THREE PROBLEMS IN SCATTERING THEORY

A THESIS SUBMITTED TO THE UNIVERSITY OF MANCHESTER
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Abstract

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ABSTRACT OF DISSERTATION submitted by **Christopher J. Luke** for the Degree of Ph. D. and entitled **Three Problems in Scattering Theory**

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We study three transmission problems. The first problem deals with the coupling of a smooth elastic body and an acoustic medium. We investigate integral equation methods for finding the scattered pressure field and the transmitted elastic displacement field produced when an acoustic wave is incident on the body. In the second problem we investigate the corner singularities in the scattered and transmitted waves when an acoustic wave is incident on an elastic polygon. Finally, we deal with the coupling of a smooth elastic body and an incommpressible fluid with a free surface. We look at the asymptotic behaviour of the scattering frequencies, as the submergence depth tends to infinity. We show that their imaginary parts are exponentially small.

Declaration

No portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institution of learning.

Dedication

To dad and the memory of mum.

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I thank my supervisor, Dr. P. A. Martin, for his help and support during the course of this degree. Without his patience and encouragement I should have never finished this work.

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The Author

After gaining a first class degree in Mathematics from the University of Cambridge, the author successfully completed Part III of the Mathematical Tripos at that university. Since 1989 he has been engaged in research at the University of Manchester under the supervision of Dr. P. A. Martin. The results of this research are contained in this thesis.

Chapter 1

Introduction

In this work we study three related transmission problems. Each chapter is concerned with a single problem. The first two are very closely related; they both deal with the coupling of a compact elastic body and an infinitely extended acoustic medium. The difference lies in the fact that in Chapter 2 only sufficiently smooth bodies are considered whereas in Chapter 3 we consider polygonal bodies. By restricting our attention to polygons in Chapter 3 we aim to isolate the feature of the transmission problem, involving bodies with edges, that makes the it different from the one studied in Chapter 2.

In both these chapters the situation is this: an incident wave in the acoustic medium causes a transmitted elastic wave within the body and a scattered acoustic wave. The problems of bodies with edges differ from the problems of smooth bodies both physically and mathematically. Physically, the presence of edges leads to transmitted and scattered waves with large gradients near the edges. This is not even a localised effect. The incident field could vanish in a neighbourhood of an edge but the transmitted and scattered waves would still, in general, have large gradients there. Mathematically, this is expressed by saying that the transmission condition does not apply. That is to say, in the problems involving sufficiently smooth bodies the transmitted and scattered waves are as smooth as the incident wave. With bodies with edges this is no longer the case.

The coupling of the two media in these problems is referred to as weak coupling. This expresses the fact that only the normal component of displacement of the elastic body is coupled to the fluid's motion. This allows, as we shall see, the possibility of elastic oscillations that do not transmit at all to the fluid. These oscillations, which we here call Jones' modes, have been noted before; they are mentioned by, *inter al.*, Jones [13], Norris [24] and Goswami *et al.* [8]. It is a mathematically interesting (although, probably not physically important) problem to try to classify the bodies that can accommodate a Jones' mode. This is discussed in Appendix A. This appendix includes a translation of [11]. In this paper it was proved that, in the class of smooth bodies, bodies that have a Jones' frequency in a given compact range are infinitely rare.

The problem in Chapter 2 has been studied elsewhere. In Goswami *et al.* [8], for example, a system of integral equations that is identical to one of the systems derived here is used. We should mention the work of Sanchez Hubert and Sanchez Palencia [27], who, like Norris [24], have studied this problem in the physically important case when the acoustic medium has a much smaller density than the elastic body; in this case, asymptotic techniques can be used.

Integral equation techniques are widely used for studying problems of this kind. The excellent book by Colton and Kress [4] deals with such techniques in acoustic and electro-magnetic wave scattering problems in great depth. Kirsch [15] collected together many of the results on the continuity properties of acoustic integral operators. In Martin [20, 21] and Martin and Rizzo [22] integral equation techniques are used in the study of elastic scattering problems. In Kupradze [16], also, integral equation techniques are used in the study of elasticity problems.

We are here interested in deriving systems of integral equations that are solvable at all frequencies. The motivation is entirely theoretical; we wish to prove the existence and uniqueness of solutions to the transmission problem and do not make any claim for these systems with respect to the degree they facilitate computation of the scattered and transmitted fields. We are interested in making rigorous the reasoning that the systems we derive can be treated exactly as Fredholm systems despite involving non-compact operators. Furthermore, in order to be as general as possible, the boundary is required to be only in the class $C^{2,\alpha}$, for $0 < \alpha < 1$.

In Section 2.1 we formulate the transmission problem. This involves determining the basic field equations, transmission conditions and the radiation condition. In Section 2.2 we prove that the transmission problem has at most one solution. The fundamental solutions of Helmholtz's equation and the elastic wave equation are introduced in the next two sections. We then introduce the single-layer and double-layer potentials for both Helmholtz's equation and the elastic wave equation. We discuss in Section 2.5 the properties of integral operators with weakly-singular, singular and hypersingular kernels. The regularization of operators with singular kernels is discussed in this section. Regularization means taking an operator that is not of the classical Fredholm form and operating on it by an operator so that the product is. The theory in Section 2.5 is heavily indebted to the books by Kupradze [16] and Zabreyko [31].

In Sections 2.6 we derive four systems of integral equations for solving the transmission problem. The first system is derived by a direct method; this means that it involves physically relevant entities. The system involves the trace of the elastic double layer potential, which has a singular kernel. We apply the regularization techniques of the previous section to show that the system is solvable at all frequencies. This leads to a proof of the existence of a solution to the transmission problem. There are frequencies at which this system is singular; these occur when an interior Dirichlet problem has non-trivial solutions and at Jones' frequencies. We shall call the eigenvalues of the interior Dirichlet problem *spurious frequencies*. The second system is indirect; it involves functions that are not physically relevant. This system is closely related to the adjoint of the first system and is shown to have identical properties to it.

The third system avoids the problem of spurious frequencies. This system also involves the fewest unknowns (three) of any of the four systems. It has, however, an operator with a hypersingular kernel. Fortunately, a special feature of the problem allows us to regularize the system. The fourth system is an indirect system and is solvable for all frequencies except Jones' frequencies.

Problems in polygonal domains, wedge-shaped domains and domains with edges have been extensively studied. For example, Ola [25] has studied the transmission problems for the scalar and vector Helmholtz operators in three-dimensional domains having edges. The normal derivative of the acoustic double layer potential was looked at by Costabel and Stephan [6]. In [7] Costabel and Stephan calculate the effects of curvature on the singularities. Costabel [5] has studied the properties of integral operators on Lipschitz domains. Von Petersdorff and Stephan [26] have looked at the regularity of solutions of Laplace's equation in polyhedra. Our approach here is similar to many of these previous studies. Mellin transform techniques are extensively used. The feature that differentiates this work from the aforesaid works is the involvement of an elastic body. This makes the actual calculation of the singularity set much more complicated, as we shall see. It also introduces the possibilty of Jones' modes.

We should note the work of Grisvard, who in his excellent book [9] studied the properties of the Laplacian in polygons and in [10, Chapter 4] studied elasto-static boundary value problems in polygons without using Mellin transform techniques.

In the first two sections of Chapter 3 we formulate the transmission problem and show that it has at most one solution except at Jones' frequencies. The work of Costabel [5] is used in Section 3.3 to extend the definition of the boundary integral operators first defined in Chapter 2 to larger function spaces. We then derive the simplest direct system of boundary integral equations — identical to the first system derived in Chapter 2.

In Section 3.4 we discuss some aspects of Mellin transforms and the Mellin convolution theorem. This section borrows heavily from the, as yet unpublished, work of Dr. Lassi Päivärinta. In Section 3.5 we utilise the important results from the previous section in a discussion of the properties of the boundary integral operators and their resolvents in wedge-shaped domains. We apply the results of this section in the following section to the transmission problem. We show that the scattered pressure and the elastic displacement field are as smooth as the incident pressure away from the corners and are supplemented by singular functions at the corners. We then demonstrate the uniqueness of the solution of the system for fnctions in $H^s(\partial\Omega)$, where s lies between 0 and $\frac{1}{2}$. We prove in Section 3.7 that the adjoint system has at most one solution in the space $H^{-s}(\partial\Omega)$ except at irregular frequencies. This leads us to the conclussion that the system of boundary integral equations we derived is solvable except at eigenvalues of an interior Dirichlet problem and at Jones' frequencies. Finally, we prove the existence of solutions to the transmission problem.

The problem in the final chapter was motivated by the work of Vullierme-Ledard [29, Chapter 2]. She studied the transmission problem of water coupled to a deeply submerged elastic body. She was interested in the complex scattering frequencies and proved that the scattering frequencies associated with simple modes have asymptotic expansions in inverse integer powers of submergence depth and that each coefficient in the expansion is real. We expand on her work here to say something about non-simple modes and, more importantly, about the imaginary parts of the scattering frequencies. We should expect that these are exponentially small and this is indeed verified. It is expected that there should be no real scattering frequencies for finite submergence depth as the existence of a real scattering frequency would imply the existence of a free oscillation at that frequency. As in the case of a rigid scatterer, no such oscillation is expected but, as yet, no proof is available. It was hoped that we could prove that for large submergence depths the imaginary parts of the scattering frequencies must be non-zero. It seems though that this may only be possible for individual geometries. Even that task may prove difficult because the algebraic manipulations required to obtain just the second term in the asymptotic expansion of the imaginary part of the scattering frequency are fiendishly complicated!

In the first two sections of this final chapter we formulate the transmission problem. In Section 4.3 we pose an exterior Neumann boundary value problem and prove that it is solvable except possibly at a set of isolated frequencies. We achieve this by showing solvability is equivalent to the non-vanishing of a function which is holomorphic in the square of the frequency; this function is shown not to vanish when the square of the frequency has positive imaginary part and thus has isolated zeros. In doing this we set up a problem which is equivalent to the exterior Neumann problem but which is set in a compact domain — called the *truncated problem*. Although this proves convenient, it is not neccessary. Other authors have solved similar exterior problems by using weighted Sobolev spaces, see, for example, Neittaanmäki and Roach [23]. Vullierme-Ledard [29] also set up an equivalent truncated problem, but she used integral equation techniques, as did Lenoir, Vullierme-Ledard and Hazard [18] for a similar problem. Here we use expansions of multi-poles to construct the truncated problem; again see [18]. The truncated problem approach has an advantage over the weighted Sobolev space approach only in so far as familiar function spaces are used throughout. There does not seem to be any distinct advantage or disadvantage in using multi-pole expansions over integral equations.

Using the results of Section 4.3 we pose an interior problem in the next section. This problem is solvable at only an isolated set of frequencies. It turns out that the problem is solvable when a particular operator, which depends on the submergence depth and holomorphically on the square of the frequency, has an eigenvalue equal to 1. We are able to determine the behaviour of the eigenvalues of this operator for large submergence depths. This allows us in Section 4.5 to see how the frequency must depend on submergence depth for the transmission problem to be solvable. In particular, we prove that

$$\omega^2(\epsilon) = \omega_0^2 + a_1\epsilon + a_2\epsilon^2 + o(\epsilon^2),$$

where ω denotes the frequency, ϵ is the inverse submergence depth and ω_0^2, a_1 and a_2 are real constants. Finally, we adapt a proof from Harrell and Simon [12] to determine the leading order behaviour of the imaginary part of a scattering frequency. We show that the imaginary part of a scattering frequency is exponentially small; that is to say, it is smaller than any power of ϵ.

Finally, we should add that each chapter in this thesis is intended to be as self-contained as possible and so a degree of repetition is unavoidable.

Chapter 2

Smooth Elastic Body — Acoustic Medium

2.1 Introduction and formulation of the problem.

Let us consider the interaction between an elastic body and a compressible, inviscid fluid. The elastic body occupies a compact open set of $\mathcal{R}^3$ of non-zero measure, Ω_i. It is coupled to a compressible fluid, which occupies the region $\mathcal{R}^3 \setminus \Omega_i$. We shall suppose that the boundary between the media, which we shall call $\partial\Omega$, is smooth. We shall state later more precisely the smoothness conditions required of the boundary. We shall call the complement of $\overline{\Omega_i}$ Ω_e.

The two media are coupled in two distinct ways. The first of these is the *kinematic boundary condition.* To ensure that a well defined boundary between the fluid and the solid persists, the normal velocity of the fluid on one side of the boundary must match the normal velocity of the solid on the other side. There is no such restriction on the tangential component of velocity because the fluid has zero viscosity and so it can slip over the surface of the solid absolutely freely.

The second coupling process is the *dynamic boundary condition.* This results from the balance of forces on all parts of the boundary. Each boundary element is, after all, massless and so a non-zero resultant force acting on it is prohibited.

We suppose that a time-harmonic acoustic wave, with frequency ω, is incident on the solid. We look here into the existence and uniqueness of any resulting scattered wave and will describe methods for determining the scattered wave. In addition to the assumptions already made, we shall assume that all motions are small — we shall, therefore, ignore all terms quadratic or higher in small quantities — and we shall assume that, before the incident wave was created, the fluid was at rest and that now all transient solutions have decayed away.

2.1.1 The velocity potential in the fluid.

The circulation of an inviscid fluid remains constant. We assume that the motion of the fluid was generated from rest and that all transient solutions have completely decayed leaving just the time-harmonic motion. The motion of the fluid must then be irrotational for all time. By a well known result of analysis, the fluid velocity, $\mathbf{v}$, can be written in the form

$$\mathbf{v}(\mathbf{x}, t) = \nabla_{\mathbf{x}} \Phi(\mathbf{x}, t), \tag{2.1}$$

where $\Phi(\mathbf{x}, t)$ is a real-valued, scalar function defined in the domain $\Omega_e \otimes \mathcal{R}$.

We know that $\Phi(\mathbf{x},t)$ is time-harmonic and, thus, we can separate the spatial and temporal dependence and write:

$$\Phi(\mathbf{x},t) = \Re(\phi(\mathbf{x})\exp(-i\omega t)), \tag{2.2}$$

where $\Re z$ represents the real part of any complex number z and where ω is the frequency.

The momentum equation is

$$\rho\frac{\partial \mathbf{v}}{\partial t} + \rho\mathbf{v}.\nabla\mathbf{v} = -\nabla p, \tag{2.3}$$

where ρ represents the fluid density and p the pressure.

The following conservation of mass equation is satisfied

$$\frac{D\rho}{Dt} + \rho\nabla.\mathbf{v} = 0, \tag{2.4}$$

where the first term is the convective derivative of ρ.

Because the pressure and density vary little with respect to the uniform and constant background pressure and density, p_0 and ρ_0, respectively, we can write

$$p = p_0 + \tilde{p}$$

and

$$\rho = \rho_0 + \tilde{\rho}$$

where $\tilde{p}$ and $\tilde{\rho}$ are small. Consequently, the fluid velocities are small.

Equations (2.3) and (2.4) become after linearisation

$$\frac{\partial \mathbf{v}}{\partial t} = -\frac{1}{\rho_0}\nabla\tilde{p} \tag{2.5}$$

and

$$\frac{\partial \tilde{\rho}}{\partial t} + \rho_0 \nabla . \mathbf{v} = 0. \tag{2.6}$$

Equations (2.1) and (2.5) imply

$$\nabla \left(\frac{\partial \phi}{\partial t} + \frac{\tilde{p}}{\rho_0} \right) = 0,$$

which implies that

$$\frac{\partial \phi}{\partial t} + \frac{\tilde{p}}{\rho_0}$$

is a constant. By suitably adjusting ϕ, we can obtain

$$-\frac{\partial \phi}{\partial t} = \frac{\tilde{p}}{\rho_0}. \tag{2.7}$$

Now write

$$\tilde{p}(\mathbf{x}; t) = \Re(\tilde{p}(\mathbf{x}) \exp(-i\omega t)).$$

Equation (2.7) implies

$$\frac{\tilde{p}}{\rho_0} = i\omega\phi. \tag{2.8}$$

Equations (2.1), (2.2) and(2.6) imply

$$i\omega\tilde{\rho} = \rho_0 \nabla^2 \phi, \tag{2.9}$$

where, once again,

$$\tilde{\rho}(\mathbf{x}; t) = \Re(\tilde{\rho}(\mathbf{x}) \exp(-i\omega t)).$$

Clearly, we need a third relationship between the three quantities $\tilde{p}$, $\tilde{\rho}$ and $\mathbf{v}$. Let us consider only barotropic fluids. That is to say,

$$p = p(\rho) \tag{2.10}$$

with

$$p_0 = p(\rho_0).$$

This implies that

$$\tilde{p} = \tilde{\rho}\left.\frac{dp}{d\rho}\right|_{\rho_0} + \frac{1}{2}\tilde{\rho}^2\left.\frac{d^2p}{d\rho^2}\right|_{\rho_0} + \ldots$$

After linearisation, this becomes

$$\tilde{p} = c^2\tilde{\rho} \tag{2.11}$$

where

$$c^2 = \left.\frac{dp}{d\rho}\right|_{\rho_0}.$$

Combining equations (2.8), (2.9) and (2.11), we obtain

$$\nabla^2\phi + k^2\phi = 0 \tag{2.12}$$

and

$$\nabla^2\tilde{p} + k^2\tilde{p} = 0, \tag{2.13}$$

where

$$k^2 = \frac{\omega^2}{c^2}.$$

From now on we shall drop the tilde over the letter p.

2.1.2 The motion of the solid.

The displacement field, $\mathbf{u}(\mathbf{x}, t)$, satisfies

$$\rho\frac{\partial^2\mathbf{u}}{\partial t^2} = \nabla.\sigma(\mathbf{u}), \tag{2.14}$$

where ρ now denotes the solid's density. (The linearisation of the problem is implicit in this formulation.)

$\sigma(\mathbf{u})$ is the stress tensor satisfying, for the class of materials in which we are mainly interested,

$$\sigma_{ij} = c_{ijkl}e_{kl},$$

where

$$e_{kl} = \frac{1}{2}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right)$$

is the strain tensor and

$$c_{ijkl} = c_{jikl} = c_{ijlk} = c_{klij}.$$

The summation convention is being followed here and will be subsequently used unless otherwise stated.

We are only interested in isotropic, homogeneous materials; for such materials

$$c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}),$$

where λ and μ are real constants (called Lamé constants) and δ_{ij} is the Kronecker delta. They satisfy

$$\lambda + \frac{2}{3}\mu > 0 \quad \text{and} \quad \mu > 0. \tag{2.15}$$

Therefore, if, as before, we write

$$\mathbf{u}(\mathbf{x};t) = \Re(\mathbf{u}(\mathbf{x})\exp(-i\omega t))$$

and use equation (2.14), we obtain

$$(\lambda + \mu)\nabla(\nabla.\mathbf{u}) + \mu\nabla^2\mathbf{u} + \rho\omega^2\mathbf{u} = \mathbf{0}. \tag{2.16}$$

Call the sum of the first two terms in the last equation $L(\mathbf{u})$.

There is another class of materials that possess a "memory" of their strain history. For such materials we have

$$\sigma(\mathbf{u};t) = \int_{-\infty}^{\infty} \mathbf{c}(\mathbf{x};\tau - t) : \mathbf{e}(\mathbf{x};\tau)d\tau. \tag{2.17}$$

Here $\mathbf{c}(\mathbf{x};\tau)$ vanishes if $\tau > 0$ — this property is a consequence of causality; the stress cannot depend on future strains!

For materials that are isotropic and homogeneous

$$c_{ijkl}(\mathbf{x};\tau - t) = \lambda(\tau - t)\delta_{ij}\delta_{kl} + \mu(\tau - t)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}),$$

where λ and μ are now functions of time. Furthermore, if the motion is time harmonic, i. e. if

$$\sigma(\mathbf{u}(\mathbf{x};t)) = \Re(\sigma(\mathbf{u}(\mathbf{x}))\exp(-i\omega t))$$

and

$$\mathbf{e}(\mathbf{x};t) = \Re(\mathbf{e}(\mathbf{x})\exp(-i\omega t)),$$

then

$$\sigma_{ij}(\mathbf{u}) = \tilde{\lambda} e_{kk}\delta_{ij} + 2\tilde{\mu} e_{ij},$$

where $\tilde{\lambda}$ and $\tilde{\mu}$ are the Fourier transforms of λ and μ respectively.

It can be shown that $\tilde{\lambda}$ and $\tilde{\mu}$ must satisfy

$$\Re(\tilde{\lambda} + \frac{2}{3}\tilde{\mu}) > 0, \qquad \Re\tilde{\mu} > 0 \tag{2.18}$$

and

$$\Im(\tilde{\lambda} + \frac{2}{3}\tilde{\mu}) \leq 0, \qquad \Im\tilde{\mu} \leq 0. \tag{2.19}$$

Such a material is effectively like an elastic material with complex Lamé constants.

2.1.3 Transmission conditions.

The kinematic condition is

$$\frac{\partial\phi}{\partial n} = -i\omega\mathbf{n}.\mathbf{u}.$$

on $\partial\Omega$. This and equation (2.8) imply

$$\frac{\partial p}{\partial n} = \rho_0\omega^2\mathbf{n}.\mathbf{u} \tag{2.20}$$

on $\partial\Omega$. The kinematic condition is neccessary to ensure that the two media remain in contact.

The dynamic condition is

$$-p\mathbf{n} = \sigma(\mathbf{u}).\mathbf{n} \tag{2.21}$$

on $\partial\Omega$. The dynamic condition ensures that the resultant force acting on a given surface element vanishes.

2.1.4 Sommerfeld's radiation condition.

We split the pressure field in Ω_e into two parts:

$$p = p_{inc} + p_s.$$

p_{inc} is the incident wave and p_s is the scattered wave. p_s satisfies the Sommerfeld radiation condition:

$$\left|\frac{\mathbf{x}}{|\mathbf{x}|}.\nabla p_s - ikp_s\right|^2 = o\left(\frac{1}{|\mathbf{x}|^2}\right) \tag{2.22}$$

as $|\mathbf{x}| \to \infty$. This must hold true wherever the origin is taken to be.

2.1.5 Function spaces and smoothness of the solutions.

We shall say that a function, $f(\mathbf{x})$, defined in a subset of $\mathcal{R}^3$, D, which may be a manifold or a set of non-zero measure, belongs to $C(D)$ if it is continuous everywhere in D.

Similarly, we shall say that it belongs to $C^p(D)$, where D is now supposed to be open, if its first p derivatives belong to $C(D)$.

$f(\mathbf{x})$ belongs to the Hölder continuous space, $C^{0,\alpha}(D)$, where α is some positive constant, if

$$\sup_{\mathbf{x}\in D} |f(\mathbf{x})|$$

exists and

$$|f(\mathbf{x}) - f(\mathbf{y})| \leq K|\mathbf{x} - \mathbf{y}|^\alpha,$$

where K is some positive constant and $\mathbf{x}$ and $\mathbf{y}$ are any two points in D. $C^{0,\alpha}(D)$ has the norm

$$\| f \|_{C^{0,\alpha}(D)} = \sup_{\mathbf{x}\in D} |f(\mathbf{x})| + \sup_{\mathbf{x},\mathbf{y}\in D} \frac{|f(\mathbf{x}) - f(\mathbf{y})|}{|\mathbf{x} - \mathbf{y}|^\alpha}.$$

$f(\mathbf{x})$ belongs to the Hölder continuous space, $C^{p,\alpha}(D)$ if

$$\frac{\partial^{p_1+p_2+p_3} f}{\partial x_1^{p_1} \partial x_2^{p_2} \partial x_3^{p_3}} \in C^{0,\alpha}(D),$$

if D is a subset of $\mathcal{R}^3$ of non-zero measure, where

$$p_1 + p_2 + p_3 \leq p,$$

or if

$$\frac{\partial^{p_1+p_2} f}{\partial x_1^{p_1} \partial x_2^{p_2}} \in C^{0,\alpha}(D),$$

if D is a two dimensional manifold and x_1 and x_2 are coordinates in it, where

$$p_1 + p_2 \leq p.$$

We shall assume that the incident wave, p_{inc}, belongs to $C^2(D)$, where D is an open subset of $\mathcal{R}^3$ that contains Ω_i; this allows for the possibility that p_{inc} is generated by, for example, a point source situated somewhere in Ω_e. We shall look for a solution $(p, \mathbf{u})$ with

$$p \in C^2(D) \bigcap C(\overline{D})$$

and

$$\mathbf{u} \in C^2(\Omega_i) \bigcap C(\overline{\Omega_i}),$$

It is also required that

$$\left.\frac{\partial p}{\partial n}\right|_{\partial\Omega}$$

and

$$\mathbf{n}.\sigma(\mathbf{u})|_{\partial\Omega}$$

exist as the limits, as h tends to zero from above, of

$$\mathbf{n}(\mathbf{x}).\nabla p(\mathbf{x} + h\mathbf{n}(\mathbf{x}))$$

and

$$\mathbf{n}(\mathbf{x}).\sigma(\mathbf{u}(\mathbf{x} - h\mathbf{n}(\mathbf{x}))$$

respectively.

2.2 Uniqueness.

Suppose that there were two solutions to the problem we have just formulated. Call these $(p_1, \mathbf{u}_1)$ and $(p_2, \mathbf{u}_2)$, with obvious notation.

Call

$$p = p_1 - p_2$$

and

$$\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2.$$

Clearly, we have

$$\nabla^2 p + k^2 p = 0 \text{ in } \Omega_e, \tag{2.23}$$

$$L(\mathbf{u}) + \rho\omega^2\mathbf{u} = \mathbf{0} \text{ in } \Omega_i, \tag{2.24}$$

the transmission conditions (2.20) and (2.21) and p satisfies the Sommerfeld radiation condition (2.22).

By an application of the divergence theorem

$$\begin{aligned}\int_{\partial\Omega_a} p\frac{\partial\overline{p}}{\partial n}dS &= \int_{\partial\Omega} p\frac{\partial\overline{p}}{\partial n}dS + \int_{\Omega^a} \nabla p.\nabla\overline{p}dV \\ &\quad + \int_{\Omega^a} p\nabla^2\overline{p}dV,\end{aligned} \tag{2.25}$$

where $\partial\Omega_a$ is the surface of the sphere of radius a, which encloses Ω_i, and Ω^a is the region between $\partial\Omega_a$ and $\partial\Omega$. Using equation (2.23) and the transmission conditions we obtain

$$\begin{aligned}\int_{\partial\Omega_a} p\frac{\partial\overline{p}}{\partial n}dS &= -\rho_0\overline{\omega}^2 \int_{\partial\Omega} \mathbf{n}.\sigma(\mathbf{u}).\overline{\mathbf{u}}dS + \int_{\Omega^a} \nabla p.\nabla\overline{p}dV \\ &\quad -\overline{k}^2 \int_{\Omega^a} |p|^2 dV.\end{aligned} \tag{2.26}$$

Let us first consider the case when ω, and hence k, are real. Take the imaginary parts of both sides of equation (2.26)

$$\Im\left(\int_{\partial\Omega_a} p\frac{\partial\overline{p}}{\partial n}dS\right) = -\rho_0\omega^2\Im\left(\int_{\partial\Omega} \mathbf{n}.\sigma(\mathbf{u}).\overline{\mathbf{u}}dS\right). \tag{2.27}$$

As a tends to infinity

$$\Im\left(\int_{\partial\Omega_a} p\frac{\partial\overline{p}}{\partial n}dS\right) \to -k\lim_{a\to\infty}\int_{\partial\Omega_a} |p|^2 dS, \tag{2.28}$$

by virtue of p satisfying equation (2.22).

Furthermore, by the divergence theorem in Ω_i and equation (2.24),

$$\begin{aligned}\int_{\partial\Omega} \mathbf{n}.\sigma(\mathbf{u}).\overline{\mathbf{u}}dS &= \int_{\Omega_i} \sigma(\mathbf{u}) : \nabla\overline{\mathbf{u}}dV \\ &\quad -\rho\omega^2 \int_{\Omega_i} \mathbf{u}.\overline{\mathbf{u}}dV.\end{aligned} \tag{2.29}$$

For purely elastic bodies,

$$\sigma(\mathbf{u}) : \nabla\overline{\mathbf{u}} = \lambda|e_{kk}|^2 + 2\mu e_{ij}\overline{e_{ij}}.$$

Thus

$$\sigma(\mathbf{u}) : \nabla\overline{\mathbf{u}}$$

is real. Therefore, equations (2.27), (2.28) and (2.29) imply that

$$\lim_{a\to\infty} \int_{\partial\Omega_a} |p|^2 dS = 0. \tag{2.30}$$

It can be easily shown that p must have the following expansion:

$$p(\mathbf{x}) = \frac{\exp(ik|\mathbf{x}|)}{|\mathbf{x}|} \sum_{n=0}^{\infty} \frac{F_n(\theta,\phi)}{|\mathbf{x}|^n}, \tag{2.31}$$

where the functions $F_n(\theta,\phi)$ are derived from the recurrence relation

$$2ikF_n = n(n-1)F_{n-1} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial F_{n-1}}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 F_{n-1}}{\partial\phi^2}.$$

θ and ϕ here represent, respectively, the usual longitudinal and latitudinal angular coordinates in spherical polars. (See, for example, Colton and Kress [4, Theorem 3.6].) Clearly,

$$\lim_{a\to\infty} \int_{\partial\Omega_a} |p|^2 dS = \int_{\partial\Omega_1} |F_0|^2 dS,$$

where $\partial\Omega_1$ is the unit sphere.

Equation (2.30) implies that

$$F_0 = 0.$$

This determines, through the recurrence relation, that each F_n is identically zero. Thus p vanishes in a neighbourhood of infinity. Any solution of Helmholtz's equation that is twice differentiable is analytic in the spatial variable, and so, by continuation, p vanishes everywhere in Ω_e. Thus,

$$\left.\frac{\partial p}{\partial n}\right|_{\partial\Omega} = 0.$$

Therefore, from equations (2.20) and (2.21),

$$\mathbf{u}.\mathbf{n}|_{\partial\Omega} = 0 \quad \text{and} \quad \mathbf{n}.\sigma(\mathbf{u})|_{\partial\Omega} = \mathbf{0}. \tag{2.32}$$

Equation (2.24) along with the boundary conditions (2.32) do not necessarily imply that $\mathbf{u}$ vanishes in Ω_i. It is known that there are, for certain geometries and for certain frequencies, non-trivial solutions to this problem. We call these *Jones' modes* and the associated frequencies *Jones' frequencies.* More is said about these in Appendix A.

Let us now consider the case when ω and k have positive imaginary parts. The expansion (2.31) implies that p decays exponentially at infinity. Clearly then

$$\lim_{a\to\infty} \int_{\partial\Omega_a} |p|^2 dS = 0.$$

This and equation (2.26) imply that

$$0 = -\rho_0\overline{\omega}^2 \int_{\partial\Omega} \mathbf{n}.\sigma(\mathbf{u}).\overline{\mathbf{u}}dS + \int_{\Omega_e} \nabla p.\nabla\overline{p}dV - \overline{k}^2 \int_{\Omega_e} |p|^2 dV. \tag{2.33}$$

Equations (2.29) and (2.33) imply

$$\begin{aligned} 0 = &-\rho_0\overline{\omega}^2 \int_{\Omega_i} \sigma(\mathbf{u}) : \nabla\overline{\mathbf{u}}dV + \rho_0\rho|\omega^2|^2 \int_{\Omega_i} \mathbf{u}.\overline{\mathbf{u}}dV \\ &+ \textstyle\int_{\Omega_e} \nabla p.\nabla\overline{p}dV - \overline{k}^2 \int_{\Omega_e} |p|^2 dV. \end{aligned} \tag{2.34}$$

Take the imaginary part of equation (2.34):

$$\Im(\omega^2)\left(\rho_0 \int_{\Omega_i} \sigma(\mathbf{u}) : \nabla\overline{\mathbf{u}}dV + \frac{1}{c^2}\int_{\Omega_e} |p|^2 dV\right) = 0. \tag{2.35}$$

Thus

$$\Im(\omega^2) = 0$$

or p vanishes in Ω_e and

$$\sigma(\mathbf{u}) : \nabla\overline{\mathbf{u}}$$

vanishes in Ω_i. In the latter case, $\mathbf{u}$ is constant in Ω_i. The transmission condition (2.20) together with the fact that p vanishes in Ω_e imply that $\mathbf{u}$ vanishes on $\partial\Omega$. Therefore, $\mathbf{u}$ vanishes in Ω_i.

If

$$\Im(\omega^2) = 0,$$

ω^2 must be negative (recall that we assumed ω is not real). Therefore, each term in equation (2.34) is positive and so

$$p = 0 \quad \text{and} \quad \mathbf{u} = \mathbf{0}.$$

We can perform a similar analysis on the problem with a visco-elastic material. For real ω^2, the analysis is identical up to equation (2.29). We have

$$\sigma(\mathbf{u}) : \nabla\overline{\mathbf{u}} = (\tilde{\lambda} + \frac{2}{3}\tilde{\mu})|e_{kk}|^2 + 2\tilde{\mu}(e_{ij} - \frac{1}{3}e_{kk}\delta_{ij})(\overline{e_{ij}} - \frac{1}{3}\overline{e_{kk}}\delta_{ij}). \tag{2.36}$$

The conditions (2.19) imply that

$$\Im\left(\int_{\Omega_i} \sigma(\mathbf{u}) : \nabla\overline{\mathbf{u}} dV\right) \leq 0. \tag{2.37}$$

Equations (2.27) and (2.28) still apply. These and equation (2.37) imply that

$$\lim_{a\to\infty} \int_{\partial\Omega_a} |p|^2 dS = 0.$$

As before,

$$p = 0$$

in Ω_e. Equations (2.28) and (2.29) imply that

$$\Im\left(\int_{\Omega_i} \sigma(\mathbf{u}) : \nabla\overline{\mathbf{u}} dV\right) = 0.$$

Assuming that the material is genuinely *visco*-elastic, equation (2.36) implies that $\mathbf{u}$ is constant in Ω_i. Since p vanishes, the transmission conditions make it clear that $\mathbf{u}$ vanishes in Ω_i. So, for real frequencies, the solution to the problem of the interaction of a visco-elastic material and an acoustic medium, if it exists, is unique.

2.3 Representation theorems and applications.

In this section we introduce the concept of the *fundamental solution* to the Helmholtz equation and state Green's second representation theorem. This will be applied to the fundamental solution along with the scattered and incident waves. The proofs will be only sketched. The reader is refered to Colton and Kress [4, Chapter 3] for more details.

It can be readily verified that the function

$$G(\mathbf{x},\mathbf{y}) = \frac{\exp(ik|\mathbf{x}-\mathbf{y}|)}{4\pi|\mathbf{x}-\mathbf{y}|},$$

which is defined for $\mathbf{x} \neq \mathbf{y}$, is a solution of the three dimensional Helmholtz equation:

$$\nabla^2 G + k^2 G = 0,$$

the differentiation being taken with respect to $\mathbf{y}$ (respectively $\mathbf{x}$), with $\mathbf{x}$ (respectively $\mathbf{y}$) fixed. $G(\mathbf{x},\mathbf{y})$ is called the fundamental solution.

If D is an open, compact subset of $\mathcal{R}^3$ whose boundary, ∂D, is $C^{2,\alpha}$ then Green's second representation theorem is true

$$\int_D (u\nabla^2 v - v\nabla^2 u)dV = \int_{\partial D} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right) dS \qquad (2.38)$$

for all functions in $C^2(D) \bigcap C(\overline{D})$. The basic idea is to first prove the result on a parallel surface to ∂D inside D and then to take the limit as the distance between this surface and ∂D itself tends to zero. In fact, we have already used Green's first representation theorem in the uniqueness proofs of the previous section.

If we apply equation (2.38) to p_{inc} and G in Ω_i, we obtain

$$\int_{\partial\Omega}\left(p_{inc}(\mathbf{y})\frac{\partial G(\mathbf{x},\mathbf{y})}{\partial n(\mathbf{y})}-G(\mathbf{x},\mathbf{y})\frac{\partial p_{inc}(\mathbf{y})}{\partial n(\mathbf{y})}\right)dS_{\mathbf{y}}=\begin{cases}0 & \text{if } \mathbf{x}\in\Omega_e\\ -p_{inc}(\mathbf{x}) & \text{if } \mathbf{x}\in\Omega_i\end{cases} \tag{2.39}$$

Here, and in what follows, $\mathbf{n}(\mathbf{y})$ means the vector normal to $\partial\Omega$ at the point $\mathbf{y}$. Similarly, $\mathbf{n}(\mathbf{x})$ means the vector normal to $\partial\Omega$ at the point $\mathbf{x}$. In the above equation the integration is carried out over all points $\mathbf{y}$ on $\partial\Omega$.

The first of the two results in equation (2.39) is clear because both p_{inc} and the fundamental solution solve Helmholtz's equation everywhere in Ω_i and so the integrand on the left hand side of equation (2.38) is zero everywhere. The second result is obtained by applying equation (2.38) to the domain $\Omega_i \setminus B$, where B denotes a small closed ball centred on $\mathbf{x}$ and then taking the limit as the radius of B tends to zero. The right hand side of equation (2.38) now involves two integrals — one over $\partial\Omega$ and one over the surface of the ball. The limit of the second integral can be found by using the mean value theorem.

Similarly, it can be shown that

$$\int_{\partial\Omega}\left(p_s(\mathbf{y})\frac{\partial G(\mathbf{x},\mathbf{y})}{\partial n(\mathbf{y})}-G(\mathbf{x},\mathbf{y})\frac{\partial p_s(\mathbf{y})}{\partial n(\mathbf{y})}\right)dS_{\mathbf{y}}=\begin{cases}p_s(\mathbf{x}) & \text{if } \mathbf{x}\in\Omega_e\\ 0 & \text{if } \mathbf{x}\in\Omega_i\end{cases} \tag{2.40}$$

This time the domain used is Ω^a (as defined in Section 2.2) and the limit is taken as a tends to infinity. The radiation condition is used to show that the limit, as a tends to infinity, of the integral over $\partial\Omega_a$ vanishes.

2.3.1 Single and double layer operators.

It will prove convenient to write equations (2.39) and (2.40) in operator notation. Let us then now define respectively the double and single layer operators, D and S. If f is a function defined on the domain $\partial\Omega$, then

$$(Df)(\mathbf{x}) = -2\int_{\partial\Omega} f(\mathbf{y})\frac{\partial G(\mathbf{x},\mathbf{y})}{\partial n(\mathbf{y})}dS_{\mathbf{y}}$$

and

$$(Sf)(\mathbf{x}) = -2\int_{\partial\Omega} f(\mathbf{y})G(\mathbf{x},\mathbf{y})dS_{\mathbf{y}}.$$

We shall say later what smoothness conditions f needs to satisfy for these integrals to exist.

With this new notation, equations (2.39) and (2.40) can be rewritten as

$$(Dp_{inc})(\mathbf{x}) - \left(S\frac{\partial p_{inc}}{\partial n}\right)(\mathbf{x}) = \begin{cases} 0 & \text{if } \mathbf{x} \in \Omega_e \\ 2p_{inc}(\mathbf{x}) & \text{if } \mathbf{x} \in \Omega_i \end{cases} \tag{2.41}$$

and

$$(Dp_s)(\mathbf{x}) - \left(S\frac{\partial p_s}{\partial n}\right)(\mathbf{x}) = \begin{cases} -2p_s(\mathbf{x}) & \text{if } \mathbf{x} \in \Omega_e \\ 0 & \text{if } \mathbf{x} \in \Omega_i \end{cases} \tag{2.42}$$

2.3.2 Limits of the single and double layer potentials as $\partial\Omega$ is approached.

If f is continuous on $\partial\Omega$ then one can show that $(Sf)(\mathbf{x})$ is defined up to and including $\partial\Omega$ and is continuous as $\mathbf{x}$ passes through the boundary. $(Df)(\mathbf{x})$, however, is not continuous as $\mathbf{x}$ passes through the boundary. In fact,

$$\lim_{\mathbf{x}\in\Omega_e\to\mathbf{x}_0\in\partial\Omega}(Df)(\mathbf{x}) = (\overline{K}^* f)(\mathbf{x}_0) - f(\mathbf{x}_0) \tag{2.43}$$

and

$$\lim_{\mathbf{x}\in\Omega_i\to\mathbf{x}_0\in\partial\Omega}(Df)(\mathbf{x}) = (\overline{K}^* f)(\mathbf{x}_0) + f(\mathbf{x}_0), \tag{2.44}$$

where

$$(\overline{K}^* f)(\mathbf{x}_0) = -2\int_{\partial\Omega} f(\mathbf{y})\frac{\partial G(\mathbf{x}_0,\mathbf{y})}{\partial n(\mathbf{y})}dS_{\mathbf{y}}$$

and

$$\mathbf{x}_0 \in \partial\Omega.$$

$\overline{K}^* f$ exists if $f \in C(\partial\Omega)$.

The normal derivative of Sf exists on $\partial\Omega$ if f is continuous. It is not continuous as the boundary is crossed. Its behaviour is similar to the behaviour of Df. We have

$$\lim_{\mathbf{x}\in\Omega_e\to\mathbf{x}_0\in\partial\Omega}\mathbf{n}(\mathbf{x}_0).\nabla(Sf)(\mathbf{x}) = (Kf)(\mathbf{x}_0) + f(\mathbf{x}_0) \tag{2.45}$$

and

$$\lim_{\mathbf{x}\in\Omega_i\to\mathbf{x}_0\in\partial\Omega}\mathbf{n}(\mathbf{x}_0).\nabla(Sf)(\mathbf{x}) = (Kf)(\mathbf{x}_0) - f(\mathbf{x}_0), \tag{2.46}$$

where

$$(Kf)(\mathbf{x}_0) = -2\int_{\partial\Omega} f(\mathbf{y})\frac{\partial G(\mathbf{x},\mathbf{y})}{\partial n(\mathbf{x})}dS_{\mathbf{y}}.$$

We can even take the normal derivative of Df, but for existence up to the boundary we require f to belong to $C^{1,\alpha}(\partial\Omega)$, where α is any positive constant. Let us call this operator N. Nf is continuous across the boundary. For details see Colton and Kress [4, p. 62].

2.4 Operators associated with the solid's displacement field.

The results and proofs of this section parallel those of the previous section. For this reason this section will be even briefer than the last, and only the salient points will be noted. See Kupradze [16, Chapter 5] for details.

Firstly, we note that we have a result analogous to Green's second representation theorem:

$$\int_{\partial\Omega}(\mathbf{u}.\sigma(\mathbf{v}).\mathbf{n}-\mathbf{v}.\sigma(\mathbf{u}).\mathbf{n})dS=\int_{\Omega_i}(\mathbf{u}.\nabla.(\sigma(\mathbf{v}))-\mathbf{v}.\nabla.(\sigma(\mathbf{u})))dV. \tag{2.47}$$

The *fundamental displacement tensor* is, for $\mathbf{x}\neq\mathbf{y}$,

$$\mathbf{G}(\mathbf{x},\mathbf{y})=\frac{1}{\mu}\Psi_s(\mathbf{x},\mathbf{y})\mathbf{I}+\frac{1}{\rho\omega^2}\nabla\nabla\left(\Psi_s(\mathbf{x},\mathbf{y})-\Psi_p(\mathbf{x},\mathbf{y})\right),$$

where

$$k_s^2=\frac{\rho\omega^2}{\mu}$$

and

$$k_p^2=\frac{\rho\omega^2}{\lambda+2\mu},$$

and where

$$\Psi_s(\mathbf{x},\mathbf{y})=\frac{\exp(ik_s|\mathbf{x}-\mathbf{y}|)}{4\pi|\mathbf{x}-\mathbf{y}|}$$

and

$$\Psi_p(\mathbf{x},\mathbf{y})=\frac{\exp(ik_p|\mathbf{x}-\mathbf{y}|)}{4\pi|\mathbf{x}-\mathbf{y}|}.$$

From equation (2.47) we have

$$\int_{\partial\Omega}(\mathbf{u}(\mathbf{y}).\sigma_{\mathbf{y}}(\mathbf{G}(\mathbf{x},\mathbf{y}))-\mathbf{G}(\mathbf{x},\mathbf{y}).\sigma_{\mathbf{y}}(\mathbf{u}(\mathbf{y})))dS_{\mathbf{y}}=\begin{cases}0 & \text{if } \mathbf{x}\in\Omega_e\\ \mathbf{u}(\mathbf{x}) & \text{if } \mathbf{x}\in\Omega_i\end{cases}, \tag{2.48}$$

where $\sigma_{\mathbf{y}}(\mathbf{G}(\mathbf{x},\mathbf{y}))$ means that the derivatives are taken with respect to the $\mathbf{y}$ variable.

Next define the elastic single layer and double layer operators to be

$$(\mathbf{S}.\mathbf{f})(\mathbf{x}) = 2\int_{\Omega_i} \mathbf{f}(\mathbf{y}).\mathbf{G}(\mathbf{x},\mathbf{y})dS_{\mathbf{y}} \tag{2.49}$$

and

$$(\mathbf{D}.\mathbf{f})(\mathbf{x}) = 2\int_{\Omega_i} \mathbf{f}(\mathbf{y}).\sigma_{\mathbf{y}}(\mathbf{G}(\mathbf{x},\mathbf{y})).\mathbf{n}(\mathbf{y})dS_{\mathbf{y}} \tag{2.50}$$

for $\mathbf{x}$ in $\Omega_i \bigcup \Omega_e$.

So equation (2.48) becomes

$$(\mathbf{D}.\mathbf{u})(\mathbf{x}) - (\mathbf{S}.\mathbf{n}.\sigma(\mathbf{u}))(\mathbf{x}) = \begin{cases} 0 & \text{if } \mathbf{x} \in \Omega_e \\ 2\mathbf{u}(\mathbf{x}) & \text{if } \mathbf{x} \in \Omega_i \end{cases} \tag{2.51}$$

$(\mathbf{S}.\mathbf{f})(\mathbf{x})$ is continuous as $\mathbf{x}$ passes through $\partial\Omega$, whereas $(\mathbf{D}.\mathbf{f})(\mathbf{x})$ and $\mathbf{n}(\mathbf{x}).\sigma_{\mathbf{x}}(\mathbf{S}.\mathbf{f})(\mathbf{x})$ exhibit jumps:

$$\mathbf{D}.\mathbf{f} = \mp\mathbf{f} + \overline{\mathbf{K}}^*.\mathbf{f} \tag{2.52}$$

and

$$\mathbf{n}.\sigma_{\mathbf{x}}(\mathbf{S}.\mathbf{f}) = \pm\mathbf{f} + \mathbf{K}.\mathbf{f}. \tag{2.53}$$

In each case the upper (resp. lower) sign corresponds to passing through $\partial\Omega$ from Ω_e (resp. Ω_i) to Ω_i (resp. Ω_e). $\mathbf{K}$ and $\overline{\mathbf{K}}^*$ are defined by

$$(\mathbf{K}.\mathbf{f})(\mathbf{x}) = 2\int_{\partial\Omega} \mathbf{f}(\mathbf{x}).\sigma_{\mathbf{x}}(\mathbf{G}(\mathbf{x},\mathbf{y})).\mathbf{n}(\mathbf{x})dS_{\mathbf{y}}$$

and

$$(\overline{\mathbf{K}}^*.\mathbf{f})(\mathbf{x}) = 2\int_{\partial\Omega} \mathbf{f}(\mathbf{x}).\sigma_{\mathbf{y}}(\mathbf{G}(\mathbf{x},\mathbf{y})).\mathbf{n}(\mathbf{x})dS_{\mathbf{x}},$$

where now $\mathbf{x} \in \partial\Omega$ and the integrals are defined in the sense of the Cauchy principal part.

