029
.18	.8485	.8545	.8504	.8665	.8725	.8786	.8846	.8906	.8964	.9021
.20	.8472	.8530	.8590	.8650	.8712	.8773	.8834	.8894	.8953	.9010
.22	.8457	.8515	.8574	.8635	.8596	.8758	.8820	.8881	.8940	.8999
.24	.8440	.8497	.8557	.8617	.8679	.8741	.8804	.8866	.8927	.8986
.26	.8422	.8479	.8538	.8598	.8660	.8723	.8785	.8849	.8911	.8972
.28	.8404	.8459	.8517	.8578	.8640	.8703	.8767	.8831	.8894	.8956
.30	.8385	.8439	.8495	.8556	.8618	.8681	.8745	.8810	.8874	.8938
.32	.8366	.8418	.8473	.8532	.8594	.8657	.8722	.8788	.8853	.8918
.34	.8348	.8397	.8450	.8507	.8568	.8632	.8597	.8764	.8830	.8896
.36	.8332	.8376	.8427	.8482	.8542	.8605	.8570	.8737	.8805	.8872
.38	.8319	.8358	.8404	.8455	.8514	.8576	.8541	.8708	.8777	.8846
.40	.8310	.8342	.8383	.8431	.8486	.8546	.8510	.8678	.8747	.8817
.42	.8305	.8331	.8365	.8408	.8459	.8516	.8578	.8645	.8714	.8785
.44	.8304	.8325	.8352	.8388	.8433	.8486	.8545	.8611	.8680	.8751
.46	.8304	.8323	.8345	.8373	.8410	.8457	.8513	.8575	.8643	.8715
.48	.8305	.8322	.8341	.8364	.8393	.8432	.8481	.8540	.8605	.8676
.50	.8309	.8323	.8340	.8359	.8383	.8413	.8454	.8505	.8567	.8635
.52	.8313	.8326	.8340	.8357	.8377	.8401	.8432	.8475	.8529	.8594
.54	.8318	.8329	.8341	.8356	.8373	.8394	.8419	.8451	.8495	.8554
.56	.8323	.8333	.8344	.8355	.8371	.8389	.8410	.8436	.8470	.8517
.58	.8329	.8337	.8347	.8358	.8371	.8386	.8404	.8426	.8453	.8488
.60	.8335	.8343	.8351	.8361	.8372	.8385	.8400	.8419	.8441	.8469
.62	.8342	.8348	.8356	.8364	.8374	.8385	.8399	.8414	.8433	.8456
.64	.8348	.8354	.8351	.8368	.8377	.8387	.8398	.8412	.8427	.8447
.66	.8355	.8361	.8366	.8373	.8380	.8389	.8399	.8410	.8424	.8440
.68	.8363	.8367	.8372	.8378	.8385	.8392	.8401	.8411	.8422	.8435
.70	.8370	.8374	.8379	.8384	.8389	.8396	.8403	.8412	.8422	.8434
.72	.8377	.8381	.8385	.8389	.8395	.8400	.8407	.8414	.8423	.8433
.74	.8385	.8388	.8392	.8396	.8400	.8405	.8411	.8417	.8425	.8433
.76	.8393	.8396	.8399	.8402	.8406	.8411	.8415	.8421	.8427	.8435
.78	.8401	.8403	.8406	.8409	.8413	.8416	.8421	.8425	.8431	.8437
.80	.8409	.8411	.8414	.8416	.8419	.8423	.8426	.8430	.8435	.8440
.82	.8417	.8419	.8421	.8424	.8425	.8429	.8432	.8436	.8440	.8444
.84	.8426	.8428	.8430	.8432	.8434	.8436	.8439	.8442	.8445	.8449
.86	.8435	.8437	.8438	.8440	.8442	.8444	.8446	.8449	.8452	.8455
.88	.8445	.8446	.8447	.8449	.8451	.8452	.8454	.8456	.8459	.8461
.90	.8455	.8456	.8457	.8459	.8460	.8462	.8463	.8465	.8467	.8469
.92	.8466	.8467	.8458	.8469	.8470	.8472	.8473	.8474	.8476	.8478
.94	.8479	.8480	.8481	.8481	.8482	.8483	.8484	.8485	.8486	.8488
.96	.8494	.8495	.8495	.8496	.8497	.8497	.8498	.8499	.8500	.8501
.98	.8514	.8515	.8515	.8516	.8516	.8516	.8517	.8518	.8518	.8519

C_0 \ a	.80	.81	.82	.83	.84	.85	.86	.87	.88	.89
0.00	.9112	.9151	.9208	.9252	.9294	.9333	.9370	.9403	.9434	.9452
.02	.9112	.9161	.9207	.9252	.9294	.9333	.9369	.9403	.9434	.9452
.04	.9110	.9160	.9206	.9251	.9293	.9332	.9369	.9403	.9434	.9452
.06	.9108	.9158	.9205	.9249	.9292	.9331	.9368	.9402	.9433	.9451