The traction of the elastic double layer potential is defined by

$$(\mathbf{N}.\mathbf{f})(\mathbf{x}) = \mathbf{n}(\mathbf{x}).\sigma_{\mathbf{x}}(\mathbf{D}.\mathbf{f})(\mathbf{x}). \tag{2.54}$$

As in the case of the analogous acoustic operator, $(\mathbf{N}.\mathbf{f})(\mathbf{x})$ is continuous across $\partial\Omega$.

2.5 Weakly singular, singular and hypersingular kernels.

Let $D \subset \mathcal{R}^3$ have non-zero measure and let ∂D be its boundary. An integral operator,

$$(Af)(\mathbf{x}) = \int_{\partial D} A(\mathbf{x},\mathbf{y})f(\mathbf{y})dS_{\mathbf{y}},$$

has a *weakly singular* kernel if

$$|A(\mathbf{x},\mathbf{y})| = O(|\mathbf{x}-\mathbf{y}|^{\gamma-2}),$$

as

$$|\mathbf{x}-\mathbf{y}| \to 0,$$

for some positive constant γ. It has a *singular* kernel if

$$|A(\mathbf{x},\mathbf{y})| = O(|\mathbf{x}-\mathbf{y}|^{-2}),$$

as

$$|\mathbf{x}-\mathbf{y}| \to 0.$$

Finally, it has a *hypersingular* kernel if

$$|A(\mathbf{x},\mathbf{y})| = O(|\mathbf{x}-\mathbf{y}|^{-\gamma-2}),$$

as

$$|\mathbf{x} - \mathbf{y}| \to 0,$$

for some positive constant γ.

It is not difficult to show that S, K, $\overline{K}^*$ and $\mathbf{S}$ have weakly singular kernels. $\mathbf{K}$ and $\overline{\mathbf{K}}^*$ have singular kernels and N and $\mathbf{N}$ have hyper-singular kernels.

For the existence of Sf, Kf, $\overline{K}^*f$ and $\mathbf{S}.\mathbf{f}$ it is required that f and $\mathbf{f}$ are continuous. It is known that for the kernels of each of these four operators the constant, γ, above equals 1. Therefore, not only are they compact on $C(\partial\Omega)$, but also on $C^{0,\beta}(\partial\Omega)$, for any β lying between zero and one. Furthermore, if f and $\mathbf{f}$ are continuous on $\partial\Omega$, then Sf, Kf, $\overline{K}^*f$ and $\mathbf{S}.\mathbf{f}$ all belong to $C^{0,\beta}(\partial\Omega)$, for any β lying between zero and one. K, $\overline{K}^*$ and S map $C^{0,\beta}(\partial\Omega)$, for any $\beta \in (0,1)$, into $C^{1,\beta}(\partial\Omega)$. (See, e. g., Kirsch [15].)

For the existence of $\mathbf{K}.\mathbf{f}$ and $\overline{\mathbf{K}}^*.\mathbf{f}$, $\mathbf{f}$ must belong to $C^{0,\beta}(\partial\Omega)$, for some positive β. $\mathbf{K}$ and $\overline{\mathbf{K}}^*$ are, however, not compact on this space.

For the existence of Nf and $\mathbf{N}.\mathbf{f}$, f and $\mathbf{f}$ must belong to a smaller space than $C^{0,\beta}(\partial\Omega)$. One can show that it is sufficient to take them in $C^{1,\beta}(\partial\Omega)$, for any β lying between zero and one. See Colton and Kress [4, p. 62].

S, $\mathbf{S}$, N and $\mathbf{N}$ are self-adjoint when the inner product is defined to be

$$< f, g >= \int_{\partial\Omega} f\overline{g}\,dS.$$

K and $\overline{K}^*$ and $\mathbf{K}$ and $\overline{\mathbf{K}}^*$ are mutually adjoint with this inner product.

2.5.1 Regularization, the symbol matrix and regularity.

Consider an operator

$$A : X \to X,$$

where X is a Banach space. The bounded operator

$$B : X \to X$$

is called a *left equivalent regularizer* if

$$BA = I + T,$$

where I denotes the identity and T is compact in X, and if the equations

$$Au = f$$

and

$$BAu = Bf$$

are equivalent. Similarly, the bounded operator

$$C : X \to X$$

is called a *right equivalent regularizer* if

$$AC = I + T',$$

where T' is compact in X, and if any solution of

$$Au = f$$

can be written as

$$u = Cv,$$

for some v in X, and *vice versa.*

The *index* of an operator is the difference between the dimension of its null space and the dimension of the null space of its adjoint. We have the following important results:

Theorem 1 *If A admits both left and right regularization, then the index of A is finite.*

Theorem 2 *If a closed operator admits a left regularization, then for the solvability of*

$$Au = f$$

it is sufficient (and of course necessary) that f be orthogonal to every element of the null space of the adjoint. We say A is normally solvable *when it has this property.*

An immediate corollary of Theorem 2 is the fact that if A admits a right regularization, then A^* is normally solvable, where A^* denotes the adjoint operator of A.

The question that concerns us here is this: Given a system of operators of the form

$$(\mathbf{A}.\mathbf{u})(\mathbf{x}) = \mathbf{u}(\mathbf{x}) + \int_{\partial\Omega} \mathbf{k}(\mathbf{x}, \mathbf{y}).\mathbf{u}(\mathbf{y}) dS_{\mathbf{y}},$$

where $\mathbf{k}(\mathbf{x}, \mathbf{y})$ is singular and $\mathbf{A}$ is considered to be acting on $L^2(\partial\Omega)$, under what conditions does a regularizer having the form

$$(\mathbf{B}.\mathbf{u})(\mathbf{x}) = \mathbf{u}(\mathbf{x}) + \int_{\partial\Omega} \mathbf{k}'(\mathbf{x}, \mathbf{y}).\mathbf{u}(\mathbf{y}) dS_{\mathbf{y}}$$

exist? To answer this question some general results will be used without being proved. Very detailed accounts of the theory of regularization of two dimensional singular integral operators are given in Zabreyko [31, Chapter 2] and in Kupradze [16, Chapter 4].

It is obvious that since $\partial\Omega$ is smooth enough, a normal to the surface can be defined at every point on the surface. If $\mathbf{n}(\mathbf{x}_0)$ denotes the normal at $\mathbf{x}_0$, then $\mathbf{n}(\mathbf{x}_0)$ is in the Hölder continuous space $C^{1,\alpha}(\partial\Omega)$. Let us define the cylinder $C_{\mathbf{x}_0}$ to be

$$C_{\mathbf{x}_0} = \{\mathbf{y} \in \mathcal{R}^3; |(\mathbf{y} - \mathbf{x}) \times \mathbf{n}(\mathbf{x}_0)| \leq d, -l \leq (\mathbf{y} - \mathbf{x}).\mathbf{n}(\mathbf{x}_0) \leq l\},$$

where l and d are chosen to be small enough so that the orthogonal projection of the intersection of $\partial\Omega$ and $C_{\mathbf{x}_0}$, which we shall refer to as $S(\mathbf{x}_0, d)$, onto the base of $C_{\mathbf{x}_0}$ is conformal. Let $\tau(\mathbf{x}_0, d)$ be the intersection of $C_{\mathbf{x}_0}$ and the tangent plane to $\partial\Omega$ at $\mathbf{x}_0$. If ζ is the image of the orthogonal projection of a point $\mathbf{x}$ in $S(\mathbf{x}_0, d)$ onto $\tau(\mathbf{x}_0, d)$ and f is any function with domain $S(\mathbf{x}_0, d)$, then we shall denote by f' the function in $\tau(\mathbf{x}_0, d)$ with

$$f'(\zeta) = f(\mathbf{x}).$$

We suppose that the point $\mathbf{x}_0$ is mapped to the origin of $\mathcal{R}^2$ under the orthogonal projection.

Suppose that

$$k'_{ij}(\zeta, \eta) = l_{ij}(\zeta, \zeta - \eta) + m_{ij}(\zeta, \eta) \tag{2.55}$$

and

$$l_{ij}(\zeta, t(\zeta - \eta)) = t^{-2} l_{ij}(\zeta, \zeta - \eta), \tag{2.56}$$

for all $t > 0$ and $\zeta \neq \eta$. Suppose, further, that $l_{ij}(\zeta, \kappa)$ and all its derivatives with respect to κ when considered as a function of ζ belong to $C^{1,\alpha}(\tau(\mathbf{x}_0, d))$, for all κ of unit modulus. Finally, suppose that $m_{ij}(\zeta, \eta)$ satisfies the following two conditions:

$$|m_{ij}(\zeta',\eta) - m_{ij}(\zeta'',\eta)| \leq M|\zeta' - \zeta''|^{\beta}(\upsilon(\zeta',\zeta'',\eta))^{-2} \tag{2.57}$$

and

$$|m_{ij}(\zeta,\eta') - m_{ij}(\zeta,\eta'')| \leq M|\eta' - \eta''|^{\beta}(\upsilon(\eta',\eta'',\zeta))^{-2}, \tag{2.58}$$

where M is a positive constant and, for example,

$$\upsilon(\zeta',\zeta'',\eta) = \min\{|\zeta' - \eta|, |\zeta'' - \eta|\}.$$

If all three of these conditions hold, then $\mathbf{A}$ is said to belong to the class $G(\beta)$.

Suppose that $\mathbf{A}$ is in the class $G(\beta)$ and that $\mathbf{t}_{\mathbf{x}_0}$ is a unit vector in $\tau(\mathbf{x}_0, d)$ that makes an angle θ with some fixed line in $\tau(\mathbf{x}_0, d)$. All the derivatives of $l_{ij}(\zeta,\kappa)$ with respect to κ were supposed to exist. $l_{ij}(\zeta, \mathbf{t}_{\mathbf{x}_0})$ may thus be expanded as a Fourier series

$$l_{ij}(\mathbf{x_0}, \mathbf{t}_{\mathbf{x}_0}) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} a_{ij}^{(n)} \exp(in\theta).$$

Define

$$\sigma_{ij}(\mathbf{x_0}, \mathbf{t}_{\mathbf{x}_0}) = \delta_{ij} + 2\pi i \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{i^{|n|}}{|n|} a_{ij}^{(n)} \exp(in\theta).$$

The terms for $n = 0$ in the two series above are missing, because, for the existence of $\mathbf{A}.\mathbf{u}$, we have to assume that

$$\int_{|\kappa|=1} l_{ij}(\zeta,\kappa)d\kappa = 0.$$

Let us define $\sigma(\mathbf{x}_0, \theta)$ to be the matrix whose entries are the $\sigma_{ij}(\mathbf{x_0}, \mathbf{t}_{\mathbf{x}_0})$'s defined in the last equation.

The main result of the general theory is given below.

Theorem 3 *If*

$$\inf_{\substack{\mathbf{x}_0 \in \partial\Omega \\ \theta \in [0, 2\pi]}} |\det \sigma(\mathbf{x}_0, \theta)| > 0,$$

then a double-sided regularizer of $\mathbf{A}$ *of the correct form exists in* $L^2(\partial\Omega)$. *Moreover, the regularizer is in the class* $G(\beta)$.

We shall need the following theorems.

Theorem 4 *If* $\mathbf{B}$ *is the regularizer of* $\mathbf{A}$ *in Theorem 3, then the index of* $\mathbf{B}$ *plus the index of* $\mathbf{A}$ *equals zero.*

Theorem 5 *If the symbol matrix is Hermitian, then the index of* $\mathbf{B}$ *is zero.*

Thus, if A satisfies the conditions of Theorem 3 and its symbol matrix is Hermitian, then its index is zero. So this and Theorem 2 imply that $\mathbf{A}$ satisfies the Fredholm properties. We follow the example of the Russian authors and say that such an operator is *quasi-Fredholm.*

Suppose that $\mathbf{A}$ is in the class $G(\beta)$. Suppose that

$$k_{ij}(\mathbf{x}, \mathbf{y}) \in C^{1,\alpha}(S(\mathbf{x}_0, \delta)), \tag{2.59}$$

as a function of its first argument uniformly in $\mathbf{y} \in S \setminus S(\mathbf{x}_0, \delta)$, where δ is any positive number less than $d/2$, and the function $m_{ij}(\zeta, \eta)$ in equation (2.55) satisfies the following property:

$$\int_{\tau(\mathbf{x}_0, d)} m_{ij}(\zeta, \eta) u(\eta) dS_\eta \in C^{1,\beta}(\tau(\mathbf{x}_0, d)) \tag{2.60}$$

whenever

$$u \in C^{0,\beta}(\tau(\mathbf{x}_0, d)),$$

for $0 < \beta \leq \alpha$. Then $\mathbf{A}$ is said to belong to class $G'(\beta)$.

We have the following important regularity result:

Theorem 6 *Let β be any positve number less than or equal to α. If*

$$\mathbf{A}.\mathbf{u} = \mathbf{f},$$

where $\mathbf{A}$ is a singular integral operator in the class $G'(\beta)$, $\mathbf{f}$ belongs to $C^{1,\beta}(\partial\Omega)$ and $\mathbf{u}$ belongs to $L^2(\partial\Omega)$, then $\mathbf{u}$ belongs to $C^{1,\beta}(\partial\Omega)$.

The effect of this theorem and the preceeding results is that any operator in $G'(\beta)$ that satisfies the conditions in Theorems 3 and 5 is quasi-Fredholm on $C^{1,\beta}(\partial\Omega)$, for any positive number β less than or equal to α.

2.6 Boundary integral equations.

In this, the main section of this chapter, we derive a series of four sets of boundary integral equations and use them to prove the existence of a solution to the coupled problem. Each subsequent set of boundary integral equations will be increasingly sophisticated. The first two will consist of four equations in four unknowns. Each of these two will exhibit spurious irregular frequencies, at which the system of integral equations is singular but the actual problem is not.

It is sometimes important not to have these irregular frequencies. If, for example, the ratio of the densities of the fluid and the solid is small, then there are scattering frequencies with negative imaginary part but which are, however, close to the real axis. (See, for example, Norris [24] and Sanchez Hubert and Sanchez Palencia [27, Chapter 9].) In this case the response curve will have peaks near the scattering frequencies and these may be difficult to distinguish from the peaks due to the irregular frequencies. It could be the case too that Jones' modes are possible and, once again, it may be difficult to distinguish between the peaks in the response due to these and the peaks due to irregular frequencies.

In view of this, the third and fourth sets of boundary integral equations derived here are designed so that the irregular frequencies do not occur. The third set consists of three equations in three unknowns. For three-dimensional problems this is likely to be optimal. However, the price to be paid is in the increased complexity of the surface potentials utilised.

2.6.1 The simplest direct boundary integral equations.

Let us begin with equations (2.41) and (2.42) and take the limits of them as we pass to a point on the surface, $\partial\Omega$. In both cases we shall take the limit as the surface is approached from the inner region. Equation (2.44) implies that

$$p_{inc} + \overline{K}^* p_{inc} - S\frac{\partial p_{inc}}{\partial n} = 2p_{inc} \tag{2.61}$$

and

$$p_s + \overline{K}^* p_s - S\frac{\partial p_s}{\partial n} = 0. \tag{2.62}$$

Adding equations (2.61) and (2.62) we obtain,

$$p + \overline{K}^* p - S\frac{\partial p}{\partial n} = 2p_{inc}. \tag{2.63}$$

Similarly, if we use equation (2.51) and equation (2.52), we obtain

$$\mathbf{u} - \overline{\mathbf{K}}^*.\mathbf{u} + \mathbf{S}(\mathbf{n}.\sigma(\mathbf{u})) = \mathbf{0}. \tag{2.64}$$

Now use the transmission conditions (2.20) and (2.21) to get

$$p + \overline{K}^* p - \rho_0 \omega^2 S(\mathbf{u}.\mathbf{n}) = 2p_{inc} \tag{2.65}$$

and

$$-\mathbf{u} + \overline{\mathbf{K}}^*.\mathbf{u} + \mathbf{S}.(\mathbf{n}p) = \mathbf{0}. \tag{2.66}$$

We look for a solution with p and $\mathbf{u}$ in $C^{1,\beta}(\partial\Omega)$, for some positive constant β.

This is the first set of boundary integral equations. It will prove helpful to write this system in the following form

$$\begin{pmatrix} I + \overline{K}^* & -\rho_0\omega^2 Sn_1 & -\rho_0\omega^2 Sn_2 & -\rho_0\omega^2 Sn_3 \\ S_{11}n_1 + S_{12}n_2 + S_{13}n_3 & -1 + \overline{K}^*_{11} & \overline{K}^*_{12} & \overline{K}^*_{13} \\ S_{21}n_1 + S_{22}n_2 + S_{23}n_3 & \overline{K}^*_{21} & -1 + \overline{K}^*_{22} & \overline{K}^*_{23} \\ S_{31}n_1 + S_{32}n_2 + S_{33}n_3 & \overline{K}^*_{31} & \overline{K}^*_{32} & -1 + \overline{K}^*_{33} \end{pmatrix} \begin{pmatrix} p \\ u_1 \\ u_2 \\ u_3 \end{pmatrix} \tag{2.67}$$

$$= \begin{pmatrix} 2p_{inc} \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

where

$$(\overline{K}^*_{ij} f)(\mathbf{x}) = 2\int_{\partial\Omega} f(\mathbf{y})(\sigma_{\mathbf{y}}(\mathbf{G}(\mathbf{x},\mathbf{y})).\mathbf{n}(\mathbf{y}))_{ij} dS_{\mathbf{y}},$$
$$(S_{ij} f)(\mathbf{x}) = 2\int_{\partial\Omega} f(\mathbf{y})(\mathbf{G}(\mathbf{x},\mathbf{y}))_{ij} dS_{\mathbf{y}}$$

and

$$(Sn_i f)(\mathbf{x}) = -2\int_{\partial\Omega} f(\mathbf{y}) n_i \mathbf{G}(\mathbf{x},\mathbf{y}) dS_{\mathbf{y}}.$$

The system (2.67) is not Fredholm because it is not of the form: identity plus a compact operator. It is under certain conditions quasi-Fredholm.

The system is quasi-Fredholm if the symbol matrix is invertible. To calculate the symbol matrix we need to identify the singular terms in equation (2.67). The only terms that can be singular are the K_{ij} terms. Let us then examine the kernels of each of these terms. After a routine calculation, one obtains

$$(2\sigma_{\mathbf{y}}(\mathbf{G}(\mathbf{x},\mathbf{y})).\mathbf{n}(\mathbf{y}))_{ij} = \gamma n_i(\mathbf{y})X_j D\Phi + (\theta\delta_{ij} + n_j(\mathbf{y})X_i)D\Psi \tag{2.68}$$
$$+\frac{2}{k_s^2}(n_i(\mathbf{y})X_j + n_j(\mathbf{y})X_i + \theta\delta_{ij})D^2(\Psi - \Phi) + \frac{2}{k_s^2}\theta X_i X_j D^3(\Psi - \Phi).$$

Here D denotes the differential operator

$$\frac{1}{R}\frac{d}{dR},$$

$$X_i = x_i - y_i,$$

$$R = |\mathbf{X}|,$$

$$\gamma = \frac{\nu}{1-\nu},$$

$$\theta = \mathbf{n}(\mathbf{y}).\mathbf{X}$$

and

$$\nu = \frac{\lambda}{2(\lambda + \mu)}$$

is Poisson's ratio. In addition,

$$\Phi = -\frac{\exp(ik_p R)}{2\pi R}$$

and

$$\Psi = -\frac{\exp(ik_s R)}{2\pi R}.$$

As R tends to zero we have

$$D^2(\Psi - \Phi) = -\frac{(k_s^2 - k_p^2)}{4\pi R^3} + \text{ smaller terms} \tag{2.69}$$

and

$$D^3(\Psi - \Phi) = \frac{3(k_s^2 - k_p^2)}{4\pi R^5} + \text{ smaller terms.} \tag{2.70}$$

We know that

$$|\theta| \leq KR^2,$$

where K is a positive constant that is independent of position (compare with [4, Theorem 2.2]). This and equation (2.70) show that the final term on the right hand side of equation (2.68) is weakly singular. Similarly,

$$\frac{2}{k_s^2}\theta\delta_{ij}D^2(\Psi - \Phi)$$

and

$$\theta\delta_{ij}D\Psi$$

are weakly singular. It is easy, if somewhat tedious, to show that the weakly singular terms in equation (2.68) and the other weakly singular operators in the system (2.67) satisfy the conditions (2.57), (2.58) and (2.60). To do this we must bear in mind that, because of the smoothness properties of the boundary, $\mathbf{n}(\mathbf{x})$ and $\mathbf{n}(\mathbf{y})$ are in $C^{1,\alpha}(\partial\Omega)$.

The remaining terms on the right hand side of equation (2.68) can be rearranged into

$$\frac{(n_i(\mathbf{y})X_j - n_j(\mathbf{y})X_i)}{4\pi R^3}\frac{(2\nu-1)}{(\nu-1)}. \tag{2.71}$$

For a particular point $\mathbf{x} \in \partial\Omega$, let us define $l(\mathbf{x})$ to be the unitary matrix that rotates the coordinate system so that the new $\mathbf{e}_3$ axis is normal to $\partial\Omega$ at $\mathbf{x}$. Let us denote vectors in the new frame with a prime. We have

$$X_i' = l_{ij}X_j$$

and

$$n_i'(\mathbf{y}') = l_{ij}n_j'(\mathbf{y}').$$

In the new frame the kernels

$$\frac{n_i'(\mathbf{y}')X_3'}{R^3} \qquad i = 1, 2, 3$$

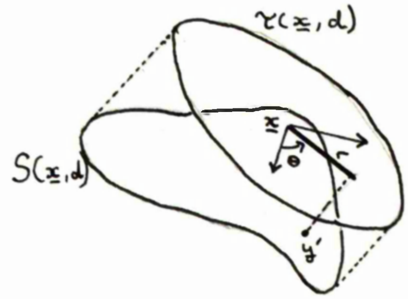

Figure 2.1: The projection of $\partial\Omega_\epsilon$ onto the tangent plane.

and

$$\frac{n_1'(\mathbf{y}')X_i'}{R^3} \text{ and } \frac{n_2'(\mathbf{y}')X_i'}{R^3} \qquad i = 1, 2, 3$$

are weakly singular.

The only singular terms are

$$\frac{n_3'(\mathbf{y}')X_1'}{R^3} \text{ and } \frac{n_3'(\mathbf{y}')X_2'}{R^3}. \tag{2.72}$$

Let us now project the functions in equation (2.72) from $S(\mathbf{x}, d)$ onto $\tau(\mathbf{x}, d)$. Let us adopt a cylindrical polar coordinate system in $\tau(\mathbf{x}, d)$, with θ the angle a line makes with the $\mathbf{e}_1$ axis and r the distance of a point from the origin. (See Figure (2.1).)

$$\frac{n_3'(\mathbf{y}')X_1'}{R^3} = \frac{(n_3'(\mathbf{x}') + (n_3'(\mathbf{y}') - n_3'(\mathbf{x}')))X_1'}{(r^2 + X_3'^2)^{3/2}}.$$

We have

$$\left(r^2 + X_3'^2\right)^{3/2} = r^3(1 + O(r^{2+2\alpha})),$$

$$|n_3'(\mathbf{y}') - n_3'(\mathbf{x}')| \leq M|\mathbf{x}' - \mathbf{y}'|^{1+\alpha},$$

for some positive constant M, and

$$n_3'(\mathbf{x}') = 1.$$

The singular term is (the l_{ij} in the notation of equation (2.55)).

$$\frac{\cos\theta}{r^2}.$$

The Hölder continuity of the normal as a function of position implies that the remaining weakly singular terms satisfy conditions (2.57), (2.58) and (2.60), for $\beta = \alpha$. The second term in equation (2.72) may be similarly analysed.

The operator on the left hand side of equation (2.67) is then in class $G'(\alpha)$. Its symbol matrix is

$$\Theta(\mathbf{x},\theta) = L^T(\mathbf{x}) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -\frac{(2\nu-1)}{2(1-\nu)} i\cos\theta \\ 0 & 0 & -1 & -\frac{(2\nu-1)}{2(1-\nu)} i\sin\theta \\ 0 & \frac{(2\nu-1)}{2(1-\nu)} i\cos\theta & \frac{(2\nu-1)}{2(1-\nu)} i\sin\theta & -1 \end{pmatrix} L(\mathbf{x}), \quad (2.73)$$

where $L(\mathbf{x})$ is the 4×4 square matrix with the entries given by

$$L_{ij}(\mathbf{x}) = \begin{cases} 1 & \text{if } i = j = 1 \\ 0 & \text{if } i = 1 \text{ and } j \neq 1 \\ 0 & \text{if } i \neq 1 \text{ and } j = 1 \\ l_{i-1,j-1}(\mathbf{x}) & \text{if } i \neq 1 \text{ and } j \neq 1 \end{cases},$$

and where $L^T(\mathbf{x})$ denotes the transpose of $L(\mathbf{x})$.

Obviously,

$$\det\Theta(\mathbf{x},\theta) = \frac{(2\nu-1)^2}{4(1-\nu)^2} - 1. \quad (2.74)$$

The condition

$$\inf|\det\Theta(\mathbf{x},\theta)| > 0,$$

where the infimum is taken over all points $\mathbf{x} \in \partial\Omega$ and over all angles θ, is fulfilled if

$$\mu \neq \frac{3}{4}.$$

The conditions on the Lamé constants in equations (2.15), (2.18) and (2.19) make a Poisson ratio of this value impossible. We conclude that the system of integral equations (2.65) and (2.66) is quasi-Fredholm when the solid is elastic or visco-elastic. The symbol matrix is Hermitian and so the index of the system is zero. Consequently, a unique solution to equations (2.65) and (2.66) exists if the following homogeneous problem has just one solution:

$$p + \overline{K}^{*}p - \rho\omega^2 S(\mathbf{n}.\mathbf{u}) = 0 \tag{2.75}$$

and

$$\mathbf{u} - \overline{\mathbf{K}}^{*}.\mathbf{u} - \mathbf{S}.(\mathbf{n}p) = \mathbf{0}. \tag{2.76}$$

Suppose that the system (2.75) and (2.76) has a non-trivial solution: $(p', \mathbf{u}')$ in $L^2(\partial\Omega)$. The fact that the operator is in $G'(\beta)$ implies that the solutions p' and $\mathbf{u}'$ are in $C^{1,\beta}(\partial\Omega)$.

Define

$$(Dp')(\mathbf{x}) - \rho_0\omega^2(S\mathbf{n}.\mathbf{u}')(\mathbf{x}) = \begin{cases} p_e(\mathbf{x}) & \text{if } \mathbf{x} \in \Omega_e \\ p_i(\mathbf{x}) & \text{if } \mathbf{x} \in \Omega_i \end{cases} \tag{2.77}$$

and

$$(\mathbf{D}.\mathbf{u}')(\mathbf{x}) + (\mathbf{S}.\mathbf{n}p')(\mathbf{x}) = \begin{cases} \mathbf{u}_e(\mathbf{x}) & \text{if } \mathbf{x} \in \Omega_e \\ \mathbf{u}_i(\mathbf{x}) & \text{if } \mathbf{x} \in \Omega_i \end{cases} \tag{2.78}$$

The continuity of the single and double layer potentials up to the boundary implies the continuity of p_e, p_i, $\mathbf{u}_e$ and $\mathbf{u}_i$ up to the boundary. The fact that their kernels are smooth and satisfy Helmholtz's equation, in the case of equation (2.77), and equation (2.16), in the case of equation (2.78), means that p_e and p_i are smooth and satisfy Helmholtz's equation in their respective domains and that $\mathbf{u}_e$ and $\mathbf{u}_i$ are smooth and satisfy equation (2.78) in their respective domains. Moreover, due to the far field behaviour of the kernels, p_e satisfies Sommerfeld's radiation condition and $\mathbf{u}_e$ satisfies the following radiation condition:

$$|\mathbf{x}| \left(\frac{\partial \mathbf{u}_e^p}{\partial |\mathbf{x}|} - ik_p \mathbf{u}_e^p \right) \rightarrow 0 \qquad (2.79)$$
$$|\mathbf{x}| \left(\frac{\partial \mathbf{u}_e^s}{\partial |\mathbf{x}|} - ik_p \mathbf{u}_e^s \right) \rightarrow 0,$$

as $|\mathbf{x}|$ tends to infinity, where

$$\mathbf{u}_e^p = -\frac{1}{k_p^2} \nabla(\nabla.\mathbf{u}_e)$$

and

$$\mathbf{u}_e^s = \mathbf{u}_e - \mathbf{u}_e^p.$$

From equations (2.75) and (2.77) and using the limit in equation (2.44) we have

$$p_i|_{\partial\Omega} = 0.$$

It is well known that for each compact domain D, there is only a countably infinite number of wave-numbers at which a non-trivial, square-integrable function, satisfying Helmholtz's equation in D and satisfying a homogeneous Dirichlet boundary condition on the surface of D, exists. (See e. g. Sanchez Hubert and Sanchez Palencia [27, Chapter 2].) We shall call the squares of such wave-numbers *eigenvalues of the interior Dirichlet problem.*

Suppose that k^2 is not an eigenvalue of the interior Dirichlet problem, then p_i vanishes in Ω_i. Thus

$$\left.\frac{\partial p_i}{\partial n}\right|_{\partial\Omega} = 0.$$

From the continuity of the normal derivative of the double layer potential across $\partial\Omega$ and the jump conditions (2.45) and (2.46) we obtain

$$\left.\frac{\partial p_e}{\partial n}\right|_{\partial\Omega} - \left.\frac{\partial p_i}{\partial n}\right|_{\partial\Omega} = -2\rho_0\omega^2 \mathbf{n}.\mathbf{u}'.$$

So,

$$\left.\frac{\partial p_e}{\partial n}\right|_{\partial\Omega} = -2\rho_0\omega^2 \mathbf{n}.\mathbf{u}'. \tag{2.80}$$

Evaluate p_e on the boundary:

$$p_e(\mathbf{x})|_{\partial\Omega} = -p' + \overline{K}^* p' - \rho_0\omega^2 S\mathbf{n}.\mathbf{u}' = -2p', \tag{2.81}$$

from equation (2.75).

From equation (2.76) and the jump conditions in equation (2.52), we have

$$\mathbf{u}_e(\mathbf{x})|_{\partial\Omega} = \mathbf{0}.$$

$\mathbf{u}_e$ satisfies equation (2.16) and the radiation condition (2.79). For the case of a purely elastic material the proof that $\mathbf{u}_e$ vanishes is given in Kupradze [16, pp. 132 - 136]. For the case of a visco-elastic material the proof of the same result is given in Appendix D. Therefore,

$$\mathbf{n}(\mathbf{x}).\sigma(\mathbf{u}_e)(\mathbf{x})|_{\partial\Omega} = \mathbf{0}.$$

Evaluating the jump in the surface tractions across the boundary we have

$$\mathbf{n}(\mathbf{x}).\sigma(\mathbf{u}_e)(\mathbf{x})|_{\partial\Omega} - \mathbf{n}(\mathbf{x}).\sigma(\mathbf{u}_i)(\mathbf{x})|_{\partial\Omega} = 2\mathbf{n}p',$$

from equation (2.53). Thus,

$$\mathbf{n}(\mathbf{x}).\sigma(\mathbf{u}_i)(\mathbf{x})|_{\partial\Omega} = -2\mathbf{n}p'. \tag{2.82}$$

After using equation (2.76) to evaluate $\mathbf{u}_i$ on the boundary, we have

$$\mathbf{u}_i(\mathbf{x})|_{\partial\Omega} = 2\mathbf{u}'(\mathbf{x}). \tag{2.83}$$

Equations (2.80), (2.81), (2.82) and (2.83) imply that $(-p_e, \mathbf{u}_i)$ solves the homogeneous transmission problem. We know, therefore, if the solid is either elastic or visco-elastic that

$$(-p_e, \mathbf{u}_i) = (0, \mathbf{0}),$$

unless a Jones' mode is possible.

If a Jones' mode is ruled out, then equations (2.81) and (2.83) imply that

$$(p', \mathbf{u}') = (0, 0).$$

The right hand sides of equations (2.65) and (2.66) are in $C^{1,\alpha}(\partial\Omega)$. In this case, a solution, p and $\mathbf{u}$ in $C^{1,\alpha}(\partial\Omega)$ of this system exists.

Now define

$$P = p_{inc} - \frac{1}{2}Dp + \frac{1}{2}\rho_0\omega^2 S(\mathbf{n}.\mathbf{u}) \tag{2.84}$$

and

$$\mathbf{U} = \frac{1}{2}\mathbf{D}.\mathbf{u} + \frac{1}{2}\mathbf{S}.(\mathbf{n}p). \tag{2.85}$$

It is clear from the smoothness of the kernels and the continuity up to the boundary of the potentials that

$$P|_{\Omega_e} \in C(\overline{D}) \bigcap C^2(D)$$

and

$$\mathbf{U}|_{\Omega_i} \in C(\overline{\Omega_i}) \bigcap C^2(\Omega_i),$$

as required. Because of the regularity result, the normal derivative of P and the surface traction of $\mathbf{U}$ on $\partial\Omega$ exist in the sense of limits mentioned earlier. P satisfies the Helmholtz equation in $\mathcal{R}^2 \setminus \partial\Omega$. It is clear from the construction of P that $P - p_{inc}$ satisfies the Sommerfeld radiation condition, and that $\mathbf{U}$ satisfies equation (2.16) in $\mathcal{R}^2 \setminus \partial\Omega$. Denote by P_- the limiting value of P as $\partial\Omega$ is approached from Ω_i, and by P_+ the limiting value of P as $\partial\Omega$ is approached from Ω_e. $\mathbf{U}_-$ and $\mathbf{U}_+$ are similarly defined.

From equation (2.65),

$$P_- = 0.$$

We have assumed that k^2 is not an eigenvalue of the interior Dirichlet problem and, so,

$$P = 0$$

in Ω_i. Therefore,

$$\frac{\partial P_-}{\partial n} = 0.$$

So, as before, the jump in the normal derivative of P across the boundary implies

$$\frac{\partial P_+}{\partial n} = \rho\omega^2 \mathbf{n}.\mathbf{u}. \tag{2.86}$$

Moreover,

$$P_+ = p. \tag{2.87}$$

Similarly,

$$\mathbf{n}.\sigma(\mathbf{U}_-) = -p\mathbf{n} \tag{2.88}$$

and

$$\mathbf{U}_- = \mathbf{u}. \tag{2.89}$$

Equations (2.86), (2.87), (2.88) and (2.89) imply that the transmission conditions (2.20) and (2.21) are satisfied.

Before we begin the next section, let us examine what happens when k^2 is an eigenvalue of the interior Dirichlet problem and/or k is a Jones' frequency. Let P_D now denote a non-trivial solution of the interior Dirichlet problem. By applying

Green's second representation theorem to P_D and the fundamental solution in the domain $\Omega_i \setminus B$, where B is a small, closed ball centred on a point $\mathbf{x}$ in Ω_i, and taking the limit as the radius of B tends to zero, we get.

$$P_D(\mathbf{x}) = -\frac{1}{2}\left(S\frac{\partial P_D}{\partial n}\right)(\mathbf{x}).$$

Use equation (2.45) to obtain

$$(I + K)\frac{\partial P_D}{\partial n} = 0.$$

On the boundary, we have

$$S\frac{\partial P_D}{\partial n} = 0.$$

Let $\{\mathbf{U}_J^{(i)}; i = 1, \ldots, n\}$ be a base of the space of Jones' modes. It is clear that

$$(\mathbf{I} - \overline{\mathbf{K}}^*).\mathbf{U}_J^{(i)} = 0.$$

From what we have already done, we know that

$$\mathbf{I} - \overline{\mathbf{K}}^*$$

is quasi-Fredholm. Therefore, the equation

$$(\mathbf{I} - \mathbf{K}).\mathbf{b} = 0$$

has at least n independent solutions. Let $\{\mathbf{b}^{(i)}; i = 1, \ldots, m\}$ be a base of

$$N(\mathbf{I} - \mathbf{K}).$$

It is easy to see that the space spanned by $\{\mathbf{S}.\mathbf{b}^{(i)}; i = 1, \ldots, m\}$ is the space of interior displacement fields with zero surface tractions. Thus, each Jones' mode can be expressed as $\mathbf{S}.\mathbf{b}$, for some $\mathbf{b}$. Clearly,

$$(\mathbf{I} - \mathbf{K}).\mathbf{b} = 0$$

and

$$\mathbf{n}.\mathbf{S}.\mathbf{b} = 0.$$

The adjoint, homogeneous version of the system (2.65) and (2.66),

$$(I + K)a - \mathbf{n}.\mathbf{S}.\mathbf{b} = 0$$

and

$$(\mathbf{I} - \mathbf{K}).\mathbf{b} - \rho_0\omega^2\mathbf{n}Sa = \mathbf{0},$$

has the solution

$$a = \left.\frac{\partial P_D}{\partial n}\right|_{\partial\Omega}$$

and with $\mathbf{b}$ as above.

Conversely, it can easily be shown that the only solution of the adjoint system has

$$\mathbf{n}.\mathbf{S}.\mathbf{b} = 0,$$

$$(\mathbf{I} - \mathbf{K}).\mathbf{b} = \mathbf{0},$$

$$Sa = 0$$

and

$$(I + K)a = 0.$$

From equation (2.39) we have

$$2p_{inc} = p_{inc} + \overline{K}^* p_{inc} - S\frac{\partial p_{inc}}{\partial n}.$$

Therefore, the inner product of $(2p_{inc}, \mathbf{0})$ with $(a, \mathbf{b})$, with a and $\mathbf{b}$ as above, equals

$$< p_{inc} + \overline{K}^* p_{inc} - S\frac{\partial p_{inc}}{\partial n}, a > .$$

This equals

$$< p_{inc}, (I+K)a > - < \frac{\partial p_{inc}}{\partial n}, Sa >= 0.$$

Therefore, the system is solvable at all frequencies. Thus, the transmission problem is solvable at all frequencies. The system is singular at eigenvalues of the interior Dirichlet problem and at Jones' frequencies. The singularities at eigenvalues of the interior Dirichlet problem are spurious because we know that the transmission problem is uniquely solvable at these frequencies, unless, of course, they happen to coincide with Jones' frequencies.

The system derived in the next section will not improve on this result. It is included only as an example of an indirect method. Indirect in this context means that the quantities found are not in themselves physically relevant. In contrast to indirect methods, direct methods, are those in which the quantities found are physically relevant. The method we have just used was direct.

2.6.2 An indirect method #1.

Look for a solution of the form

$$\mathbf{u} = \mathbf{S}.\mathbf{g} \tag{2.90}$$

and

$$p = S\mu + p_{inc}. \tag{2.91}$$

We require that both $\mathbf{g}$ and μ belong to $C^{0,\beta}(\partial\Omega)$, for some positive constant β.

Equation (2.90) implies

$$\mathbf{n}.\sigma(\mathbf{u})|_{\partial\Omega} = -\mathbf{g} + \mathbf{K}.\mathbf{g} \tag{2.92}$$

and equation (2.91) implies

$$\left.\frac{\partial p}{\partial n}\right|_{\partial\Omega} = \mu + K\mu + \left.\frac{\partial p_{inc}}{\partial n}\right|_{\partial\Omega}. \tag{2.93}$$

Furthermore,

$$\mathbf{u}|_{\partial\Omega} = \mathbf{S}.\mathbf{g} \tag{2.94}$$

and

$$p|_{\partial\Omega} = S\mu + p_{inc}. \tag{2.95}$$

Using the transmission conditions (2.20) and (2.21), we obtain

$$\mu + K\mu - \rho_0\omega^2\mathbf{n}.\mathbf{S}.\mathbf{g} = -\frac{\partial p_{inc}}{\partial n} \tag{2.96}$$

and

$$-\mathbf{g} + \mathbf{K}.\mathbf{g} + \mathbf{n}S\mu = -p_{inc}\mathbf{n}. \tag{2.97}$$

We discovered in the previous section that the system

$$\begin{pmatrix} I + \overline{K}^* & -\rho_0\omega^2 S\mathbf{n} \\ -\mathbf{S}.\mathbf{n} & \mathbf{I} - \overline{\mathbf{K}}^* \end{pmatrix} \tag{2.98}$$

is quasi-Fredholm. This implies that the system

$$\begin{pmatrix} I + \overline{K}^* & -S\mathbf{n} \\ -\rho_0\omega^2\mathbf{S}.\mathbf{n} & \mathbf{I} - \overline{\mathbf{K}}^* \end{pmatrix} \tag{2.99}$$

is quasi-Fredholm too, because its singular part is identical to that of the former system. Because the index is zero, the relationship between this system and its adjoint is symmetric. Therefore,

$$\begin{pmatrix} I + K & -\rho_0\omega^2\mathbf{n}.\mathbf{S} \\ \mathbf{n}S & \mathbf{I} - \mathbf{K} \end{pmatrix} \tag{2.100}$$

is quasi-Fredholm.

Thus, the system in equations (2.96) and (2.97) is quasi-Fredholm. In addition to this, the system is solvable if and only if k^2 is not an eigenvalue of the interior Dirichlet problem and a Jones' mode is not possible. To see this, note that, due to the vanishing index, the null-space of the system (2.100) has the same dimension as the null space of the system (2.99). We have the easily verifiable identity

$$\begin{pmatrix} 1 & 0 \\ 0 & \rho_0\omega^2\mathbf{I} \end{pmatrix}\begin{pmatrix} I+\overline{K}^* & -\rho_0\omega^2 S\mathbf{n} \\ -\mathbf{S}.\mathbf{n} & \mathbf{I}-\overline{\mathbf{K}}^* \end{pmatrix}$$
$$= \begin{pmatrix} I+\overline{K}^* & -S\mathbf{n} \\ -\rho_0\omega^2\mathbf{S}.\mathbf{n} & \mathbf{I}-\overline{\mathbf{K}}^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \rho_0\omega^2\mathbf{I} \end{pmatrix}.$$

Clearly the null space of system (2.98) has the same dimension as the null space of system (2.99). The claim then follows.

If k^2 is not an eigenvalue of the interior Dirichlet problem and ω is not a Jones' frequency, then the system (2.96) and (2.97) has the unique solution $(\mu, \mathbf{g})$.

The pressure and displacement fields are then given by the equations (2.90) and (2.91). As for the direct formulation, it is easily verified that this is indeed the solution of the transmission problem.

2.6.3 Single integral equation.

In this section, we shall derive a system of three equations in three unknowns that has no irregular frequencies.

We have the *ansatz*

$$\mathbf{u} = \mathbf{S}.\mathbf{n}g + \mathbf{S}.\mathbf{f}, \tag{2.101}$$

where

$$\mathbf{f}.\mathbf{n} = 0$$

and g and $\mathbf{f}$ belong to $C^{0,\beta}(\partial\Omega)$, for some positive constant β.

From Green's second representation theorem and the jump conditions (2.44) and (2.46), we have

$$p + \overline{K}^* p - S\frac{\partial p}{\partial n} = 2p_{inc} \tag{2.102}$$

and

$$\frac{\partial p}{\partial n} - K\frac{\partial p}{\partial n} + Np = 2\frac{\partial p_{inc}}{\partial n}. \tag{2.103}$$

Use the transmission conditions and equation (2.102) to obtain

$$\mathbf{n}.\sigma(\mathbf{u}).\mathbf{n} + \overline{K}^*(\mathbf{n}.\sigma(\mathbf{u}).\mathbf{n}) + \rho_0\omega^2 S(\mathbf{u}.\mathbf{n}) = -2p_{inc}. \tag{2.104}$$

From equation (2.101) we have

$$\mathbf{n}.\sigma(\mathbf{u}) = -g\mathbf{n} - \mathbf{f} + \mathbf{K}.\mathbf{n}g + \mathbf{K}.\mathbf{f}. \tag{2.105}$$

Now substitute equation (2.105) into equation (2.104) to give

$$\begin{aligned} g - \mathbf{n}.\mathbf{K}.\mathbf{n}g - \mathbf{n}.\mathbf{K}.\mathbf{f} + \overline{K}^* g - \overline{K}^*(\mathbf{n}.\mathbf{K}.\mathbf{n})g - \overline{K}^*(\mathbf{n}.\mathbf{K}).\mathbf{f} \\ -\rho_0\omega^2 S(\mathbf{n}.\mathbf{S}.\mathbf{n})g - \rho_0\omega^2 S(\mathbf{n}.\mathbf{S}).\mathbf{f} = 2p_{inc}. \end{aligned} \tag{2.106}$$

Similarly, by using equations (2.101) and (2.103) and the transmission conditions one can obtain

$$\begin{aligned} \rho_0\omega^2\mathbf{n}.\mathbf{S}.\mathbf{n}g + \rho_0\omega^2\mathbf{n}.\mathbf{S}.\mathbf{f} - \rho_0\omega^2 K(\mathbf{n}.\mathbf{S}.\mathbf{n})g - \rho_0\omega^2 K(\mathbf{n}.\mathbf{S}).\mathbf{f} \\ +Ng - N(\mathbf{n}.\mathbf{K}.\mathbf{n})g - N(\mathbf{n}.\mathbf{K}).\mathbf{f} = 2\frac{\partial p_{inc}}{\partial n}. \end{aligned} \tag{2.107}$$

Now let us add equation (2.106) to $i\eta$ times equation (2.107), where η is a constant to be chosen later. We have

$$g - \mathbf{n}.\mathbf{K}.\mathbf{n}g - \mathbf{n}.\mathbf{K}.\mathbf{f} + \overline{K}^* g - \overline{K}^*(\mathbf{n}.\mathbf{K}.\mathbf{n})g - \overline{K}^*(\mathbf{n}.\mathbf{K}).\mathbf{f} \tag{2.108}$$

$$-\rho_0\omega^2 S(\mathbf{n}.\mathbf{S}.\mathbf{n})g - \rho_0\omega^2 S(\mathbf{n}.\mathbf{S}).\mathbf{f} + i\eta\rho_0\omega^2\mathbf{n}.\mathbf{S}.\mathbf{n}g$$
$$+i\eta\rho_0\omega^2\mathbf{n}.\mathbf{S}.\mathbf{f} - i\eta\rho_0\omega^2 K(\mathbf{n}.\mathbf{S}.\mathbf{n})g - i\eta\rho_0\omega^2 K(\mathbf{n}.\mathbf{S}).\mathbf{f}$$
$$+i\eta Ng - i\eta N(\mathbf{n}.\mathbf{K}.\mathbf{n})g - i\eta N(\mathbf{n}.\mathbf{K}).\mathbf{f} = 2p_{inc} + 2i\eta\frac{\partial p_{inc}}{\partial n}.$$

Moreover, since

$$\sigma(\mathbf{u}).\mathbf{n} - \mathbf{n}\mathbf{n}.\sigma(\mathbf{u}).\mathbf{n} = \mathbf{0},$$

$$\mathbf{f} - (\mathbf{K}.\mathbf{n})g + \mathbf{n}(\mathbf{n}.\mathbf{K}.\mathbf{n})g - \mathbf{K}.\mathbf{f} + \mathbf{n}(\mathbf{n}.\mathbf{K}).\mathbf{f} = \mathbf{0}. \tag{2.109}$$

We now require that g and $\mathbf{f}$ belong to $C^{1,\beta}(\partial\Omega)$, for some $\beta \in (0,1)$.

Before we prove that the system (2.108) and (2.109) is quasi-Fredholm, let us first see whether the homogeneous version of the system has a non-trivial solution. Suppose $(g', \mathbf{f}')$ is a solution of the homogeneous system. In addition g' and $\mathbf{f}'$ belong to $C^{1,\beta}(\partial\Omega)$, for some $\beta \in (0,1)$. Let

$$\mathbf{u} = \mathbf{S}.\mathbf{n}g' + \mathbf{S}.\mathbf{f}' \tag{2.110}$$

and

$$p = \frac{\rho_0\omega^2}{2}S(\mathbf{n}.\mathbf{S}.\mathbf{n})g' + \frac{\rho_0\omega^2}{2}S(\mathbf{n}.\mathbf{S}).\mathbf{f}' - \frac{1}{2}Dg' + \frac{1}{2}D(\mathbf{n}.\mathbf{K}.\mathbf{n})g' - \frac{1}{2}D\mathbf{n}.\mathbf{f}' + \frac{1}{2}D(\mathbf{n}.\mathbf{K}).\mathbf{f}'. \tag{2.111}$$

Equations (2.109) and (2.110) imply that

$$\sigma(\mathbf{u}).\mathbf{n} - \mathbf{n}\mathbf{n}.\sigma(\mathbf{u}).\mathbf{n} = \mathbf{f}'.\mathbf{n}\mathbf{n}.$$

Therefore,

$$\mathbf{f}'.\mathbf{n} = 0$$

and

$$\mathbf{n}\mathbf{n}.\sigma(\mathbf{u}).\mathbf{n} = \sigma(\mathbf{u}).\mathbf{n}.$$

Let us once again adopt the notation that a negative (resp. positive) subscript denotes that a limit has been taken on $\partial\Omega$ as the boundary is approached from Ω_i (resp. Ω_e).

Equation (2.111) implies that

$$\begin{aligned} p_- = {} & \frac{\rho_0\omega^2}{2}S(\mathbf{n}.\mathbf{S}.\mathbf{n})g' + \frac{\rho_0\omega^2}{2}S(\mathbf{n}.\mathbf{S}).\mathbf{f}' - \frac{1}{2}g' - \frac{1}{2}\overline{K}^* g' \\ & + \frac{1}{2}\mathbf{n}.\mathbf{K}.\mathbf{n}g' + \frac{1}{2}\overline{K}^*(\mathbf{n}.\mathbf{K}.\mathbf{n})g' + \frac{1}{2}\mathbf{n}.\mathbf{K}.\mathbf{f}' + \frac{1}{2}\overline{K}^*(\mathbf{n}.\mathbf{K}).\mathbf{f}' \end{aligned} \tag{2.112}$$

and

$$\begin{aligned} \frac{\partial p_-}{\partial n} = {} & -\frac{\rho_0\omega^2}{2}\mathbf{n}.\mathbf{S}.\mathbf{n}g' - \frac{\rho_0\omega^2}{2}\mathbf{n}.\mathbf{S}.\mathbf{f}' + \frac{\rho_0\omega^2}{2}K(\mathbf{n}.\mathbf{S}.\mathbf{n})g' \\ & + \frac{\rho_0\omega^2}{2}K(\mathbf{n}.\mathbf{S}).\mathbf{f}' - \frac{1}{2}Ng' + \frac{1}{2}N(\mathbf{n}.\mathbf{K}.\mathbf{n})g' + \frac{1}{2}N(\mathbf{n}.\mathbf{K}).\mathbf{f}'. \end{aligned} \tag{2.113}$$

Equations (2.112) and (2.113), and the fact that $(g', \mathbf{f}')$ is a solution of the homogeneous version of equation (2.108), imply that

$$p_- + i\eta\frac{\partial p_-}{\partial n} = 0.$$

Therefore,

$$0 = \int_{\partial\Omega} \overline{p_-}\left(p_- + i\eta\frac{\partial p_-}{\partial n}\right) dS.$$

By the divergence theorem and Helmholtz's equation

$$0 = \int_{\partial\Omega} |p_-|^2 dS - i\eta k^2 \int_{\Omega_i} |p|^2 dV + i\eta \int_{\Omega_i} \nabla p.\nabla\overline{p} dV.$$

Assuming that $\Im(k^2) \geq 0$ then choose η to equal

$$\frac{\overline{k}^2}{|k|^2}.$$

Now take the real part of the last equation. Clearly p_- vanishes. If $\Im(k^2) > 0$ then the gradient of p vanishes in Ω_i. Consequently, p itself vanishes.

If k^2 is real then p_- still vanishes. To complete the analysis, let us note that we obtain a very similar equation by considering

$$\int_{\partial\Omega} \frac{\partial \overline{p_-}}{\partial n}\left(p_- + i\eta\frac{\partial p_-}{\partial n}\right) dS.$$

Now we have

$$0 = i\eta\int_{\partial\Omega}\left|\frac{\partial p_-}{\partial n}\right|^2 dS - k^2\int_{\Omega_i}|p|^2 dV + \int_{\Omega_i}\nabla p.\nabla\overline{p}dV.$$

Taking the imaginary part of this last equation, with η equal to 1 or -1, we have that

$$\frac{\partial p_-}{\partial n} = 0.$$

No non-trivial solution of Helmholtz's equation in Ω_i which vanishes on $\partial\Omega$ and whose normal derivative vanishes on $\partial\Omega$ exists. Therefore, p vanishes in Ω_i.

So

$$\frac{\partial p_-}{\partial n} = 0.$$

The jump conditions (2.45) and (2.46) imply that

$$\frac{\partial p_+}{\partial n} = \rho_0\omega^2\mathbf{n}.\mathbf{S}.\mathbf{n}g' + \rho_0\omega^2\mathbf{n}.\mathbf{S}.\mathbf{f}'. \tag{2.114}$$

Moreover,

$$\mathbf{u}_- = \mathbf{S}.\mathbf{n}g' + \mathbf{S}.\mathbf{f}'. \tag{2.115}$$

We have

$$\sigma(\mathbf{u}_-).\mathbf{n} = -g'\mathbf{n} - \mathbf{f}' + \mathbf{K}.\mathbf{n}g' + \mathbf{K}.\mathbf{f}' \tag{2.116}$$

and

$$p_+ = g' - \mathbf{n}.\mathbf{K}.\mathbf{n}g' - \mathbf{n}.\mathbf{K}.\mathbf{f}', \tag{2.117}$$

since p_- equals 0.

The transmission conditions are satisfied by $p|_{\Omega_e}$ and $\mathbf{u}|_{\Omega_i}$. Clearly the field equations are also satisified and $p|_{\Omega_e}$ satisfies the Sommerfeld radiation condition. Therefore,

$$(p|_{\Omega_e}, \mathbf{u}|_{\Omega_i}) = (0, \mathbf{0}),$$

unless a Jones' mode is possible. If not then $\mathbf{u}_-$ vanishes. The continuity of the single layer potential implies that $\mathbf{u}_+$ vanishes. Therefore, by Kupradze [16],

$$\mathbf{u}|_{\Omega_e} = \mathbf{0}.$$

Clearly,

$$\mathbf{0} = \sigma(\mathbf{u}_+) - \sigma(\mathbf{u}_-) = 2g'\mathbf{n} + 2\mathbf{f}'.$$

Since $\mathbf{f}'.\mathbf{n}$ vanishes then

$$g' = 0$$

and

$$\mathbf{f}' = \mathbf{0}.$$

So the system is unique unless ω is a Jones' frequency.

We must now prove the existence of solutions to the system (2.108) and (2.109). Let us write

$$N = N_0 + (N - N_0),$$

where N_0 is to k_0, a wavenumber whose square is neither an eigenvalue of the interior Dirichlet problem nor an eigenvalue of the interior Neumann problem — i. e. the interior problem with homogeneous Neumann conditions — what N is to k.

N_0 is invertible with

$$N_0^{-1} = S_0(I + K_0)^{-1}(-I + K_0)^{-1},$$

where S_0 and K_0 have analogous definitions to N_0. See Colton and Kress [4, p. 90]. It is clear from the last formula that N_0^{-1} is compact on $C^{0,\beta}(\partial\Omega)$ and that N_0^{-1} maps $C^{0,\beta}(\partial\Omega)$ into $C^{1,\beta}(\partial\Omega)$.

$(N - N_0)$ is compact on $C^{0,\beta}(\partial\Omega)$ since its kernel is weakly singular. Its kernel also satisfies the conditions (2.57), (2.58) and (2.60).

By operating on the left on equation (2.108) with N_0^{-1}, we obtain

$$\begin{aligned} N_0^{-1}g - N_0^{-1}(\mathbf{n}.\mathbf{K}.\mathbf{n})g - N_0^{-1}(\mathbf{n}.\mathbf{K}).\mathbf{f} + N_0^{-1}\overline{K}^*g - N_0^{-1}\overline{K}^*(\mathbf{n}.\mathbf{K}.\mathbf{n})g \quad (2.118) \\ -N_0^{-1}(\overline{K}^*\mathbf{n}.\mathbf{K}).\mathbf{f} - N_0^{-1}\mathbf{n}.\mathbf{K}.\mathbf{f} - \rho_0\omega^2 N_0^{-1}S(\mathbf{n}.\mathbf{S}.\mathbf{n})g - \rho_0\omega^2 N_0^{-1}S(\mathbf{n}.\mathbf{S}).\mathbf{f} \\ +i\eta\rho_0\omega^2 N_0^{-1}(\mathbf{n}.\mathbf{S}.\mathbf{n})g + i\eta\rho_0\omega^2 N_0^{-1}(\mathbf{n}.\mathbf{S}).\mathbf{f} - i\eta\rho_0\omega^2 N_0^{-1}K(\mathbf{n}.\mathbf{S}.\mathbf{n})g \\ -i\eta\rho_0\omega^2 N_0^{-1}K(\mathbf{n}.\mathbf{S}).\mathbf{f} + i\eta g - i\eta\mathbf{n}.\mathbf{K}.\mathbf{n}g - i\eta\mathbf{n}.\mathbf{K}.\mathbf{f} + i\eta N_0^{-1}(N - N_0)g \\ -i\eta N_0^{-1}(N - N_0(\mathbf{n}.\mathbf{K}.\mathbf{n})g - i\eta N_0^{-1}(N - N_0(\mathbf{n}.\mathbf{K}).\mathbf{f} = 2N_0^{-1}\left(p_{inc} + i\eta\frac{\partial p_{inc}}{\partial n}\right). \end{aligned}$$

By dividing equation (2.118) through by $i\eta$, we can see that

$$g - (\mathbf{n}.\mathbf{K}).\mathbf{f} + \text{ compact terms } = \frac{2}{\eta}N_0^{-1}\left(\eta\frac{\partial p_{inc}}{\partial n} - ip_{inc}\right). \qquad (2.119)$$

We can see that the system (2.109) is in $G'(\beta)$, for $0 < \beta \leq \alpha$. It is easy to calculate the symbol for this system. It is

$$\Theta(\mathbf{x},\theta) = l^T(\mathbf{x})\begin{pmatrix} 1 & \frac{(2\nu-1)}{2(1-\nu)}i\cos\theta & \frac{(2\nu-1)}{2(1-\nu)}i\sin\theta \\ -\frac{(2\nu-1)}{2(1-\nu)}i\cos\theta & 1 & 0 \\ -\frac{(2\nu-1)}{2(1-\nu)}i\sin\theta & 0 & 1 \end{pmatrix} l(\mathbf{x}),$$

where $l(\mathbf{x})$ is the rotation matrix introduced in subsection 2.6.1. Clearly,

$$\sup_{\substack{\mathbf{x}\in\partial\Omega \\ \theta\in[0,2\pi]}} |\det\Theta(\mathbf{x},\theta)| > 0$$

for all feasible values of Poisson's ratio.

We know from this that the only solution of the homogeneous system is the trivial one. Thus, the Fredholm property of the system (2.109) and (2.118) implies the existence of a solution in $C^{1,\alpha}(\partial\Omega)$. Once the solution, $(g, \mathbf{f})$, is known, the solution to the transmission problem is

$$\mathbf{u} = \mathbf{S}.(\mathbf{n}g) + \mathbf{S}.\mathbf{f}$$

and

$$\begin{aligned} p = \frac{1}{2}\rho_0\omega^2 S(\mathbf{n}.\mathbf{S}.\mathbf{n})g + \frac{1}{2}\rho_0\omega^2 S(\mathbf{n}.\mathbf{S}).\mathbf{f} - \frac{1}{2}Dg \\ + \frac{1}{2}(\mathbf{n}.\mathbf{K}.\mathbf{n})g + \frac{1}{2}(\mathbf{n}.\mathbf{K}).\mathbf{f} + p_{inc}. \end{aligned}$$

2.6.4 An indirect method #2.

We now conclude this chapter by describing another indirect method.

Let us begin by representing p and $\mathbf{u}$ as

$$p = S\mu + iD\mu + p_{inc} \tag{2.120}$$

and

$$\mathbf{u} = \mathbf{S}.\mathbf{n}g + \mathbf{S}.\mathbf{f}. \tag{2.121}$$

We require that μ belongs to $C^{1,\beta}(\partial\Omega)$ and that g and $\mathbf{f}$ belong to $C^{0,\beta}(\partial\Omega)$, for some positive β.

Now apply the transmission conditions (2.20) and (2.21). We obtain

$$\mu + K\mu + iN\mu - \rho_o\omega^2(\mathbf{n}.\mathbf{S}.\mathbf{n})g - \rho_o\omega^2(\mathbf{n}.\mathbf{S}).\mathbf{f} = -\frac{\partial p_{inc}}{\partial n}, \tag{2.122}$$

$$g - (\mathbf{n}.\mathbf{K}.\mathbf{n})g - \mathbf{n}.\mathbf{K}.\mathbf{f} + i\mu - i\overline{K}^*\mu - S\mu = p_{inc} \tag{2.123}$$

and

$$\mathbf{f} - \mathbf{K}.\mathbf{f} - (\mathbf{K}.\mathbf{n})g + \mathbf{n}(\mathbf{n}.\mathbf{K}).\mathbf{f} + \mathbf{n}(\mathbf{n}.\mathbf{K}.\mathbf{n})g = \mathbf{0}. \tag{2.124}$$

Equation (2.124) comes from

$$\sigma(\mathbf{u}).\mathbf{n} - \mathbf{n}(\mathbf{n}.\sigma(\mathbf{u}).\mathbf{n}) = \mathbf{0}.$$

Let μ' in $C^{1,\beta}(\partial\Omega)$ and $(g', \mathbf{f}')$ in $C^{0,\beta}(\partial\Omega)$ solve the homogeneous version of the system (2.122) to (2.124).

Let

$$P = S\mu' + iD\mu'$$

and

$$\mathbf{U} = \mathbf{S}.\mathbf{n}g' + \mathbf{S}.\mathbf{f}'.$$

Equation (2.124) implies that

$$\mathbf{n}.\mathbf{f}' = 0.$$

This and the homogeneous version of equation (2.122) imply that

$$\frac{\partial P_+}{\partial n} = \rho_0\omega^2\mathbf{U}_-.\mathbf{n}. \tag{2.125}$$

The homogeneous version of equation (2.123) implies that

$$\sigma(\mathbf{U}_-).\mathbf{n} = -P_+\mathbf{n}. \tag{2.126}$$

So unless ω is a Jones' frequency, $P|_{\Omega_e}$ and $\mathbf{U}|_{\Omega_i}$ both vanish. So assume that ω is not a Jones' frequency. The continuity of the single layer potential implies that $\mathbf{U}_+$ vanishes. This in turn implies that $\mathbf{U}|_{\Omega_e}$ vanishes. The jump in the surface tractions due to the displacement fields in Ω_e and Ω_i clearly vanishes. From the jump conditions (2.53) we have

$$\sigma(\mathbf{U}_+).\mathbf{n} - \sigma(\mathbf{U}_-).\mathbf{n} = 2g'\mathbf{n} + 2\mathbf{f}'.$$

Clearly then both g' and $\mathbf{f}'$ vanish.

Furthermore,

$$P_- = P_+ + 2i\mu' = 2i\mu'$$

and

$$\frac{\partial P_-}{\partial n} = \frac{\partial P_+}{\partial n} - 2\mu' = -2\mu'.$$

So,

$$P_- + i\frac{\partial P_-}{\partial n} = 0. \tag{2.127}$$

Therefore,

$$\begin{aligned} 0 &= \textstyle\int_{\partial\Omega} P\left(\overline{P}_- - i\frac{\partial \overline{P}_-}{\partial n}\right) dS \\ &= \textstyle\int_{\partial\Omega} |P_-|^2 dS - i\int_{\Omega_i} \nabla P.\nabla\overline{P} dV + i\overline{k}^2 \int_{\Omega_i} |P|^2 dV. \end{aligned} \tag{2.128}$$

Taking the real part of equation (2.128) implies

$$0 = \int_{\partial\Omega} |P_-|^2 dS + \Im(k^2)\int_{\Omega_i} |P|^2 dV. \tag{2.129}$$

We assumed that $\Im(k^2) \geq 0$. Then P_- vanishes and from equation (2.127),

$$\frac{\partial P_-}{\partial n} = 0.$$

Green's second representation theorem implies that P vanishes in Ω_i. Therefore, μ' vanishes and the homogeneous version of equations (2.122) to (2.124) has only the trivial solution when μ lies in $C^{1,\beta}(\partial\Omega)$.

We must now prove the existence of solutions to equations (2.122) to (2.124). The first of these three equations clearly has a hypersingular term. To find a regularizer for this, we write

$$N = N_0 + (N - N_0),$$

where, once again, N_0 represents the gradient of the double layer potential for a wavenumber that is an eigenvalue of neither the interior Neumann nor the interior Dirichlet problems. Thus it is invertible. Its inverse and $(N - N_0)$ are compact on $C^{0,\beta}(\partial\Omega)$. Equation (2.122) becomes, when operated on on the left by N_0^{-1},

$$i\mu + iN_0^{-1}(N - N_0)\mu + N_0^{-1}\mu + N_0^{-1}K\mu - \rho_0 \qquad (2.130)$$
$$\omega^2 N_0^{-1}(\mathbf{n}.\mathbf{S}.\mathbf{n})g - \rho_0\omega^2 N_0^{-1}(\mathbf{n}.\mathbf{S}).\mathbf{f} = -N_0^{-1}\frac{\partial p_{inc}}{\partial n}.$$

Let us rewrite the equations (2.122) to (2.124) as

$$\mu + \text{ compact terms } = iN_0^{-1}\frac{\partial p_{inc}}{\partial n},$$

$$g - (\mathbf{n}.\mathbf{K}.\mathbf{n})g - (\mathbf{n}.\mathbf{K}).\mathbf{f} + i\mu + \text{ compact terms } = p_{inc}$$

and

$$\mathbf{f} - \mathbf{K}.\mathbf{n}g - \mathbf{K}.\mathbf{f} + \mathbf{n}(\mathbf{n}.\mathbf{K}.\mathbf{n})g + \mathbf{n}(\mathbf{n}.\mathbf{K}).\mathbf{f} = 0.$$

As before, $\mathbf{n}.\mathbf{K}.\mathbf{n}$ is actually compact on $C^{0,\beta}(\partial\Omega)$. The singular part of $\mathbf{n}.\mathbf{K}$ is

$$-\frac{(2\nu - 1)X_k}{4\pi(1-\nu)R^3} \qquad k = 1, 2, 3.$$

The singular part of $\mathbf{K}.\mathbf{n}$ is

$$\frac{(2\nu - 1)X_k}{4\pi(1-\nu)R^3} \qquad k = 1, 2, 3.$$

The singular part of $\mathbf{K} - \mathbf{nn}.\mathbf{K}$ is

$$\frac{(2\nu - 1)n_m(\mathbf{y})X_k}{4\pi(1-\nu)R^3} \qquad k, m = 1, 2, 3.$$

Here the coordinates have been rotated so the normal to $\partial\Omega$ at $\mathbf{x}$ points in the $\mathbf{e}_3$ direction. The symbol matrix is, once again, simple to calculate and the result is identical to the preceeding results. The system is, thus, quasi-Fredholm with index zero, for all feasible values of Poisson's ratio.

We must now turn our attention to the regularity of the solutions to the system. Specifically, we must show that the function μ in $C^{0,\beta}(\partial\Omega)$, that is a solution of the system, is actually in the smaller space $C^{1,\beta}(\partial\Omega)$, for some positive β. To do this, note that we have

$$\mu + \text{ compact terms } = iN_0^{-1}\frac{\partial p_{inc}}{\partial n}$$

and that the label "compact terms" refers to terms in equation (2.130) that are of the form

$$N_0^{-1}\phi,$$

where ϕ belongs to $C^{0,\beta}(\partial\Omega)$. The mapping propeties of N_0^{-1} indicate that the image of ϕ under N_0^{-1} belongs to $C^{1,\beta}(\partial\Omega)$. Obviously, μ also belongs to this class.

This regularity property of the solutions, the uniqueness property of the homogeneous version of the system and the quasi-Fredholm property of the system with the vanishing of the index imply that the system is always solvable unless ω is a Jones' frequency.

2.7 Conclusion.

We have seen four different approaches to tackling the transmission problem by means of integral equation methods. The first two systems involved integral operators that had considerably simpler kernels than the final two systems. This latter pair had the advantage of not having had spurious frequencies at which the system of integral equations was singular but the real problem was not.

The main theoretical result of this chapter was the proof of the existence of a solution to the transmission problem at all frequencies, for both the elastic and visco-elastic cases, and the proof of uniqueness at all frequencies other than Jones' frequencies.

Chapter 3

Elastic Polygon — Acoustic Medium

3.1 Introduction.

In this chapter we shall consider a problem that is very similar to the one considered in the previous chapter. We are interested now in the effect that edges have on the solutions to the transmission problem. We know that when the elastic body occupies a sufficiently smooth domain the solution is as smooth as the datum, that is to say, the incident wave. This, as we shall see, is no longer the case when the body has an edge.

For simplicity, we shall consider only two-dimensional polygonal domains. By this means the essential feature of the problem will be isolated. The extension to three-dimensional problems of bodies with edges is straightforward.

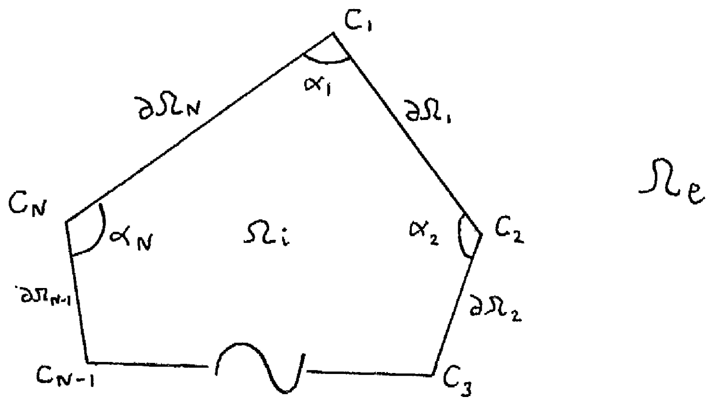

Figure 3.1: The polygonal domain.

3.2 Preliminaries.

We shall denote by Ω_i the compact, open subset of $\mathcal{R}^2$ that is occupied by the elastic body. Ω_e will denote the set $\mathcal{R}^2 \setminus \overline{\Omega_i}$ and $\partial\Omega$, the boundary, is the complement of $\Omega_i \bigcup \Omega_e$. Let us suppose that the polygonal domain, Ω_i, has N corners. Let us label these

$$\{C_1, C_2, \ldots, C_N\}.$$

Let us denote by $\partial\Omega_j$, for $j \in \{1, 2, \ldots, N-1\}$, the edge that joins C_j to C_{j+1}. $\partial\Omega_N$ is the edge that joins C_N to C_1. Let us denote by α_i the interior angle at the corner C_i. See Figure (3.1).

The equations satisfied by the displacement field in Ω_i and by the pressure field in Ω_e are unchanged. This is true of the transmission conditions too. Recalling equations (2.13) and (2.16), we have

$$\nabla^2 p + k^2 p = 0 \text{ in } \Omega_e \tag{3.1}$$

and

$$(\lambda + \mu)\nabla(\nabla.\mathbf{u}) + \mu\nabla^2\mathbf{u} + \rho\omega^2\mathbf{u} = \mathbf{0} \text{ in } \Omega_i. \tag{3.2}$$

Let us rewrite the transmission conditions (2.20) and (2.21)

$$\left.\frac{\partial p}{\partial n}\right|_{\partial\Omega} = \rho_0\omega^2 \mathbf{n}.\mathbf{u}|_{\partial\Omega} \tag{3.3}$$

and

$$-(p\mathbf{n})|_{\partial\Omega} = (\sigma(\mathbf{u}).\mathbf{n})|_{\partial\Omega}. \tag{3.4}$$

We split the pressure field up into an incident part, p_{inc}, and a scattered part, p_s. The scattered part satisfies the Sommerfeld radiation condition

$$\left|\frac{\mathbf{x}}{|\mathbf{x}|}.\nabla p_s - ikp_s\right|^2 = o\left(\frac{1}{|\mathbf{x}|}\right) \tag{3.5}$$

as $|\mathbf{x}| \to \infty$. This is a modified version of the radiation condition of equation (2.22). The modification is due to the reduction of dimension from three to two.

We shall search for a solution of the transmission problem expressed in equations (3.1) to (3.5) with

$$p \in H^s_{loc}(\Omega_e)$$

and

$$\mathbf{u} \in H^s(\Omega_i),$$

for s lying in the range $(\frac{1}{2}, \frac{3}{2})$.

For $p \in H^s_{loc}(\Omega_e)$, with $\frac{1}{2} < s < \frac{3}{2}$, one can define the trace of p on $\partial\Omega$. The trace map is continuous from $H^s_{loc}(\Omega_e)$ into $H^{s-\frac{1}{2}}(\partial\Omega)$. This is Gagliardo's trace lemma. The same, of course, applies to the trace map from $H^s(\Omega_i)$ into $H^{s-\frac{1}{2}}(\partial\Omega)$. These trace maps have right continuous inverses, which are sometimes called lifting operators.

Suppose now that $p \in H^1_{loc}(\Omega_e)$ and p satisfies Helmholtz's equation. Then one can define the normal derivative of p on $\partial\Omega$ to be the unique element of

$H^{-\frac{1}{2}}(\partial\Omega)$ satisfying

$$< \frac{\partial p}{\partial n}, \phi >_{H^{-\frac{1}{2}}(\partial\Omega)H^{\frac{1}{2}}(\partial\Omega)} = k^2 \int_{\Omega_e} p\overline{\Phi}dV - \int_{\Omega_e} \nabla p.\nabla\overline{\Phi}dV,$$

where ϕ is any member of $H^{\frac{1}{2}}(\partial\Omega)$, Φ is a lifting of ϕ that vanishes in a neighbourhood of infinity, $\overline{\Phi}$ denotes the complex conjugate of Φ and the angled brackets denote the duality product. Similarly, for any solution $\mathbf{u}$, of equation (3.2) in $H^1(\Omega_i)$, its surface traction can be uniquely defined as an element of $H^{-\frac{1}{2}}(\partial\Omega)$ through the equation

$$< \mathbf{n}.\sigma(\mathbf{u}), \mathbf{v} >_{H^{-\frac{1}{2}}(\partial\Omega)H^{\frac{1}{2}}(\partial\Omega)} = \int_{\Omega_i} \sigma(\mathbf{u}) : \nabla\mathbf{v}'dV - \rho\omega^2 \int_{\Omega_i} \mathbf{u}.\mathbf{v}'dV,$$

where now $\mathbf{v}'$ is a lifting of $\mathbf{v}$.

We shall later require the following uniqueness result.

Theorem 7 *Let $\Re\omega^2 \geq 0$. Suppose p_{inc} vanishes and that $\mathbf{u} \in H^1(\Omega_i)$ and $p \in H^1_{loc}(\Omega_e)$ satisfy equations (3.1) to (3.5). Then $\mathbf{u}$ and p vanish identically in their respected domains unless a Jones' mode is possible.*

Proof: We first note that the transmission conditions actually make sense since the normal derivative of p on $\partial\Omega$ and the surface traction of $\mathbf{u}$ exist. Let Σ be the boundary of a circle that completely encloses Ω_i. Let $\Omega_e{}'$ be the open domain bounded by the curves Σ and $\partial\Omega$. The transmission conditions imply that

$$< \frac{\partial p}{\partial n}, p >_{H^{-\frac{1}{2}}(\partial\Omega)H^{\frac{1}{2}}(\partial\Omega)} = -\rho_0\omega^2 < \mathbf{n}.\sigma(\mathbf{u}), \mathbf{u} >_{H^{-\frac{1}{2}}(\partial\Omega)H^{\frac{1}{2}}(\partial\Omega)} . \tag{3.6}$$

It is evident that the left hand side of this equation equals

$$k^2 \int_{\Omega_e} |p|^2\chi dV - \int_{\Omega_e} \nabla p.\nabla\overline{p}\chi dV,$$

where χ is a smooth function that takes the value 1 within $\Omega_e{}'$ and that vanishes in a neighbourhood of infinity. It is well known from regularity theory that p is smooth in any open subset of Ω_e. Therefore, we have

$$k^2 \int_{\mathcal{R}^2 \setminus \Omega_e'} |p|^2 \chi dV - \int_{\mathcal{R}^2 \setminus \Omega_e'} \nabla p . \nabla \overline{p} \chi dV = \int_\Sigma \frac{\partial p}{\partial n} \overline{p} dS.$$

Therefore,

$$< \frac{\partial p}{\partial n}, p >_{H^{-\frac{1}{2}}(\partial\Omega) H^{\frac{1}{2}}(\partial\Omega)} = k^2 \int_{\Omega_e'} |p|^2 dV - \int_{\Omega_e'} \nabla p . \nabla \overline{p} dV + \int_\Sigma \frac{\partial p}{\partial n} \overline{p} dS. \quad (3.7)$$

Similarly,

$$-\rho_0 \omega^2 < \mathbf{n}.\sigma(\mathbf{u}), \mathbf{u} >_{H^{-\frac{1}{2}}(\partial\Omega) H^{\frac{1}{2}}(\partial\Omega)} = \rho\rho_0 |\omega|^4 \int_{\Omega_i} \mathbf{u}.\overline{\mathbf{u}} dV - \rho_0 \omega^2 \int_{\Omega_i} \nabla \mathbf{u}. \nabla \overline{\mathbf{u}} dV. \quad (3.8)$$

Equations (3.6), (3.7) and (3.8) imply that

$$\begin{aligned} & k^2 \int_{\Omega_e'} |p|^2 dV - \int_{\Omega_e'} \nabla p . \nabla \overline{p} dV + \int_\Sigma \frac{\partial p}{\partial n} \overline{p} dS \\ &= \rho\rho_0 |\omega|^4 \int_{\Omega_i} \mathbf{u}.\overline{\mathbf{u}} dV - \rho_0 \omega^2 \int_{\Omega_i} \nabla \mathbf{u}. \nabla \overline{\mathbf{u}} dV. \end{aligned} \quad (3.9)$$

Suppose first that ω^2 and k^2 are real. Then

$$\Im \int_\Sigma \frac{\partial p}{\partial n} \overline{p} dS = 0. \quad (3.10)$$

Equation (3.10) is true regardless of the radius of the circle. Write

$$\frac{\partial p}{\partial n} = ikp + g.$$

Equation (3.10) implies that

$$\int_\Sigma |p|^2 dS = -\Im \left(\frac{1}{k} \int_\Sigma g \overline{p} dS \right). \quad (3.11)$$

Thus, we have

$$|k| \int_\Sigma |p|^2 dS \leq \| g \|_{L^2(\Sigma)} \| p \|_{L^2(\Sigma)} . \quad (3.12)$$

The radiation condition and inequality (3.12) imply that

$$\| p \|_{L^2(\Sigma)} \to 0 \tag{3.13}$$

as the circle's radius tends to infinity. As in the three-dimensional problem of the previous chapter, this implies that p vanishes in a neighbourhood of infinity. By analytic continuation p vanishes everywhere in Ω_e. The transmission conditions imply that $\mathbf{u}$ vanishes unless a Jones' mode is possible.

If ω^2, and hence k^2, have positive imaginary parts, then, as in the three-dimensional case, p decays exponentially at infinity. Therefore, the limit of the integral in equation (3.10) as the radius tends to infinity is zero. So the imaginary part of each term on the left hand side of equation (3.9), with Ω_e' replaced by Ω_e, is positive and the imaginary part of each term on the right hand side is negative. Thus, p vanishes and $\mathbf{u}$ is constant in Ω_i. The transmission conditions imply that $\mathbf{u.n}$ vanishes on $\partial\Omega$. Therefore, $\mathbf{u}$ must vanish in Ω_i.

3.3 Integral equations.

Let us recall the integral operators introduced in the previous chapter. We here define analogous operators. For the problem in two dimensions the fundamental solutions are now

$$-\frac{1}{4} i H_0^1(kR)$$

for Helmholtz's equation and

$$-\frac{1}{4\mu} i H_0^1(k_s R)\mathbf{I} - \frac{1}{4\rho\omega^2} i \nabla\nabla(H_0^1(k_s R) - H_0^1(k_p R))$$

for the elastic wave equation, where $H_0^1(z)$ is a Hankel function and

$$R = |\mathbf{x} - \mathbf{y}|.$$

These operators were previously defined on Hölder continuous spaces; we now wish to extend their definitions to larger spaces. From Costabel [5], if $\sigma \in (-\frac{1}{2}, \frac{1}{2})$, the operators

$$\begin{aligned} S &: H^{-\frac{1}{2}+\sigma}(\partial\Omega) \to H^{1+\sigma}_{loc}(\mathcal{R}^2) \\ D &: H^{\frac{1}{2}+\sigma}(\partial\Omega) \to H^{1+\sigma}(\Omega_i) \\ D &: H^{\frac{1}{2}+\sigma}(\partial\Omega) \to H^{1+\sigma}(\Omega_e) \\ S &: H^{-\frac{1}{2}+\sigma}(\partial\Omega) \to H^{\frac{1}{2}+\sigma}(\partial\Omega) \\ K &: H^{-\frac{1}{2}+\sigma}(\partial\Omega) \to H^{-\frac{1}{2}+\sigma}(\partial\Omega) \\ \overline{K}^* &: H^{\frac{1}{2}+\sigma}(\partial\Omega) \to H^{\frac{1}{2}+\sigma}(\partial\Omega) \\ N &: H^{\frac{1}{2}+\sigma}(\partial\Omega) \to H^{-\frac{1}{2}+\sigma}(\partial\Omega) \end{aligned} \tag{3.14}$$

as well as their elasticity counterparts are bounded extensions of the previously defined operators. We shall not distinguish between the single layer operator that maps $H^{-\frac{1}{2}+\sigma}(\partial\Omega)$ to $H^{1+\sigma}_{loc}(\mathcal{R}^2)$ and the operator that maps $H^{-\frac{1}{2}+\sigma}(\partial\Omega)$ to $H^{\frac{1}{2}+\sigma}(\partial\Omega)$. Nor shall we distiguish between the double layer operator that maps $H^{\frac{1}{2}+\sigma}(\partial\Omega)$ to $H^{1+\sigma}(\Omega_i)$ and the one that maps $H^{\frac{1}{2}+\sigma}(\partial\Omega)$ to $H^{1+\sigma}(\Omega_e)$.

Furthermore, if p_s belongs to $H^1_{loc}(\Omega_e)$ and satisfies Helmholtz's equation,

$$\frac{1}{2}(Dp_s)(\mathbf{x}) - \frac{1}{2}(S\frac{\partial p_s}{\partial n})(\mathbf{x}) = \begin{cases} -p_s(\mathbf{x}) & \text{if } \mathbf{x} \in \Omega_e \\ 0 & \text{if } \mathbf{x} \in \Omega_i \end{cases}. \tag{3.15}$$

For p_{inc} we have

$$\frac{1}{2}(Dp_{inc})(\mathbf{x}) - \frac{1}{2}(S\frac{\partial p_{inc}}{\partial n})(\mathbf{x}) = \begin{cases} 0 & \text{if } \mathbf{x} \in \Omega_e \\ p_{inc}(\mathbf{x}) & \text{if } \mathbf{x} \in \Omega_i \end{cases}. \tag{3.16}$$

Similarly, for $\mathbf{u}$ in $H^1(\Omega_i)$ satisfying equation (3.2) in the sense of distributions

$$\frac{1}{2}(\mathbf{D}.\mathbf{u})(\mathbf{x}) - \frac{1}{2}(\mathbf{S}.\sigma(\mathbf{u}).\mathbf{n})(\mathbf{x}) = \begin{cases} 0 & \text{if } \mathbf{x} \in \Omega_e \\ \mathbf{u}(\mathbf{x}) & \text{if } \mathbf{x} \in \Omega_i \end{cases}. \tag{3.17}$$

From equations (3.15) and (3.16) we see that

$$\frac{1}{2}(Dp)(\mathbf{x}) - \frac{1}{2}(S\frac{\partial p}{\partial n})(\mathbf{x}) - p_{inc}(\mathbf{x}) = \begin{cases} -p(\mathbf{x}) & \text{if } \mathbf{x} \in \Omega_e \\ 0 & \text{if } \mathbf{x} \in \Omega_i \end{cases} . \tag{3.18}$$

Moreover, for ψ belonging to $H^{-\frac{1}{2}}(\partial\Omega)$,

$$(S\psi)_+ - (S\psi)_- = 0 \text{ and } \left(\frac{\partial S\psi}{\partial n}\right)_+ - \left(\frac{\partial S\psi}{\partial n}\right)_- = -2\psi, \tag{3.19}$$

where the + subscript denotes taking the limit onto $\partial\Omega$ from the exterior and the − subscript denotes the limit taken from the interior. For ψ belonging to $H^{\frac{1}{2}}(\partial\Omega)$,

$$(D\psi)_+ - (D\psi)_- = 2\psi \text{ and } \left(\frac{\partial D\psi}{\partial n}\right)_+ - \left(\frac{\partial D\psi}{\partial n}\right)_- = 0. \tag{3.20}$$

This is true of the elasticity potentials too. The proof of all these claims may be found in Costabel [5]. Define the operators $\overline{K}^*$ and $\mathbf{K}$ through the identities

$$\overline{K}^*\psi = -\psi + (D\psi)_+$$

and

$$\overline{\mathbf{K}}^*.\mathbf{v} = \mathbf{v} + (\mathbf{D}.\mathbf{v})_+.$$

Equation (3.18) implies that

$$p + \overline{K}^*p - S\frac{\partial p}{\partial n} = 2p_{inc} \tag{3.21}$$

and equation (3.17) implies that

$$\mathbf{u} - \overline{\mathbf{K}}^*.\mathbf{u} + \mathbf{S}.\sigma(\mathbf{u}).\mathbf{n} = \mathbf{0}. \tag{3.22}$$

Now substitute the transmission conditions into equations (3.21) and (3.22) to obtain

$$p + \overline{K}^*p - \rho_0\omega^2 S(\mathbf{u}.\mathbf{n}) = 2p_{inc} \tag{3.23}$$

and

$$\mathbf{u} - \overline{\mathbf{K}}^*.\mathbf{u} - \mathbf{S}.\mathbf{n}p = 0. \tag{3.24}$$

3.4 Mellin transforms and the convolution theorem.

Mellin transforms will play a large role in what follows. The Mellin transform of a function f, belonging to $C_0^\infty(0,\infty)$, is

$$\mathcal{M}f(z) = \int_0^\infty t^{z-1} f(t)dt. \tag{3.25}$$

The definition of the Mellin transform can be extended to functions in $L^2(0,\infty)$. In fact, the map

$$\mathcal{M} : L^2(0,\infty) \to L^2(\Re z = \frac{1}{2})$$

is an isomorphism. The inversion formula is

$$f(t) = \frac{1}{2\pi i}\int_{\Re z=\frac{1}{2}} t^{-z}\tilde{f}(z)dz \tag{3.26}$$

for $f \in L^2(0,\infty)$, where, of course, $\tilde{f}$ represents the Mellin transform of f.

The Mellin transform can be extended to a still wider class of functions. We have

Lemma 1 *Let $f \in L^2_{loc}(0,\infty)$ and let the numbers a and b be given by*

$$a = \sup\{\alpha; f(t) = O(t^{-\alpha}) \text{ as } t \to 0_+\}$$

$$b = \sup\{\beta; f(t) = O(t^{-\beta}) \text{ as } t \to \infty\}$$

with $b > a$. Then the integral in equation (3.25) converges uniformly for $a < \Re z < b$ and defines a holomorphic function there.

For $a < x < b$ we have

$$\lim_{y\to\pm\infty} \tilde{f}(x+iy) = 0,$$

for any subinterval I of (a,b) the function

$$N(f,I,y) = \sup_{x\in I} |\tilde{f}(x+iy)|$$

is continuous with respect to y and satisfies

$$\lim_{y\to\pm\infty} N(f,I,y) = 0.$$

The inversion formula (3.26) is valid along any line $\Re z = c$ for $a < c < b$.

The proof of this may be found in, for example, Bleistein and Handelsman [3].

Now we introduce the space $L^{2,c}(\frac{dt}{t})$, by defining that a function f belongs to it if

$$\int_0^\infty t^{2c-1}|f(t)|^2 dt$$

exists. Equipped with the obvious norm these spaces become Banach spaces. We can prove that

$$\mathcal{M} : L^{2,c}\left(\frac{dt}{t}\right) \to L^2(\Re z = c)$$

is an isomorphism.

The inversion formula (3.26) has to be modified slightly. For $f \in L^{2,c}(\frac{dt}{t})$

$$\lim_{M\to\infty} \frac{1}{2\pi i}\int_{c-iM}^{c+iM} t^{-z}\tilde{f}(z)dz = f(t), \tag{3.27}$$

for almost every $t \in (0,\infty)$.

The Mellin transform is useful for solving integral equations of the form

$$\int_0^\infty g\left(\frac{x}{t}\right) f(t)\frac{dt}{t} = h(x). \tag{3.28}$$

The left hand side of equation (3.28) is called the Mellin convolution of f and g. Formally the application of Mellin transforms yields

$$\tilde{g}(z)\tilde{f}(z) = \tilde{h}(z). \tag{3.29}$$

For $f \in L^{2,c}(\frac{dt}{t})$ and $\tilde{g} \in L^2(\Re z = c)$ this can be rigorously proved. The inversion formula (3.28) then enables us to write

$$f(t) = \lim_{M\to\infty} \frac{1}{2\pi i} \int_{c-iM}^{c+iM} t^{-z} \frac{\tilde{h}(z)}{\tilde{g}(z)} dz.$$

Let us call $\mathcal{S}_c$ the subspace of $\mathcal{D}(0,\infty)$, the dual space of $C_0^\infty(0,\infty)$, defined by

$$\mathcal{S}_c = \{f \in \mathcal{D}(0,\infty); e^{cx} f(e^x) \in \mathcal{S}(-\infty,\infty)\}.$$

The Mellin transform of any $u \in \mathcal{S}_c$ can be defined on the line $\Re z = c$. Here $\mathcal{S}(-\infty,\infty)$ stands for the space of distributions for which a Fourier transform can be defined. $e^{cx} f(e^x)$ is the distribution g defined by

$$< g, \phi > = < f, t^{c-1}\phi(\ln t) > .$$

Let us state the convolution theorem for distributions and restate the convolution theorem for functions:

Theorem 8 *If $f \in L^{2,c}(0,\infty)$ and $g \in \mathcal{S}_c$ is such that $\tilde{g}$ is bounded, then*

$$\int_0^\infty g\left(\frac{s}{t}\right) f(t) \frac{dt}{t} = \lim_{M\to\infty} \frac{1}{2\pi i} \int_{c-iM}^{c+iM} s^{-z} \tilde{f}(z)\tilde{g}(z) dz. \tag{3.30}$$

If $f \in L^{2,c}(0,\infty)$ and g is such that $\tilde{g} \in L^2(\Re z = c)$, then

$$\int_0^\infty g\left(\frac{s}{t}\right) f(t) \frac{dt}{t} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} s^{-z} \tilde{f}(z)\tilde{g}(z) dz. \tag{3.31}$$

3.5 Properties in the wedge.

Here we shall concern ourselves with the properties of the boundary integral operators on the boundary of a wedge-shaped region

$$\Gamma = \{(x,0); x \geq 0\} \bigcup \{(s\cos\alpha, s\sin\alpha); s \geq 0\},$$

where α is some constant in the range $(0, 2\pi)$. To this end, let us introduce the space $Z^s_{\mathcal{P}}(0,\infty)$, for $s > 0$ and $s - \frac{1}{2}$ not equal to an integer. It is defined to be the space consisting of elements of the form

$$u = u_0 + \sum_{k=1}^{n} \sum_{j=0}^{m_k} c_{kj} \omega(x) x^{p_k} \ln^j x.$$

$\mathcal{P}$ denotes the set

$$\{(p_k, m_k) \in \mathcal{R} \otimes \mathcal{N} \bigcup \{0\}; k = 1, 2, \ldots, n\},$$

$$0 \leq p_1 < p_2 < \ldots < p_n < [s - \frac{1}{2}],$$

x is a variable that measures the distance from any point on Γ to the origin, u_0 belongs to $\tilde{H}^s(0,\infty)$ and $\omega(x)$ is a smooth function which takes the value 1 for x in a neighbourhood of zero, and which vanishes in a neighbourhood of infinity and, finally, c_{kj} are constants. $u \in \tilde{H}^s(0,\infty)$ if the continuation of u by zero belongs to $H^s(0,\infty)$. We define the norm of u in $Z^s_{\mathcal{P}}(0,\infty)$ by

$$\| u \|_{Z^s_{\mathcal{P}}(0,\infty)} = \| u_0 \|_{H^s(0,\infty)} + \sum_{k=1}^{n} \sum_{j=0}^{m_k} |c_{kj}|^2.$$

This is similar to the space defined by Costabel and Stefan [6]. See also Ola [25] for a similar singularity space.

$Z^s_{\mathcal{P}}(0,\infty)$ is independent of the choice of $\omega(x)$.

Let us define the space $Z^s_{\mathcal{P}}(\Gamma)$ to be the space of functions on the boundary of the wedge, Γ, with the property that the restriction of the functions to each arm of the wedge belongs to $Z^s_{\mathcal{P}}(0,\infty)$.

Let

$$\mathcal{P}^{(1)} = \{(p_k^{(1)}, m_k^{(1)}); k = 1, 2 \ldots, n^{(1)}\}$$

and

$$\mathcal{P}^{(2)} = \{(p_k^{(2)}, m_k^{(2)}); k = 1, 2 \ldots, n^{(2)}\}$$

denote two singularity sets. We define the sum of these

$$\mathcal{P} = \mathcal{P}^{(1)} + \mathcal{P}^{(2)}$$

to be the set

$$\{(p_k, m_k); k = 1, 2, \ldots, n\},$$

where

$$\{p_k\} = \{p_k^{(1)}\} \bigcup \{p_k^{(2)}\}$$

and

$$m_k = m_{k'}^{(j)},$$

for $j = 1, 2$, if

$$p_k = p_{k'}^{(j)}$$

and if p_k does not belong to $\{p_{k'}^{(3-j)}; k' = 1, 2, \ldots, n^{(3-j)}\}$. On the other hand,

$$m_k = m_{k'}^{(1)} + m_{k''}^{(2)} + 1$$

if

$$p_k = p_{k'}^{(1)} = p_{k''}^{(2)}.$$

The first result of this section is the following:

Lemma 2 *Let T be an integral operator with a smooth kernel $k(t,s)$. Let $u \in Z_{\mathcal{P}}^k(0,\infty)$ have compact support in $[0,L]$ for some constant L. If $\chi(t)$ is a smooth cut-off function with support in $[0,L]$, then*

$$\chi(t) \int_0^L k(t,s) u(s) ds$$

is smooth and we have the estimate

$$\| \chi T u \|_{C^\infty(0,\infty)} \leq M \| u \|_{Z_p^k(0,\infty)},$$

for some constant M independent of u, where

$$\| \cdot \|_{C^\infty(0,\infty)}$$

denotes the supremum norm.