.08	.9105	.9155	.9202	.9247	.9290	.9330	.9367	.9401	.9432	.9451
.10	.9101	.9151	.9199	.9245	.9287	.9328	.9365	.9399	.9431	.9450
.12	.9095	.9147	.9195	.9241	.9284	.9325	.9363	.9398	.9430	.9459
.14	.9090	.9142	.9191	.9237	.9281	.9322	.9360	.9396	.9428	.9457
.16	.9083	.9135	.9185	.9232	.9277	.9318	.9357	.9393	.9426	.9456
.18	.9075	.9128	.9179	.9227	.9272	.9314	.9354	.9390	.9424	.9454
.20	.9065	.9120	.9171	.9220	.9266	.9309	.9350	.9387	.9421	.9452
.22	.9055	.9110	.9163	.9213	.9260	.9304	.9345	.9383	.9418	.9450
.24	.9044	.9100	.9153	.9204	.9253	.9298	.9340	.9379	.9415	.9447
.26	.9031	.9088	.9143	.9195	.9244	.9291	.9334	.9374	.9411	.9444
.28	.9015	.9075	.9131	.9184	.9235	.9283	.9327	.9369	.9406	.9441
.30	.9000	.9060	.9117	.9173	.9225	.9274	.9320	.9362	.9401	.9437
.32	.8981	.9043	.9103	.9159	.9213	.9264	.9312	.9355	.9396	.9432
.34	.8961	.9025	.9085	.9145	.9201	.9253	.9302	.9348	.9389	.9427
.36	.8939	.9004	.9067	.9128	.9186	.9241	.9292	.9339	.9382	.9422
.38	.8914	.8981	.9047	.9110	.9170	.9227	.9280	.9329	.9374	.9415
.40	.8887	.8956	.9024	.9089	.9152	.9211	.9266	.9318	.9365	.9408
.42	.8857	.8928	.8998	.9065	.9132	.9193	.9251	.9305	.9355	.9400
.44	.8825	.8898	.8970	.9041	.9109	.9174	.9235	.9291	.9343	.9391
.46	.8789	.8864	.8939	.9012	.9083	.9151	.9215	.9275	.9330	.9380
.48	.8751	.8827	.8904	.8980	.9055	.9126	.9194	.9257	.9315	.9368
.50	.8709	.8787	.8865	.8945	.9023	.9098	.9169	.9236	.9293	.9355
.52	.8665	.8744	.8824	.8906	.8987	.9065	.9141	.9212	.9278	.9339
.54	.8622	.8697	.8778	.8862	.8945	.9029	.9109	.9185	.9256	.9320
.56	.8573	.8650	.8730	.8814	.8901	.8988	.9073	.9154	.9229	.9299
.58	.8537	.8602	.8678	.8763	.8851	.8942	.9031	.9117	.9199	.9274
.60	.8505	.8558	.8627	.8708	.8797	.8890	.8984	.9076	.9163	.9245
.62	.8485	.8523	.8579	.8652	.8738	.8833	.8930	.9028	.9122	.9211
.64	.8470	.8500	.8540	.8599	.8678	.8770	.8870	.8972	.9074	.9170
.66	.8460	.8484	.8515	.8557	.8620	.8705	.8804	.8910	.9017	.9122
.68	.8452	.8472	.8497	.8529	.8574	.8642	.8733	.8839	.8952	.9064
.70	.8447	.8464	.8484	.8510	.8543	.8590	.8663	.8753	.8877	.8997
.72	.8444	.8458	.8475	.8495	.8522	.8557	.8606	.8686	.8794	.8915
.74	.8443	.8455	.8450	.8486	.8507	.8534	.8570	.8622	.8709	.8828
.76	.8443	.8453	.8454	.8478	.8493	.8517	.8544	.8592	.8638	.8734
.78	.8444	.8452	.8452	.8474	.8487	.8505	.8526	.8565	.8594	.8654
.80	.8445	.8453	.8451	.8471	.8482	.8496	.8513	.8535	.8564	.8605
.82	.8449	.8455	.8452	.8470	.8479	.8490	.8504	.8521	.8543	.8573
.84	.8453	.8458	.8454	.8470	.8478	.8487	.8497	.8511	.8528	.8550
.86	.8458	.8462	.8457	.8472	.8478	.8485	.8494	.8504	.8517	.8534
.88	.8464	.8467	.8471	.8475	.8480	.8486	.8492	.8500	.8510	.8523
.90	.8471	.8474	.8477	.8480	.8484	.8488	.8493	.8499	.8507	.8515
.92	.8479	.8481	.8484	.8486	.8489	.8492	.8496	.8501	.8506	.8512
.94	.8489	.8491	.8492	.8494	.8496	.8499	.8501	.8504	.8508	.8513
.96	.8502	.8503	.8504	.8505	.8507	.8508	.8510	.8512	.8514	.8517
.98	.8519	.8520	.8520	.8521	.8522	.8523	.8524	.8525	.8526	.8527