The proof of this lemma is straightforward. We have

$$\frac{d}{dt}(\chi(t)(Tu)(t)) = \frac{d\chi(t)}{dt}(Tu)(t) + \chi(t)\frac{d(Tu)(t)}{dt}.$$

The second term equals

$$\lim_{\Delta t \to 0} \chi(t) \frac{1}{\Delta t} \int_0^L (k(t + \Delta t, s) - k(t, s))u(s)ds. \tag{3.32}$$

By the mean value theorem,

$$k(t + \Delta t, s) - k(t, s) = \Delta t \frac{\partial k(t,s)}{\partial t} + \Delta t \left(\frac{\partial k(t', s)}{\partial t} - \frac{\partial k(t,s)}{\partial t} \right),$$

for some t' lying between t and $t + \Delta t$. Again using the mean value theorem this equals

$$\Delta t \frac{\partial k(t,s)}{\partial t} + \Delta t (t' - t) \frac{\partial^2 k(t'', s)}{\partial t^2},$$

for some t'' in $[t, t']$.

$$\frac{\partial^2 k(t'', s)}{\partial t^2}$$

is bounded for all (t'', s) in $[0, L] \otimes [0, L]$. Thus the limit in (3.32) equals

$$\chi(t) \int_0^L \frac{\partial k(t,s)}{\partial t} u(s)ds.$$

By induction we can see that all higher order derivatives exist.

The estimate is readily obtained.

Denote by $Z^s_{\mathcal{P}}(\Gamma; N)$ the subspace of $Z^s_{\mathcal{P}}(\Gamma)$ consisting of functions whose supports are contained in a fixed compact neighbourhood, N, of the corner of the wedge.

Lemma 3 *Suppose that u belongs to $Z^s_{\mathcal{P}}(\Gamma; N)$, where*

$$\mathcal{P} = \{(p_k, m_k); k = 1, \ldots, n\},$$

$s > 0$ and $s - \frac{1}{2}$ is not an integer. Then $\chi S u$, where χ is a smooth cut-off function, belongs to $Z^{s+1}_{\mathcal{P}'' + \mathcal{P}'}(\Gamma)$, where

$$\mathcal{P}' = \{(k, 0); k = 0, 1, 2, 3, \ldots, [s - \frac{1}{2}]\}$$

and

$$\mathcal{P}'' = \{(p_k + l, m_k); k = 1, \ldots, n \text{ and } l = 1, 3, 5, \ldots\}.$$

Moreover,

$$\| \chi S u \|_{Z^{s+1}_{\mathcal{P}'' + \mathcal{P}'}(\Gamma)} \leq M \| u \|_{Z^s_{\mathcal{P}}(\Gamma)},$$

for some positive constant M.

$\mathbf{S}$ *has the same properties.*

Proof: We shall prove the lemma only for S. The proof for $\mathbf{S}$ is identical.

If we write u, a function on Γ, as

$$\begin{pmatrix} u_+ \\ u_- \end{pmatrix},$$

where u_+ and u_- denote the values of u taken on the upper and lower arms of the wedge respectively, we may write Su as

$$\begin{pmatrix} S_{++} & S_{+-} \\ S_{-+} & S_{--} \end{pmatrix} \begin{pmatrix} u_+ \\ u_- \end{pmatrix}. \tag{3.33}$$

Here we have

$$(S_{++}u)(t) = \frac{1}{\pi}\int_0^\infty \ln(|\frac{s}{t}-1|)J_0(k|s-t|)u(s)ds + \frac{1}{\pi}\ln t\int_0^\infty J_0(k|s-t|)u(s)ds$$
$$+\int_0^\infty k_1(t,s)u(s)ds = (S_{--}u)(t)$$

and

$$(S_{+-}u)(t) = \int_0^\infty \ln((\frac{s}{t}-\cos\alpha)^2+\sin^2\alpha)J_0(k|s-t|)u(s)ds$$
$$+\ln t\int_0^\infty J_0(k|s-t|)u(s)ds + \int_0^\infty k_2(t,s)u(s)ds = (S_{-+}u)(t),$$

where $k_1(t,s)$ and $k_2(t,s)$ are smooth and $J_0(z)$ is a Bessel function.

Denote the integral operator with kernel

$$\frac{1}{\pi}\begin{pmatrix} \ln(|\frac{s}{t}-1|) & \ln((\frac{s}{t}-\cos\alpha)^2+\sin^2\alpha) \\ \ln((\frac{s}{t}-\cos\alpha)^2+\sin^2\alpha) & \ln(|\frac{s}{t}-1|)\end{pmatrix} J_0(k|s-t|)$$

by S_0. $(S_0u)(t)$ equals

$$\frac{1}{\pi}\int_0^\infty \begin{pmatrix} \ln(|\frac{s}{t}-1|) & \ln((\frac{s}{t}-\cos\alpha)^2+\sin^2\alpha) \\ \ln((\frac{s}{t}-\cos\alpha)^2+\sin^2\alpha) & \ln(|\frac{s}{t}-1|)\end{pmatrix}\begin{pmatrix} u_+(s) \\ u_-(s)\end{pmatrix}$$
$$\times\sum_{m=0}^\infty s^{2m}\frac{(-1)^m k^{2m}}{4^m(m!)^2}\left(1-\frac{t}{s}\right)^{2m} ds.$$

See, for example, Abramowitz and Stegun [1, Chapter 9] for the small argument asymptotics of Bessel functions. Let us call the square matrix in the previous equation $M(t,s;\alpha)$. Thus, $(S_0u)(t)$ equals

$$\frac{1}{\pi}\sum_0^\infty \frac{(-1)^m k^{2m}}{4^m(m!)^2}\int_0^\infty M(t,s;\alpha)s^{2m}u(s)\sum_{n=0}^{2m}\frac{(2m)!}{(2m-n)!n!}\left(-\frac{t}{s}\right)^{2m-n} ds.$$

The reversal of the order of integration and summation is justifiable.

It is easily verifiable that

$$\ln\left|\frac{1}{t}-1\right| t^{2m-n}$$

and

$$\ln\left(\left(\frac{1}{t}-\cos\alpha\right)^2+\sin^2\alpha\right)t^{2m-n}$$

belong to $L^{2,-2m+n+c}(0,\infty)$, for $0<c<1$. It is clear that since all the exponents in the singular parts of u_+ and u_- are greater than -1, then $t^{2m}u_\pm(t)\in L^{2,-2m+c}(0,\infty)$, for c in the range

$$(\max\{-s+\frac{1}{2},0\},1).$$

Thus from the Mellin convolution theorem and Lemma 2, we can represent χSu by

$$\sum_{m=0}^{\infty}\frac{(-1)^m k^{2m}}{4^m(m!)^2}\sum_{n=0}^{2m}\frac{(2m)!}{(2m-n)!n!}\frac{\chi(x)}{2\pi i}\int_{\Re z=c-2m+n}x^{-z} \tag{3.34}$$
$$\times\begin{pmatrix}\frac{\cos\pi(z+2m-n)}{(z+2m-n)\sin\pi(z+2m-n)} & \frac{\cos(\pi-\alpha)(z+2m-n)}{(z+2m-n)\sin\pi(z+2m-n)}\\ \frac{\cos(\pi-\alpha)(z+2m-n)}{(z+2m-n)\sin\pi(z+2m-n)} & \frac{\cos\pi(z+2m-n)}{(z+2m-n)\sin\pi(z+2m-n)}\end{pmatrix}\begin{pmatrix}\tilde{u}_+(z+2m+1)\\ \tilde{u}_-(z+2m+1)\end{pmatrix}dz$$
$$+\chi(x)(S_1u)(x),$$

where $\tilde{u}$ represents the Mellin transform of u and where

$$\chi(x)(S_1u)(x)=\frac{\chi(x)}{\pi}\ln x\int_0^\infty\begin{pmatrix}1&1\\1&1\end{pmatrix}\begin{pmatrix}u_+(s)\\u_-(s)\end{pmatrix}J_0(k|s-x|)ds$$

plus smooth functions. The integral in equation (3.34) is to be interpreted in the sense of equation (3.27).

$\tilde{u}_\pm(z+2m+1)$ are meromorphic to the right of $\Re=-s-2m$ with poles of order m_k at $-p_k-2m-1$. Thus, if we move each contour of integration in equation (3.34) to the left to the line $\Re z=-s-2m-\frac{1}{2}$ we pick up contributions from the poles of $\tilde{u}_\pm(z+2m+1)$ at $-p_k-2m-1$ and from the simple poles of the matrix in equation (3.34), which are situated at $z=-N+n-2m$, where N is a positive integer that is smaller than $[s+\frac{1}{2}]$ and from the double poles situated $z=n-2m$. The log singularity due to the double poles exactly cancels the log singularity in S_1. The residue at a pole, not due to a singularity of the square matrix in equation (3.34), is proportional to the residue of $\tilde{u}_+$ and $\tilde{u}_-$ there. This is proportional to the coefficient of the singular part of $\tilde{u}_+$ and $\tilde{u}_-$. Suppose that $\tilde{u}_\pm(z)$ do not have poles at $z = -N + n - 2m$, where N takes one of the values $N = 0, 1, 2, \ldots, [s - \frac{1}{2}]$, then there is a simple pole in the integrand in equation (3.34) at this point. The residue at this point is proportional to

$$\frac{1}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^m k^{2m}}{4^m (m!)^2} \sum_{n=2m-N+1}^{2m} \frac{(2m)!}{(2m-n)!n!} \frac{(-1)^n x^N}{-N+2m-n} \tag{3.35}$$
$$\times \begin{pmatrix} 1 & \frac{\cos(\pi-\alpha)(-N+n-2m)}{\cos\pi(-N+n-2m)} \\ \frac{\cos(\pi-\alpha)(-N+n-2m)}{\cos\pi(-N+n-2m)} & 1 \end{pmatrix} \begin{pmatrix} \tilde{u}_+(-N+2m+1) \\ \tilde{u}_-(-N+2m+1) \end{pmatrix}$$
$$+ \frac{1}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^m k^{2m}}{4^m (m!)^2} (-x)^N$$
$$\times \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \frac{\partial}{\partial z} \begin{pmatrix} \tilde{u}_+(z) \\ \tilde{u}_-(z) \end{pmatrix} \Bigg|_{z=-N+2m+1} .$$

The part of the residue due to the singular parts of $u_\pm$ is proportional to the coefficients of the singular parts of $u_\pm$. The large m behaviour of the Mellin transform of the singular part is $O(L^{2m})$, where L is a constant that depends only on the compact neighbourhood N. The part of the residue due to the non-singular parts of $u_\pm$ is proportional to

$$\int_0^\infty t^{-N+2m} u_{0\pm}(t) dt.$$

and

$$\int_0^\infty t^{-N+2m} \ln t u_{0\pm}(t) dt.$$

It can easily be shown that

$$\int_0^\infty |t^{-N+2m} u_{0\pm}(t)| dt$$

and

$$\int_0^\infty |t^{-N+2m} \ln t u_{0\pm}(t)| dt$$

are less than

$$ML^{2m} \parallel u_{0\pm} \parallel_{\tilde{H}^s(0,\infty)},$$

where M and L are constants that depend only on the compact neighbourhood N.

Therefore, the term in equation (3.35) exists and has a modulus less than or equal to a constant times

$$\parallel u_{\pm} \parallel_{Z^s_P(0,\infty)} .$$

The more complicated cases of multiple poles are dealt with in a similar way. Finally, if we bear in mind Lemma 2, we have the sum of squares of coefficients of the singular terms of $\chi S u$ is less than or equal to a positive constant multiplied by the sum of squares of coefficients of the singular terms of u plus the norm in $\tilde{H}^s(\Gamma)$ of the non-singular part of u. ($\tilde{H}^s(\Gamma)$ denotes the space of distributions on Γ whose component on each arm of the wedge belongs to $\tilde{H}^s(0,\infty)$.)

The non-singular part of $\chi S u$ is

$$\frac{\chi(x)}{\pi i} \sum_{m=0}^{\infty} \frac{(-1)^m k^{2m}}{4^m (m!)^2} \sum_{n=\max\{0,[2m-s-\frac{1}{2}]\}}^{2m} (-1)^n \int_{\Re z=-2m-s-\frac{1}{2}} x^{-z} \tag{3.36}$$

$$\times \begin{pmatrix} \frac{\cos\pi(z+2m-n)}{(z+2m-n)\sin\pi(z+2m-n)} & \frac{\cos(\pi-\alpha)(z+2m-n)}{(z+2m-n)\sin\pi(z+2m-n)} \\ \frac{\cos(\pi-\alpha)(z+2m-n)}{(z+2m-n)\sin\pi(z+2m-n)} & \frac{\cos\pi(z+2m-n)}{(z+2m-n)\sin\pi(z+2m-n)} \end{pmatrix} \begin{pmatrix} \tilde{u}_+(z+2m+1) \\ \tilde{u}_-(z+2m+1) \end{pmatrix} dz$$

plus smooth terms. This is the term that belongs to $\tilde{H}^{s+1}(\Gamma)$. To see this we note that, when $t-\frac{1}{2}$ is not an integer and u belongs to $\tilde{H}^t(0,\infty)$,

$$\parallel u \parallel_{\tilde{H}^t(0,\infty)} = \int_{\Re z=-t+\frac{1}{2}} (1+|z|^2)^t |\tilde{u}(z)|^2 d|z|,$$

where $\tilde{u}$ represents the Mellin transform of u. This is Parseval's identity. Thus, the term in integral (3.36) belongs to $\tilde{H}^{s+1}(\Gamma)$. Furthermore, it is easy to show that the norm in $\tilde{H}^{s+1}(\Gamma)$ of the term in integral (3.36) is less than the norm in

$\tilde{H}^s(\Gamma)$ of the part of u in $\tilde{H}^s(\Gamma)$. This results from the fact that all the terms in the matrix in the integral (3.36) are bounded on the lines $\Re z = -2m - s - \frac{1}{2}$. This, together with the result about the singular part of $\chi S u$ and Lemma 2, is enough to show that χS is continuous between $Z^s_{\mathcal{P}}(\Gamma)$ and $Z^{s+1}_{\mathcal{P}''+\mathcal{P}'}(\Gamma)$. This completes the proof. $\square$

The kernel of $\overline{K}^*$ consists of a singular part and a continuous part. Let $\overline{K}_0^*$ be the integral operator whose kernel is the singular part of the kernel of $\overline{K}^*$. Similarly, let $\overline{\mathbf{K}}_0^*$ be the integral operator whose kernel is the singular part of the kernel of $\overline{\mathbf{K}}^*$.

Let us now prove the following lemma:

Lemma 4 *Suppose that f belongs to $Z^k_{\mathcal{P}}(\Gamma)$, where $k > \frac{1}{2}$, $k - \frac{1}{2}$ is not an integer and does not equal*

$$\frac{m\pi}{\alpha}$$

or

$$\frac{m\pi}{2\pi - \alpha},$$

for any integer m. Let u, belonging to $H^s(\Gamma)$ and having compact support, with $0 < s < \frac{1}{2}$, satisfy

$$(I + \overline{K}_0^*)u = f.$$

Then u belongs to $Z^k_{\mathcal{P}'}(\Gamma)$ and we have the estimate

$$\| u \|_{Z^k_{\mathcal{P}'}(\Gamma)} \leq M \| f \|_{Z^k_{\mathcal{P}}(\Gamma)}, \tag{3.37}$$

where M is a positive constant and

$$\mathcal{P}' = \mathcal{P} + \{(\beta, 0); \beta = \frac{m\pi}{\alpha} \text{ or } \beta = \frac{m\pi}{2\pi - \alpha} \text{ and } m \in \mathcal{Z}\}$$

if the ratio of α to π is irrational. Otherwise,

$$\mathcal{P}' = \mathcal{P} + \{(\beta, 1); \beta = \frac{m\pi}{\alpha} = \frac{m'\pi}{2\pi - \alpha} \text{ and } m, m' \in \mathcal{Z}\}.$$

Proof: Let us begin by writing $(I + \overline{K_0^*})u$ as

$$\begin{pmatrix} 1 & \overline{K}^*_{+-} \\ \overline{K}^*_{-+} & 1 \end{pmatrix} \begin{pmatrix} u_+ \\ u_- \end{pmatrix}, \tag{3.38}$$

where

$$(\overline{K}^*_{+-} g)(t) = \frac{1}{\pi} \int_0^\infty \frac{\frac{s}{t} \sin \alpha}{(\frac{s}{t} - \cos \alpha)^2 + \sin^2 \alpha} g(s) \frac{ds}{s} = (\overline{K}^*_{-+} g)(t).$$

u belongs to $L^{2,c}\left(\frac{dt}{t}\right)$ for $0 < c < 1$. It is easy to see that

$$\frac{\frac{1}{t}}{\left(\frac{1}{t} - \cos \alpha\right)^2 + \sin^2 \alpha}$$

belongs to $L^{2,c}\left(\frac{dt}{t}\right)$ for $0 < c < 1$. By the convolution theorem we can write the term (3.38) as

$$\frac{1}{2\pi i} \int_{\Re z = -s + \frac{1}{2}} t^{-z} \begin{pmatrix} 1 & \frac{\sin(\pi - \alpha)z}{\sin \pi z} \\ \frac{\sin(\pi - \alpha)z}{\sin \pi z} & 1 \end{pmatrix} \begin{pmatrix} \tilde{u}_+(z) \\ \tilde{u}_-(z) \end{pmatrix} dz. \tag{3.39}$$

The Mellin transform of f is meromorphic to the right of $\Re z = -k + \frac{1}{2}$. We may write

$$\begin{pmatrix} f_+(t) \\ f_-(t) \end{pmatrix} = \frac{1}{2\pi i} \int_{\Re z = -s + \frac{1}{2}} t^{-z} \begin{pmatrix} \tilde{f}_+(z) \\ \tilde{f}_-(z) \end{pmatrix} dz. \tag{3.40}$$

Comparison of the integral (3.39) and the integral in equation (3.40) implies that

$$\begin{pmatrix} 1 & \frac{\sin(\pi - \alpha)z}{\sin \pi z} \\ \frac{\sin(\pi - \alpha)z}{\sin \pi z} & 1 \end{pmatrix} \begin{pmatrix} \tilde{u}_+(z) \\ \tilde{u}_-(z) \end{pmatrix} = \begin{pmatrix} \tilde{f}_+(z) \\ \tilde{f}_-(z) \end{pmatrix}. \tag{3.41}$$

Thus the Mellin transform of u is

$$\frac{\sin \pi z}{\sin \alpha z \sin(2\pi - \alpha) z} \begin{pmatrix} \sin \pi z & -\sin(\pi - \alpha) z \\ -\sin(\pi - \alpha) z & \sin \pi z \end{pmatrix} \begin{pmatrix} \tilde{f}_+(z) \\ \tilde{f}_-(z) \end{pmatrix}. \tag{3.42}$$

We have

$$u(t) = \frac{1}{2\pi i} \int_{\Re z = -s + \frac{1}{2}} t^{-z} \tilde{u}(z) dz. \tag{3.43}$$

We may move the contour in equation (3.43) to the left to the line $\Re z = -k + \frac{1}{2}$. By doing this we pick up contributions due to the poles of $\tilde{f}$ and contributions due to the singularities (poles) of the square matrix in (3.42). These latter poles occur when

$$\sin \alpha z = 0,$$

or when

$$\sin(2\pi - \alpha) z = 0.$$

Therefore, if the ratio of α to π is irrational, the poles of the matrix are all simple. If, on the other hand, the ratio of α to π is rational the matrix may (and, generically, will) have double poles.

So $u(t)$ is the sum of the singular terms due to the poles and the integral

$$\frac{1}{2\pi i} \int_{\Re z = -k + \frac{1}{2}} t^{-z} \tilde{u}(z) dz. \tag{3.44}$$

It is readily verified that this last term is bounded in the $\tilde{H}^k(\Gamma)$ norm and the truth of inequality (3.37) is also evident. This is what we wanted to prove. □

The analysis of the equation

$$(\mathbf{I} - \overline{\mathbf{K}}_0^*)\mathbf{u} = \mathbf{f}$$

is essentially the same. The difference lies in the increased complexity of the kernel of $\overline{\mathbf{K}}_0^*$. We have the following lemma:

Lemma 5 *Let* $\mathbf{f}$ *belong to* $Z^k_{\mathcal{P}}(\Gamma)$*, where* $k > \frac{1}{2}$*,* $k - \frac{1}{2}$ *is not an integer and is not a root of the equation (3.45). If* $\mathbf{u}$ *belonging to* $H^s(\Gamma)$*, with* $0 < s < \frac{1}{2}$*, satisfies*

$$(\mathbf{I} - \overline{\mathbf{K}}^*)\mathbf{u} = \mathbf{f},$$

and $\mathbf{u}$ *has compact support, then* $\mathbf{u}$ *belongs to* $Z^k_{\mathcal{P}+\mathcal{P}'}(\Gamma)$*. Here we have*

$$\mathcal{P}' = \{(p_k, m_k)\},$$

where, for each k*,* $-p_k$ *is a root of the equation*

$$\frac{1}{\sin^4 \pi z}(z^2 \sin^2 \alpha - \sin^2 z\alpha)(z^2 \sin^2 \alpha - \frac{(\lambda + 3\mu)^2}{(\lambda + \mu)^2} \sin^2 z(2\pi - \alpha)) = 0 \qquad (3.45)$$

lying in the strip

$$-k + \frac{1}{2} < \Re z < 0$$

and m_k *is the order of that root.*

Proof: Let us first identify the singular terms in the kernel of $\overline{\mathbf{K}}^*$. A routine calculation shows that for the (i,j)th term these are

$$\frac{\mu}{\lambda + 2\mu}(-n_i(\mathbf{y})X_j + X_i n_j(\mathbf{y}) + \theta\delta_{ij})\frac{1}{\pi R^2} + \frac{2\lambda + 2\mu}{\lambda + 2\mu}\frac{X_i X_j \theta}{\pi R^4}, \qquad (3.46)$$

where

$$X_i = y_i - x_i,$$

$$R = |\mathbf{y} - \mathbf{x}|$$

and

$$\theta = (\mathbf{y} - \mathbf{x}).\mathbf{n}(\mathbf{y}).$$

Let us write $(\mathbf{I} - \overline{\mathbf{K}}^*_0).\mathbf{u}$ as

$$\begin{pmatrix} \mathbf{I} - \overline{\mathbf{K}}^*_{++} & -\overline{\mathbf{K}}^*_{+-} \\ -\overline{\mathbf{K}}^*_{-+} & \mathbf{I} - \overline{\mathbf{K}}^*_{--} \end{pmatrix} . \begin{pmatrix} \mathbf{u}_+ \\ \mathbf{u}_- \end{pmatrix}, \qquad (3.47)$$

where, as in the acoustic case, the $+$ (resp. $-$) subscript denotes the function defined on the upper (resp. lower) arm of the wedge. Here $\overline{\mathbf{K}}^*_{++}$, $\overline{\mathbf{K}}^*_{+-}$, $\overline{\mathbf{K}}^*_{-+}$ and $\overline{\mathbf{K}}^*_{--}$ are two by two matrices.

It will prove convenient to write $\mathbf{u}_+$ as

$$u_+^{(1)}\mathbf{e}_+ + u_+^{(2)}\mathbf{f}_+,$$

where $\mathbf{e}_+$ denotes a unit vector parallel to the upper arm of the wedge, and where $\mathbf{f}_+$ denotes a unit vector perpendicular to the upper arm and pointing into the wedge. Similarly, we shall write

$$\mathbf{u}_- = u_-^{(1)}\mathbf{e}_- + u_-^{(2)}\mathbf{f}_-,$$

where $\mathbf{e}_-$ and $\mathbf{f}_-$ denote unit vectors parallel to the lower arm of the wedge and perpendicular to it respectively. (See Figure (3.2).)

An elementary calculation shows that

$$\overline{\mathbf{K}}_{++}.\mathbf{u} = \overline{\mathbf{K}}_{--}.\mathbf{u} = \int_0^\infty \left(\frac{-\mu}{\lambda+2\mu} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \frac{1}{\pi(y-x)} \right) \begin{pmatrix} u_1(y) \\ u_2(y) \end{pmatrix} dy,$$

where the integral is meant in the sense of the Cauchy principal value, and

$$\overline{\mathbf{K}}_{+-}.\mathbf{u} = \overline{\mathbf{K}}_{-+}.\mathbf{u} = \int_0^\infty (A(x,y)\frac{1}{\pi R^2} - B(x,y)\frac{x\sin\alpha}{\pi R^4},) \begin{pmatrix} u_1(y) \\ u_2(y) \end{pmatrix} dy,$$

where

$$A(x,y) = \frac{\mu}{\lambda+2\mu} \begin{pmatrix} y\sin\alpha & x - y\cos\alpha \\ x - y\cos\alpha & -y\sin\alpha \end{pmatrix}$$

and

$$B_{11}(x,y) = \frac{2\lambda+2\mu}{\lambda+2\mu}(x^2\cos\alpha - 2xy\cos^2\alpha - xy\sin^2\alpha + y^2\cos\alpha),$$

$$B_{12}(x,y) = \frac{2\lambda+2\mu}{\lambda+2\mu}(x^2\sin\alpha - xy\sin\alpha\cos\alpha),$$

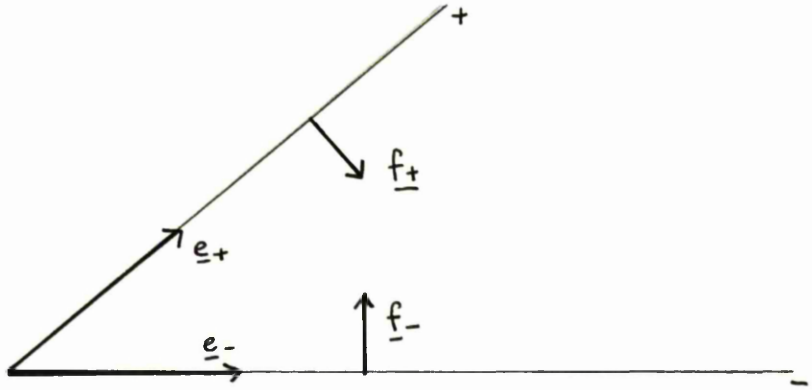

Figure 3.2: The coordinate system around the wedge.

$$B_{21}(x,y) = \frac{2\lambda + 2\mu}{\lambda + 2\mu}(y^2 \sin\alpha - xy\cos\alpha\sin\alpha)$$

and

$$B_{22}(x,y) = -\frac{2\lambda + 2\mu}{\lambda + 2\mu} xy \sin^2\alpha,$$

and where

$$R = \sqrt{x^2 - 2xy\cos\alpha + y^2}.$$

The distribution sending $u(t) \in C_0^\infty(0,\infty)$ to

$$\int_0^\infty \frac{1}{t-1} u(t) dt,$$

where the integral is meant as a Cauchy principal part, belongs to $\mathcal{S}^c$ for $c < 1$. $A(t,1)$ belongs to $L^{2,c}\left(\frac{dt}{t}\right)$ for $0 < c < 1$ and $B(t,1)$ belongs to $L^{2,c}\left(\frac{dt}{t}\right)$ for $-1 < c < 1$. As $\mathbf{u} \in H^s(\Gamma)$ for $0 < s < \frac{1}{2}$ we may use the convolution theorem for Mellin transforms. We see that

$$(\mathbf{I} - \overline{\mathbf{K}_0^*}).\mathbf{u}$$

may be written as

$$\frac{1}{2\pi i}\int_{\Re z = -s + \frac{1}{2}} \begin{pmatrix} I - A(z) & -B(z) \\ -B(z) & I - A(z) \end{pmatrix} \begin{pmatrix} \tilde{\mathbf{u}}_{(+)}(z) \\ \tilde{\mathbf{u}}_{(-)}(z) \end{pmatrix} t^{-z} dz,$$

where

$$A(z) = \frac{\mu}{\lambda + 2\mu} \cot \pi z \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and

$$B_{11}(z) = -\frac{1}{\lambda + 2\mu} \frac{1}{\sin \pi z}((\lambda + 2\mu) \sin z(\alpha - \pi) \cos \alpha - \mu \cos z(\alpha - \pi) \sin \alpha + (\lambda + \mu) z \cos z(\alpha - \pi) \sin \alpha),$$

$$B_{22}(z) = -\frac{1}{\lambda + 2\mu} \frac{1}{\sin \pi z}(-(\lambda + 2\mu) \sin z(\alpha - \pi) \cos \alpha + \mu \cos z(\alpha - \pi) \sin \alpha + (\lambda + \mu) z \cos z(\alpha - \pi) \sin \alpha),$$

$$B_{12}(z) = -\frac{1}{\lambda + 2\mu} \frac{1}{\sin \pi z}((\lambda + 2\mu) \sin z(\alpha - \pi) \sin \alpha + \mu \cos z(\alpha - \pi) \cos \alpha + (\lambda + \mu) z \sin z(\alpha - \pi) \sin \alpha)$$

and

$$B_{21}(z) = -\frac{1}{\lambda + 2\mu} \frac{1}{\sin \pi z}((\lambda + 2\mu) \sin z(\alpha - \pi) \sin \alpha + \mu \cos z(\alpha - \pi) \cos \alpha - (\lambda + \mu) z \sin z(\alpha - \pi) \sin \alpha).$$

Let us denote

$$\begin{pmatrix} I - A(z) & -B(z) \\ -B(z) & I - A(z) \end{pmatrix}$$

by $\mathcal{M}K(z)$. We can show that its determinant is

$$\frac{(\lambda + \mu)^2}{(\lambda + 2\mu)^4} \frac{1}{\sin^4 \pi z}(z^2 \sin^2 \alpha - \sin^2 z\alpha)((\lambda + \mu)^2 z^2 \sin^2 \alpha - (\lambda + 3\mu)^2 \sin^2 z(\alpha - 2\pi)). \tag{3.48}$$

Define

$$L(z) = (\mathcal{M}K(z))^{-1}.$$

Thus we have

$$\begin{pmatrix} \tilde{\mathbf{u}}_+(z) \\ \tilde{\mathbf{u}}_-(z) \end{pmatrix} = L(z) \begin{pmatrix} \tilde{\mathbf{f}}_+(z) \\ \tilde{\mathbf{f}}_-(z) \end{pmatrix}. \tag{3.49}$$

$\tilde{\mathbf{u}}(z)$ is meromorphic to the right of the line

$$\Re z = -k + \frac{1}{2}$$

with poles of order m_k at p_k and at the poles of $\tilde{f}(z)$. It is clear that $L(z)$ is bounded on any line parallel to the imaginary axis that does not contain a zero of the determinant (3.48). Let us call the set of zeros of the determinant (3.48) Q. We have

$$\begin{pmatrix} \mathbf{u}_+(z) \\ \mathbf{u}_-(z) \end{pmatrix} = \frac{1}{2\pi i} \int_{\Re z = -s + \frac{1}{2}} t^{-z} L(z) \begin{pmatrix} \tilde{\mathbf{f}}_+(z) \\ \tilde{\mathbf{f}}_-(z) \end{pmatrix} dz. \tag{3.50}$$

As before, we may move the contour to the left to line

$$\Re z = -k + \frac{1}{2}.$$

Additional terms, due to the poles of $L(z)$ and the poles of $\mathbf{f}_+(z)$ and $\mathbf{f}_-(z)$, are picked up.

It is clear why the points $z = \pm 1$ are not poles of the matrix $L(z)$. It is because the zero in the denominator of the determinant of $L(z)$ is cancelled by the zero in its numerator due to the $\sin^4 z\pi$ term.

Double poles (and perhaps poles of higher orders) are possible. Double poles are possible when α satisfies

$$\frac{\sin \alpha}{\alpha} = \cos u,$$

where u satisfies

$$u = \tan u.$$

We show in Lemma 6 that there is only a finite number of poles between the lines $\Re z = -k + \frac{1}{2}$ and $\Re z = -s + \frac{1}{2}$.

Lemma 6 *There are only finitely many real roots of the equation (3.45). There is only a finite number of roots of this equation within the strip*

$$a < \Re z < b,$$

for any real numbers a and b. $\frac{1}{2}$ *is the greatest lower bound for the set of modulii of the real parts of the roots.*

Proof: For the first assertion we only need show that the equation

$$a^2 z^2 = \sin^2 z,$$

where a is a constant that is smaller than 1, has finitely many real solutions. Obviously, this equation has no real solutions for which

$$|z| > a^{-1}.$$

Thus all the real solutions are contained in the region

$$[-a^{-1}, a^{-1}].$$

If there were an infinite number of roots in this region, we should be able to find a non-isolated zero of

$$a^2 z^2 - \sin^2 z.$$

As this is clearly a holomorphic function everywhere, by a well known result of analysis it would vanish everywhere. This is patently absurd. Therefore, there must be a finite number of real roots.

The second assertion will be proved for the equation

$$az = \sin z.$$

If we write

$$z = x + iy,$$

where x and y are both real, then we have

$$ax = \sin x \cosh y$$

and

$$ay = \sinh y \cos x.$$

The second equation implies that

$$0 < \cos x < a.$$

Let us call the solution of

$$\cos x = a,$$

lying between 0 and $\frac{\pi}{2}$, b. The real part of a root could lie between

$$2n\pi + b \text{ and } (2n + \frac{1}{2})\pi,$$

where n is an integer. In this case, the allowable range of values $\cosh y$ is

$$\frac{a(2n\pi + b)}{\sin b} \leq \cosh y \leq 2na\pi.$$

Alternatively, x could lie between

$$(2n - \frac{1}{2})\pi \text{ and } 2n\pi - b,$$

where, once again, n is an integer. In this second region, however, the terms

$$ax \text{ and } \sin x \cosh y$$

are of different signs and so, in fact, no root with real part lying in the aforesaid range exists.

Thus, the roots of

$$az = \sin z,$$

lying in the region

$$c < \Re z < d,$$

for real constants c and d, are contained in a finite number of finite area rectangles. Thus, if there were an infinite number of roots, at least one rectangle would contain an infinite number of roots. This would mean that we could find a non-isolated zero of the holomorphic function

$$az - \sin z.$$

As before, this is impossible.

We can perform a similar analysis for the equation

$$az = -\sin z$$

and prove an identical result.

These results apply to the equation (3.45) because any root of that equation is a root of

$$z^2 \sin^2 \alpha = \sin^2 z\alpha \tag{3.51}$$

or

$$\frac{(\lambda+\mu)^2}{(\lambda+3\mu)^2} z^2 \sin^2 \alpha = \sin^2 z(\alpha - 2\pi), \tag{3.52}$$

and each of these is of the form we have just analysed for all α in the range $(0, 2\pi)$ and for all allowable values of the Lamé constants.

All that remains to be done is the proof of the third assertion. Clearly,

$$\sin x\alpha \cosh y\alpha$$

is convex as a function of x over the range $(0, \frac{1}{2})$, for all admissible α. We have

$$\sin\frac{\alpha}{2}\cosh y\alpha \geq |\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}| = |\frac{1}{2}\sin\alpha|.$$

This and the convexity imply that

$$|\sin x\alpha \cosh y\alpha| > |x \sin \alpha|,$$

over the entire range $(0, \frac{1}{2})$. Thus, equation (3.51) has no roots with $\Re z \leq \frac{1}{2}$. By identical reasoning, the same is true of equation (3.52).

Finally, we note that, if $\{\alpha_n\}$ is a sequence that tends to the limit 2π and if z_n is the root with the smallest positive real part corresponding to α_n then

$$\lim_{n\to\infty} \Re z_n = \frac{1}{2}.$$

This completes the proof. □

An immediate consequence of Lemma 6 is the following.

Lemma 7 *If* $\mathbf{f}_{\pm}$ *in Lemma 5 belong to* $H^{\frac{1}{2}+\delta}(0,\infty)$ *for some constant* δ *greater than zero and*

$$\mathbf{f}_{+}(0) = \mathbf{f}_{-}(0),$$

then $\mathbf{u}$ *belongs to* $H^{\frac{1}{2}+\delta'}(\Gamma)$, *where* δ' *is a constant greater than zero.*

An identical result is true for the equation

$$(I + \overrightarrow{K_0^*})u = f.$$

Proof: All that needs to be verified is that $\mathbf{u}_{+}$ and $\mathbf{u}_{-}$ each belong to $H^{\frac{1}{2}+\delta'}(0,\infty)$ and that

$$\mathbf{u}_{+}(0) = \mathbf{u}_{-}(0). \tag{3.53}$$

$\tilde{\mathbf{f}}(z)$ is meromorphic to the right of the line $\Re z = -\delta$ with simple poles situated at the points

$$z = 0, -1, -2, \ldots, -[\delta].$$

Furthermore, the residues of $\tilde{\mathbf{f}}_+$ and $\tilde{\mathbf{f}}_-$ at $z = 0$ are identical. Thus $\tilde{\mathbf{u}}$ has poles at

$$z = 0, -1, -2, \ldots, -[\delta],$$

and at the zeros of the determinant (3.45). Since there are no zeros between the lines $\Re z = -\frac{1}{2}$ and $\Re z = \frac{1}{2}$, the first two poles of $\tilde{\mathbf{u}}$ occurs at $z = 0$ and at $z = -\frac{1}{2} - \delta''$, say. Thus

$$\mathbf{u}(t) = \mathbf{u}_0(t) + \mathbf{u}_1(t),$$

where $\mathbf{u}_0(t)$ is a smooth function that is constant in a neighbourhood of the origin and vanishes in a neighbourhood of infinity and where $\mathbf{u}_1$ belongs to $\tilde{H}^{\frac{1}{2}+\Re\delta''}(\Gamma)$. The residues of $\tilde{\mathbf{u}}_+$ and $\tilde{\mathbf{u}}_-$ at the origin are equal since the residues of $\tilde{\mathbf{f}}_+$ and $\tilde{\mathbf{f}}_-$ at the origin are and since $L(z)$ is meromorphic at the origin. Thus equation (3.53) is verified. Finally, choose δ' to lie in the region

$$(0, \min\{\Re\delta'', \frac{3}{2}\})$$

and we are done.

Similarly, in the acoustic case no singularities of the Mellin transform of $I + \overline{K_0^*}$ occur in the region $[-\frac{1}{2}, \frac{1}{2}]$ and, therefore, the same result holds. □

Before proceeding to study the problem in the polygon we shall need one more lemma.

Lemma 8 *Let $u \in Z^k_{\mathcal{P}}(\Gamma; N)$, where*

$$\mathcal{P} = \{(p_k, m_k); k = 1, 2, \ldots, m\},$$

$k - \frac{1}{2}$ is not an integer and $k > 0$. $\overline{K_0^*}u$ belongs to $Z^k_{\mathcal{P}'}(\Gamma)$, where

$$\mathcal{P}' = \mathcal{P} + \{(p, 0); p = 0, 1, 2, \ldots, [k - \frac{1}{2}]\},$$

and $\chi\overline{K}_1^*u \equiv \chi(\overline{K}^* - \overline{K}_0^*)u$ *belongs to* $Z_{\mathcal{P}''}^{k+1}(\Gamma)$, *where*

$$\mathcal{P}'' = \{(p_k + l, m_k); l = 2, 4, 6, \ldots\} + \{(p, 0); p = 0, 1, 2, \ldots, [k - \frac{1}{2}]\}$$

and where χ *is a smooth cut off function.*

Moreover, the operators are bounded.

The same result holds for the equivalent elasticity operators.

This is proved in a similar way to Lemma 3.

3.6 Properties in the polygon.

We can apply the results of the last section to the polygon. We can split the problem of solving equations (3.23) and (3.24) into N wedge problems.

We wish to determine the behaviour of a solution, which we know to be in $H^s(\partial\Omega)$, where $0 < s < \frac{1}{2}$, of the system (3.23) to (3.24) in a neighbourhood of the corner C_i, say, given that we know that p_{inc} is contained in $\mathcal{H}^k(\partial\Omega)$, where

$$\mathcal{H}^k(\partial\Omega) = \prod_{i=1}^{N} H^k(\partial\Omega_i),$$

for some number k. To this end we introduce a new open, compact region $\Omega_i{}'$. This region is delineated by the curve $\partial\Omega'$, which consists of that part of $\partial\Omega$ outside a sufficiently small open neighbourhood of C_i, call this part $\partial\Omega_1'$, together with a smooth curve $\partial\Omega_2'$ whose endpoints join smoothly onto the endpoints of $\partial\Omega_1'$ but which does not intersect $\partial\Omega$ anywhere else. (See Figure (3.3).)

Let us denote by $\Omega_e{}'$ the complement of $\overline{\Omega_i{}'}$ and let us label the points where

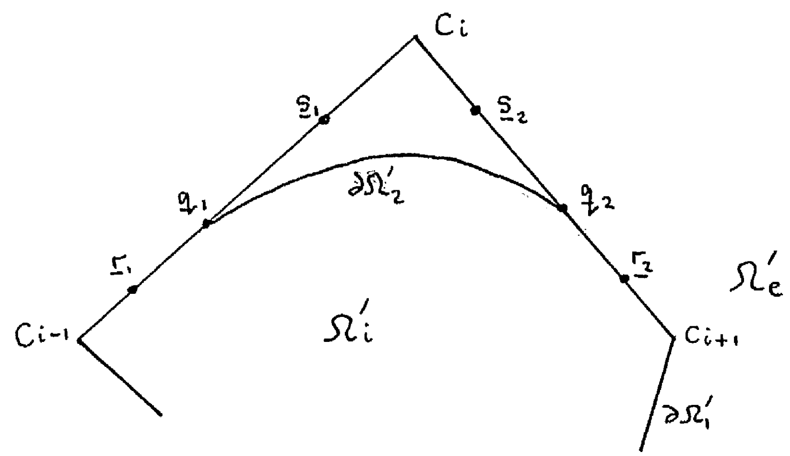

Figure 3.3: The curve $\partial\Omega'$.

$\partial\Omega_1'$ and $\partial\Omega_2'$ meet on $\partial\Omega_{i-1}$ and $\partial\Omega_i$ by q_1 and q_2 respectively. Let r_1 and r_2 be points on $\partial\Omega'$ situated between q_1 and C_{i-1} and between q_2 and C_{i+1} respectively.

Suppose that p and $\mathbf{u}$, belonging to $H^s(\partial\Omega)$, solve equations (3.23) and (3.24). Define p_1 on $\partial\Omega'$ to be equal to p on that part of $\partial\Omega_1'$ between r_1 and r_2 to vanish on $\partial\Omega_2'$ and to equal

$$\omega_j(\mathbf{x})p(\mathbf{x}) \qquad j = 1, 2$$

in the intervals (q_j, r_j), where each $\omega_j(\mathbf{x})$ is a smooth function which takes the value 1 in a neighbourhood of r_j and which vanishes in a neighbourhood of q_j. Let us define $\mathbf{u}_1$ in an analogous way. Let us suppose, without loss of generality, that the corner point C_i is contained in $\Omega_e{}'$. Nothing essential changes in the following analysis if C_i is contained in $\Omega_i{}'$.

The functions p_1 and $\mathbf{u}_1$ belong to $H^s(\partial\Omega')$. According to equations (3.14), Sp_1 and $\mathbf{S}.\mathbf{u}_1$ belong to $H^t_{loc}(\mathcal{R}^2)$, for all $t \in (\frac{1}{2}, \frac{3}{2})$. We note that the equations (3.14) are indeed applicable to $\partial\Omega'$ as well as $\partial\Omega$. The restrictions of Dp_1 and $\mathbf{D}.\mathbf{u}_1$ to $\Omega_e{}'$ belong to $H^1(\Omega_e{}')$. By construction, Sp_1, $\mathbf{S}.\mathbf{u}_1$, Dp_1 and $\mathbf{D}.\mathbf{u}_1$ satisfy Helmholtz's or the elastic wave equation in $\Omega_e{}'$. By interior regularity theory, each of these four functions is smooth on (C_i, q_1) and (C_i, q_2).

Let s_1 and s_2 be points on $\partial\Omega_{i-1}$ and $\partial\Omega_i$, lying between C_i and q_1 and q_2, respectively. Let p_2 equal p on $\partial\Omega$ between C_i and q_1, between C_i and q_2, let it equal

$$(1-\omega_j(\mathbf{x}))p(\mathbf{x}) \qquad j=1,2,$$

between q_j and r_j, and let it be equal to 0 everywhere else on $\partial\Omega$. Define $\mathbf{u}_2$ analogously. Let $\chi(\mathbf{x})$ be a smooth function equal to 1 on any point of $\partial\Omega_i \bigcup \partial\Omega_{i+1}$ between C_i and s_j, where $j=1,2$, and let it vanish between q_j and r_j.

Let us rewrite equations (3.23) to (3.24) as

$$\gamma(Dp_2+Dp_1)-\rho_0\omega^2\gamma(S\mathbf{n}.\mathbf{u}_2+S\mathbf{n}.\mathbf{u}_1)=2\gamma p_{inc} \tag{3.54}$$

and

$$\gamma(\mathbf{D}.\mathbf{u}_2+\mathbf{D}.\mathbf{u}_1)+\gamma(\mathbf{S}.\mathbf{n}p_2+\mathbf{S}.\mathbf{n}p_1)=0, \tag{3.55}$$

where, for example,

$$Dp_1(\mathbf{x})=2\int_{\partial\Omega_1'}\frac{\partial G(\mathbf{x},\mathbf{y})}{\partial n(\mathbf{y})}p_1(\mathbf{y})d|\mathbf{y}|$$

and

$$Dp_2(\mathbf{x})=2\int_{\partial\Omega}\frac{\partial G(\mathbf{x},\mathbf{y})}{\partial n(\mathbf{y})}p_1(\mathbf{y})d|\mathbf{y}|.$$

In equation (3.54) γ denotes the trace operator from Ω_e to $(C_i,r_1)\bigcup(C_i,r_2)$ and in equation (3.55) it denotes the trace operator from Ω_i to $(C_i,r_1)\bigcup(C_i,r_2)$.

Let us now multiply equations (3.54) and (3.55) by $\chi(\mathbf{x})$ to get

$$\chi(I+\overline{K}^*)p_2-\rho_0\omega^2\chi S\mathbf{n}.\mathbf{u}_2=\chi\gamma(-Dp_1+S\mathbf{n}.\mathbf{u}_1)+2\chi\gamma p_{inc} \tag{3.56}$$

and

$$\chi(\mathbf{I}-\overline{\mathbf{K}}^*).\mathbf{u}_2-\chi\mathbf{S}.\mathbf{n}p_2=-\chi\gamma(-\mathbf{D}.\mathbf{u}_1+\mathbf{S}.\mathbf{n}p_1). \tag{3.57}$$

Let us call the right-hand sides of equations (3.56) and (3.57) f and $\mathbf{g}$, respectively. We shall now rearrange equations (3.56) and (3.57) thus

$$(I+\overline{K_0^*})\chi p_2-\rho_0\omega^2\chi S\mathbf{n}.\mathbf{u}_2=(1-\chi)\overline{K_0^*}\chi p_2-\chi\overline{K_0^*}(1-\chi)p_2 \qquad (3.58)$$
$$-\chi\overline{K_1^*}p_2+f$$

and

$$(\mathbf{I}-\overline{\mathbf{K}_0^*}).\chi\mathbf{u}_2-\chi\mathbf{S}.\mathbf{n}p_2=-(1-\chi)\overline{\mathbf{K}_0^*}.\chi\mathbf{u}_2+\chi\overline{\mathbf{K}_0^*}.(1-\chi)\mathbf{u}_2 \qquad (3.59)$$
$$+\chi\overline{\mathbf{K}_1^*}.\mathbf{u}_2+\mathbf{g},$$

where

$$\overline{\mathbf{K}_1^*}=\overline{\mathbf{K}^*}-\overline{\mathbf{K}_0^*}.$$

We know from our previous analysis and from the assumptions about the smoothness of the datum p_{inc} that f' and $\mathbf{g}'$, where now, for example, f' denotes the continuation by zero of f on the wedge Γ of angle α_i, belong to $Z^k_{\mathcal{P}}(\Gamma)$, with

$$\mathcal{P}=\{(l,0);l=0,1,2,\ldots[k-\frac{1}{2}]\}.$$

As in Chapter 2, it is easy to see that the solution is as smooth as the incident wave away from the corners. Therefore, p_2 belongs to $Z^k_{\mathcal{P}_1^{(i)}}(\Gamma)$ and $\mathbf{u}_2$ belongs to $Z^k_{\mathcal{P}_2^{(i)}}(\Gamma)$ for some unknown singularity sets $\mathcal{P}_1^{(i)}$ and $\mathcal{P}_2^{(i)}$. By Lemma 8,

$$(1-\chi)\overline{K_0^*}\chi p_2-\chi\overline{K_0^*}(1-\chi)p_2-\chi\overline{K_1^*}p_2$$

belongs to $Z^k_{\mathcal{P}_3^{(i)}}(\Gamma)$, and

$$-(1-\chi)\overline{\mathbf{K}_0^*}.\chi\mathbf{u}_2+\chi\overline{\mathbf{K}_0^*}.(1-\chi)\mathbf{u}_2+\chi\overline{\mathbf{K}_1^*}.\mathbf{u}_2$$

belongs to $Z^k_{\mathcal{P}_4^{(i)}}(\Gamma)$, with $\mathcal{P}_3^{(i)}$ and $\mathcal{P}_4^{(i)}$, as in Lemma 8.

Let us re-label the terms in equations (3.58) and (3.59). Denote by p the continuation by zero of χp_2, by $\mathbf{u}$ the continuation by zero of $\chi \mathbf{u}_2$, by f the right-hand side of equation (3.58) and by g the right-hand side of equation (3.59). Now we have a problem in the infinite wedge of angle α_i.

$$(I + \overline{K}_0^*)p - \rho_0 \omega^2 \chi S \mathbf{n}.\mathrm{u}_2 = f \tag{3.60}$$

and

$$(\mathbf{I} - \overline{\mathbf{K}}_0^*).\mathbf{u} + \chi \mathbf{S}.\mathbf{n} p_2 = \mathbf{g}. \tag{3.61}$$

Let us denote by $\{\beta_1, \ldots, \beta_n\}$ the set of values of

$$\frac{m\pi}{\alpha_i} \text{ and } \frac{m\pi}{(2\pi - \alpha_i)},$$

for integer m, lying in $[0, [k - \frac{1}{2}]]$. Let m_i be zero if the equation

$$\sin \alpha_i z \sin(2\pi - \alpha_i) z = 0$$

has a simple root at $z = \beta_i$ and let it equal one if the equation has a double root there. Denote by $\{\gamma_1, \ldots, \gamma_q\}$ the set of roots of the determinant (3.48) with real part lying in the interval $[0, [k - \frac{1}{2}]]$. Let p_i equal zero if γ_i is a simple root, one if it is a root of order two, and so on.

If we let

$$\mathcal{P}_1^{(i)} = \{(p_k^{(1)}, s_k^{(1)}); k = 1, \ldots, t_1\},$$

and

$$\mathcal{P}_2^{(i)} = \{(p_k^{(2)}, s_k^{(2)}); k = 1, \ldots, t_2\},$$

then, according to Lemma 8, f has a singular part characterized by the set

$$\mathcal{P}_3^{(i)} = \{(p_k^{(1)} + 2l, s_k^{(1)}); k = 1, \ldots, t_1; l = 1, 2, 3, \ldots\} + \{(l, 0); l = 1, 2, 3, \ldots\}.$$

According to Lemma 3, the singular part of $\rho_0\omega^2\chi S\mathbf{n}.\mathbf{u}_2$ is characterized by

$$\mathcal{P}_5^{(i)} = \{(p_k^{(2)} + 2l - 1, s_k^{(2)}); k = 1, \ldots, t_2; l = 1, 2, 3, \ldots\} + \{(l, 0); l = 1, 2, 3, \ldots\}.$$

The sum

$$f + \rho_0\omega^2\chi S\mathbf{n}.\mathbf{u}_2$$

is characterized by

$$\mathcal{P}_6^{(i)} = \mathcal{P}_3^{(i)} \bigcup \mathcal{P}_5^{(i)}.$$

From equation (3.60) and Lemma 4, the singular part of p is characterized by

$$\{(\beta_i, m_i)\} + \mathcal{P}_6^{(i)}.$$

For consistency it is required that this equals $\mathcal{P}_1^{(i)}$. There is a similar relationship between $\mathcal{P}_2^{(i)}$ and $\{(\gamma_i, p_i)\}$.

It is easy to calculate the singularity sets of p and $\mathbf{u}$ for particular sets $\{(\beta_i, m_i)\}$ and $\{(\gamma_i, p_i)\}$. For example, suppose that neither β_i nor γ_i are integers, for all i. Furthermore, suppose that

$$\beta_i + 2l \neq \gamma_j + 2m + 1,$$

for all i and j and for all non-negative integers l and m, such that

$$\beta_i + 2l \text{ and } \gamma_j + 2m + 1$$

lie in $[0, [k - \frac{1}{2}]]$, and

$$\beta_i + 2l + 1 \neq \gamma_j + 2m$$

for all i and j and for all non-negative integers l and m such that

$$\beta_i + 2l + 1 \text{ and } \gamma_j + 2m.$$

The singularity set $\mathcal{P}_1^{(i)}$ then consists of the elements

$$(\beta_1, m_1), (\beta_1 + 2, m_1), (\beta_1 + 4, m_1), \dots$$
$$\vdots$$
$$(\beta_n, m_n), (\beta_n + 2, m_n), (\beta_n + 4, m_n), \dots$$
$$(\gamma_1 + 1, p_1), (\gamma_1 + 3, p_1), (\gamma_1 + 5, , p_1), \dots$$
$$\vdots$$
$$(\gamma_q + 1, p_q), (\gamma_q + 3, p_q), (\gamma_q + 5, p_q), \dots$$
$$(0,0), (1,1), (2,2), (3,3), (4,4), \dots$$

$\mathcal{P}_2^{(i)}$ consists of analogous terms.

We now apply this reasoning to each corner in turn, but first w eneed a definition.

Let

$$\mathcal{P} = \mathcal{P}^{(1)} \otimes \mathcal{P}^{(2)} \otimes \dots \otimes \mathcal{P}^{(N)},$$

where each $\mathcal{P}^{(j)}$ denotes a singularity set. Let $Z^k_{\mathcal{P}}(\partial\Omega)$ denote the space of functions u, such that

$$u = u_0 + u_1,$$

where the restriction of u_0 to $\partial\Omega_i$ belongs to $\tilde{H}^k(\partial\Omega_i)$, for each $i = 1, 2, \dots, N$, where u_1 is function with singular behaviour in the ith corner, characteristic of functions with exponents in $\mathcal{P}^{(i)}$. $Z^k_{\mathcal{P}}(\partial\Omega)$ is equipped with the obvious norm.

Theorem 9 *Let p and $\mathbf{u}$ in $H^s(\partial\Omega)$, where $0 < s < \frac{1}{2}$, solve equations (3.23) and (3.24) with p_{inc} belonging to $\mathcal{H}^k(\partial\Omega)$ with $k > \frac{1}{2}$ and $k - \frac{1}{2}$ not a root of the* *equation*

$$\prod_{i=1}^{N} A_i(z)B_i(z) = 0, \tag{3.62}$$

where

$$A_i(z) = \frac{\sin\alpha_i z \sin(2\pi - \alpha_i)z}{\sin\pi z}$$

and

$$B_i(z) = \frac{1}{\sin^4\pi z}(z^2\sin^2\alpha_i - \sin^2\alpha_i z)(z^2\sin^2\alpha_i - \frac{(\lambda+3\mu)^2}{(\lambda+\mu)^2}\sin^2\alpha_i z),$$

for $i = 1, \ldots, N$. p *and* $\mathbf{u}$ *belong to* $Z^k_{\mathcal{P}'}(\partial\Omega)$ *and* $Z^k_{\mathcal{P}''}(\partial\Omega)$, *respectively, where*

$$\mathcal{P}' = \prod_{i=1}^{N} \mathcal{P}_1^{(i)}$$

and

$$\mathcal{P}'' = \prod_{i=1}^{N} \mathcal{P}_2^{(i)},$$

with $\mathcal{P}_1^{(i)}$ *and* $\mathcal{P}_2^{(i)}$ *as above.*

Moreover, we have the estimate

$$\| p \|_{Z^k_{\mathcal{P}'}(\partial\Omega)} + \| \mathbf{u} \|_{Z^k_{\mathcal{P}''}(\partial\Omega)} \leq M(\| p_{inc} \|_{\mathcal{H}^k(\partial\Omega)} + \| p \|_{H^s(\partial\Omega)} + \| \mathbf{u} \|_{H^s(\partial\Omega)}),$$

for some constant M.

As a consequence of Lemma 7 we have:

Lemma 9 *If* p *and* $\mathbf{u}$ *belong to* $H^s(\partial\Omega)$ *and solve equations (3.23) and (3.24) with* p_{inc} *in* $H^k(\partial\Omega)$ *for* $k > \frac{1}{2}$, *then* p *and* $\mathbf{u}$ *belong to* $H^t(\partial\Omega)$ *for some* t *lying in* $(\frac{1}{2}, \frac{3}{2})$.

Let us now prove the following lemma:

Lemma 10 *If p and $\mathbf{u}$ belong to $H^s(\partial\Omega)$, where $0 < s < \frac{1}{2}$, and solve equations (3.23) and (3.24) with $p_{inc} = 0$, then p and $\mathbf{u}$ vanish as long as k is not an eigenvalue of the interior Dirichlet problem and ω is not a Jones' frequency.*

Proof: By the previous lemma, p and $\mathbf{u}$ belong to $H^t(\partial\Omega)$ for some t in $(\frac{1}{2}, \frac{3}{2})$. By the Sobolev imbedding theorem, p and $\mathbf{u}$ belong to $C^{0,t-\frac{1}{2}}(\partial\Omega)$ (see, e. g., Sanchez Hubert and Sanchez Palencia [27, p. 33]). Define

$$\frac{\rho\omega^2}{2} S\mathbf{n}.\mathbf{u} - \frac{1}{2} Dp = \begin{cases} P_i(\mathbf{x}) \text{ if } \mathbf{x} \in \Omega_i \\ P_e(\mathbf{x}) \text{ if } \mathbf{x} \in \Omega_e \end{cases}$$

and

$$\frac{1}{2}\mathbf{D}.\mathbf{u} + \frac{1}{2}\mathbf{S}.\mathbf{n}p = \begin{cases} \mathbf{U}_i(\mathbf{x}) \text{ if } \mathbf{x} \in \Omega_i \\ \mathbf{U}_e(\mathbf{x}) \text{ if } \mathbf{x} \in \Omega_e \end{cases} .$$

The jump conditions imply that

$$P_i|_{\partial\Omega} = 0. \tag{3.63}$$

Since k is not an eigenvalue of the interior Dirichlet problem, P_i vanishes in its domain. Thus,

$$\left.\frac{\partial P_i}{\partial n}\right|_{\partial\Omega} = 0. \tag{3.64}$$

Since p is Hölder continuous, the normal derivative of the double layer potential is continuous across $\partial\Omega$. Therefore,

$$\left.\frac{\partial P_e}{\partial n}\right|_{\partial\Omega} = \rho\omega^2\mathbf{n}.\mathbf{u}. \tag{3.65}$$

From the jump conditions and equation (3.61), we have

$$P_e|_{\partial\Omega} = p. \tag{3.66}$$

We have

$$\mathbf{U}_e|_{\partial\Omega} = 0. \tag{3.67}$$

By construction $\mathbf{U}_e$ satisfies the elastic wave equation in Ω_e and an appropriate radiation condition. Therefore, from Kupradze [16, pp. 132–136], $\mathbf{U}_e$ vanishes in Ω_e. So

$$\sigma(\mathbf{U}_e).\mathbf{n}|_{\partial\Omega} = \mathbf{0}. \tag{3.68}$$

As before, since $\mathbf{u}$ is Hölder continuous, the tractions corresponding to $\mathbf{D}.\mathbf{u}$ are continuous across $\partial\Omega$. Therefore, from the the jump conditions and equation (3.68), we have

$$\sigma(\mathbf{U}_i).\mathbf{n}|_{\partial\Omega} = p\mathbf{n}. \tag{3.69}$$

Clearly,

$$\mathbf{U}_i|_{\partial\Omega} = \mathbf{u}. \tag{3.70}$$

Equations (3.65), (3.66), (3.69) and (3.70) imply that the transmission conditions are satisfied by P_e and $\mathbf{U}_i$. $\mathbf{U}_i$ clearly satisfies the elastic wave equation in Ω_i and belongs to $H^1(\Omega_i)$. P_e satisfies Helmholtz's equation in Ω_e with the Sommerfeld radiation condition and belongs to $H^1_{loc}(\Omega_e)$. Lemma 1 implies that P_e and $\mathbf{U}_i$ vanish identically if ω is not a Jones' frequency. Therefore, p and $\mathbf{u}$ vanish. This completes the proof. □

3.7 The adjoint problem and bijectivity of the system.

Now let us consider the equations

$$f + Kf - \mathbf{n}.\mathbf{S}.\mathrm{g} = 0 \tag{3.71}$$

and

$$\mathrm{g} - \mathbf{K}.\mathrm{g} - \rho_0\omega^2\mathbf{n}Sf = 0. \tag{3.72}$$

The system in equations (3.71) and (3.72) is the adjoint of the one in equations (3.23) and (3.24). Suppose that these equations have a non-trivial solution in $H^{-s}(\partial\Omega)$, for $0 < s < \frac{1}{2}$. Let

$$P = \frac{1}{2} S f$$

and

$$\mathbf{U} = \frac{1}{2} \mathbf{S}.\mathbf{g}.$$

We know that P and $\mathbf{U}$ belong to $H^1_{loc}(\mathcal{R}^2)$ and that P satisfies Helmholtz's equation in Ω_i and in Ω_e and that $\mathbf{U}$ satisfies the elastic wave equation in Ω_i and in Ω_e. Now equation (3.71) implies that

$$\left.\frac{\partial P_+}{\partial n}\right|_{\partial\Omega} = \mathbf{n}.\mathbf{U}|_{\partial\Omega}, \tag{3.73}$$

where the $+$ subscript refers to the limit as the surface is approached from the exterior. Later the $-$ will refer to the limit as the surface is approached from the interior. Equation (3.72) implies that

$$(\sigma(\mathbf{U}_-).\mathbf{n})|_{\partial\Omega} = -\rho_0 \omega^2 P \mathbf{n}|_{\partial\Omega}. \tag{3.74}$$

By amending the proof of Theorem 7 slightly we can show that P vanishes in Ω_e and $\mathbf{U}$ vanishes in Ω_i unless ω is a Jones' mode. If P does vanish in Ω_e then it vanishes on $\partial\Omega$. In this case it vanishes within Ω_i unless k is an eigenvalue of the interior Dirichlet problem. Thus,

$$p = \left.\frac{\partial P_+}{\partial n}\right|_{\partial\Omega} - \left.\frac{\partial P_-}{\partial n}\right|_{\partial\Omega} = 0.$$

If $\mathbf{U}$ vanishes within Ω_i, then it vanishes on $\partial\Omega$. From Kupradze [16, pp.132–136], then, $\mathbf{U}$ vanishes everywhere. Clearly, this implies that $\mathbf{u}$ vanishes.

We are now in a position to prove the following theorem:

Theorem 10 *If k is not an eigenvalue of the interior Dirichlet problem and ω is not a Jones' frequency, then the system*

$$\begin{pmatrix} I+\overline{K}^* & -\rho_0\omega^2 S\mathbf{n}. \\ \mathbf{S}.\mathbf{n} & (\mathbf{I}-\overline{\mathbf{K}}^*). \end{pmatrix}$$

is bijective in $H^s(\partial\Omega)$ for $0 < s < \frac{1}{2}$.

Proof: We have already proved that the system is injective. We shall suppose that it is not subjective and show that this leads to a contradiction. If it were not subjective, then there would exist elements of the space $H^{-s}(\partial\Omega)$ that are perpendicular to the image of $H^s(\partial\Omega)$ under the system. That is to say, there would exist f and g belonging to $H^{-s}(\partial\Omega)$ with the property that

$$< f, (I+\overline{K}^*)p - \rho_0\omega^2 S\mathbf{n}.\mathbf{u} > + < \mathbf{g}, (\mathbf{I}-\overline{\mathbf{K}}^*).\mathbf{u} + \mathbf{S}.\mathbf{n}p >= 0,$$

for all p and $\mathbf{u}$ in $H^s(\partial\Omega)$, where the angled brackets denote the duality product between $H^{-s}(\partial\Omega)$ and $H^s(\partial\Omega)$. Therefore,

$$< (I+K)f + \mathbf{n}.\mathbf{S}.\mathbf{g}, p > + < (\mathbf{I}-\mathbf{K}).\mathbf{g} - \rho_0\omega^2\mathbf{n}Sf, \mathbf{u} >= 0.$$

This would imply the existence of a non-trivial solution of equations (3.71) and (3.72). As we have already seen, this is impossible under the given assumptions. Therefore, the system is subjective. □

3.8 Conclusions.

We have proved the existence of a solution of the transmission problem, at least for frequencies that are neither eigenvalues of the interior Dirichlet problem nor Jones' frequencies; we have $p \in H^s_{loc}(\Omega_e)$ and $\mathbf{u} \in H^s(\Omega_i)$, for $s > 1$. As in the case of a smooth elastic body, which was studied in the previous chapter, the non-solvability of the system at eigenvalues of the interior Dirichlet is spurious; a solution exists at all frequencies including Jones' frequencies. Furthermore, the solution is unique at all frequencies except Jones' frequencies.

In this chapter we showed what kind of singularity behaviour is to be expected near the corners. Similar results would hold in the three-dimensional problem of a body with edges (see Ola [25] for the three-dimensional transmission problem involving a field satisfying Helmholtz's equation coupled to another field satisfying Helmholtz's equation). If the body had curved sides then we should expect the leading singular behaviour to be unchanged, but with modifications appearing at higher orders. (See Costabel and Stephan [7] for the effects of curvature in a related problem.)

Chapter 4

Asymptotics of Scattering Frequencies

4.1 Introduction

In this chapter we consider the problem of an elastic body deeply submerged in an incompressible, inviscid fluid. The fluid is subjected to gravity and has a free surface. The body occupies a compact region of $\mathcal{R}^3$ of non-zero measure. The solid–fluid interface is infinitely smooth.

We consider here only *free* oscillations of the system. That is to say, we will not study the problem in which some external forcing term is present. Furthermore, the oscillations are assumed to be small. This means that we will ignore all non-linear terms. This assumption also means that the positions of surfaces (e. g. the free surface and the solid–fluid interface) do not change with time.

The elastic body and the fluid are coupled in two separate ways. Firstly, the normal component of the velocity of the solid must match the normal component of velocity of the fluid at the interface between the two. Secondly, the surface traction is to be continuous across the interface. The first of these two couplings, which we will call the *kinematic boundary condition*, is necessary to ensure that the fluid and the solid remain in contact. We note that there is no link between the tangential components of the velocities across the interface. This is because we are assuming that the fluid is inviscid, and so it can slip over the surface of the elastic solid. The second matching condition (the so-called *dynamic boundary condition*) results from the force balance across elements of the interface.

The motion is assumed to be time-harmonic with frequency ω. We will generalise the problem to allow the possibility of ω having non-zero imaginary part. We will then show that there is a countably infinite number of values of ω for which the generalised problem has a non-trivial solution. We call these *scattering frequencies*.

We shall treat the problem as a perturbation to the problem of an elastic body surrounded by an incompressible and inviscid fluid that is unbounded in all directions. That is to say, in the unperturbed problem there is no free surface. It is easily shown that there is a countably infinite number of values of ω for which the unperturbed problem has a non-trivial solution. The problem of finding the scattering frequencies of the perturbed problem was studied by Vullierme–Ledard [29]. She showed that the scattering frequencies associated with simple modes have purely real asymptotic expansions in inverse powers of submergence depth.

The presence of the free surface will allow waves to be generated. These waves radiate energy away. If waves appear, then a free oscillation with real ω will, therefore, be an impossibility. We intuitively expect that waves will always be

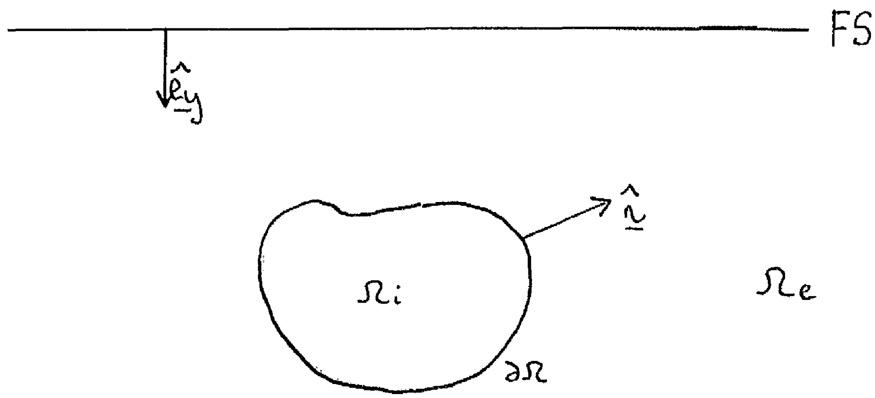

Figure 4.1: The coupled system.

generated and, so, we concentrate here on the imaginary parts of the scattering frequencies. We shall confirm that they are non-zero for finite but very large submergence depth and we shall show that they are "exponentially small"; i. e. they have zero asymptotic expansion in inverse powers of submergence depth to all orders. This is consistent with Vullierme–Ledard's result.

4.2 The formulation of the problem.

The reader is refered to Figure (4.1). The elastic solid occupies the compact region Ω_i and the fluid occupies the region

$$\Omega_e = \{(x, y, z); y > -1/\epsilon\} \setminus \Omega_i.$$

The common boundary of Ω_e and Ω_i is $\partial\Omega$ and the free-surface is the set

$$FS = \{(x, -1/\epsilon, z); x \in \mathcal{R}, z \in \mathcal{R}\}.$$

The vector normal to $\partial\Omega$ is denoted by $\mathbf{n}$ and the origin is assumed to be contained in Ω_i.

In what follows we shall denote by $\mathbf{x}$ any position vector in $\Omega_i \cup \partial\Omega \cup \Omega_e \cup FS$ and t will be the temporal variable.

4.2.1 The velocity potential in the fluid.

The circulation of an inviscid fluid remains constant. We assume that the motion of the fluid was generated from rest and that all transient solutions have completely decayed leaving just the time-harmonic motion. The motion of the fluid must then be irrotational for all time. By a well known result of analysis, the fluid velocity, $\mathbf{v}$, can be written in the form

$$\mathbf{v}(\mathbf{x}, t) = \nabla_{\mathbf{x}} \Phi(\mathbf{x}, t), \tag{4.1}$$

where $\Phi(\mathbf{x}, t)$ is a real-valued, scalar function defined in the domain $\Omega_e \otimes \mathcal{R}$.

We know that $\Phi(\mathbf{x}, t)$ is time-harmonic and, thus, we can separate the spatial and temporal dependence and write:

$$\Phi(\mathbf{x}, t) = \Re(\phi(\mathbf{x}) \exp(-i\omega t)), \tag{4.2}$$

where $\Re z$ represents the real part of any complex number z.

If ρ_0 is the density of the fluid, then the following conservation of mass equation is satisfied:

$$\frac{D\rho_0}{Dt} + \rho_0 \nabla.\mathbf{v} = 0,$$

where the first term is the convective derivative of ρ_0. The assumption of incompressibility means that the first term in this equation vanishes. Therefore,

$$\nabla.\mathbf{v} = 0 \qquad \text{in } \Omega_e. \tag{4.3}$$

Equations (4.1) to (4.3) imply that ϕ satisfies

$$\nabla^2\phi = 0.$$

The equation satisfied by ϕ on FS — the so-called linearised free surface condition — is

$$\left.\left(\frac{\partial\phi}{\partial y} + \frac{\omega^2}{g}\phi\right)\right|_{FS} = 0,$$

where g represents the acceleration due to gravity. This results from the combination of a kinematic boundary condition and a dynamic boundary condition on FS.

4.2.2 The motion of the solid.

Our starting point is the basic equation of motion of a homogeneous, isotropic elastic body:

$$\nabla.\sigma = \rho\frac{\partial^2\mathbf{u}}{\partial t^2}, \tag{4.4}$$

where the constant ρ represents the density of the solid in its undeformed state.

σ is called the stress tensor and is given in terms of the displacement vector $\mathbf{u}$, by

$$\sigma(\mathbf{u}) = \lambda(\nabla.\mathbf{u})\mathbf{I} + \mu(\nabla\mathbf{u} + (\nabla\mathbf{u})^T). \tag{4.5}$$

$\mathbf{I}$ represents the identity matrix and if $\mathbf{A}$ is any matrix then $\mathbf{A}^T$ is its transpose. λ and μ are the Lamé coefficients. They are independent of both $\mathbf{x}$ and t.

Equations (4.4) and (4.5) together imply that $\mathbf{u}$ satisfies

$$L(\mathbf{u}) \equiv \mu\nabla^2\mathbf{u} + (\mu+\lambda)\nabla(\nabla.\mathbf{u}) = \rho\frac{\partial^2\mathbf{u}}{\partial t^2}. \tag{4.6}$$

As with $\Phi(\mathbf{x},\mathbf{t})$, we write

$$\mathbf{u}(\mathbf{x},t)=\Re(\mathbf{u}(\mathbf{x})\exp(-i\omega t)).$$

Equation (4.6) implies

$$L(\mathbf{u})+\rho\omega^2\mathbf{u}=\mathbf{0}. \tag{4.7}$$

4.2.3 The matching conditions across $\partial\Omega$

The kinematic boundary condition implies

$$-i\omega\mathbf{u}.\mathbf{n}=\frac{\partial\phi}{\partial n} \text{ on } \partial\Omega_a, \tag{4.8}$$

and the dynamic boundary condition implies

$$\mathbf{n}.\sigma(\mathbf{u})=i\rho_0\omega\phi\mathbf{n} \text{ on } \partial\Omega_a. \tag{4.9}$$

Equation (4.9) comes from substituting equations (4.1) and (4.2) into the linearised Stokes equation. We see that the part of the fluid pressure that is varying harmonically is given by $i\rho_0\omega\phi$.

4.2.4 Radiation condition.

To complete the formulation of the problem, we must add a radiation condition. This ensures that any solution is physically relevant — that is to say, that energy is radiated *away*. The radiation condition is given by

$$\lim_{R\to\infty}\int\left|\frac{\partial\phi}{\partial R}-i\frac{\omega^2}{g}\phi\right|^2 dS=0, \tag{4.10}$$

where the integral is taken over the surface of a vertical cylinder of radius R. This is the Rellich radiation condition.

4.3 The exterior problem.

We now proceed to solve for ϕ with $\mathbf{u}.\mathbf{n}|_{\partial\Omega}$ in equation (4.8) given as datum. Once ϕ has been found in terms of $\mathbf{u}.\mathbf{n}|_{\partial\Omega}$, equations (4.6) and (4.9) are utilized in order to derive a single equation for $\mathbf{u}$. This equation will be solvable for only a discrete set of (in general complex) values of ω^2.

We shall first deal with the case in which the frequency, ω, is real. Later, we shall see how the problem for non-real frequencies can have meaning.

The exterior problem is: Find $\phi \in H^1_{loc}(\Omega_e)$ such that

$$\nabla^2\phi = 0 \qquad \text{in } \Omega_e, \tag{4.11}$$

$$\left(\frac{\partial\phi}{\partial y} + \tfrac{\omega^2}{g}\phi\right)\bigg|_{FS} = 0, \tag{4.12}$$

$$\frac{\partial\phi}{\partial n}|_{\partial\Omega} = -i\omega\mathbf{u}.\mathbf{n}|_{\partial\Omega} \equiv f \in L^2(\partial\Omega), \tag{4.13}$$

$$\lim_{R\to\infty}\int\left|\frac{\partial\phi}{\partial R} - i\tfrac{\omega^2}{g}\phi\right|^2 dS = 0. \tag{4.14}$$

Theorem 11 *The exterior problem has a unique solution for every $f \in L^2(\partial\Omega)$ except possibly at a set of isolated values of ω^2.*

The first step to proving the theorem is to consider the problem below. Given $h \in H^{\frac{1}{2}}(\partial\Omega_a)$, find $\psi \in H^1_{loc}(\Omega_a)$ satisfying

$$\nabla^2\psi = 0 \qquad \text{in } \Omega_a, \tag{4.15}$$

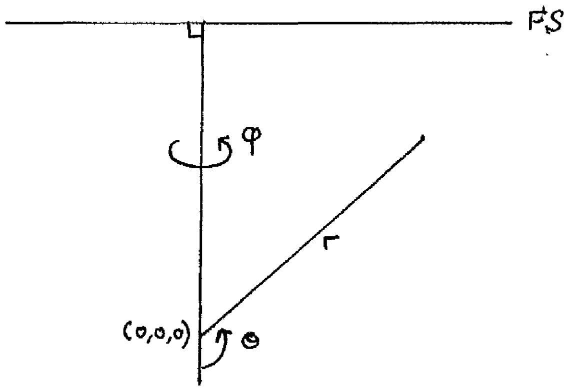

Figure 4.2: Spherical polar coordinates.

$$\psi|_{\partial\Omega_a} = h, \tag{4.16}$$

$$\left(\frac{\partial\psi}{\partial y} + \tfrac{\omega^2}{g}\psi\right)\Big|_{FS} = 0, \tag{4.17}$$

$$\lim_{R\to\infty}\int\left|\frac{\partial\phi}{\partial R} - i\tfrac{\omega^2}{g}\phi\right|^2 dS = 0. \tag{4.18}$$

Ω_a is the set $\{\mathbf{x}\in\Omega_e; |\mathbf{x}| > a\}$ and $\partial\Omega_a = \{\mathbf{x}; |\mathbf{x}| = a\}$. We choose a so that $\partial\Omega_a \subset \Omega_e$.

Firstly, we express h as a sum of Legendre polynomials and trigonometric functions:

$$h = \sum_{m=0}^{\infty}\sum_{n=m}^{\infty} P_n^m(\cos\theta)(c_{mn}\cos m\phi + d_{mn}\sin m\phi), \tag{4.19}$$

where θ and ϕ are coordinates on $\partial\Omega_a$— see Figure (4.2).

Lemma 11 *h belongs to $H^{\frac{1}{2}}(\partial\Omega_a)$ if and only if*

$$\sum_{m=0}^{\infty}\sum_{n=m}^{\infty}(|c_{mn}|^2 + |d_{mn}|^2)\frac{(n+m)!}{(n-m)!} \text{ exists.} \tag{4.20}$$

Proof: We shall start by showing that the condition is necessary. Let us suppose that the condition is not necessary. h clearly belongs to a smaller space than

$L^2(\partial\Omega_a)$, so suppose

$$h \in H^s(\partial\Omega_a), \text{ where } 0 < s < \frac{1}{2}.$$

Denote by B_a the interior of the sphere of radius a. Let f be any member of $H^{1-s}(\partial\Omega_a)$ and let f^* be a continuous lifting from $H^{1-s}(\partial\Omega_a)$ to $H^{\frac{3}{2}-s}(B_a)$; i. e.

$$f^*|_{\partial\Omega_a} = f \text{ and } \| f^* \|_{H^{\frac{3}{2}-s}(B_a)} \leq K \| f \|_{H^{1-s}(\partial\Omega_a)},$$

for some positive constant K independent of f.

We can assert the existence of a distribution v, belonging to $H^{\frac{1}{2}+s}(B_a)$ with the properties

$$v|_{\partial\Omega_a} = h,$$

$$\nabla^2 v = 0 \qquad \text{in } B_a.$$

Furthermore,

$$\| v \|_{H^{\frac{1}{2}+s}(B_a)} \leq C \| h \|_{H^s(\partial\Omega_a)}, \tag{4.21}$$

for some constant C. This is a result of the theory of Lions and Magenes [19, Chapter 2]. In their terms, the problem with the operator $-\nabla^2$, with the Dirichlet boundary, condition is a *properly elliptical problem.*

We define the normal derivative $\frac{\partial v}{\partial n}$, of v on $\partial\Omega_a$, via the formula

$$\int_{\partial\Omega_a} \frac{\partial v}{\partial n} f dS = \int_{B_a} \nabla v . \nabla f^* dV. \tag{4.22}$$

The integral on the right hand side of equation (4.22) exists for all pairs (v, f^*) because $\nabla v \in H^{-\frac{1}{2}+s}(B_a)$, $\nabla f^* \in H^{\frac{1}{2}-s}(B_a)$ and $H^{-\frac{1}{2}+s}(B_a)$ is the dual space of $H^{\frac{1}{2}-s}(B_a)$, since s lies between zero and one half. Therefore, $\frac{\partial v}{\partial n}$ is a well defined member of $H^{s-1}(\partial\Omega_a)$.

The following inequalities hold:

$$\left|\int_{B_a} \nabla v . \nabla f^* dV\right| \leq \| \nabla v \|_{H^{-\frac{1}{2}+s}(B_a)} \| \nabla f^* \|_{H^{\frac{1}{2}-s}(B_a)} \tag{4.23}$$
$$\leq \| v \|_{H^{\frac{1}{2}+s}(B_a)} \| f^* \|_{H^{\frac{3}{2}-s}(B_a)} .$$

The first inequality is a direct consequence of the definition of the norm of an element of $H^{-\frac{1}{2}+s}(B_a)$. Remember that if $y \in X'$, the dual space of some Banach space X, then

$$\| y \|_{X'} = \sup_{x \in X, x \neq 0} \frac{< y, x >_{X', X}}{\| x \|_X}.$$

Inequalities (4.21), (4.23) and the fact that the lifting from $H^{1-s}(\partial\Omega_a) \rightarrow H^{\frac{3}{2}-s}(B_a)$ is continuous imply a third inequality, which, once again, is a consequence of the definition of the norm of an element of a dual space:

$$\left\|\frac{\partial v}{\partial n}\right\|_{H^{s-1}(\partial\Omega_a)} \leq M \| h \|_{H^s(\partial\Omega_a)} . \tag{4.24}$$

If we define the injection V as the operator that sends h to $\frac{\partial v}{\partial n}$, then inequality (4.24) implies that V is bounded when thought of as an operator between $H^s(\partial\Omega_a)$ and $H^{s-1}(\partial\Omega_a)$. So the range of V is closed in $H^{s-1}(\partial\Omega_a)$.

We must now prove that the range of V is dense in the subspace $H^{s-1}(\partial\Omega_a)\backslash L$, where L is the space of constant functions on $\partial\Omega_a$.

Suppose this were not true. Then there would exist a non-trivial element f, belonging to $H^{1-s}(\partial\Omega_a) \setminus L$, with the property that

$$\int_{\partial\Omega_a} (Vh) f dS = 0 \text{ for all } h \in H^s(\partial\Omega_a). \tag{4.25}$$

Since $H^{1-s}(\partial\Omega_a) \subset H^s(\partial\Omega_a)$ (remember that $s < 1-s$), we can choose $h = \overline{f}$, where $\overline{f}$ denotes the complex conjugate of f. Equations (4.22) and (4.25) imply

$$0 = \int_{\partial\Omega_a} (V\overline{f}) f dS = \int_{B_a} \nabla\overline{f^*}.\nabla f^* dV. \tag{4.26}$$

f^* is the solution of the Dirichlet problem: Find $f^* \in H^{\frac{3}{2}-s}(\partial\Omega_a)$, with $\nabla^2 f^* = 0$ and $f^*|_{\partial\Omega_a} = f$.

Equation (4.26) implies that f^* is constant. Clearly, then, f is constant. Since f is perpendicular to L, $f = 0$.

So we have proved that the range of V is closed and dense in $H^{s-1}(\partial\Omega_a) \setminus L$. Thus the range of $V = H^{s-1}(\partial\Omega_a) \setminus L$.

We now go on to prove that the range of V is in $H^{-s}(\partial\Omega_a)$. Let h and g be defined as

$$h = \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} P_n^m(\cos\theta)(c_{mn}\cos m\phi + d_{mn}\sin m\phi) \tag{4.27}$$

$$g = \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} P_n^m(\cos\theta)(a_{mn}\cos m\phi + b_{mn}\sin m\phi), \tag{4.28}$$

where the coefficient pairs (c_{mn}, d_{mn}) and (a_{mn}, b_{mn}) satisfy the condition (4.20). Thus h and g belong to $H^s(\partial\Omega_a)$.

If v is the solution of the interior Dirichlet problem, with $v = h$ on $\partial\Omega_a$, then

$$v = \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} P_n^m(\cos\theta)(c_{mn}\cos m\phi + d_{mn}\sin m\phi)(r/a)^n,$$

where r is the distance from the origin. Hence Vh, the normal derivative of v on $\partial\Omega_a$, is given by

$$\frac{\partial v}{\partial n} = \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} na^{-1} P_n^m(\cos\theta)(c_{mn}\cos m\phi + d_{mn}\sin m\phi).$$

Therefore

$$\left|\int_{\partial\Omega_a} (Vh) g dS\right| \le C \left|\sum_{m=0}^{\infty} \sum_{n=m}^{\infty} (c_{mn}a_{mn} + d_{mn}b_{mn}) \frac{n}{n+\frac{1}{2}} \frac{(n+m)!}{(n-m)!}\right|,$$

for some constant C. This comes from the orthogonality property of Legendre polynomials:

$$\int_{-1}^{1} P_n^m(c)P_s^m(c)dc = \frac{\delta_{ns}}{n+\frac{1}{2}}\frac{(n+m)!}{(n-m)!},$$

where δ_{ns} denotes the Kronecker delta. Clearly then

$$\left|\int_{\partial\Omega_a}(Vh)gdS\right| \le K\sum_{m=0}^{\infty}\sum_{n=m}^{\infty}(|c_{mn}||a_{mn}|+|d_{mn}||b_{mn}|)\frac{(n+m)!}{(n-m)!},$$

for some constant K. Schwarz's and the triangle inequalities imply

$$\left|\int_{\partial\Omega_a}(Vh)gdS\right| \le$$

$$K\left(\sum_{m=0}^{\infty}\sum_{n=m}^{\infty}(|c_{mn}|^2+|d_{mn}|^2)\frac{(n+m)!}{(n-m)!}\right)^{\frac{1}{2}}\left(\sum_{m=0}^{\infty}\sum_{n=m}^{\infty}(|a_{mn}|^2+|b_{mn}|^2)\frac{(n+m)!}{(n-m)!}\right)^{\frac{1}{2}}.$$

Hence the inner product of Vh and g exists. Since h and g were arbitrarily chosen, the range of V is a subset of $H^{-s}(\partial\Omega_a)$. This implies that $H^{s-1}(\partial\Omega_a) \subseteq H^{-s}(\partial\Omega_a)$, which, in turn, implies that $-s \le s-1$, or $s \ge \frac{1}{2}$. We assumed at the beginning that $s < \frac{1}{2}$. This is clearly absurd. Therefore, $s = \frac{1}{2}$. □

We shall now prove that the condition $h \in H^{\frac{1}{2}}(\partial\Omega_a)$ is sufficient for the condition (4.20) to hold true, where the notation of equation (4.19) is followed.

We know that there exists a unique solution of the interior Dirichlet problem: Find $v \in H^1(B_a)$ such that

$$\nabla^2 v = 0 \quad \text{in } B_a,$$

$$v|_{\partial\Omega_a} = h.$$

Clearly, the solution for v is

$$v = \sum_{m=0}^{\infty}\sum_{n=m}^{\infty} P_n^m(\cos\theta)(c_{mn}\cos m\phi + d_{mn}\sin m\phi)(r/a)^n.$$

The normal derivative of v on $\partial\Omega_a$ is defined by the equation

$$\int_{\partial\Omega_a} \frac{\partial v}{\partial n} f dS = \int_{B_a} \nabla v . \nabla f^* dV,$$

where f is any member of $H^{\frac{1}{2}}(\partial\Omega_a)$, and f^* is a continuous lifting of f to $H^1(B_a)$. $\frac{\partial v}{\partial n}$ is, therefore, a well defined member of $H^{-\frac{1}{2}}(\partial\Omega_a)$. Hence, the inner product of h and $\frac{\partial v}{\partial n}$ must exist. That is to say,

$$\int_{\partial\Omega_a} \left(\frac{\partial v}{\partial n}\right) h dS = \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} c_m (|c_{mn}|^2 + |d_{mn}|^2) a \frac{n}{n+\frac{1}{2}} \frac{(n+m)!}{(n-m)!},$$

where $c_m = \pi$ if $m \neq 0$ and $c_m = 2\pi$ if $m = 0$, exists since

$$\frac{\partial v}{\partial n} = \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} n a^{-1} P_n^m(\cos\theta)(c_{mn} \cos m\phi + d_{mn} \sin m\phi).$$

It is obvious, then, that condition (4.20) is true. □

Lemma 12 *The exterior Dirichlet problem described in equations (4.15)—(4.18) is uniquely solvable for all $h \in H^{\frac{1}{2}}(\partial\Omega_a)$ except possibly at a set of isolated values of ω^2.*

Proof: We first note that there is at present no uniqueness theorem for this problem when the submergence depth is arbitrary.

Regularity theory implies that ψ is infinitely smooth in the closure of Ω_a. Consequently, ψ can be written as a multi-pole expansion. Multi-poles are solutions of Laplace's equation everywhere except the origin. They satisfy the radiation condition (4.10) and the free surface condition. The leading order asymptotic behaviour of the (m, n)th symmetric (resp. anti-symmetric) multi-pole for small radial distances from the origin is given as

$$P_n^m(\cos\theta) \cos m\phi \text{ (resp. } \sin m\phi) \; r^{-(n+1)}. \tag{4.29}$$

θ measures the angle from the vertical and ϕ measures the azimuthal angle. (See Figure (4.2) for further explanation.)

Let us briefly see how expressions for the multi-poles are obtained. The (m,n)th symmetric (resp. anti-symmetric) multi-pole is written as

$$(\psi_n^m(r,\theta) + i\chi_n^m(r,\theta))\cos m\phi \text{ (resp. } \sin m\phi).$$

For real ω, $\psi_n^m(r,\theta)$ and $\chi_n^m(r,\theta)$ are real valued functions. The multi-poles satisfy Laplace's equation and they are infinitely smooth functions everywhere in $y > -1/\epsilon$, except at the origin where their asymptotic behaviour is given in (4.29). Bearing all this in mind, we write

$$\psi_n^m(r,\theta) + i\chi_n^m(r,\theta) = \frac{P_n^m(\cos\theta)}{r^{n+1}} + \int_C f(k)J_m(kr\sin\theta)\exp(-kr\cos\theta)dk, \quad (4.30)$$

where $f(k)$ is a function which is to be found, J_m is the mth Bessel function of the first kind and the contour C is chosen so that the radiation condition (4.10) is satisfied.

The trick that is commonly used is to write

$$\frac{P_n^m(\cos\theta)}{r^{n+1}} = (-1)^{m+n}\frac{P_n^m(\cos\theta')}{r'^{n+1}} \quad \text{on } FS$$

and

$$\frac{\partial(P_n^m(\cos\theta)r^{-(n+1)})}{\partial y} = (-1)^{m+n+1}\frac{\partial(P_n^m(\cos\theta')r'^{-(n+1)})}{\partial y} \quad \text{on } FS,$$

where r' is the distance to the image point of the origin when reflected in the plain FS, and θ' is the angle subtended by the line joining the point on FS to the image point to the vertical. (See Figure (4.3).)

For $r\cos\theta + 2/\epsilon > 0$ we may use the identity

$$\frac{P_n^m(\cos\theta')}{r'^{n+1}} = \frac{1}{(n-m)!}\int_0^\infty k^n J_m(kr\sin\theta)\exp(-k(r\cos\theta + 2/\epsilon))dk. \quad (4.31)$$

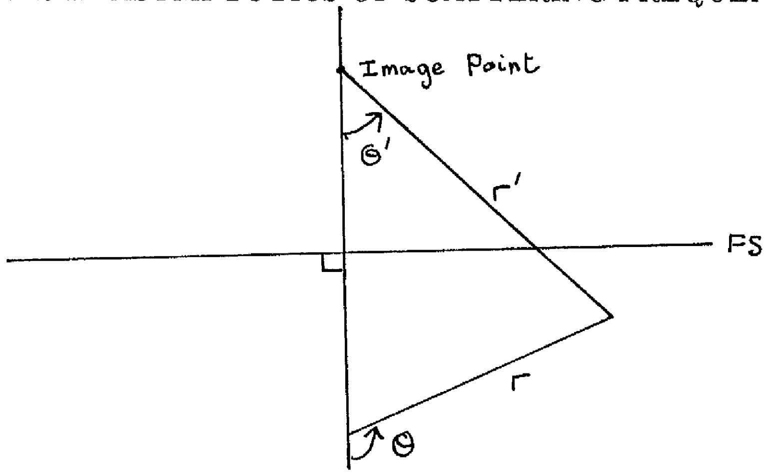

Figure 4.3: The image point.

See, for example, Wang [30].

From equations (4.30), (4.31) and the free surface condition we have

$$\begin{aligned}\frac{(-1)^{m+n}}{(n-m)!}\int_0^\infty (k+\tfrac{\omega^2}{g})k^n J_m(kr\sin\theta)\exp(-k/\epsilon)dk \\ +\int_C(\tfrac{\omega^2}{g}-k)f(k)J_m(kr\sin\theta)\exp(k/\epsilon)dk \quad &= 0.\end{aligned} \tag{4.32}$$

Equation (4.32) implies that we must take

$$f(k)=\frac{(-1)^{m+n}}{(n-m)!}\frac{\left(k+\frac{\omega^2}{g}\right)}{\left(k-\frac{\omega^2}{g}\right)}k^n\exp(-2k/\epsilon)$$

and C as an open contour with one end at the origin and the other at plus infinity. C must not pass through the point $\frac{\omega^2}{g}$. We choose C as in Figure (4.4). The contour is chosen to pass underneath $\frac{\omega^2}{g}$ in order that the radiation condition is satisfied.

Thus,

$$\begin{aligned}\psi_n^m(r,\theta) &= P_n^m(\cos\theta)r^{-(n+1)} \\ &+\frac{(-1)^{m+n}}{(n-m)!}\int_0^\infty \frac{k+\frac{\omega^2}{g}}{k-\frac{\omega^2}{g}}k^n J_m(kr\sin\theta)\exp(-k(r\cos\theta+2/\epsilon))dk\end{aligned} \tag{4.33}$$

and

$$\chi_n^m(r,\theta)=\frac{(-1)^{m+n}}{(n-m)!}2\pi(\tfrac{\omega^2}{g})^{n+1}J_m(\tfrac{\omega^2}{g}r\sin\theta)\exp(-\tfrac{\omega^2}{g}(r\cos\theta+2/\epsilon)), \tag{4.34}$$

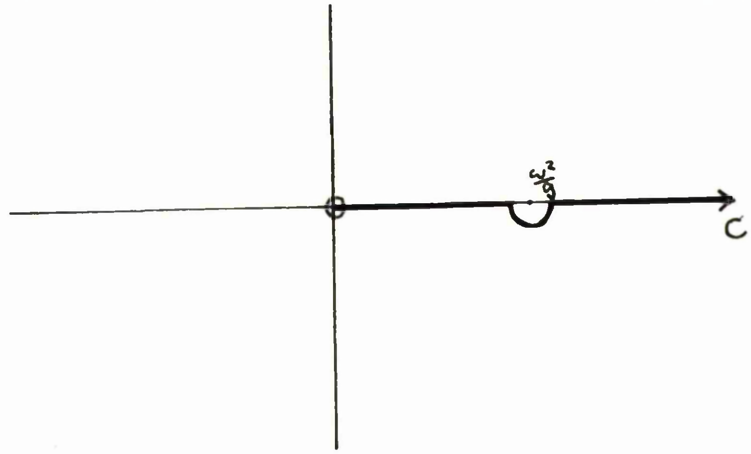

Figure 4.4: The contour in the k plane.

where the integral in (4.33) is to be interpreted as a Cauchy principal value integral.

Finally, we use the identity

$$J_m(kr\sin\theta)\exp(-kr\cos\theta) \equiv \sum_{s=m}^{\infty} \frac{(-1)^{s+m}(kr)^s}{(s+m)!} P_s^m(\cos\theta)$$

to write

$$\psi_n^m(r,\theta) = P_n^m(\cos\theta) r^{-(n+1)} + \sum_{s=m}^{\infty} \frac{(-1)^{s+n}}{(n-m)!(s+m)!} P_s^m(\cos\theta) r^s \int_0^{\infty} \frac{k+\frac{\omega^2}{g}}{k-\frac{\omega^2}{g}} k^{s+n} \exp(-2k/\epsilon)\, dk, \tag{4.35}$$

and

$$\chi_n^m(r,\theta) = \sum_{s=m}^{\infty} 2\pi\left(\frac{\omega^2}{g}\right)^{s+n+1} r^s \frac{(-1)^{s+n}}{(n-m)!} \exp\left(-2\frac{\omega^2}{g}/\epsilon\right) P_s^m(\cos\theta). \tag{4.36}$$

Once again, refer to [30].

We must now write the surface distribution h in the statement of Lemma 12 as

$$h = \sum_{m=0}^{\infty}\sum_{n=m}^{\infty} P_n^m(\cos\theta)(c_{mn}\cos m\phi + d_{mn}\sin m\phi),$$

where, of course, the condition (4.20) is satisfied. We wish to write h as the trace on $\partial\Omega_a$ of a sum of multi-poles. This means, we want to be able to write

$$h = \sum_{m=0}^{\infty}\sum_{n=m}^{\infty} (\psi_n^m(a,\theta) + i\chi_n^m(a,\theta))(a_{mn}\cos m\phi + b_{mn}\sin m\phi)a^{n+1}. \tag{4.37}$$

The following lemma is very useful:

Lemma 13 *Any element $h \in H^{\frac{1}{2}}(\partial\Omega_a)$ can be written in the form of equation (4.37) except possibly at a set of isolated values of ω^2. The coefficients a_{mn} and b_{mn} are meromorphic in ω^2 and have no poles in the upper half plane.*

Proof: We have

$$\begin{aligned}
&\sum_{m=0}^{\infty}\sum_{n=m}^{\infty} a_{mn}P_n^m(\cos\theta)\cos m\phi \qquad (4.38)\\
&\quad+\sum_{m=0}^{\infty}\sum_{n=m}^{\infty}\sum_{s=m}^{\infty} a_{mn}\frac{(-1)^{s+n}}{(n-m)!(s+m)!}a^{n+s+1}P_s^m(\cos\theta)\cos m\phi\\
&\quad\times\int_0^\infty \frac{k+\frac{\omega^2}{g}}{k-\frac{\omega^2}{g}}k^{n+s}\exp(-2k/\epsilon)dk\\
&\quad+i\sum_{m=0}^{\infty}\sum_{n=m}^{\infty}\sum_{s=m}^{\infty} a_{mn}\frac{(-1)^{s+n}}{(n-m)!(s+m)!}2\pi a^{n+s+1}(\frac{\omega^2}{g})^{s+n+1}\\
&\quad\times\exp(-2\frac{\omega^2}{g}/\epsilon)P_s^m(\cos\theta)\cos m\phi\\
&\quad=\sum_{m=0}^{\infty}\sum_{n=m}^{\infty} c_{mn}P_n^m(\cos\theta)\cos m\phi.
\end{aligned}$$

There is a similar relationship between the coefficients of the anti-symmetric part.

From equation (4.38) and the orthogonality property of the associated Legendre functions, we have

$$\begin{aligned}
&a_{mn} \qquad (4.39)\\
&\quad+\sum_{s=m}^{\infty} a_{ms}\frac{(-1)^{s+n}}{(s-m)!(n+m)!}a^{n+s+1}\int_0^\infty \frac{k+\frac{\omega^2}{g}}{k-\frac{\omega^2}{g}}k^{s+n}\exp(-2k/\epsilon)dk\\
&\quad+i\sum_{s=m}^{\infty} a_{ms}2\pi\frac{(-1)^{s+n}}{(s-m)!(n+m)!}a^{n+s+1}(\frac{\omega^2}{g})^{s+n+1}\exp(-2\frac{\omega^2}{g}/\epsilon)=c_{mn}.
\end{aligned}$$

Let us now multiply each a_{mn} and each c_{mn} by a factor

$$\left(\frac{(n+m)!}{(n-m)!}\right)^{\frac{1}{2}}.$$

That is to say, let

$$\alpha_{mn} \equiv a_{mn} \left(\frac{(n+m)!}{(n-m)!} \right)^{\frac{1}{2}}$$

and

$$\gamma_{mn} \equiv c_{mn} \left(\frac{(n+m)!}{(n-m)!} \right)^{\frac{1}{2}}.$$

Equation (4.39) becomes

$$\sum_{s=m}^{\infty} (\delta_{ns} + A_{ns}^{m}(\omega^2;\epsilon))\alpha_{ms} = \gamma_{mn}, \tag{4.40}$$

where

$$A_{ns}^{m}(\omega^2;\epsilon) = \left(\frac{1}{(s+m)!(s-m)!(n+m)!(n-m)!} \right)^{\frac{1}{2}} a^{n+s+1}(-1)^{s+n} \tag{4.41}$$
$$\times \left(\int_0^{\infty} \frac{k+\frac{\omega^2}{g}}{k-\frac{\omega^2}{g}} k^{s+n} \exp(-2k/\epsilon)dk + 2\pi i (\frac{\omega^2}{g})^{s+n+1} \exp(-2\frac{\omega^2}{g}/\epsilon) \right).$$

We have to show under what conditions equation (4.40) is uniquely solvable, given that

$$\sum_{m=0}^{\infty} \sum_{n=m}^{\infty} |\gamma_{mn}|^2$$

exists.

Firstly, we can say that it is solvable if

$$\sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \sum_{s=m}^{\infty} |A_{ns}^{m}(\omega^2;\epsilon)|^2 < 1.$$

This is a consequence of the following iterative procedure: Let $\alpha_{mn}^{(0)} = \gamma_{mn}$ and

$$\alpha_{mn}^{(N+1)} = \gamma_{mn} - \sum_{s=m}^{\infty} A_{ns}^{m}(\omega^2;\epsilon)\alpha_{ms}^{(N)},$$

where $N = 0, 1, 2, 3, \ldots$

Clearly,

$$(\alpha_{mn}^{(N+1)} - \alpha_{mn}^{(N)}) = \sum_{s=m}^{\infty} A_{ns}^{m}(\omega^2;\epsilon)(\alpha_{ms}^{(N)} - \alpha_{ms}^{(N-1)}).$$

Therefore,

$$\sum_{m=0}^{\infty}\sum_{n=m}^{\infty}|\alpha_{mn}^{(N+1)}-\alpha_{mn}^{(N)}|^2 \leq$$
$$\left(\sum_{m=0}^{\infty}\sum_{n=m}^{\infty}\sum_{s=m}^{\infty}|A_{ns}^{m}(\omega^2;\epsilon)|^2\right)\left(\sum_{m=0}^{\infty}\sum_{s=m}^{\infty}|\alpha_{ms}^{(N)}-\alpha_{ms}^{(N-1)}|^2\right).$$

Hence, $\alpha_{mn}^{(N+1)}-\alpha_{mn}^{(N)}$ tends to zero in $l^2\otimes l^2$ — the product of the space of all square-summable series with itself. Since $l^2\otimes l^2$ is complete,

$$\lim_{N\to\infty}\alpha_{mn}^{(N)}\equiv\alpha_{mn} \text{ exists}$$

and, furthermore, α_{mn} belongs to $l^2\otimes l^2$. It is easily verified that α_{mn} is a solution to equation (4.40). The uniqueness of the solution is also plain to see.

Suppose next that

$$\sum_{m=0}^{\infty}\sum_{n=m}^{\infty}\sum_{s=m}^{\infty}|A_{ns}^{m}(\omega^2;\epsilon)|^2$$

exists, but is greater than one. It must be the case that we can choose N such that

$$\sum_{m=0}^{\infty}\sum_{n=m'}^{\infty}\sum_{s=m'}^{\infty}|A_{ns}^{m}(\omega^2;\epsilon)|^2<1,$$

where m' is the maximum of $N+1$ and m. For the moment, consider all those α_{mn}'s with m and n each less than $N+1$ as given, and solve the remaining equations:

$$\alpha_{mn}+\sum_{s=N+1}^{\infty}A_{ns}^{m}(\omega^2;\epsilon)\alpha_{ms}=\gamma_{mn}-\sum_{s=m}^{N}A_{ns}^{m}(\omega^2;\epsilon)\alpha_{ms} \tag{4.42}$$

when $m<N+1$ and

$$\alpha_{mn}+\sum_{s=m}^{\infty}A_{ns}^{m}(\omega^2;\epsilon)\alpha_{ms}=\gamma_{mn} \tag{4.43}$$

when $m>N$. Both (4.42) and (4.43) are solvable from the previous analysis. Suppose that the solution to equation (4.42) is given as

$$\alpha_{mn}=\sum_{s=m}^{\infty}B_{ns}^{m}(\omega^2;\epsilon)(\gamma_{ms}-\sum_{t=m}^{N}A_{st}^{m}(\omega^2;\epsilon)\alpha_{mt}). \tag{4.44}$$

Let us denote by $D^m_{nt}(\omega^2;\epsilon)$ the term

$$\sum_{s=m}^{\infty} B^m_{ns}(\omega^2;\epsilon)A^m_{st}(\omega^2;\epsilon).$$

Equation (4.44) becomes

$$\alpha_{mn} + \sum_{t=m}^{N} D^m_{nt}(\omega^2;\epsilon)\alpha_{mt} = \sum_{s=m}^{\infty} B^m_{ns}(\omega^2;\epsilon)\gamma_{ms}. \tag{4.45}$$

Equation (4.45) is a square matrix equation in a finite number of unknowns. If the homogeneous version of equation (4.45) has just one solution, then equation (4.45) is solvable whatever the right hand side of it is. If the homogeneous equation has a non-trivial solution then equation (4.45) will, in general, not be solvable. The range of the matrix will be perpendicular to the space spanned by the solutions of the homogeneous adjoint equation.

Let us denote by $K(\omega^2;\epsilon)$ the matrix operator whose (m,n,s)th entry is $A^m_{ns}(\omega^2;\epsilon)$. Equation (4.40) can be re-written as

$$(I + K(\omega^2;\epsilon))a = c, \tag{4.46}$$

where a and c denote elements of $l^2 \otimes l^2$ whose (m,n)th entries are α_{mn} and γ_{mn} respectively. The above analysis shows that $I + K(\omega^2;\epsilon)$ is a *Fredholm* operator on $l^2 \otimes l^2$. Thus the necessary and sufficient condition for solvability of equation (4.40) with any right hand side is that the only solution of the homogeneous equation,

$$(I + K(\omega^2;\epsilon))a = 0, \tag{4.47}$$

is the trivial solution.

4.3.1 Proof that $A^m_{ns}(\omega^2;\epsilon)$ is square-summable.

We must show that

$$\sum_{m=0}^{\infty}\sum_{n=m}^{\infty}\sum_{s=m}^{\infty}|A^m_{ns}(\omega^2;\epsilon)|^2<\infty. \tag{4.48}$$

$$|A^m_{ns}(\omega^2;\epsilon)|^2=\frac{a^{2n+2s+2}}{(n-m)!(n+m)!(s-m)!(s+m)!} \tag{4.49}$$
$$\times\left(\left|\int_0^\infty\frac{k+\frac{\omega^2}{g}}{k-\frac{\omega^2}{g}}k^{s+n}\exp(-2k/\epsilon)dk\right|^2+\left|2\pi(\frac{\omega^2}{g})^{s+n+1}\exp(-2\frac{\omega^2}{g}/\epsilon)\right|^2\right).$$

Let us consider the two terms on the right hand side of equation (4.49) individually. First of all, we shall find a bound for

$$S_1(\epsilon)\equiv\sum_{m=0}^{\infty}\sum_{n=m}^{\infty}\sum_{s=m}^{\infty}\frac{a^{2n+2s+2}}{(s-m)!(s+m)!(n-m)!(n+m)!}$$
$$\times\left(\int_0^\infty\frac{k+\frac{\omega^2}{g}}{k-\frac{\omega^2}{g}}k^{s+n}\exp(-2k/\epsilon)dk\right)^2.$$

The leading order term in the asymptotic expansion for large $s+n$ and for fixed ϵ of the integral

$$I(\epsilon)\equiv\int_0^\infty\frac{k+\frac{\omega^2}{g}}{k-\frac{\omega^2}{g}}k^{s+n}\exp(-2k/\epsilon)dk,$$

is

$$(s+n)!\frac{\epsilon^{s+n+1}}{2^{s+n+1}}.$$

This is true because the part of the interval that dominates the integral for very large $s+n$ is well away from the singularity at $\frac{\omega^2}{g}$.

It is, therefore, true that $S_1(\epsilon)$ exists if and only if

$$S_2(\epsilon)\equiv\sum_{m=0}^{\infty}\sum_{n=m}^{\infty}\sum_{s=m}^{\infty}\frac{\epsilon^{2s+2n+2}a^{2n+2s+2}((s+n)!)^2}{2^{2s+2n+2}(s-m)!(s+m)!(n-m)!(n+m)!}$$

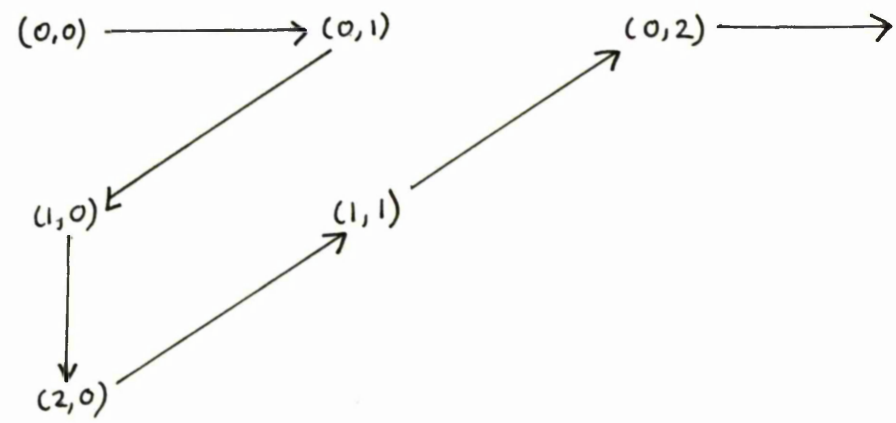

Figure 4.5: The rearrangement of the series.

exists. We have

$$S_1(\epsilon) \leq M(\omega^2) S_2(\epsilon), \tag{4.50}$$

for sufficiently small ϵ, where $M(\omega^2)$ is independent of ϵ. This can be deduced from the asymptotic behaviour of each term of $S_1(\epsilon)$ for small ϵ. Now change the counting variables

$$s \mapsto s - m$$

$$n \mapsto n - m.$$

Thus,

$$S_2(\epsilon) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{s=0}^{\infty} \left(\frac{a\epsilon}{2}\right)^{2s+2n+4m+2} \frac{((s+n+2m)!)^2}{s!(s+2m)!n!(n+2m)!}.$$

All the terms in this series are positive and so it exists if and only if any rearrangement of it exists. Instead of summing rows of columns, we sum the series by taking the terms in the order indicated in Figure (4.5). Hence,

$$S_2(\epsilon) = \sum_{m=0}^{\infty} \sum_{r=0}^{\infty} \sum_{t=0}^{r} \left(\frac{a\epsilon}{2}\right)^{2r+4m+2} \frac{((r+2m)!)^2}{t!(t+2m)!(r-t)!(r-t+2m)!}. \tag{4.51}$$

If $0 \leq t \leq r$, then

$$\frac{1}{t!(r-t)!} \leq \frac{1}{(([r/2])!)^2} \text{ and}$$

$$\frac{1}{(r-t+2m)!(t+2m)!} \leq \frac{1}{(([r/2]+2m)!)^2},$$

where $[r/2]$ denotes the integer part of $r/2$.

Each term in the series for $S_2(\epsilon)$ is, therefore, bounded by

$$\left(\frac{a\epsilon}{2}\right)^{2r+4m+2} \frac{((r+2m)!)^2}{(([r/2])!)^2(([r/2]+2m)!)^2}. \tag{4.52}$$

Equations (4.51) and (4.52) imply that $S_2(\epsilon)$ exists if

$$S_3(\epsilon) = \frac{a^2\epsilon^2}{4}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\left(\frac{a\epsilon}{2}\right)^{2r+4m} \frac{(r+1)((r+2m)!)^2}{(([r/2])!)^2(([r/2]+2m)!)^2}$$

exists. Furthermore,

$$0 \leq S_2(\epsilon) < S_3(\epsilon). \tag{4.53}$$

Claim:

$$\lim_{r\to\infty} \frac{(r+1)((r+2m)!)^2}{2^{2r+4m}(([r/2])!)^2(([r/2]+2m)!)^2} = \frac{2}{\pi}.$$

The claim is proved by considering the large r asymptotic behaviour of each of the factorial functions in the left hand side.

$$(r+2m)! \sim r^{2m} r! \sim r^{2m} r^{r+\frac{1}{2}} \exp(-r)(2\pi)^{\frac{1}{2}},$$

$$([r/2])! \sim (r/2)^{\frac{r+1}{2}} \exp(-r/2)(2\pi)^{\frac{1}{2}}$$

and

$$\begin{aligned}([r/2]+2m)! &\sim (r/2)^{2m}([r/2])! \\ &\sim (r/2)^{\frac{r+4m+1}{2}} \exp(-r/2)(2\pi)^{\frac{1}{2}}.\end{aligned}$$

See, for example, Abramowitz and Stegun [1, p. 257]. Therefore, each term in $S_3(\epsilon)$ is bounded by

$$C(\epsilon a)^{2r+4m+2},$$

where C is some positive constant. That is to say, $S_3(\epsilon)$ exists if

$$S_4(\epsilon) = C \sum_{m=0}^{\infty} \sum_{r=0}^{\infty} (\epsilon a)^{2r+4m+2} \tag{4.54}$$

exists. It is clear that $S_4(\epsilon)$ exists if $|\epsilon a| < 1$. Furthermore,

$$S_3(\epsilon) \leq S_4(\epsilon). \tag{4.55}$$

Thus (4.50), (4.53), (4.54) and (4.55) imply

$$S_1(\epsilon) \leq M(\omega^2) C \frac{(\epsilon a)^2}{(1-(\epsilon a)^2)(1-(\epsilon a)^4)}. \tag{4.56}$$

It is easy to see that a bound for the second term in equaton (4.49) is

$$2\pi a^2 \frac{\omega^2}{g} \exp(2\frac{\omega^2}{g}(a - 1/\epsilon)).$$

Hence,

$$S_5(\epsilon) \leq N(\omega^2) a^2 \exp(-2\frac{\omega^2}{g}\epsilon), \tag{4.57}$$

where

$$S_5(\epsilon) \equiv 4\pi^2 \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \sum_{s=m}^{\infty} \frac{(a\frac{\omega^2}{g})^2}{(n-m)!(n+m)!(s-m)!(s+m)!} \exp(-2\frac{\omega^2}{g}/\epsilon)$$

and $N(\omega^2)$ is independent of ϵ.

The bounds (4.56) and (4.57) imply the condition (4.48).

4.3.2 The analytic continuation of $A^m_{ns}(\omega^2;\epsilon)$ and the proof of Lemma 13.

Before we prove Lemma 13, we shall consider the extension of equation (4.40) to non-real values of ω^2.

The function $A^m_{ns}(\omega^2;\epsilon)$ can be analytically continued into $\mathcal{C}\setminus\mathcal{R}_-$, where $\mathcal{R}_-$ denotes the negative reals. For $\Im\omega^2 > 0$ the analytic continuation of $A^m_{ns}(\omega^2;\epsilon)$ is

$$A^m_{ns}(\omega^2;\epsilon) = \left(\frac{1}{(s+m)!(s-m)!(n+m)!(n-m)!}\right)^{1/2} \times(-a)^{n+s+1}\int_0^\infty \frac{k+\frac{\omega^2}{g}}{k-\frac{\omega^2}{g}}k^{n+s}\exp(-2k/\epsilon)dk, \tag{4.58}$$

and for $\Im\omega^2 < 0$ it is

$$A^m_{ns}(\omega^2;\epsilon) = \left(\frac{1}{(s+m)!(s-m)!(n+m)!(n-m)!}\right)^{1/2}(-a)^{n+s+1} \times\left(\int_0^\infty \frac{k+\frac{\omega^2}{g}}{k-\frac{\omega^2}{g}}k^{n+s}\exp(-2k/\epsilon)dk + 4\pi i(\frac{\omega^2}{g})^{n+s+1}\exp(-2\frac{\omega^2}{g}/\epsilon)\right). \tag{4.59}$$

It can be shown that, just as in the case of real ω^2, the condition (4.48) holds true for complex ω^2.

The operator $K(\omega^2;\epsilon)$ is, for fixed ϵ, bounded-holomorphic in ω^2 everywhere in $\mathcal{C}\setminus\mathcal{R}_-$. This is because the inner product

$$(K(\omega^2;\epsilon)a,b)_{l^2\otimes l^2}$$

is holomorphic with respect to ω^2 for all a and b in $l^2\otimes l^2$. It is easy to see that this last assertion is true if we bear in mind the fact that, because

$$(K(\omega^2;\epsilon)a,b)_{l^2\otimes l^2} = \sum_{m=0}^{\infty}\sum_{n=m}^{\infty}\sum_{s=m}^{\infty} A^m_{ns}(\omega^2;\epsilon)\alpha_{ms}\beta_{mn},$$

where α_{ms} and β_{mn} are the entries of the matrices a and b respectively, the inner product is uniformly bounded in a neighbourhood of every point of ω^2-space, and each term in the above identity is holomorphic in ω^2.

Let us denote by $S_1(\omega^2;\epsilon)$ the inverse of $I+K(\omega^2;\epsilon)$, when it exists. We shall now show that $S_1(\omega^2;\epsilon)$ is meromorphic with respect to ω^2. For this we need the following theorem, which is Theorem 1.3 in Kato [14, Chapter 7].

Theorem 12 *Let $T(x) : X \to X$ be a closed operator which is defined at $x = 0$ and let ζ belong to the resolvent set of $T(0)$. Then $T(x)$ is holomorphic at $x = 0$ if and only if ζ belongs to the resolvent set of $T(x)$ and the resolvent $R(\zeta; x) = (T(x) - \zeta)^{-1}$ is bounded-holomorphic for sufficiently small $|x|$. $R(\zeta; x)$ is even bounded-holomorphic in the two variables on the set of all (ζ, x) such that ζ belongs to the resolvent set of $T(x)$ and $|x|$ is sufficiently small (depending on ζ).*

It should be noted that we have not yet said what it means for an unbounded operator to be holomorphic. It is enough to note that for our present application of Theorem 12 we deal with bounded-holomorphic operators only. The point $x = 0$ is not essential in the theorem — we could have chosen any point.

We apply the first part of this theorem with $K(\omega^2; \epsilon)$ taking the role of $T(x)$, X is the product of the square-summable series with itself and $\zeta = -1$.

The theorem tells us, if we bear in mind the Fredholm property of $I + K(\omega^2; \epsilon)$ and the holomorphicity of $K(\omega^2; \epsilon)$, that $S_1(\omega^2; \epsilon)$ is holomorphic with respect to ω^2 if and only if the homogeneous equation (4.47) has only one solution — the trivial one.

We must now define the projection operator onto the subspace of eigenvectors of $K(\omega^2; \epsilon)$ associated with the eigenvalue (-1). Let us call this subspace M. Let C be a curve in the complex ζ plane that encloses $\zeta = -1$ but no other eigenvalue of $K(\omega^2; \epsilon)$. The projection operator $P(\omega^2; \epsilon)$, is defined by

$$P(\omega^2; \epsilon) = \frac{-1}{2\pi i} \int_C (K(\omega^2; \epsilon) - \zeta)^{-1} d\zeta.$$

We wish now to prove that the set of points Q, for which (-1) is an eigenvalue of $K(\omega^2;\epsilon)$, consists of isolated points. Suppose that it does not. Let $\{\omega_m^2\}$ be a sequence of such points that converge to a point ω_∞^2 not in the sequence. We do not yet know whether or not ω_∞^2 is in Q. The second part of Theorem 12 implies that $P(\omega^2;\epsilon)$ is holomorphic in ω^2. Hence, without loss of generality, we can assume that

$$|P(\omega^2;\epsilon) - P(\omega_\infty^2;\epsilon)| < 1,$$

whenever ω^2 is in some neighbourhood of ω_∞^2. Hence, the dimension of the ranges of $P(\omega^2;\epsilon)$ and $P(\omega_m^2;\epsilon)$ are equal whenever ω^2 is close enough to ω_∞^2, and we can choose an invertible holomorphic operator $U(\omega^2;\epsilon)$, with the property that

$$U(\omega^2;\epsilon)P(\omega^2;\epsilon)U(\omega^2;\epsilon)^{-1} = P(\omega_\infty^2;\epsilon).$$

The proof of this can be found in Kato [14, Section 4.6, Chapter 1].

The implication of all of this is that the eigenvalue problem for points enclosed by C for $K(\omega^2;\epsilon)$, is equivalent to the eigenvalue problem for the holomorphic matrix given by

$$K_1(\omega^2;\epsilon) = P(\omega_\infty^2;\epsilon)U(\omega^2;\epsilon)K(\omega^2;\epsilon)U(\omega^2;\epsilon)^{-1}P(\omega_\infty^2;\epsilon). \tag{4.60}$$

$K_1(\omega^2;\epsilon)$ operates in the fixed and finite dimensional subspace of eigenvectors of $K(\omega_\infty^2;\epsilon)$ associated with the eigenvector (-1). Whether ω^2 belongs to Q or not depends on whether

$$\det(K_1(\omega^2;\epsilon) + I) = 0$$

or not.

Clearly $\det(K_1(\omega^2;\epsilon) + I)$ is a holomorphic function of ω^2. It vanishes at the points $\{\omega_m^2\}$ which accumulate at ω_∞^2. By a well known result of complex analysis, this implies that the determinant vanishes identically inside C. By the equivalence of the eigenvalue problems for $K(\omega^2;\epsilon)$ and for the matrix, this implies that every point enclosed by C is in Q, and by continuation every point in $\mathcal{C} \setminus \mathcal{R}_-$ is in Q. We only have to find a point not in Q, therefore, to prove that Q consists of isolated points.

The fact that every singularity of $S_1(\omega^2;\epsilon)$ is a pole can be seen by constructing the projection operator and the analogy of identity (4.60) for the singularity. The singular parts of $S_1(\omega^2;\epsilon)$ and $(K_1(\omega^2;\epsilon)+1)^{-1}$ are equivalent. The latter is the quotient of two holomorphic functions (by Cramer's rule).

We shall now prove that every point in the upper half plane is not in Q. Equivalently, when $\Im\omega^2 > 0$ the equation (4.47) has only the trivial solution.

Suppose this were not true. That is to say, there exists an element of $l^2 \otimes l^2$ with

$$\sum_{n=m}^{\infty} (\delta_{sn} + A^m_{ns}(\omega^2;\epsilon))\alpha_{mn} = 0,$$

for all m and s greater than m. Now construct the function

$$\psi = \sum_{m=0}^{\infty}\sum_{n=m}^{\infty}\sum_{s=m}^{\infty} a^{n+1} P^m_s(\cos\theta) \tag{4.61}$$
$$\times \left(\frac{\delta_{sn}}{r^{n+1}} + \frac{r^s}{a^{n+s+1}} A^m_{ns}(\omega^2;\epsilon)\right)\left(\frac{(s-m)!}{(s+m)!}\right)^{1/2} \alpha_{mn}\cos m\phi.$$

A glance at equation (4.58) tells us that $A^m_{ns}(\omega^2;\epsilon)$ decays faster at infinity when the imaginary part of ω^2 is positive than when ω^2 is real. A more detailed analysis (see Appendix B) shows that ψ in equation (4.61) satisfies

$$(1+r^2)^{-1/2}\psi \in L^2(\Omega_a),$$

$$\nabla\psi \in (L^2(\Omega_a))^3,$$

and

$$\psi|_{FS} \in L^2(FS).$$

Most important of all is the property that

$$\psi|_{\partial\Omega_a} = 0.$$

This comes about because the α_{mn}'s are solutions to the homogeneous equation above. These conditions imply that ψ vanishes — the proof of this can be found in Lenoir and Martin [17]. This, in turn, implies that each α_{mn} is zero.

We know that the spectrum of $K(\omega^2;\epsilon)$ is either the whole plane, or consists of a set of isolated points. The above result implies that the latter alternative is true. This completes the proof of Lemma 13. □

Once we have Lemma 13 it is evident that the solution of the problem described in equations (4.15) to (4.18) is

$$\psi = \sum_{m=0}^{\infty}\sum_{n=m}^{\infty}(\psi_n^m(r,\theta) + i\chi_n^m(r,\theta))(a_{mn}\cos m\phi + b_{mn}\sin m\phi)a^{n+1}. \tag{4.62}$$

This is because the solution, ψ, must have the form given in equation (4.62) and, to ensure that its trace on $\partial\Omega_a$ belongs to $L^2(\partial\Omega_a)$, the coefficients a_{mn} and b_{mn} must satisfy the condition in the statement of Lemma 13. Thus we have proved Lemma 12. □

It can easily be shown that the normal derivative of ψ in equation (4.62) on $\partial\Omega_a$ belongs to $H^{-\frac{1}{2}}(\partial\Omega_a)$. Let $T(\omega^2;\epsilon)$ be the operator that maps h to the normal derivative of ψ on $\partial\Omega_a$. $T(\omega^2;\epsilon)$ acts on ψ in the following way:

$$\begin{aligned} T(\omega^2;\epsilon)\psi = &\sum_{m=0}^{\infty}\sum_{n=m}^{\infty}\sum_{s=m}^{\infty}(a_{mn}\cos m\phi + b_{mn}\sin m\phi) \\ &\times\left(-\tfrac{n+1}{a}\delta_{ns} + \tfrac{s}{a}\left(\tfrac{(s-m)!(n+m)!}{(s+m)!(n-m)!}\right)^{1/2} A_{ns}^m(\omega^2;\epsilon)\right)P_s^m(\cos\theta). \end{aligned} \tag{4.63}$$

Of course, $T(\omega^2;\epsilon)$ can be continued into $\mathcal{C}\backslash\mathcal{R}_-$. It is, according to Lemma 13, a meromorphic function of ω^2 with no poles in the upper half plane. $T(\omega^2;\epsilon)$ is bounded when considered as acting between $H^{\frac{1}{2}}(\partial\Omega_a)$ and $H^{-\frac{1}{2}}(\partial\Omega_a)$.

4.3.3 The truncated problem.

We are now in a position to define a problem equivalent to the exterior problem. This problem will be called the *truncated problem.* It is equivalent to the exterior problem in two senses. Firstly, if either problem is unique, then so is the other and, secondly, the solutions of the two problems are related in a very simple way. Before we proceed, we shall define the domain Ω^a as the open set between the surfaces $\partial\Omega_a$ and $\partial\Omega$. The truncated problem is:

Find ψ in $H^1(\Omega^a)$ with the following properties:

$$\nabla^2\psi = 0 \qquad \text{in } \Omega^a, \tag{4.64}$$

$$\left.\frac{\partial\psi}{\partial n}\right|_{\partial\Omega} = f \in L^2(\partial\Omega_a), \tag{4.65}$$

$$\left.\frac{\partial\psi}{\partial n}\right|_{\partial\Omega_a} = T(\omega^2;\epsilon)(\psi|_{\partial\Omega_a}), \tag{4.66}$$

where f is considered as some given datum, just as in the exterior problem. We shall assume that $T(\omega^2;\epsilon)$ is defined.

Lemma 14 *The truncated problem has a unique solution for all f in $L^2(\partial\Omega_a)$ except possibly at a set of isolated points in the complex ω^2 plane.*

Proof: The equations (4.64) to (4.66) are equivalent to the weak formulation: Find ψ in $H^1(\Omega^a)$, such that

$$\int_{\Omega^a} \nabla\psi.\nabla\overline{\phi}dV - \int_{\partial\Omega_a}(T(\omega^2;\epsilon)\psi)\overline{\phi}dS = -\int_{\partial\Omega} f\overline{\phi}dS, \tag{4.67}$$

for any ϕ in $H^1(\Omega^a)$. Let us denote by $A(\omega^2;\epsilon)(.,.)$ the sesquilinear form on the left hand side of equation (4.67). To any sesquilinear form we can associate an operator. Let us denote by $A(\omega^2;\epsilon)$ the operator associated with $A(\omega^2;\epsilon)(.,.)$. That is to say,

$$(A(\omega^2;\epsilon)\psi,\phi)_{H^1(\Omega^a)} = A(\omega^2;\epsilon)(\psi,\phi).$$

Taking ϕ equal to $A(\omega^2;\epsilon)\psi$ implies

$$\begin{aligned}
\| A(\omega^2;\epsilon)\psi \|^2_{H^1(\Omega^a)} \leq & \| \psi \|_{H^1(\Omega^a)} \| A(\omega^2;\epsilon)\psi \|_{H^1(\Omega^a)} \\
& + \| T(\omega^2;\epsilon)\psi \|_{H^{-\frac{1}{2}}(\partial\Omega_a)} \| A(\omega^2;\epsilon)\psi \|_{H^{\frac{1}{2}}(\partial\Omega_a)} \\
\leq & \left(\| \psi \|_{H^1(\Omega^a)} + \| T(\omega^2;\epsilon)\psi \|_{H^{-\frac{1}{2}}(\partial\Omega_a)} \right) \| A(\omega^2;\epsilon)\psi \|_{H^1(\Omega^a)} \\
\leq & \left(\| \psi \|_{H^1(\Omega^a)} + M \| \psi \|_{H^{\frac{1}{2}}(\partial\Omega_a)} \right) \| A(\omega^2;\epsilon)\psi \|_{H^1(\Omega^a)} \\
\leq & (1+M) \| \psi \|_{H^1(\Omega^a)} \| A(\omega^2;\epsilon)\psi \|_{H^1(\Omega^a)} .
\end{aligned}$$

The second and fourth inequalities are due to the continuity of the trace map from $H^1(\Omega^a)$ into $H^{\frac{1}{2}}(\partial\Omega_a)$ and the third inequality is due to the continuity of $T(\omega^2;\epsilon)$ from $H^{\frac{1}{2}}(\partial\Omega_a)$ into $H^{-\frac{1}{2}}(\partial\Omega_a)$. So $A(\omega^2;\epsilon)$ is bounded in $H^1(\Omega^a)$.

That $A(\omega^2;\epsilon)$ is holomorphic with respect to ω^2, except at a set consisting of isolated points, is seen by observing that the sesquilinear form $A(\omega^2;\epsilon)(\psi,\phi)$ is holomorphic with respect to ω^2, for all ψ and ϕ in some fundamental subset of $H^1(\Omega^a)$. This is a result of the properties of $T(\omega^2;\epsilon)$.

Let us call the inverse of $A(\omega^2;\epsilon)$, when it exists, $S_2(\omega^2;\epsilon)$. We will use the following result

Lemma 15 *$A(\omega^2;\epsilon)$ is a Fredholm operator of index zero. Consequently, $S_2(\omega^2;\epsilon)$ exists if and only if the homogeneous equation*

$$A(\omega^2;\epsilon)\psi = 0$$

has only the trivial solution.

Proof: We shall demonstrate the existence of a double-sided regularizer of $A(\omega^2;\epsilon)$. To do this we will write $A(\omega^2;\epsilon)$ in the form of the identity plus an operator whose real part is positive plus a compact operator. The inverse of the identity plus the positive operator exists; this is the aforesaid regularizer.

If

$$\psi = \sum_{m=0}^{\infty}\sum_{n=m}^{\infty}(\psi_n^m(a,\theta)+i\chi_n^m(a,\theta))(a_{mn}\cos m\phi + b_{mn}\sin m\phi)a^{n+1}$$

then

$$\begin{aligned}
-\int_{\partial\Omega_a}(T(\omega^2;\epsilon)\psi)\overline{\psi}dS = & \qquad\qquad (4.68)\\
& a\pi\textstyle\sum_{m=0}^{\infty}\sum_{n=m}^{\infty}C_m(|a_{mn}|^2+|b_{mn}|^2)\frac{(n+1)(n+m)!}{(n+1/2)(n-m)!}\\
& -a\pi\textstyle\sum_{m=0}^{\infty}\sum_{n=m}^{\infty}\sum_{s=m}^{\infty}C_m(a_{mn}\overline{a_{ms}}+b_{mn}\overline{b_{ms}})\frac{s}{s+1/2}\left(\frac{(s+m)!(n+m)!}{(s-m)!(n-m)!}\right)^{1/2}A_{ns}^m(\omega^2;\epsilon)\\
& +a\pi\textstyle\sum_{m=0}^{\infty}\sum_{n=m}^{\infty}\sum_{s=m}^{\infty}C_m(a_{mn}\overline{a_{ms}}+b_{mn}\overline{b_{ms}})\frac{n+1}{n+1/2}\left(\frac{(n+m)!(s+m)!}{(n-m)!(s-m)!}\right)^{1/2}\overline{A_{ns}^m(\omega^2;\epsilon)}\\
& -a\pi\textstyle\sum_{m=0}^{\infty}\sum_{n=m}^{\infty}\sum_{s=m}^{\infty}\sum_{t=m}^{\infty}C_m(a_{mn}\overline{a_{ms}}+b_{mn}\overline{b_{ms}})\\
& \times\left(\frac{(n+m)!(s+m)!}{(n-m)!(s-m)!}\right)^{1/2}A_{ns}^m(\omega^2;\epsilon)\overline{A}_{st}^m(\omega^2;\epsilon),
\end{aligned}$$

where

$$C_m = \begin{cases} 2 & \text{if } m=0 \\ 1 & \text{otherwise}\end{cases}$$

We know from equation (4.41) and its extension that

$$A_{ns}^m(\omega^2;\epsilon)\sim\left(\frac{1}{(s+m)!(s-m)!(n+m)!(n-m)!}\right)^{1/2}(n+s)!\left(-\frac{a\epsilon}{2}\right)^{n+s+1}$$

as $n+s$ tends to infinity. As in the proof of Lemma 13,

$$\lim_{n+s\to\infty}\frac{A_{ns}^m(\omega^2;\epsilon)}{(a\epsilon)^{n+s+1}\left(\frac{1}{2\pi(n+s-m+1)}\right)^{1/2}} = k,$$

where k is less than or equal to one. As $a\epsilon < 1$, it is clear that if n or s is sufficiently large $|A_{ns}^m(\omega^2;\epsilon)|$ is less than any given positive number. It is clear then that the analogous term to the right hand side in equation (4.68) with n, s and t all greater than some number N, chosen so that $A^m_{ns}(\omega^2;\epsilon)$ is less than one quarter when n or s is greater than it, always has a positive real part. The remaining sum consists of a finite number of terms — the sum over m from zero to N and the sums over n, s and t from m to N. This term is associated with an operator whose range is of finite dimension, which is therefore compact.

The second term in the definition of $A(\omega^2;\epsilon)(.,.)$ is

$$\int_{\Omega^a} \nabla\psi.\nabla\overline{\phi}dV.$$

This can be written as

$$(\psi,\phi)_{H^1(\Omega^a)} - (\psi,\phi)_{L^2(\Omega^a)}.$$

The former term is associated with the identity and the latter term is associated with the imbedding operator from $H^1(\Omega^a)$ into $L^2(\Omega^a)$, which is compact.

Let us write

$$A(\omega^2;\epsilon) = I + K_1(\omega^2;\epsilon) + K_2(\omega^2;\epsilon),$$

where I denotes the identity, $K_1(\omega^2;\epsilon)$ satisfies

$$\Re(K_1(\omega^2;\epsilon)\psi,\psi)_{L^2(\Omega^a)} \geq 0,$$

for all ψ belonging to $H^1(\Omega^a)$, and $K_2(\omega^2;\epsilon)$ is compact. $I + K_1(\omega^2;\epsilon)$ is clearly a coercive, bounded operator in $H^1(\Omega^a)$. The Lax–Milgram Theorem (see, e. g. Sanchez Hubert and Sanchez Palencia [27, p. 76]) tells us that $I + K_1(\omega^2;\epsilon)$ has a bounded inverse in $H^1(\Omega^a)$, let us call it $B(\omega^2;\epsilon)$. $B(\omega^2;\epsilon)$ is a *left equivalent regularizer* of $A(\omega^2;\epsilon)$, since

$$B(\omega^2;\epsilon)A(\omega^2;\epsilon) = I + K_3(\omega^2;\epsilon),$$

where $K_3(\omega^2;\epsilon)$ is a compact operator in $H^1(\Omega^a)$. (Recall that the product of a bounded operator and a compact operator is compact.) It is called equivalent because

$$A(\omega^2;\epsilon)\psi = F$$

and

$$B(\omega^2;\epsilon)A(\omega^2;\epsilon)\psi = B(\omega^2;\epsilon)F$$

are equivalent for all $F \in H^1(\Omega^a)$,

$B(\omega^2;\epsilon)$ is also a *right equivalent regularizer* since

$$A(\omega^2;\epsilon)B(\omega^2;\epsilon) = I + K_4(\omega^2;\epsilon),$$

where $K_4(\omega^2;\epsilon)$ is compact in $H^1(\Omega^a)$. It is called equivalent because

$$A(\omega^2;\epsilon)\psi = F \tag{4.69}$$

and

$$A(\omega^2;\epsilon)B(\omega^2;\epsilon)\phi = F \tag{4.70}$$

are equivalent in the following sense. Equations (4.69) and (4.70) are either both solvable or both unsolvable. To any solution ψ of equation (4.69) there is a corresponding solution ϕ of equation (4.70) with

$$\psi = B(\omega^2;\epsilon)\phi.$$

The result of all of this is that $A(\omega^2;\epsilon)$ is a Fredholm operator with index zero and Lemma 15 is proved. □

We have then that $A(\omega^2;\epsilon)$ has a bounded inverse if and only if the homogeneous equation,

$$A(\omega^2;\epsilon)\psi = 0, \tag{4.71}$$

has only one solution. We now utilise Theorem 12 with $A(\omega^2;\epsilon)$ taking the role of $T(x)$, $H^1(\Omega^a)$ is X and ω^2 is x. Thus $S_2(\omega^2;\epsilon)$ exists and is bounded-holomorphic at ω^2 if and only if the equation (4.71) has only one solution.

As with the case of the Dirichlet problem in the region outside the sphere, it can be proved that equation (4.71) has just one solution if $\Im\omega^2 > 0$. In fact, this is proved in an identical way as before. That is to say, we show that the existence of a non-trivial solution of equation (4.71) implies the existence of a distribution ψ, with

$$(1+r^2)^{-1/2}\psi \in L^2(\Omega_e),$$

$$\nabla\psi \in (L^2(\Omega_e))^3,$$

$$\psi|_{FS} \in L^2(FS)$$

and

$$\psi|_{\partial\Omega} = 0.$$

As with $S_1(\omega^2;\epsilon)$, we can show that $S_2(\omega^2;\epsilon)$ is meromorphic in $\mathcal{C}\setminus\mathcal{R}_-$.

Now write equation (4.67) as

$$A(\omega^2;\epsilon)\psi = F(f),$$

where

$$(F(f),\phi)_{H^1(\Omega^a)} = -\int_{\partial\Omega} f\overline{\phi}dS, \tag{4.72}$$

for all ϕ belonging to $H^1(\Omega^a)$. It is clear that F is continuous from $L^2(\partial\Omega)$ into $H^1(\Omega^a)$. The solution of the truncated problem given in equations (4.64) to (4.66) is then

$$\psi = S_2(\omega^2;\epsilon)F(f).$$

We know that $S_2(\omega^2;\epsilon)F$ is defined everywhere except at a set consisting of isolated points and that it is bounded from $L^2(\partial\Omega)$ into $H^1(\Omega^a)$. Thus Lemma 14 is proved. □

Before we proceed further let us define the operator $S_3(\omega^2;\epsilon)$ by

$$S_3(\omega^2;\epsilon)f = \gamma S_2(\omega^2;\epsilon)F(f),$$

where γ is the trace operator on $\partial\Omega$ between $H^1(\Omega^a)$ and $H^{\frac{1}{2}}(\partial\Omega)$.

4.3.4 The relationship between the exterior problem and the truncated problem.

We have said that the exterior problem for complex values of ω^2 can be thought of as being defined from the truncated problem together with the Dirichlet problem in the exterior of the sphere. It will be helpful, however, to explicitly write down the exterior problem in the case when the imaginary part of ω^2 is positive. Any solution can be expressed as a sum of multi-poles. A glance at the expressions for the extended multi-poles (equation (4.73) below) will tell us that the solution will decay faster at infinity than is the case for real ω^2.

$$\begin{aligned}\psi_n^m(r,\theta) + i\chi_n^m(r,\theta) &= P_n^m(\cos\theta)r^{-(n+1)} \\ &+\tfrac{(-1)^{m+n}}{(n-m)!}\int_0^\infty \frac{k+\frac{\omega^2}{g}}{k-\frac{\omega^2}{g}} k^n J_m(kr\sin\theta)\exp(-k(r\cos\theta+2/\epsilon))dk.\end{aligned} \tag{4.73}$$

Whereas, for real ω^2 the solution belonged to $H^1_{loc}(\Omega_e)$ we might expect that, for the case of $\Im\omega^2$ greater than zero, the solution will belong to a smaller space, W, if its trace on $\partial\Omega$ belongs to $H^{\frac{1}{2}}(\partial\Omega)$. Appendix B shows that ψ belongs to W, defined below:

Define

$$W = \{\psi; (1+r^2)^{-1/2}\psi \in L^2(\Omega_e),\ \nabla\psi \in (L^2(\Omega_e))^3 \text{ and } \psi|_{FS} \in L^2(FS)\}.$$

The exterior problem in this case is: Given f belonging to $L^2(\partial\Omega)$, find $\psi \in W$ satisfying

$$\nabla^2\psi = 0 \text{ in } \Omega_e, \tag{4.74}$$

$$\left.\frac{\partial\psi}{\partial n}\right|_{\partial\Omega} = f, \tag{4.75}$$

$$\left.\left(\frac{\partial\psi}{\partial n} + \frac{\omega^2}{g}\psi\right)\right|_{FS} = 0. \tag{4.76}$$

Lenoir and Martin [17] have given a proof of the existence of a solution to this problem.

Let us denote by $S_4(\omega^2;\epsilon)$ the operator that maps f to ψ and let

$$S_5(\omega^2;\epsilon)f = \gamma S_4(\omega^2;\epsilon)f.$$

It looks as if the poles of $S_3(\omega^2;\epsilon)$ will depend upon the construction of the truncated problem. That is to say, they depend on the radius of the chosen sphere. We can see that this is not the case because the operators $S_3(\omega^2;\epsilon)$ and $S_5(\omega^2;\epsilon)$ are identical in the upper half plane. This is a consequence of the equivalence between the exterior problem and the truncated problem. $S_5(\omega^2;\epsilon)$ is clearly independent of the construction of the truncated problem. $S_3(\omega^2;\epsilon)$ is, therefore, equal to the continuation of $S_5(\omega^2;\epsilon)$ everywhere, and the poles of this operator are independent of the construction of the truncated problem.

4.3.5 The large submergence depth limit.

In this part we shall show that if K is any compact subset of $\mathcal{C} \setminus \mathcal{R}_-$ and if ϵ is smaller than some positive number depending on K, then no poles of $S_3(\omega^2;\epsilon)$ are in K.

We know from the bounds (4.56) and (4.57) that $T(\omega^2;\epsilon)$ exists for any ω^2 in K if ϵ is smaller than a certain number — call it $\epsilon_1(K)$. This is true because if ϵ is sufficiently small then

$$\sum_{m=0}^{\infty}\sum_{n=m}^{\infty}\sum_{s=m}^{\infty}|A_{ns}^m(\omega^2;\epsilon)|^2 < 1.$$

Thus the poles of $S_1(\omega^2;\epsilon)$, which were defined on page 20, lie outside K.

Furthermore, equation (4.68) tells us that we can choose a number $M > 0$ such that

$$-\Re\int_{\partial\Omega_a}(T(\omega^2;\epsilon)\psi)\overline{\psi}dS \geq M \parallel \psi \parallel^2_{H^{\frac{1}{2}}(\partial\Omega_a)}$$

if ϵ is sufficiently small — smaller than $\epsilon_2(K)$, say, which is smaller than $\epsilon_1(K)$. To see this note that the modulus of each $A_{ns}^m(\omega^2;\epsilon)$ in equation (4.68) can be made as small as we like. Thus the final three terms on the right-hand side of that equation can be made as small as desired. Then the equivalence of the $H^{\frac{1}{2}}(\partial\Omega_a)$ norm and the norm in the first term on the right-hand side of equation (4.68) is a consequence of Lemma 11.

We have already seen that $S_3(\omega^2;\epsilon)$ has a pole if and only if the truncated problem with zero boundary datum has a non-trivial solution. Let us consider, then, equation (4.67) with $f = 0$. Putting ϕ equal to ψ and taking the real part we have

$$\int_{\Omega^a}\nabla\psi.\nabla\overline{\psi}dV - \Re\int_{\partial\Omega_a}(T(\omega^2;\epsilon)\psi)\overline{\psi}dS = 0.$$

If ϵ is less than $\epsilon_2(K)$, then

$$\int_{\Omega^a} \nabla\psi.\nabla\overline{\psi}dV + M \parallel \psi \parallel^2_{H^{\frac{1}{2}}(\partial\Omega_a)} = 0.$$

As $M > 0$, the last equation implies that ψ vanishes on $\partial\Omega_a$ and, thus, the gradient of ψ is zero in Ω^a. The condition $\psi = 0$ on $\partial\Omega_a$ means then that ψ vanishes inside Ω^a. Clearly then $S_3(\omega^2;\epsilon)$ has no poles in K if ϵ is less than $\epsilon_2(K)$.

4.3.6 Summary of exterior problem.

The exterior Neumann problem was described for the case of real ω^2. To solve this problem we set up a Dirichlet boundary value problem in the exterior of a sphere. We extended the problem to non-real frequencies and showed that this problem is solvable for every value of ω^2 except those contained in a set consisting of isolated points. This lead to the definition of a problem that is equivalent to the exterior problem when the frequency is real, set in a bounded domain — the truncated problem. This has the advantage of involving only familiar function spaces. Using this problem, extended exterior problems were defined.

The truncated problem was shown to be solvable except at a set of isolated frequencies. Furthermore, the operator connecting the Neumann datum to the solution was proved to be meromorphic with respect to ω^2 with no poles in the upper half plane. As the depth of submergence increases the poles of the operator were shown to "tend to infinity". The physical interpretaion of this is as the body is more deeply submerged the effect of the free surface lessens and so the situation more and more "resembles" the case of a body surrounded by an incompressible, inviscid fluid that extends infinitely in *all* directions; the Neumann problem in this case is known to be uniquely solvable at all real frequencies. The point about this problem being solvable in the large submergence depth limit for all frequencies is a truism since the frequency dependence has disappeared in this limit.

4.4 The interior problem.

Let us now assume that ω^2 lies in some compact region K, of $\mathcal{C} \setminus \mathcal{R}_-$, and that ϵ is smaller than $\epsilon_2(K)$. Thus $S_3(\omega^2;\epsilon)$ exists.

We know that the trace of the solution on $\partial\Omega$ of the exterior problem, given Neumann boundary datum, f, is

$$S_3(\omega^2;\epsilon)f.$$

If we return to the matching conditions across $\partial\Omega$ given in equations (4.8) and (4.9) we see that the surface traction and the surface displacement of the elastic body must be related in the following way

$$\mathbf{n}.\sigma(\mathbf{u}) = \rho_0\omega^2\mathbf{n}S_3(\omega^2;\epsilon)(\mathbf{u}.\mathbf{n}). \tag{4.77}$$

Thus the interior problem is: Find $\mathbf{u}$ belonging to $H^1(\Omega_i)$ such that the boundary condition (4.77) holds as well as the equation (4.7), which shall be re-written below

$$L(\mathbf{u}) + \rho\omega^2\mathbf{u} = 0$$

in the sense of distributions. Here we adopt the notation that, for example, $H^1(\Omega_i)$ stands for $(H^1(\Omega_i))^3$ — the space of all three-dimensional vector distributions, each component of which belongs to $H^1(\Omega_i)$. No confusion should occur as it will be obvious when we are talking about vectors and when we are talking about scalars.

The above formulation is equivalent to the weak formulation: Find $\mathbf{u} \in H^1(\Omega_i)$ such that

$$\rho_0\omega^2 \int_{\partial\Omega} \overline{\mathbf{v}}.\mathbf{n}S_3(\omega^2;\epsilon)(\mathbf{u}.\mathbf{n})dS + \rho\omega^2 \int_{\Omega_i} \mathbf{u}.\overline{\mathbf{v}}dV - \int_{\Omega_i} \sigma(\mathbf{u}) : \nabla\overline{\mathbf{v}}dV = 0 \tag{4.78}$$

for all $\mathbf{v} \in H^1(\Omega_i)$.

Let us denote by $B(\omega^2;\epsilon)$ the operator defined by

$$(B(\omega^2;\epsilon)\mathbf{u},\mathbf{v})_{H^1(\Omega_i)} = \rho_0\omega^2 \int_{\partial\Omega} \overline{\mathbf{v}}.\mathbf{n}S_3(\omega^2;\epsilon)(\mathbf{u}.\mathbf{n})dS + (1+\rho\omega^2) \int_{\Omega_i} \mathbf{u}.\overline{\mathbf{v}}dV. \tag{4.79}$$

$S_3(\omega^2;0)$ is actually independent of ω^2. We have

$$S_3(\omega^2;0) = \gamma(A(\omega^2;0))^{-1}F.$$

The dependence of $A(\omega^2;0)$ on ω^2 comes from $T(\omega^2;0)$. This can be seen not to depend on ω^2 from equation (4.63). This is true because each $A^m_{ns}(\omega^2;\epsilon)$ vanishes in the limit as ϵ tends to zero according to equation (4.41).

Before we proceed further we note that

$$\int_{\Omega_i} \sigma(\mathbf{u}) : \nabla\overline{\mathbf{v}}dV + \int_{\Omega_i} \mathbf{u}.\overline{\mathbf{v}}dV$$

defines an inner product in $H^1(\Omega_i)$.

4.4.1 The spectrum of $B(\omega^2; 0)$.

We want to show that $B(\omega^2; 0)$, when considered as acting in $H^1(\Omega_i)$, is compact. Therefore, for fixed ω^2, $B(\omega^2; 0)$ has a countable number of eigenvalues. Each of these eigenvalues is real when ω^2 is real and passes through the value one as ω^2 is increased from zero.

From the definition of $B(\omega^2; 0)$ in equation (4.79) with $\mathbf{v}$ equal to $B(\omega^2; 0)\mathbf{u}$, we have

$$\begin{aligned}
&\| B(\omega^2; 0)\mathbf{u} \|^2_{H^1(\Omega_i)} \\
&\quad \leq \rho_0\omega^2 \| \mathbf{n}.B(\omega^2; 0)\mathbf{u} \|_{L^2(\partial\Omega)} \| S_3(\omega^2; 0)(\mathbf{u}.\mathbf{n}) \|_{L^2(\partial\Omega)} \\
&\qquad +(1+\rho\omega^2) \| \mathbf{u} \|_{L^2(\Omega_i)} \| B(\omega^2; 0)\mathbf{u} \|_{L^2(\Omega_i)} \\
&\qquad \leq M\rho_0\omega^2 \| B(\omega^2; 0)\mathbf{u} \|_{H^1(\Omega_i)} \| \mathbf{u} \|_{L^2(\partial\Omega)} \\
&\qquad +(1+\rho\omega^2) \| \mathbf{u} \|_{L^2(\Omega_i)} \| B(\omega^2; 0)\mathbf{u} \|_{H^1(\Omega_i)},
\end{aligned}$$

where M is independent of ω^2 and $\mathbf{u}$. We have used the boundedness of $S_3(\omega^2; 0)$ and the boundedness of the trace operator from $H^1(\Omega_i)$ into $L^2(\partial\Omega)$.

The compact imbedding of $H^1(\Omega_i)$ into $L^2(\Omega_i)$ and into $L^2(\partial\Omega)$ shows that $B(\omega^2; 0)$ is compact in $H^1(\Omega_i)$.

$B(\omega^2; 0)$ is self-adjoint. (We are identifying the dual of $H^1(\Omega_i)$ with $H^1(\Omega_i)$.) The self-adjointness results from the Hermitian-symmetry of the form in equation (4.79) when ϵ is zero. The Hermitian-symmetry of

$$\int_{\partial\Omega} \overline{\mathbf{v}}.\mathbf{n} S_3(\omega^2; 0)(\mathbf{u}.\mathbf{n})\, dS$$

is a consequence of the Hermitian-symmetry of

$$\int_{\partial\Omega_a} \overline{\phi} T(\omega^2; 0)\psi\, dS \tag{4.80}$$

$$= -2a^2\pi \textstyle\sum_{m=0}^{\infty} \sum_{n=m}^{\infty} (a_{mn}\overline{c_{mn}} + b_{mn}\overline{d_{mn}}) \frac{(n+1)(n+m)!}{(2n+1)(n-m)!} C_m,$$

where

$$C_m = \begin{cases} 2 & \text{if } m = 0 \\ 1 & \text{if } m \neq 0 \end{cases}$$

and where

$$\psi = \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} P_n^m(\cos\theta)(a_{mn}\cos m\phi + b_{mn}\sin m\phi)$$

and

$$\phi = \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} P_n^m(\cos\theta)(c_{mn}\cos m\phi + d_{mn}\sin m\phi).$$

Call

$$\phi = A(\omega^2;\epsilon)^{-1}F(\mathbf{u}.\mathbf{n}).$$

From equation (4.67),

$$(F(\mathbf{u}.\mathbf{n}), A(\omega^2;\epsilon)^{-1}F(\mathbf{u}.\mathbf{n}))_{H^1(\Omega^a)} = \int_{\Omega_i} \nabla\phi.\nabla\overline{\phi}dV - \int_{\partial\Omega_a} (T(\omega^2;0)\phi)\overline{\phi}dS.$$

Now from the definition of F in equation (4.72), the left hand side of this last equation is

$$-\int_{\partial\Omega} \mathbf{u}.\mathbf{n}S_3(\omega^2;\epsilon)(\overline{\mathbf{u}}.\mathbf{n})dS$$

since

$$S_3(\omega^2;\epsilon) = \gamma A(\omega^2;\epsilon)^{-1}F$$

by definition. Now, from equation (4.80),

$$\int_{\partial\Omega} \overline{\mathbf{u}}.\mathbf{n}S_3(\omega^2;0)(\mathbf{u}.\mathbf{n})dS$$

is always positive whenever $\mathbf{u} \neq 0$. Therefore,

$$(B(\omega^2;0)\mathbf{u},\mathbf{u})_{H^1(\Omega_i)}$$

is always positive.

The compactness, self-adjointness and positive properties of $B(\omega^2; 0)$ ensure that its spectrum consists of a countably infinite number of real points which have no accumulation point other than zero. The maximum distance from any point in the spectrum to zero is finite and every point in the spectrum is an eigenvalue of $B(\omega^2; 0)$ except for zero. This result is well known and a proof of it can be found in Sanchez Hubert and Sanchez Palencia [27, Chapter 1].

A useful result is the max-min principle. This is closely related to the so-called Minimax principle to be found in Propostion 7.1 in Sanchez Hubert and Sanchez Palencia [27, Chapter 1].

Lemma 16 *The nth largest eigenvalue of $B(\omega^2; 0)$ is given by*

$$\lambda_n = \min(\mu(\{\mathbf{w}_1, \ldots, \mathbf{w}_{n-1}\})), \tag{4.81}$$

where

$$\mu(\{\mathbf{w}_1, \ldots, \mathbf{w}_{n-1}\}) = \sup \frac{(B(\omega^2; 0)\mathbf{u}, \mathbf{u})_{H^1(\Omega_i)}}{\| \mathbf{u} \|^2_{H^1(\Omega_i)}}, \tag{4.82}$$

where in equation (4.82) the supremum is taken over all elements in the subspace perpendicular to the span of

$$\{\mathbf{w}_1, \ldots, \mathbf{w}_{n-1}\}$$

and in equation (4.81) $\mu(\{\mathbf{w}_1, \ldots, \mathbf{w}_{n-1}\})$ is minimised over all possible choices of the $(n-1)$ elements $\{\mathbf{w}_1, \ldots, \mathbf{w}_{n-1}\}$ of $H^1(\Omega_i)$.

This is proved by adapting the proof of the Minimax principle given in [27].

An immediate consequence of Lemma 16 and equation (4.79) is that each eigenvalue of $B(\omega^2; 0)$ increases monotonically as ω^2 is increased. In fact, it

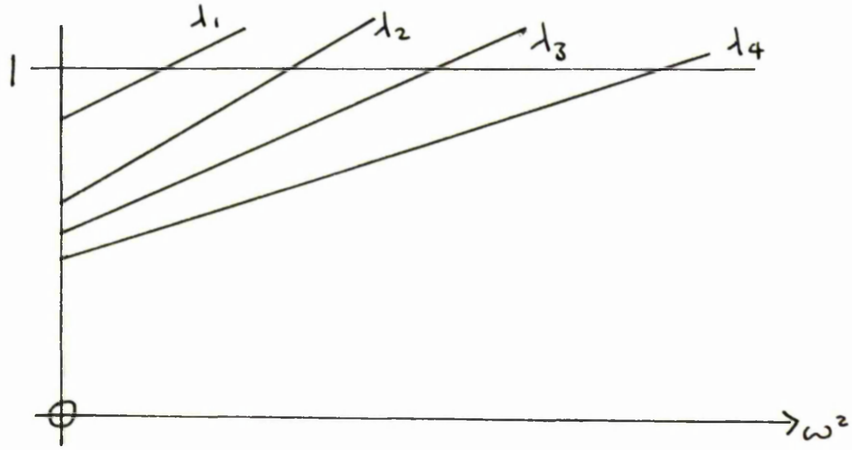

Figure 4.6: The growth of the eigenvalues of $B(\omega^2; 0)$.

is clear that the dependence of each eigenvalue on ω^2 will be linear in ω^2. It is evident then that there is a countably infinite set of values of ω^2 for which $B(\omega^2; 0)$ has an eigenvalue equal to one — call this set Q. See Figure (4.6). The figure also makes it clear that Q has no finite accumulation point and that there is no largest point in Q.

4.4.2 The eigenvalues of $B(\omega^2; \epsilon)$ and their connection to the eigenvalues of $B(\omega^2; 0)$

We begin this section by noting that $B(\omega^2; \epsilon)$ is holomorphic with respect to ω^2 for fixed ϵ. We have

$$(B(\omega^2; \epsilon)\mathbf{u}, \mathbf{v})_{H^1(\Omega_i)}$$
$$= (1 + \rho_0\omega^2) \int_{\Omega_i} \mathbf{u}\overline{\mathbf{v}} dV + \rho\omega^2 \int_{\partial\Omega} \overline{\mathbf{v}}.\mathbf{n} S_3(\omega^2; \epsilon)(\mathbf{u}.\mathbf{n}) dS.$$

We have, by definition,

$$S_3(\omega^2; \epsilon) = \gamma(A(\omega^2; \epsilon))^{-1} F.$$

So $S_3(\omega^2; \epsilon)$ and therefore $B(\omega^2; \epsilon)$ are holomorphic with respect to ω^2 for fixed ϵ because $A(\omega^2; \epsilon)$ is invertible and holomorphic in ω^2.

$B(\omega^2;\epsilon)$ has an asymptotic expansion in integer powers of ϵ. This expansion is uniform in ω^2. Let us begin by showing that $A(\omega^2;\epsilon)$ has an asymptotic expansion in powers of ϵ and that this is uniform in ω^2.

$$
\begin{aligned}
&(A(\omega^2;\epsilon)\chi,\psi)_{H^1(\Omega^a)} \qquad (4.83)\\
&= -\textstyle\int_{-1}^{1}\int_0^{2\pi} a^2 dc d\phi \left(\sum_{m=0}^{\infty}\sum_{n=m}^{\infty}\sum_{s=m}^{\infty}(\overline{\gamma_{mn}}\cos m\phi + \overline{\delta_{mn}}\sin m\phi)\right.\\
&\qquad \times \left.\left(\delta_{ns} + \sqrt{\tfrac{(s-m)!(n+m)!}{(s+m)!(n-m)!}}\overline{A^m_{ns}(\omega^2;\epsilon)}\right) P^m_n(c)\right)\\
&\qquad \times \left(\textstyle\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{s=0}^{\infty}(\alpha_{mn}\cos m\phi + \beta_{mn}\sin m\phi)\frac{1}{a}\right.\\
&\qquad \times \left.\left(-(n+1)\delta_{ns} + s\sqrt{\tfrac{(s-m)!(n+m)!}{(s+m)!(n-m)!}}A^m_{ns}(\omega^2;\epsilon)\right) P^m_n(c)\right)\\
&\qquad + \textstyle\int_{\Omega^a}\nabla\chi.\nabla\overline{\psi}dV,
\end{aligned}
$$

where

$$
\begin{aligned}
\chi = &\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{s=0}^{\infty}(\alpha_{mn}\cos m\phi + \beta_{mn}\sin m\phi)\\
&\times\tfrac{1}{a}\left(\delta_{ns} + \sqrt{\tfrac{(s-m)!(n+m)!}{(s+m)!(n-m)!}}A^m_{ns}(\omega^2;\epsilon)\right) P^m_n(\cos\theta)
\end{aligned}
$$

and

$$
\begin{aligned}
\psi = &\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{s=0}^{\infty}(\gamma_{mn}\cos m\phi + \delta_{mn}\sin m\phi)\\
&\times\tfrac{1}{a}\left(\delta_{ns} + \sqrt{\tfrac{(s-m)!(n+m)!}{(s+m)!(n-m)!}}A^m_{ns}(\omega^2;\epsilon)\right) P^m_n(\cos\theta).
\end{aligned}
$$

Of course, if χ and ψ are to be independent of ω^2 and ϵ then, for each m and n, α_{mn}, β_{mn}, γ_{mn} and δ_{mn} will depend on ω^2 and ϵ. The dependence of α_{mn} on ω^2 and ϵ is shown in the equation (4.40), which is rewritten below:

$$\sum_{s=0}^{\infty}(\delta_{ns} + A^m_{ns}(\omega^2;\epsilon))\alpha_{ms} = a_{mn}, \qquad (4.84)$$

where a_{mn} is the constant given by

$$a_{mn} = \frac{n+1/2}{C_m a^2\pi}\sqrt{\frac{(n-m)!}{(n+m)!}}\int_{\partial\Omega_a}\chi(\theta,\phi)P^m_n(\cos\theta)\cos m\phi dS.$$

Similar results hold true for β_{mn}, γ_{mn} and δ_{mn}.

We know that the series in equation (4.83) are absolutely convergent and, so, we can exchange the order of integration and summation to obtain

$$
\begin{aligned}
&(A(\omega^2;\epsilon)\chi,\psi)_{H^1(\Omega^a)} \qquad (4.85)\\
&= a\pi \sum_{m=0}^{\infty}\sum_{n=m}^{\infty}\sum_{s=m}^{\infty}\sum_{t=m}^{\infty} C_m(\alpha_{mn}\overline{\gamma_{mt}}+\beta_{mn}\overline{\delta_{mt}})\\
&\times\Bigg((n+1)\delta_{ns}\delta_{ts}+(n+1)\delta_{ns}\sqrt{\tfrac{(s-m)!(n+m)!}{(s+m)!(n-m)!}}\overline{A_{ts}^m(\omega^2;\epsilon)}\\
&\qquad -s\delta_{ts}\sqrt{\tfrac{(s-m)!(t+m)!}{(s+m)!(t-m)!}}A_{ns}^m(\omega^2;\epsilon)\\
&-s\tfrac{(s-m)!}{(s+m)!}\sqrt{\tfrac{(n-m)!(t+m)!}{(n+m)!(t-m)!}}A_{ns}^m(\omega^2;\epsilon)\overline{A_{ts}^m(\omega^2;\epsilon)}\Bigg)\tfrac{(s+m)!}{(s+1/2)(s-m)!}\\
&\qquad + \int_{\Omega^a}\nabla\chi.\nabla\overline{\psi}dV.
\end{aligned}
$$

It is can be verified that

$$
\begin{aligned}
A_{ns}^m(\omega^2;\epsilon) &= \frac{(n+s)!\left(-a\frac{\omega^2}{g}\right)^{n+s+1}}{\sqrt{(n+m)!(n-m)!(s+m)!(s-m)!}}\left(\frac{g\epsilon}{2\omega^2}\right)^{n+s+1} \qquad (4.86)\\
&-2\sum_{q=1}^{p}\frac{(n+s+q)!\left(-a\frac{\omega^2}{g}\right)^{n+s+1}}{\sqrt{(n+m)!(n-m)!(s+m)!(s-m)!}}\left(\frac{g\epsilon}{2\omega^2}\right)^{n+s+q+1}\\
&+o(\epsilon^{n+s+p+1}).
\end{aligned}
$$

So each $A_{ns}^m(\omega^2;\epsilon)$ has an asymptotic expansion in integer powers of ϵ and each term in the expansion is real if ω^2 is real. Therefore, equation (4.85) and the expressions for α_{mn}, β_{mn}, γ_{mn} and δ_{mn} show that the sesquilinear form

$$(A(\omega^2;\epsilon)\chi,\psi)_{H^1(\Omega^a)}$$

has an asymptotic expansion in ϵ:

$$(A(\omega^2;\epsilon)\chi,\psi)_{H^1(\Omega^a)} = A_0(\chi,\psi)+\epsilon A_1(\chi,\psi)+\epsilon^2 A_2(\omega^2)(\chi,\psi)+\dots \qquad (4.87)$$

Each of the sesquilinear forms $A_n(\omega^2)(.,.)$ is bounded in $H^1(\Omega^a)$. It is a feature of this problem that the first two sesquilinear forms are independent of ω^2. In fact, we have

$$A_0(\chi,\psi) = \int_{\Omega^a} \nabla\chi.\nabla\overline{\psi}dV \tag{4.88}$$

$$+a\pi \sum_{m=0}^{\infty}\sum_{n=m}^{\infty} C_m(a_{mn}\overline{c_{mn}} + b_{mn}\overline{d_{mn}})\frac{n+1}{n+1/2} \tag{4.89}$$

and

$$A_1(\chi,\psi) = 6a^2\pi(a_{00}\overline{c_{00}} + b_{00}\overline{d_{00}}). \tag{4.90}$$

Let us denote by $A_n(\omega^2)$ the operator associated with the form $A_n(\omega^2)(.,.)$. Equation (4.87) can be rewritten as

$$A(\omega^2;\epsilon) = A_0 + \epsilon A_1 + \epsilon^2 A_2(\omega^2) + \dots \tag{4.91}$$

A_0 is invertible and its inverse is bounded in $H^1(\Omega^a)$. The first few terms of its asymptotic expansion are

$$A(\omega^2;\epsilon)^{-1} = A_0^{-1} - \epsilon A_0^{-1}A_1A_0^{-1} + \epsilon^2(\frac{1}{2}A_0^{-1}A_1A_0^{-1}A_1A_0^{-1} - A_0^{-1}A_2(\omega^2)A_0^{-1}) + \dots \tag{4.92}$$

Let us recall the relationship between $B(\omega^2;\epsilon)$ and $A(\omega^2;\epsilon)^{-1}$:

$$(B(\omega^2;\epsilon)\mathbf{u},\mathbf{v})_{H^1(\Omega_i)} = (1+\rho_0\omega^2)\int_{\Omega_i}\mathbf{u}.\overline{\mathbf{v}}dV$$
$$+\rho\omega^2\int_{\partial\Omega}\gamma A(\omega^2;\epsilon)^{-1}F(\mathbf{u}.\mathbf{n})\overline{\mathbf{v}}.\mathbf{n}dS.$$

This implies that $B(\omega^2;\epsilon)$ has an asymptotic expansion in powers of ϵ:

$$B(\omega^2;\epsilon) = B_0(\omega^2) + \epsilon B_1(\omega^2) + \epsilon^2 B_2(\omega^2) + \dots$$

From equation (4.92) it is clear that

$$(B_1(\omega^2)\mathbf{u},\mathbf{v})_{H^1(\Omega_i)} = -\rho\omega^2\int_{\partial\Omega}\gamma A_0^{-1}A_1A_0^{-1}F(\mathbf{u}.\mathbf{n})\overline{\mathbf{v}}.\mathbf{n}dS. \tag{4.93}$$

The definition of F and the equation (4.93) imply that

$$(B_1(\omega^2)\mathbf{u},\mathbf{v})_{H^1(\Omega_i)} = \rho\omega^2(A_0^{-1}A_1A_0^{-1}F(\mathbf{u}.\mathbf{n}), F(\overline{\mathbf{v}}.\mathbf{n}))_{H^1(\Omega^a)}. \tag{4.94}$$

Equation (4.88) implies that A_0 is self-adjoint and, therefore, so is A_0^{-1}. This fact together with equation (4.94) imply

$$(B_1(\omega^2)\mathbf{u}, \mathbf{v})_{H^1(\Omega_i)} = \rho\omega^2(A_1 A_0^{-1} F(\mathbf{u}.\mathbf{n}), A_0^{-1} F(\overline{\mathbf{v}}.\mathbf{n}))_{H^1(\Omega^a)}. \tag{4.95}$$

Let λ_0 be an eigenvalue of $B(\omega_0^2; 0)$. Denote by $P(\omega^2; \epsilon)$ the operator

$$P(\omega^2; \epsilon) = -\frac{1}{2\pi i} \int_C \left(B(\omega^2; \epsilon) - \zeta\right)^{-1} d\zeta, \tag{4.96}$$

where C is a simple positively orientated curve that encloses λ_0 but no other eigenvalue of $B(\omega^2; 0)$ and that is contained entirely within the resolvent set of $B(\omega^2; 0)$. $P(\omega^2; \epsilon)$ is the sum of the eigenprojections associated with all the eigenvalues of $B(\omega^2; \epsilon)$ enclosed by C. (See, for example, Kato [14, p. 67].) Since

$$(B(\omega^2; 0)\mathbf{u}, \mathbf{v})_{H^1(\Omega_i)}$$

is holomorphic with respect to ω^2 in a neighbourhood of ω_0^2 contained in K for all $\mathbf{u}$ and $\mathbf{v}$, then $B(\omega^2; 0)$ is holomorphic with respect to ω^2. Theorem 12 and the fact that the curve C is contained in the resolvent set of $B(\omega_0^2; 0)$ imply that the resolvent operator,

$$(B(\omega^2; 0) - \zeta)^{-1},$$

is holomorphic at ω_0^2 for all points of C. Clearly, since the spectrum of $B(\omega_0^2; 0)$ consists of isolated points,

$$\sup |(B(\omega_0^2; 0) - \zeta)^{-1}| \tag{4.97}$$

is finite, where the supremum is taken over all points of C. The identity

$$\begin{aligned}(B(\omega^2; 0) - \zeta)^{-1} \qquad\qquad & \qquad\qquad\qquad\qquad (4.98)\\ = (I + (B(\omega_0^2; 0) - \zeta)^{-1}(B(\omega^2; 0) - B(\omega_0^2; 0)))^{-1}&(B(\omega_0^2; 0) - \zeta)^{-1},\end{aligned}$$

the holomorphicity of $B(\omega^2;0)$ and the fact that the supremum (4.97) is finite imply that

$$|(B(\omega^2;0)-\zeta)^{-1}| < M, \tag{4.99}$$

for some constant M, for all ζ on C and for all ω^2 in some neighbourhood $N(\omega_0^2)$ of ω_0^2, contained in K.

In a similar way we can say that, because

$$\begin{aligned}&(B(\omega^2;\epsilon)-\zeta)^{-1}\\&\quad=(I+(B(\omega^2;0)-\zeta)^{-1}(B(\omega^2;\epsilon)-B(\omega^2;0)))^{-1}(B(\omega^2;0)-\zeta)^{-1}\end{aligned}$$

and using the fact $B(\omega^2;\epsilon)$ is continuous at $\epsilon=0$ uniformly in ω^2,

$$|(B(\omega^2;\epsilon)-\zeta)^{-1}| < M' \tag{4.100}$$

for a constant M' and for all ζ on C, for all ω^2 in N and for all ϵ smaller than some positive number $\epsilon_3(K)$.

Since $B(\omega^2;\epsilon)$ has an asymptotic expansion with respect to ϵ, which is uniform in ω^2, then the resolvent operator has an asymptotic expansion with respect to ϵ, which is also uniform in ω^2:

$$(B(\omega^2;\epsilon)-\zeta)^{-1}=(B(\omega^2;0)-\zeta)^{-1}+\epsilon R_1(\omega^2;\zeta)+\ldots+\epsilon^p R_p(\omega^2;\zeta)+o(\epsilon^{p+1}). \tag{4.101}$$

This expansion is also uniform in ζ. The bound (4.100) implies that

$$|R_p(\omega^2;\zeta)| \le M_p(\omega^2), \tag{4.102}$$

for all points on C and where each $M_p(\omega^2)$ is a function of ω^2 only. Thus, the bounds (4.102) imply that if the resolvent in equation (4.96) is replaced by its asymptotic expansion (4.101), then each term can be integrated separately and, finally, the projection operator can be expanded asymptotically around ϵ=0:

$$P(\omega^2;\epsilon) = P_0(\omega^2) + \epsilon P_1(\omega^2) + \ldots + \epsilon^p P_p(\omega^2) + o(\epsilon^{p+1}), \tag{4.103}$$

where

$$P_0(\omega^2) = -\frac{1}{2\pi i}\int_C (B(\omega^2;0)-\zeta)^{-1}\,d\zeta.$$

The bound (4.100) implies that $P(\omega^2;\epsilon)$ is holomorphic with respect to ω^2 in N and for all ϵ belonging to the interval $[0,\epsilon_3(K)]$.

Equation (4.103) and the holomorphicity of $P(\omega^2;\epsilon)$ imply that

$$|P(\omega^2;\epsilon) - P_0(\omega_0^2)| < 1,$$

if ϵ is smaller than some positive number $\epsilon_4(K)$, say. This implies that, just as in Subsection 4.3.2, we can (non-uniquely) construct a transformation function $U(\omega^2;\epsilon)$, that has the property

$$U(\omega^2;\epsilon)P_0(\omega_0^2)U(\omega^2;\epsilon)^{-1} = P(\omega^2;\epsilon). \tag{4.104}$$

$U(\omega^2;\epsilon)$ and $U(\omega^2;\epsilon)^{-1}$ can be chosen to be holomorphic with respect to ω^2 and to have asymptotic expansions with respect to ϵ.

Since

$$P(\omega^2;\epsilon)B(\omega^2;\epsilon)P(\omega^2;\epsilon) = U(\omega^2;\epsilon)P_0(\omega_0^2)\tilde{B}(\omega^2;\epsilon)P_0(\omega_0^2)U(\omega^2;\epsilon)^{-1},$$

where

$$\tilde{B}(\omega^2;\epsilon) = U(\omega^2;\epsilon)^{-1}B(\omega^2;\epsilon)U(\omega^2;\epsilon), \tag{4.105}$$

finding the eigenvalues of $B(\omega^2;\epsilon)$ in the subspace

$$M(\omega^2;\epsilon) = P(\omega^2;\epsilon)H^1(\Omega_i),$$

is equivalent to finding the eigenvalues of $\tilde{B}(\omega^2;\epsilon)$ in the subspace

$$M(\omega_0^2) = P_0(\omega_0^2)H^1(\Omega_i).$$

$M(\omega_0^2)$ is a fixed subspace and is finite dimensional. This is because it is the eigenspace associated with the eigenvalue λ_0 of $B(\omega^2;0)$ and these eigenspaces are known to be finite dimensional, as has already been noted.

The eigenvalues of $B(\omega^2;\epsilon)$ in $N \otimes [0, \epsilon_4(K)]$ are precisely the solutions of the polynomial equation

$$\det(P_0(\omega_0^2)\tilde{B}(\omega^2;\epsilon)P_0(\omega_0^2) - \lambda) = 0. \tag{4.106}$$

4.4.3 The expansion of $P_0(\omega_0^2)\tilde{B}(\omega^2;\epsilon)P_0(\omega_0^2)$.

We know that

$$B(\omega^2;\epsilon) = B_0(\omega^2) + \epsilon B_1(\omega^2) + \ldots + \epsilon^p B_p(\omega^2) + C_p(\omega^2;\epsilon),$$

where

$$C_p(\omega^2;\epsilon) = o(\epsilon^p).$$

Denote by $f(\omega^2;\epsilon)$ the product

$$\frac{1}{\epsilon}((B(\omega^2;\epsilon) - B_0(\omega^2))\mathbf{u}, \mathbf{v})_{H^1(\Omega_i)},$$

where $\mathbf{u}$ and $\mathbf{v}$ are any two elements of $H^1(\Omega_i)$. Clearly,

$$(B_1(\omega^2)\mathbf{u}, \mathbf{v})_{H^1(\Omega_i)} = \lim_{\epsilon \to 0} f(\omega^2;\epsilon) \equiv f(\omega^2;0).$$

We have

$$\left|\int_\Delta (f(\omega^2;\epsilon) - f(\omega^2;0))d\omega^2\right| \leq \int_\Delta |f(\omega^2;\epsilon) - f(\omega^2;0)|\, d\omega^2$$
$$= \int_\Delta \tfrac{1}{\epsilon}\left|(C_1(\omega^2;\epsilon)\mathbf{u}, \mathbf{v})_{H^1(\Omega_i)}\right| d\omega^2$$
$$\to 0$$

as $\epsilon \to 0$, where Δ is any triangle in N. Therefore,

$$0 = \lim_{\epsilon \to 0} \int_\Delta f(\omega^2;\epsilon)d\omega^2 = \int_\Delta B_1(\omega^2)d\omega^2,$$

by Cauchy's theorem. By Morera's theorem, $B_1(\omega^2)$ is holomorphic in N. We can continue this process to prove that $B_p(\omega^2)$ is holomorphic in N for all p. Therefore,

$$\begin{aligned} B(\omega^2;\epsilon) \quad &= B_{00} + (\omega^2-\omega_0^2)B_{01} + (\omega^2-\omega_0^2)^2 B_{02} + \dots \\ &+\epsilon B_{10} + \epsilon(\omega^2-\omega_0^2)B_{11} + \epsilon(\omega^2-\omega_0^2)^2 B_{12} + \dots \\ &+\epsilon^2 B_{20} + \epsilon^2(\omega^2-\omega_0^2)B_{21} + \epsilon^2(\omega^2-\omega_0^2)^2 B_{22} + \dots \\ &+\vdots \\ &+C_p(\omega^2;\epsilon). \end{aligned}$$

The series converges absolutely in N and, therefore, it can be rearranged.

$$\begin{aligned} B(\omega^2;\epsilon) \quad &= B_{00} + (\omega^2-\omega_0^2)B_{01} + \epsilon B_{10} \\ &+(\omega^2-\omega_0^2)^2 B_{02} + \epsilon(\omega^2-\omega_0^2)^2 B_{12} + \epsilon^2 B_{20} \\ &+\vdots \\ &+\textstyle\sum_{q=0}^{p} \epsilon^p(\omega^2-\omega_0^2)^{p-q} B_{q\,p-q} \\ &+D(\omega^2;\epsilon). \end{aligned} \tag{4.107}$$

The same is, of course, true for $P(\omega^2;\epsilon)$.

The transformation function $U(\omega^2;\epsilon)$ can be chosen to be

$$(I-(P_0(\omega_0^2)-P(\omega^2;\epsilon))^2)^{-1/2}(P_0(\omega_0^2)P(\omega^2;\epsilon)+(I-P_0(\omega_0^2))(I-P(\omega^2;\epsilon))), \tag{4.108}$$

where

$$(I-(P_0(\omega_0^2)-P(\omega^2;\epsilon))^2)^{-1/2}$$

is defined as

$$\sum_{n=0}^{\infty} \begin{pmatrix} -1/2 \\ n \end{pmatrix} (-(P_0(\omega_0^2) - P(\omega^2;\epsilon))^2)^n.$$

This makes sense if

$$\| P_0(\omega_0^2) - P(\omega^2;\epsilon) \| < 1.$$

The proof of this is in Kato [14, Section 4.2, Chapter 2].

Evidently, $U(\omega^2;\epsilon)$ can be expanded as a double power series:

$$U(\omega^2;\epsilon) = I + (\omega^2 - \omega_0^2)U_{01} + \epsilon U_{10} + \ldots$$

This all implies that there is a double power series expansion for $\tilde{B}(\omega^2;\epsilon)$:

$$\begin{aligned} \tilde{B}(\omega^2;\epsilon) = \tilde{B}_{00}(\omega_0^2) + (\omega^2 - \omega_0^2)\tilde{B}_{01}(\omega_0^2) + \epsilon \tilde{B}_{10}(\omega_0^2) \\ +(\omega^2 - \omega_0^2)^2 \tilde{B}_{02}(\omega_0^2) + \epsilon(\omega^2 - \omega_0^2)\tilde{B}_{11}(\omega_0^2) + \epsilon^2 \tilde{B}_{20}(\omega_0^2) \\ +\vdots \\ + \textstyle\sum_{q=0}^{p} \epsilon^p (\omega^2 - \omega_0^2)^{p-q} \tilde{B}_{q\,p-q}(\omega_0^2) \\ +\tilde{D}(\omega^2;\epsilon). \end{aligned} \tag{4.109}$$

It is immediately clear that

$$\tilde{B}_{00}(\omega_0^2) = B(\omega_0^2;0).$$

From equation (4.108)

$$U_{01} = P_0(\omega_0^2)P_{01}(\omega_0^2) - P_{01}(\omega_0^2)P_0(\omega_0^2), \tag{4.110}$$

where

$$P_{01} = \lim_{\omega^2 \to \omega_0^2} \frac{P(\omega^2;0) - P_0(\omega_0^2)}{\omega^2 - \omega_0^2}.$$

Equation (4.105) implies

$$P_0(\omega_0^2)\tilde{B}_{01}(\omega_0^2)P_0(\omega_0^2) = P_0(\omega_0^2)B_{01}(\omega_0^2)P_0(\omega_0^2) \tag{4.111}$$
$$-P_0(\omega_0^2)U_{01}(\omega_0^2)B_{00}(\omega_0^2)P_0(\omega_0^2) + P_0(\omega_0^2)B_{00}(\omega_0^2)U_{01}(\omega_0^2)P_0(\omega_0^2).$$

It is proved in Kato [14, Chapter 2] that

$$P_{01} = -P_0(\omega_0^2)B_{01}(\omega_0^2)S(\omega_0^2) - S(\omega_0^2)B_{01}(\omega_0^2)P_0(\omega_0^2),$$

where $S(\omega_0^2)$ is the reduced resolvent of $B(\omega_0^2;0)$; i. e.

$$S(\omega_0^2)\mathbf{u} = \sum_{j\in Q'} \frac{1}{\lambda_j - \lambda} u_j \mathbf{e}_j,$$

where the λ_j's are the eigenvalues of $B(\omega_0^2;0)$, the $\mathbf{e}_j$'s are the corresponding normalized eigenvectors, λ is the particular eigenvalue for which $P_0(\omega_0^2)$ is the projection operator, Q' is the set of j's for which λ_j does not equal λ and

$$u_j = (\mathbf{u}, \mathbf{e}_j)_{H^1(\Omega_i)}.$$

Given this it is easy to verify that the last two terms on the right hand side of equation (4.111) vanish.

Therefore,

$$P_0(\omega_0^2)\tilde{B}_{01}(\omega_0^2)P_0(\omega_0^2) = P_0(\omega_0^2)B_{01}(\omega_0^2)P_0(\omega_0^2). \tag{4.112}$$

Similarly,

$$P_0(\omega_0^2)\tilde{B}_{10}(\omega_0^2)P_0(\omega_0^2) = P_0(\omega_0^2)B_{10}(\omega_0^2)P_0(\omega_0^2). \tag{4.113}$$

Thus,

$$(P_0(\omega_0^2)\tilde{B}_{01}(\omega_0^2)P_0(\omega_0^2)\mathbf{u}, \mathbf{v})_{H^1(\Omega_i)} = \rho \int_{\Omega_i} \mathbf{u}'.\overline{\mathbf{v}'}dV \tag{4.114}$$
$$+\rho_0 \int_{\partial\Omega} S(\omega_0^2;0)(\mathbf{u}'.\mathbf{n})\overline{\mathbf{v}'}.\mathbf{n}dS$$

where

$$\mathbf{u}' = P_0(\omega_0^2)\mathbf{u}$$

and

$$\mathbf{v}' = P_0(\omega_0^2)\mathbf{v}.$$

From equations (4.90) and (4.95)

$$(P_0(\omega_0^2)\tilde{B}_{10}(\omega_0^2)P_0(\omega_0^2)\mathbf{u},\mathbf{v})_{H^1(\Omega_i)} = 6a^2\pi\rho\omega_0^2(a_{00}\overline{c}_{00} + b_{00}\overline{d}_{00}), \tag{4.115}$$

with obvious notation.

4.4.4 The imaginary parts of the eigenvalues for real frequencies.

The eigenvalues of $B(\omega^2;0)$ for real ω^2 are all real. The eigenvalues of $B(\omega^2;\epsilon)$ for real ω^2 and for non-zero ϵ need not be real. Our intuition tells us that they are never real regardless of the shape of the body and the value of ϵ except for the special case of Jones' modes. This, however, has yet to be proved.

Suppose that ω^2 is real and that $\lambda(\omega^2;\epsilon)$ is an eigenvalue of $B(\omega^2;\epsilon)$, with $\mathbf{u}(\omega^2;\epsilon)$ the corresponding normalized eigenvector. Define

$$\psi(\omega^2;\epsilon) = A(\omega^2;\epsilon)^{-1}F(\mathbf{u}(\omega^2;\epsilon).\mathbf{n}).$$

$\psi(\omega^2;\epsilon)$ satisfies

$$\nabla^2\psi(\omega^2;\epsilon) = 0 \text{ in } \Omega^a,$$
$$\left.\frac{\partial\psi(\omega^2;\epsilon)}{\partial n}\right|_{\partial\Omega} = \mathbf{u}(\omega^2;\epsilon).\mathbf{n}$$

and

$$\left.\frac{\partial\psi(\omega^2;\epsilon)}{\partial n}\right|_{\partial\Omega_a} = T(\omega^2;\epsilon)\psi(\omega^2;\epsilon)\Big|_{\partial\Omega_a}.$$

$\psi(\omega^2;\epsilon)$ belongs to $H^1(\Omega^a)$. Furthermore, from the definition of $S_3(\omega^2;\epsilon)$, we have

$$\psi(\omega^2;\epsilon)\Big|_{\partial\Omega} = S_3(\omega^2;\epsilon)(\mathbf{u}(\omega^2;\epsilon).\mathbf{n}).$$

Therefore,

$$\int_{\partial\Omega} \overline{\mathbf{u}(\omega^2;\epsilon).\mathbf{n}} S_3(\omega^2;\epsilon)(\mathbf{u}(\omega^2;\epsilon).\mathbf{n})dS = \int_{\Omega^a} \nabla\psi(\omega^2;\epsilon).\nabla\overline{\psi(\omega^2;\epsilon)}dV$$
$$+\int_{\partial\Omega_a} \overline{T(\omega^2;\epsilon)\psi(\omega^2;\epsilon)}\psi(\omega^2;\epsilon)dS.$$

Finally,

$$\Im\int_{\partial\Omega} \overline{\mathbf{u}(\omega^2;\epsilon).\mathbf{n}} S_3(\omega^2;\epsilon)(\mathbf{u}(\omega^2;\epsilon).\mathbf{n})dS = \Im\int_{\partial\Omega_a} \overline{T(\omega^2;\epsilon)\psi(\omega^2;\epsilon)}\psi(\omega^2;\epsilon)dS. \tag{4.116}$$

Clearly,

$$\lambda(\omega^2;\epsilon) = (B(\omega^2;\epsilon)\mathbf{u}(\omega^2;\epsilon),\mathbf{u}(\omega^2;\epsilon))_{H^1(\Omega_i)}. \tag{4.117}$$

Therefore,

$$\Im\lambda(\omega^2;\epsilon) = \Im\rho\omega^2\int_{\partial\Omega} \overline{\mathbf{u}(\omega^2;\epsilon).\mathbf{n}} S_3(\omega^2;\epsilon)(\mathbf{u}(\omega^2;\epsilon).\mathbf{n})dS. \tag{4.118}$$

Equations (4.116) and (4.118) imply

$$\Im\lambda(\omega^2;\epsilon) = \Im\int_{\partial\Omega_a} \overline{T(\omega^2;\epsilon)\psi(\omega^2;\epsilon)}\psi(\omega^2;\epsilon)dS. \tag{4.119}$$

Now suppose that

$$\psi(\omega^2;\epsilon)\Big|_{\partial\Omega_a} = \sum_{m=0}^{\infty}\sum_{n=m}^{\infty}\sum_{s=m}^{\infty}(\alpha_{mn}(\omega^2;\epsilon)\cos m\phi + \beta_{mn}(\omega^2;\epsilon)\sin m\phi) \tag{4.120}$$
$$\times\left(\delta_{ns} - \frac{(-a)^{n+s+1}}{(n-m)!(n+m)!}\int_0^\infty \frac{k+\frac{\omega^2}{g}}{k-\frac{\omega^2}{g}}k^{s+n}\exp(-2k/\epsilon)dk\right.$$
$$\left.-2\pi i\frac{(-a\frac{\omega^2}{g})^{n+s+1}}{(n-m)!(n+m)!}\exp(-2\frac{\omega^2}{g}da/\epsilon)\right)P_s^m(\cos\theta),$$

then

$$
\begin{aligned}
&T(\omega^2;\epsilon)(\psi(\omega^2;\epsilon)\Big|_{\partial\Omega_a}) \qquad (4.121)\\
&= -\textstyle\sum_{m=0}^{\infty}\sum_{n=m}^{\infty}\sum_{s=m}^{\infty}(\alpha_{mn}(\omega^2;\epsilon)\cos m\phi + \beta_{mn}(\omega^2;\epsilon)\sin m\phi)\\
&\quad\times\left(\tfrac{(n+1)}{a}\delta_{ns} - \tfrac{s(-a)^{n+s}}{(n-m)!(n+m)!}\int_0^\infty \frac{k+\frac{\omega^2}{g}}{k-\frac{\omega^2}{g}}k^{s+n}\exp(-2k/\epsilon)dk\right.\\
&\qquad\left. - \tfrac{2\pi i}{a}\tfrac{s(-\frac{\omega^2}{g})^{n+s+1}}{(n-m)!(n+m)!}\exp(-2\tfrac{\omega^2}{g}/\epsilon)\right)P_s^m(\cos\theta).
\end{aligned}
$$

Equations (4.119), (4.120) and (4.121) yield the following

$$
\begin{aligned}
&\Im\lambda(\omega^2;\epsilon) \qquad (4.122)\\
&= 8\pi^2 a\textstyle\sum_{m=0}^{\infty}\sum_{n=m}^{\infty}\sum_{s=m}^{\infty}C_m(\alpha_{mn}(\omega^2;\epsilon)\overline{\alpha_{ms}(\omega^2;\epsilon)} + \beta_{mn}(\omega^2;\epsilon)\overline{\beta_{ms}(\omega^2;\epsilon)})\\
&\qquad\times\tfrac{(-a\frac{\omega^2}{g})^{n+s+1}}{(s-m)!(n-m)!}\exp(-2\tfrac{\omega^2}{g}/\epsilon).
\end{aligned}
$$

Equation (4.122) can be rewritten as

$$
\begin{aligned}
&\Im\lambda(\omega^2;\epsilon) \qquad (4.123)\\
&= -8\pi^2 a^2\omega^2\textstyle\sum_{m=0}^{\infty}C_m\left(\left|\sum_{n=m}^{\infty}\alpha_{mn}(\omega^2;\epsilon)\frac{(-a\frac{\omega^2}{g})^{n+s}}{(n-m)!}\right|^2\right.\\
&\qquad\left.+\left|\textstyle\sum_{n=m}^{\infty}\beta_{mn}(\omega^2;\epsilon)\frac{(-a\frac{\omega^2}{g})^{n+s}}{(n-m)!}\right|^2\right)\\
&\qquad\times\exp(-2\tfrac{\omega^2}{g}/\epsilon).
\end{aligned}
$$

$\alpha_{mn}(\omega^2;\epsilon)$ and $\beta_{mn}(\omega^2;\epsilon)$ are continuous in ϵ in the interval

$$[0,\epsilon_4(K)].$$

To see this recall equation (4.84), with

$$a_{mn}(\omega^2;\epsilon) = \frac{n+1/2}{C_m a^2\pi}\sqrt{\frac{(n-m)!}{(n+m)!}}\int_{\partial\Omega_a} A(\omega^2;\epsilon)^{-1}F(\mathbf{u}(\omega^2;\epsilon).\mathbf{n})P_n^m(\cos\theta)\cos m\phi dS$$

and similarly for $\beta_{mn}(\omega^2;\epsilon)$. The eigenvector $\mathbf{u}(\omega^2;\epsilon)$ is continuous in ϵ and so the result is clear. Equation (4.123) tells us that the imaginary part of any eigenvalue of $B(\omega^2;\epsilon)$, for real ω^2, is non-positive and is bounded by an exponentially small term.

Similarly, for any element of $H^1(\Omega_i)$,

$$\Im(B(\omega^2;\epsilon)\mathbf{u},\mathbf{u})_{H^1(\Omega_i)} = o(\epsilon^p), \tag{4.124}$$

for all powers p when ω^2 is real. Therefore, it is clear that in the expansion of $B(\omega^2;\epsilon)$ in equation (4.107) each coefficient, B_{ij}, is real. This implies that each coefficient in the equivalent expansion for $P(\omega^2;\epsilon)$ is real. This determines, through equation (4.108), that the U_{ij}, the coefficients in the expansion of $U(\omega^2;\epsilon)$, are real. Thus, due to equation (4.105), the coefficients in the expansion of $\tilde{B}(\omega^2;\epsilon)$ in equation (4.109) are all real.

4.5 Scattering frequencies.

Suppose we now look for *scattering frequencies* — i. e. those values of ω^2 for which, for a given value of ϵ, one is an eigenvalue of $B(\omega^2;\epsilon)$ — then we must solve

$$\det(P_0(\omega_0^2)\tilde{B}(\omega^2;\epsilon)P_0(\omega_0^2) - 1) = 0 \tag{4.125}$$

for ω^2. We shall look for the scattering frequencies in a neighbourhood of ω_0^2, where ω_0^2 is such that

$$B(\omega_0^2;0)$$

has eigenvalue 1. Let us call the multiplicity of this eigenvalue m. The holomorphicity of $B(\omega^2;\epsilon)$ and $U(\omega^2;\epsilon)$ with respect to ω^2 for fixed ϵ implies that $\tilde{B}(\omega^2;\epsilon)$ is holomorphic with respect to ω^2 for fixed ϵ. Thus the left hand side of equation (4.125) is holomorphic in ω^2. The left hand side of equation (4.125) is continuous with respect to ϵ uniformly for ω^2 on some simple closed curve, C', that encloses ω_0^2 but no other scattering frequency. Thus

$$
\begin{aligned}
&|\det(P_0(\omega_0^2)\tilde{B}(\omega^2;\epsilon)P_0(\omega_0^2) - 1) - \det(P_0(\omega_0^2)\tilde{B}(\omega^2;0)P_0(\omega_0^2) - 1)| \\
&\quad < |\det(P_0(\omega_0^2)\tilde{B}(\omega^2;0)P_0(\omega_0^2) - 1)|
\end{aligned}
$$

on C'.

Using Rouché's theorem (see e. g. Ahlfors [2, p.152]) equation (4.125) has, counting multiplicty, m solutions. Thus, there are m not necessarily distinct scattering frequencies in the vicinity of ω_0^2.

4.5.1 Uniqueness theorem for frequencies with positive imaginary part.

We wish to prove the following result

Lemma 17 *There are no scattering frequencies with*

$$\Im\omega^2(\epsilon) > 0.$$

Proof: Suppose the lemma were not true. Let $\mathbf{u}(\epsilon)$ be the eigenvector of $B(\omega^2(\epsilon);\epsilon)$ associated with the eigenvalue 1. Define

$$\psi'(\epsilon) = -i\omega(\epsilon)A(\omega^2(\epsilon);\epsilon)^{-1}F(\mathbf{u}(\epsilon).\mathbf{n}).$$

Let $\psi(\epsilon)$ be the function that satisfies Laplace's equation in Ω_e, the free surface condition and the Rellich radiation condition and whose restriction to Ω^a equals $\psi'(\epsilon)$. Lemma 12 guarantees the existence of such a function. We already know that, since the imaginary part of $\omega^2(\epsilon)$ is positive,

$$\psi(\epsilon)|_{FS} \in L^2(FS),$$

and

$$\nabla\psi(\epsilon) \in L^2(\Omega_e). \tag{4.126}$$

Equation (4.126) and the radiation condition imply that

$$\lim_{b\to\infty}\int_{\Sigma_b}\frac{\partial\overline{\psi}(\epsilon)}{\partial n}\psi(\epsilon)dS = 0,$$

where Σ_b is the semi-infinite cylinder of radius b that lies below the free surface. Thus,

$$\int_{\partial\Omega}\frac{\partial\overline{\psi}(\epsilon)}{\partial n}\psi(\epsilon)dS = -\int_{\Omega_e}\nabla\overline{\psi}(\epsilon).\nabla\psi(\epsilon)dV - \frac{\overline{\omega^2(\epsilon)}}{g}\int_{FS}|\psi(\epsilon)|^2dS.$$

$\mathbf{u}(\epsilon)$ satisfies

$$\begin{aligned}0 = \rho\omega^2(\epsilon)\int_{\Omega_i}\overline{\mathbf{u}(\epsilon)}.\mathbf{u}(\epsilon)dV - \int_{\Omega_i}\sigma(\mathbf{u}(\epsilon)) : \nabla\mathbf{u}(\epsilon)dV\\ -\rho_0\omega^2(\epsilon)\textstyle\int_{\partial\Omega} S_3(\omega^2(\epsilon);\epsilon)(\mathbf{u}(\epsilon).\mathbf{n})\overline{\mathbf{u}(\epsilon)}.\mathbf{n}dS.\end{aligned}$$

Finally, we have

$$\begin{aligned}0 = \rho\omega^2(\epsilon)\int_{\Omega_i}\overline{\mathbf{u}(\epsilon)}.\mathbf{u}(\epsilon)dV - \int_{\Omega_i}\sigma(\mathbf{u}(\epsilon)) : \nabla\mathbf{u}(\epsilon)dV\\ +\rho_0\tfrac{\omega^2(\epsilon)}{|\omega^2(\epsilon)|}\textstyle\int_{\Omega_e}\nabla\overline{\psi(\epsilon)}.\nabla\psi(\epsilon)dV + \rho_0\frac{|\omega^2(\epsilon)|}{g}\int_{FS}|\psi(\epsilon)|^2dS.\end{aligned} \tag{4.127}$$

By taking the imaginary part of equation (4.127) we can see that $\mathbf{u}(\epsilon)$ must vanish. This contradicts what was said earlier and so the assumption that there exists a scattering frequency with positive imaginary part must be false. □

Vullierme-Ledard [29] showed that, when the algebraic multiplicity of the scattering frequency is 1 (that is to say, when m equals 1), the scattering frequency has an asymptotic expansion in integer powers of ϵ and all the coefficients in the expansion are real.

We aim to extend this to look at scattering frequencies whose algebraic multiplicity is greater than 1 and to examine the behaviour of the imaginary part of each scattering frequency.

Firstly, let us note that the eigenvalues of $B(\omega^2;0)$ are *semi-simple*, that is to say,

$$(B(\omega^2;0)-\lambda)P(\omega^2)=0, \tag{4.128}$$

where λ is any eigenvalue of $B(\omega^2;0)$ and $P(\omega^2)$ is the projection operator onto the eigenspace associated with this eigenvalue. It has already been noted that $B(\omega^2;0)$ is self-adjoint. So the eigenvectors corresponding to different eigenvalues are orthogonal and by the Gram-Schmidt orthogonalization process the eigenvectors corresponding to the same eigenvalue can be selected to be mutually orthogonal. Therefore, a complete set of eigenvectors can be chosen to be mutually orthogonal and of unit modulus. Call this set

$$\{\mathbf{e}_1,\mathbf{e}_2,\ldots\}.$$

This set spans the whole of $H^1(\Omega_i)$.

So, if an element of $H^1(\Omega_i)$ is

$$\mathbf{u}=\sum_{j=1}^{\infty}u_j\mathbf{e}_j,$$

then

$$B(\omega^2;0)\mathbf{u}=\sum_{j=1}^{\infty}\lambda_j(\omega^2)u_j\mathbf{e}_j.$$

If $\{\mathbf{e}_i,\mathbf{e}_{i+1},\ldots,\mathbf{e}_{i+m-1}\}$ are the m eigenvectors associated with the eigenvalue λ, then the projection operator is given by

$$P(\omega^2)\mathbf{u}=\sum_{j=0}^{m-1}u_{i+j}\mathbf{e}_{i+j}.$$

Thus

$$(B(\omega^2;0)-\lambda)P(\omega^2)\mathbf{u}=(B(\omega^2;0)-\lambda)\sum_{j=0}^{m-1}u_{i+j}\mathbf{e}_{i+j}=0$$

and equation (4.128) is verified.

This leads us to the following result:

Lemma 18 *The scattering frequencies have expansions in ϵ that begin like*

$$\omega^2(\epsilon) = \omega_0^2 + a_1\epsilon + a_2\epsilon^2 + o(\epsilon^2),$$

where a_1 and a_2 are real.

Proof: Begin by writing

$$P_0(\omega_0^2)(\tilde{B}(\omega^2;\epsilon) - 1)P_0(\omega_0^2)$$

as

$$P_0(\omega_0^2)(B_{00} - 1 - (\omega^2 - \omega_0^2)\tilde{B}_{01} + \epsilon\tilde{B}_{10} + \tilde{D}(\omega^2;\epsilon))P_0(\omega_0^2).$$

Equation (4.128) implies that

$$P_0(\omega_0^2)(\tilde{B}(\omega^2;\epsilon) - 1)P_0(\omega_0^2) = P_0(\omega_0^2)((\omega^2 - \omega_0^2)\tilde{B}_{01} + \epsilon\tilde{B}_{10} + \tilde{D}(\omega^2;\epsilon))P_0(\omega_0^2). \tag{4.129}$$

At a scattering frequency,

$$\det(P_0(\omega_0^2)(\tilde{B}(\omega^2;\epsilon) - 1)P_0(\omega_0^2)) = 0.$$

This implies that

$$f(\omega^2;\epsilon) \equiv \det(P_0(\omega_0^2)((\omega^2 - \omega_0^2)\tilde{B}_{01} + \epsilon\tilde{B}_{10} + \tilde{D}(\omega^2;\epsilon))P_0(\omega_0^2)) = 0. \tag{4.130}$$

We already know that there must be m solutions of this equation. For a first approximation, try

$$\omega^2 - \omega_0^2 = x_j\epsilon,$$

where x_j is the jth solution of the polynomial

$$Q(x) \equiv \det(P_0(\omega_0^2)(x\tilde{B}_{01} + \tilde{B}_{10})P_0(\omega_0^2)) = 0. \tag{4.131}$$

Equation (4.131) is equivalent to

$$\det(P_0(\omega_0^2)(xI + \tilde{B}_{01}^{-1}\tilde{B}_{10})P_0(\omega_0^2)) = 0. \tag{4.132}$$

$\tilde{B}_{01}^{-1}$ exists because from equation (4.112) its non-existence would imply the existence of a non-trivial solution to

$$B_{01}\mathbf{u} = 0.$$

From equation (4.114),

$$\rho \int_{\Omega_i} \mathbf{u}.\overline{\mathbf{v}} dV + \rho_0 \int_{\partial\Omega} S(\omega_0^2; 0)(\mathbf{u}.\mathbf{n})\overline{\mathbf{v}}.\mathbf{n} dS = 0,$$

for all $\mathbf{v}$ in $H^1(\Omega_i)$ and for $\mathbf{v}$=$\mathbf{u}$ in particular. Finally, we know that the second integral never vanishes when $\mathbf{v}$ is $\mathbf{u}$ and, therefore, $\mathbf{u}$ must be zero. This contradicts what was said before and so the initial asumption that $\tilde{B}_{01}^{-1}$ does not exist must be false.

Equations (4.114) and (4.115) imply that

$$\tilde{B}_{01}^{-1}\tilde{B}_{10}$$

is self-adjoint. Therefore, equation (4.132) has m real solutions — we have called these $\{x_1, \ldots, x_m\}$.

Again, since

$$\tilde{B}_{01}^{-1}\tilde{B}_{10}$$

is self-adjoint its eigenvalues must be at worst semi-simple. Let P' be the projection operator onto the eigenspace spanned by one of the eigenvalues and suppose the dimension of this space is p. Near this particular eigenvalue — call it x_j — equation (4.130) becomes

$$\det(P'P_0(\omega_0^2)((\omega^2 - \omega_0^2 - x_j\epsilon)\tilde{B}_{01} + \tilde{D}(\omega^2; \epsilon))P_0(\omega_0^2)P') = 0. \tag{4.133}$$

Now write $\tilde{D}(\omega^2;\epsilon)$ as

$$\epsilon^2\tilde{B}_{20} + \epsilon(\omega^2 - \omega_0^2)\tilde{B}_{11} + \tilde{E}(\omega^2;\epsilon),$$

and write this as

$$\epsilon^2(\tilde{B}_{20} + x_j\tilde{B}_{11}) + \epsilon(\omega^2 - \omega_0^2 - x_j\epsilon)\tilde{B}_{11} + \tilde{E}(\omega^2;\epsilon).$$

Define the new variable

$$\zeta = \frac{\omega^2 - \omega_0^2 - x_j\epsilon}{\epsilon^2}.$$

Equation (4.133) becomes

$$\det(P'P_0(\omega_0^2)(\zeta\tilde{B}_{01} + \tilde{B}_{20} + x_j\tilde{B}_{11} + F(\zeta;\epsilon))P_0(\omega_0^2)P') = 0, \tag{4.134}$$

with obvious notation. Denote by $g(\zeta;\epsilon)$ the left hand side of equation (4.134).

We have

$$g(\zeta;0) = \det(P'P_0(\omega_0^2)(\zeta\tilde{B}_{01} + \tilde{B}_{20} + x_j\tilde{B}_{11})P_0(\omega_0^2)P').$$

This has zeros at $\{y_1,\ldots,y_p\}$, say. Let C be a simple positively orientated curve enclosing just one of these points, y_k, say. Now,

$$\inf|g(\zeta;0)| = s > 0,$$

where the infimum is taken over all points ζ on C. Furthermore,

$$\sup|g(\zeta,\epsilon) - g(\zeta;0)| < s,$$

for all ϵ smaller than some suitably small positive number. So, by Rouché's Theorem, $g(\zeta;\epsilon)$ has the same number of zeros inside C as $g(\zeta;0)$. The smaller we take ϵ to be, the smaller the curve can be allowed to be. Consequently, the kth zero of $g(\zeta;\epsilon)$ is

$$y_k + o(1).$$

Finally, each y_k is real. To see this we note that there must be a $\mathbf{u}$ satisfying

$$(y_k\tilde{B}_{01} + \tilde{B}_{20} + x_j\tilde{B}_{11})\mathbf{u} = \mathbf{0}.$$

So,

$$y_k = -\frac{((\tilde{B}_{20} + x_j\tilde{B}_{11})\mathbf{u}, \mathbf{u})_{H^1(\Omega_i)}}{(\tilde{B}_{01}\mathbf{u}, \mathbf{u})_{H^1(\Omega_i)}}.$$

We know that each of these terms is positive.

We have

$$\omega^2(\epsilon) = \omega_0^2 + x_j\epsilon + y_k\epsilon^2 + o(\epsilon^2)$$

and we know that x_j and y_k are real. So the lemma is proved. □

We cannot go any further with the expansion because we cannot use the reduction process any more. That is to say, the eigenvalues of

$$y_k\tilde{B}_{01} + \tilde{B}_{20} + x_j\tilde{B}_{11}$$

need not be semi-simple.

If the eigenvalue we started with had been simple, we could have proceeded indefinitely and re-captured Vullierme-Ledard's result.

4.5.2 The imaginary parts of the scattering frequencies.

We now wish to investigate the imaginary part of the scattering frequencies. The scattering frequency is a solution of

$$\lambda(\omega^2; \epsilon) = 0,$$

where $\lambda(\omega^2; \epsilon)$ is the eigenvalue of $B(\omega^2; \epsilon)$. We adapt a proof from Harrell and xSimon [12] to prove that the imaginary part of each scattering frequency is

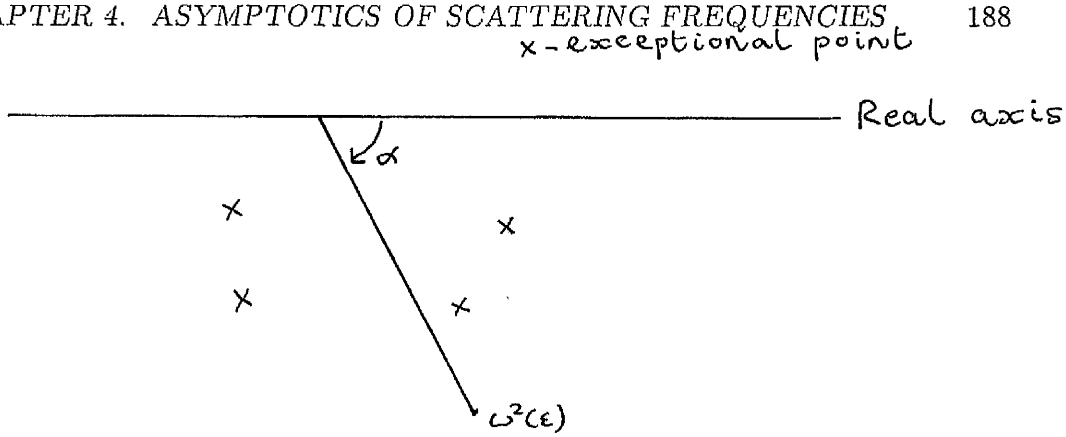

Figure 4.7: Line joining $\omega^2(\epsilon)$ to real axis.

bounded by an exponentially small term. We are interested in showing that it is non-zero but, unfortunately, this has not been done completely.

$\lambda(\omega^2;\epsilon)$ is a root of

$$\det(\tilde{B}(\omega^2;\epsilon) - \lambda I) = 0.$$

For fixed ϵ, $\lambda(\omega^2;\epsilon)$ is holomorphic with respect to ω^2 except at a discrete set of *exceptional points.* In any compact domain, there is only a finite number of exceptional points. This is found in Kato [14, p. 64]. So we can draw a straight line from the point $\omega^2(\epsilon)$ to the real axis that avoids every singularity. Suppose this line makes an angle α with the real axis, as indicated in the Figure (4.7).

Denote by $\Upsilon_\epsilon(x)$ the function

$$\Im\lambda(\Re\omega^2(\epsilon) - x\cos\alpha + i\Im\omega^2(\epsilon) - ix\sin\alpha;\epsilon).$$

We shall use the mean value theorem on $\Upsilon_\epsilon(x)$ to find $\Im\omega^2(\epsilon)$ in terms of the value taken by $\Upsilon_\epsilon(x)$ at the endpoints of its range and in terms of its derivative with respect to x somewhere in the range. We then find the leading order behaviour of this derivative as ϵ tends to zero.

Clearly,

$$\Upsilon_\epsilon(0) = 0.$$

Equation (4.123) implies that

$$\begin{aligned}\Upsilon_\epsilon\left(\frac{\Im\omega^2(\epsilon)}{\sin\alpha}\right) &= \Im\lambda(\Re\omega^2(\epsilon) - \Im\omega^2(\epsilon)\cot\alpha;\epsilon) \\ &= f(\Re\omega^2(\epsilon) - \Im\omega^2(\epsilon)\cot\alpha;\epsilon)\exp(-2(\Re\omega^2(\epsilon) - \Im\omega^2(\epsilon)\cot\alpha)/\epsilon),\end{aligned} \tag{4.135}$$

where

$$\begin{aligned}&f(\Re\omega^2(\epsilon) - \Im\omega^2(\epsilon)\cot\alpha;\epsilon) \\ &= -8\pi^2 a^2\omega^2 \textstyle\sum_{m=0}^{\infty} C_m \left(\left|\sum_{n=m}^{\infty} \alpha_{mn}\frac{(-a\frac{\omega^2}{g})^{n+s}}{(n-m)!}\right|^2 + \left|\sum_{n=m}^{\infty} \beta_{mn}\frac{(-a\frac{\omega^2}{g})^{n+s}}{(n-m)!}\right|^2\right).\end{aligned}$$

From the holomorphicity of $\lambda(\omega^2;\epsilon)$ at all points on the line and the mean value theorem, we have

$$\begin{aligned}\left.\frac{\partial\Upsilon_\epsilon}{\partial x}\right|_c \left(\frac{\Im\omega^2(\epsilon)}{\sin\alpha} - 0\right) &= \Upsilon_\epsilon\left(\frac{\Im\omega^2(\epsilon)}{\sin\alpha}\right) - 0 \\ &= f(\Re\omega^2(\epsilon) - \Im\omega^2(\epsilon)\cot\alpha;\epsilon)\exp(-2(\Re\omega^2(\epsilon) - \Im\omega^2(\epsilon)\cot\alpha)/\epsilon),\end{aligned} \tag{4.136}$$

where c lies in the interval

$$[0, \frac{\Im\omega^2(\epsilon)}{\sin\alpha}].$$

We have

$$\left.\frac{\partial\Upsilon_\epsilon}{\partial x}\right|_c = \sin\alpha \left.\frac{\partial\Im\lambda(z;\epsilon)}{\partial\Im z}\right|_{c'},$$

where

$$c' = \Re\omega^2(\epsilon) - c\cos\alpha + i\Im\omega^2(\epsilon) - ic\sin\alpha.$$

Since the line contains no exceptional points, $\lambda(z;\epsilon)$ must have a Taylor expansion around every point. This implies that $\Upsilon_\epsilon(x)$ is twice differentiable at every point on the line. Therefore,

$$\left.\frac{\partial \Upsilon_\epsilon}{\partial x}\right|_c = \left.\frac{\partial \Upsilon_\epsilon}{\partial x}\right|_{\frac{\Im\omega^2(\epsilon)}{\sin\alpha}} + o(1).$$

We have

$$\left.\frac{\partial \Upsilon_\epsilon}{\partial x}\right|_{\frac{\Im\omega^2(\epsilon)}{\sin\alpha}} = \sin\alpha \left.\frac{\partial \Im\lambda(z;\epsilon)}{\partial \Im z}\right|_{c''},$$

where

$$c'' = \Re\omega^2(\epsilon) - \Im\omega^2(\epsilon)\cot\alpha.$$

So, by this and the Cauchy-Riemann equations, equation (4.136) becomes

$$\begin{aligned}\left.\frac{\partial \Re\lambda(z;\epsilon)}{\partial \Re z}\right|_{c''} \Im\omega^2(\epsilon) &= f(\Re\omega^2(\epsilon) - \Im\omega^2(\epsilon)\cot\alpha;\epsilon) \\ &\times \exp(-2(\Re\omega^2(\epsilon) - \Im\omega^2(\epsilon)\cot\alpha)/\epsilon)(1+o(1)).\end{aligned} \tag{4.137}$$

Clearly,

$$\frac{\partial \Re\lambda(z;\epsilon)}{\partial \Re z} = \frac{\partial\lambda(z;\epsilon)}{\partial z} - i\frac{\partial \Im\lambda(z;\epsilon)}{\partial \Re z}.$$

Equation (4.124) implies that

$$\left.\frac{\partial \Im\lambda(z;\epsilon)}{\partial \Re z}\right|_{c''} = o(\epsilon^p),$$

for all powers p. $\lambda(z;\epsilon)$ can be expanded around the point $\epsilon = 0$:

$$\lambda(z;\epsilon) = \lambda(z;0) + o(1).$$

This expansion is uniform in z and, therefore,

$$\left.\frac{\partial\lambda(z;\epsilon)}{\partial z}\right|_{c''} = \left.\frac{\partial\lambda(z;0)}{\partial z}\right|_{c''} + o(1).$$

Furthermore, since

$$\begin{aligned}&P_0(\omega_0^2)(\tilde{B}(z;0) - 1)P_0(\omega_0^2) \\ &\qquad = P_0(\omega_0^2)(\tilde{B}(z;0) - \tilde{B}(\omega_0^2;0))P_0(\omega_0^2) \\ &\qquad \equiv (z-\omega_0^2)P_0(\omega_0^2)\tilde{B}_{01}P_0(\omega_0^2),\end{aligned}$$

(recall that, since $\tilde{B}(\omega_0^2;0)$ is self-adjoint,

$$P_0(\omega_0^2)(\tilde{B}(\omega_0^2;0)-1)P_0(\omega_0^2)=0)$$

then

$$\lambda(z;0)=1+x(z-\omega_0^2),$$

for some constant x — the eigenvalue of $P_0(\omega_0^2)\tilde{B}_{01}P_0(\omega_0^2)$ — and so

$$\left.\frac{\partial\lambda(z;0)}{\partial z}\right|_{c''}=\left.\frac{\partial\lambda(z;0)}{\partial z}\right|_{\omega_0^2}=x.$$

(Recall the linear dependence of $\lambda(z;0)$ on z.) Thus, we have,

$$\left.\frac{\partial\Re\lambda(z;\epsilon)}{\partial\Re z}\right|_{c'}=x(1+o(1)). \tag{4.138}$$

Equations (4.137) and (4.138) imply that to leading order

$$\Im\omega^2(\epsilon)=\frac{f(\omega_0^2;0)}{x}\exp(-2a_1/g)\exp(-2\frac{\omega_0^2}{g}/\epsilon), \tag{4.139}$$

where a_1 is the order one term in the expansion of $\omega^2(\epsilon)$, which, according to Lemma 18, has the form:

$$\Re\omega^2(\epsilon)=\omega_0^2+a_1\epsilon+o(\epsilon).$$

We note that $f(\omega_0^2;0)$ is never positive. This is consistent with the fact that no scattering frequencies lie in the upper half plain. If $f(\omega_0^2;0)$ is non-zero, then, of course, the scattering frequency has a non-zero imaginary part. Even if $f(\omega_0^2;0)$ vanishes it seems likely that a higher order term of $\Im\omega^2(\epsilon)$ does not vanish; this appears to be difficult to prove, however.

Appendix A

Jones' modes and Jones' frequencies.

A.1 What is a Jones' mode?

Let Ω be an open, compact domain of non-zero measure. A Jones' mode — named after Professor Douglas Jones, who emphasized their importance — is a non-trivial solution of the equation

$$\nabla.\sigma(\mathbf{u}) + k^2\mathbf{u} = 0$$

that satisfies

$$\mathbf{u}.\mathbf{n}|_{\partial\Omega} = 0$$

and

$$\sigma(\mathbf{u}).\mathbf{n}|_{\partial\Omega} = \mathbf{0},$$

where $\partial\Omega$ denotes the surface of Ω and $\mathbf{n}$ denotes the outward pointing normal to $\partial\Omega$. k is called a Jones' frequency.

A.2 Examples of Jones' modes.

A.2.1 The cylinder.

Suppose that we look for a Jones' mode in a cylinder of radius a. We write $\mathbf{u}$ in the form

$$\mathbf{u} = \nabla \times (\Psi \mathbf{e}_z),$$

where Ψ is a function that depends only on the radial variable R and $\mathbf{e}_z$ is a unit vector parallel to the axis of the cylinder. It is not difficult to show that Ψ must satisfy Helmholtz's equation

$$\nabla^2 \Psi + k_s^2 \Psi = 0,$$

where

$$k_s = \frac{k}{\mu}.$$

Within a multiplicative constant the only regular solution is

$$\Psi(R) = J_0(k_s R),$$

where $J_0(z)$ represents the zeroth Bessel function. Thus,

$$\mathbf{u} = -k_s J_0'(k_s R)\mathbf{e}_\phi,$$

where $\mathbf{e}_\phi$ denotes the angular unit vector. Immediately we see that the normal component of dispacement vanishes on $\partial\Omega$.

The surface traction on $\partial\Omega$ is given by

$$\mu(-k_s^2 J_0''(k_s a) + \frac{k_s}{a} J_0'(k_s a))\mathbf{e}_\phi.$$

This vanishes if and only if

$$2J_1(k_s a) = k_s a J_0(k_s a)$$

whre $J_1(z)$ denotes the first Bessel function. Since the first and second Bessel functions have interlacing zeros this equation has infinitely many solutions.

A.2.2 The sphere.

Consider a sphere of radius a. Let (r, θ, ϕ) represent a spherical coordinate system. Let us look for a Jones' mode of the form

$$\mathbf{u} = \nabla \times (\Psi(\mathbf{e}_r \cos\theta - \mathbf{e}_\theta \sin\theta)),$$

where Ψ is a function of r only.

As with the cylinder, Ψ satisfies Helmholtz's equation. Thus,

$$\Psi = j_0(k_s r),$$

where $j_0(z)$ denotes the zeroth spherical Bessel function. Therefore,

$$\mathbf{u} = k_s j_0'(k_s r) \sin\theta \mathbf{e}_\phi.$$

The normal component of displacement automatically vanishes on $\partial\Omega$. The surface traction is given by

$$\mu(k_s^2 j_0''(k_s a) - \frac{k_s}{a} j_0'(k_s a)) \sin\theta \mathbf{e}_\phi.$$

A neccessary and sufficient condition for this to vanish is that

$$3j_1(k_s a) = k_s a j_0(k_s a),$$

where $j_1(z)$ denotes the first spherical Bessel function. There are infinitely many values of k_s for which this is true.

A.3 Bodies of rotation.

Suppose that Ω is the body formed by rotating the two dimensional body S about some axis. Let the boundary of S consist of a finite number of smooth pieces joined at non-zero angles. Let (R, ϕ, z) denote the usual cylindrical polar coordinates and let us look for a Jones' mode of the form

$$\mathbf{u} = u\mathbf{e}_\phi, \tag{A.1}$$

where u depends only on R and z. Let us stipulate that u must vanish on the axis of rotation.

The condition of vanishing normal component of displacement on the surface is automatically satisfied. The surface traction is given by

$$\mu\left(\frac{\partial u}{\partial R}\mathbf{e}_R + \frac{\partial u}{\partial z}\mathbf{e}_z\right).\mathbf{n}\mathbf{e}_\phi.$$

The equation satisfied by u in the interior is

$$\frac{\partial^2 u}{\partial R^2} + \frac{2}{R}\frac{\partial u}{\partial R} + \frac{\partial^2 u}{\partial z^2} + k_s^2 u = 0. \tag{A.2}$$

Let us define the Hilbert space V by supposing that a distribution u belongs to V if

$$\int_S R|u|^2 dS$$

exists. The inner product between two elements u and v of this space is

$$\int_S RuvdS.$$

The Hilbert space H is the space of those distributions belonging to V for which

$$\int_S R\left(\left|\frac{\partial u}{\partial R}\right|^2 + \left|\frac{\partial u}{\partial z}\right|^2\right) dS$$

exists. The inner product of u and v is given by

$$\int_S RuvdS + \int_S R\left(\frac{\partial u}{\partial R}\frac{\partial v}{\partial R} + \frac{\partial u}{\partial z}\frac{\partial v}{\partial z}\right) dS.$$

We have

$$H \subset V \equiv V' \subset H'.$$

Furthermore, the imbeddings are compact. To see this let us consider a sequence $\{u_i\}$ in H with

$$\| u_i \|_H = 1 \text{ for } i \in \mathcal{N}.$$

Let

$$\mathbf{u}_i = u_i \mathbf{e}_\phi,$$

where each u_i is now considered as a function in Ω that does not depend on ϕ. Clearly,

$$\| \mathbf{u}_i \|_{H^1(\Omega)} = 2\pi \| u_i \|_H$$

and

$$\| \mathbf{u}_i \|_{L^2(\Omega)} = 2\pi \| u_i \|_V .$$

From the compact imbedding from $H^1(\Omega)$ into $L^2(\Omega)$ (see, for example, Sanchez-Palencia and Sanchez-Hubert [27, Chapter 1]) there exists a convergent subsequence of $\{\mathbf{u}_i\}$ in $L^2(\Omega)$. Consequently, there exists a convergent subsequence of $\{u_i\}$ in V. Thus, the imbedding from H to V is compact. From this, one can show that the imbedding from V to H' is compact.

Let us take the function u in equation (A.1) and consider it as a function in S. From equation (A.2), we have

$$0 = \int_S Rv\left(\frac{\partial^2 u}{\partial R^2} + \frac{2}{R}\frac{\partial u}{\partial R} + \frac{\partial^2 u}{\partial z^2} + k_s^2 u\right) dS,$$

where v is any smooth function in S. By integrating by parts we can see that

$$(u, v)_H = (k_s^2 + 1)(u, v)_V. \tag{A.3}$$

It is well known that there is a countably infinite number of values of k_s for which there is a non-trivial solution to equation (A.3). These solutions are, in fact, smooth within S. Thus a large class of rotationally symmetric bodies can support Jones' modes.

A.4 Thierry Hargé's work.

We might guess that a body that has no axis of symmetry cannot support a Jones' mode. It seems likely that Jones' modes are always of the form in equation (A.1). That is to say, they are always a torsional mode. This is pure speculation. We do know, though, that, in a sense, the class of smooth bodies having a Jones' frequency in any finite range is infinitely rare. This was proved in a paper by Thierry Hargé. The author's own translation of this work is included here as the work does not seem to appear in English anywhere else. We note that what we have called a Jones' frequency Hargé calls an exceptional eigenvalue.

FREE OSCILLATIONS OF AN ELASTIC BODY.

Preliminaries Denote by L the usual elasticity operator, that is to say

$$L\mathbf{u} = (\lambda + \mu)\underline{\nabla}(\underline{\nabla}.\mathbf{u}) + \mu\nabla^2\mathbf{u}, \tag{A.4}$$

where λ and μ are the usual Lamé constants. Let Ω be an open, bounded and simply conected subset of $\mathcal{R}^3$ that has a "smooth" boundary. Define an operator $A(\Omega)$ defined on a domain $D(A(\Omega)) = \{\mathbf{u} \in H^2(\Omega); B(\Omega)\mathbf{u} = \mathbf{0}\}$ by

$$A(\Omega)\mathbf{u} = -L\mathbf{u}. \tag{A.5}$$

B is the surface traction operator and is given by

$$B(\Omega)\mathbf{u} = \lambda \underline{\nabla}.\mathbf{u}\mathbf{n} \mid_{\partial\Omega} + \mu(\underline{\nabla}\mathbf{u} + (\underline{\nabla}\mathbf{u})^T).\mathbf{n} \mid_{\partial\Omega},$$

where $\mathbf{n}$ is the outward pointing normal to the surface $\partial\Omega$.

If considered as an operator from $L^2(\Omega) \to L^2(\Omega)$, $A(\Omega)$ is unbounded, self-adjoint, positive and anti-compact (i. e. its resolvent is compact).

We call eigenvalue ω of $A(\Omega)$ *exceptional* if the associated eigenfunction $\mathbf{u}$ is such that $\mathbf{u}.\mathbf{n} \mid_{\partial\Omega} = 0$.

Fix Ω and let $E = \{\phi \in C^\infty(\overline{\Omega}, \mathcal{R}^3)\}$. Let Ω_ϕ be the open domain of $\mathcal{R}^3$ that consists of points $\mathbf{x}$ with $\mathbf{x} = \phi(\xi)$, where $\xi \in \Omega$.

Theorem 13 *There exists a countable ensemble of open subsets of E, $\{G_n\}_{n\in\mathcal{N}}$, with each G_n dense in E and with $G_{n+1} \subseteq G_n$, such that for any $\phi \in G_n$, $A(\Omega_\phi)$ does not have an exceptional eigenvalue in the range $[0, n]$. Consequently, if we restrict ourselves to a compact region of $\mathcal{R}_+$, then almost every body with a boundary of class C^∞ will not have an exceptional eigenvalue in this region.*

Proof: We call A_ϕ the unbounded operator on $L^2(\Omega)$ defined in the domain

$$D(A_\phi) = \{\mathbf{u} \in H^2(\Omega); B(\Omega_\phi)(\phi_*\mathbf{u}) = \mathbf{0}\}$$

by

$$A_\phi(\mathbf{u}) = \phi^*(A(\Omega_\phi)(\phi_*\mathbf{u})) \tag{A.6}$$

In this notation, if $\mathbf{u}$ is a vector field defined in Ω, then $\phi_*\mathbf{u}$ is the vector field defined in Ω_ϕ that is given by

$$(\phi_*\mathbf{u})(\mathbf{x}) = \mathbf{u}(\xi), \tag{A.7}$$

where $\mathbf{x} = \phi(\xi)$.

Similarly,

$$(\phi^* \mathbf{u})(\xi) = \mathbf{u}(\mathbf{x}). \tag{A.8}$$

A_ϕ is self-adjoint for the scalar product

$$L^2(\Omega)(f,g)_\phi \equiv (\phi_* f, \phi_* g)_{L^2(\Omega_\phi)}. \tag{A.9}$$

A_ϕ and $A(\Omega_\phi)$ have identical spectra and for every exceptional eigenvalue of $A(\Omega_\phi)$ there exists an eigenfunction $\mathbf{u}$ of A_ϕ such that

$$\mathbf{u}.({}^t d\phi)^{-1}(\mathbf{n}) \mid_{\partial\Omega} = 0, \tag{A.10}$$

where $({}^t d\phi)^{-1}$ denotes the transpose of

$$(d\phi)^{-1} = \left(\frac{\partial \psi_i}{\partial x_j} \right).$$

If we define $B_\phi(\mathbf{u}) \equiv \phi^* B(\Omega_\phi)(\phi_* \mathbf{u})$, then B_ϕ is a first order linear differential operator whose coefficients depend linearly on $(d\phi)^{-1}$.

For simplicity, we write A_I and B_I as A and B respectively. Let J be a lifting operator from $H^{\frac{1}{2}}(\partial\Omega)$ into $H^2(\Omega)$, such that $B.J = I$.

For any $\mathbf{g} \in L^2(\Omega)$ and $\lambda \in C$ the equation

$$(A_\phi - \lambda)\mathbf{u} = \mathbf{g} \tag{A.11}$$

is equivalent to the equation for $(\mathbf{u}, \mathbf{v}) \in D(A_\phi) \otimes D(A)$

$$\begin{aligned} \mathbf{v} = &\ \mathbf{u} - J((B - B_\phi)\mathbf{u}) \\ (A - \lambda)\mathbf{v} = &\ \mathbf{g} + (A - A_\phi)\mathbf{v} - (A_\phi - \lambda)J(B - B_\phi)\mathbf{u}, \end{aligned} \tag{A.12}$$

which shows that the spectrum of A_ϕ depends continuously on $\phi \in E$.

An eigenvalue is said to be *stable* if there exists a neighbourhood U of ω_0 and a neighbourhood V of I in E such that, for all $\phi \in V$, the operator A_ϕ has only one eigenvalue in U.

Simple eigenvalues are, of course, stable.

A function $\psi \in C^\infty(\overline{\Omega}; \mathcal{R}^3)$ is chosen and $I + s\psi$ is called ϕ_s, where $\mid s \mid$ is small. We denote $\frac{dA_{\phi_s}}{ds} \mid_{s=0}$ and $\frac{dB_{\phi_s}}{ds} \mid_{s=0}$ as $\dot{A}$ and $\dot{B}$ respectively. Then $\dot{A}$ (resp. $\dot{B}$) depends linearly on ψ' and ψ'' (resp. ψ'). As a result of equation (A.12), we have the following lemma.

Lemma 19 *Let ω_0 be stable eigenvalue of A, $\omega(s)$ the corresponding eigenvalue of A_{ϕ_s}, and $F(s)$ the associated space of eigenfunctions. Then $\omega(s)$ and $F(s)$ are analytic in s for small $|s|$ and if $\mathbf{u}(s) \in F(s)$ belongs to the class C^1 we have*

$$p\dot{\mathbf{u}}(0) = (A-\omega_0)^{-1}[-p\dot{A}\mathbf{u}(0)] - pJ\dot{B}\mathbf{u}(0) + (A-\omega_0)^{-1}[p(A-\omega_0)J\dot{B}\mathbf{u}(0)], \quad \text{(A.13)}$$

where p the orthogonal projection operator in $L^2(\Omega_s; \mathcal{R}^3)$ of the space $F(0)$.

Lemma 20 *Under the hypotheses of Lemma 19, a function ψ can be chosen so that, for all $\mathbf{u}(s)$ $F(s)$ with $\mathbf{u}(0) \neq 0$,*

$$\frac{d}{ds}[\mathbf{u}(s).({}^t d\phi_s)^{-1}(\mathbf{n})]_{s=0}\Big|_{\partial\Omega} \not\equiv 0. \quad \text{(A.14)}$$

Proof: Suppose $\mathbf{x}_0 \in \partial\Omega$; we construct ψ in a small neighbourhood of $\mathbf{x}_0$. Let $d = \dim F(0)$, and let $\{\mathbf{u}_j\}$, for $1 \leq j \leq d$, be a base of $F(0)$. $\{v_j\}$, for $1 \leq j \leq l \leq d$, is a base of the space spanned by $\{\mathbf{u}_j.\mathbf{n} \mid_{\partial\Omega}\}$. There exists a neighbourhood W of $\mathbf{x_0}$ in $\partial\Omega$ in which the vectors $\{v_j \mid_{\partial\Omega\backslash W}\}_{1\leq j\leq l}$ are linearly independent. By the theorem of Holmgren, the vectors $\{\mathbf{u}_j \mid_W\}_{1\leq j\leq d}$ are linearly independent. There then exists d points, $\mathbf{x}_1, \ldots, \mathbf{x}_d$, in W and d vectors in $\mathcal{R}^3$, $\mathbf{a}_1, \ldots, \mathbf{a}_d$, such that the matrix $(\mathbf{u}_j(\mathbf{x}_k).\mathbf{a}_k)_{j,k}$ is invertible. Without loss of generality, we can suppose that $\mathbf{a}_k.\mathbf{n}(\mathbf{x}_k) \neq 0$. Let $\nu \in (0, \frac{1}{2})$ and let $\epsilon > 0$ be small. Define

$$\psi_\epsilon(\mathbf{x}) = \sum_{k=1}^{d} \theta_k(\mathbf{x})\epsilon^{2+\nu} \sin\left(\frac{\mathbf{a}_k.\mathbf{x}}{\epsilon}\right) \mathbf{n}(\mathbf{x}_k), \tag{A.15}$$

where the $\theta_k \in C_0^\infty(\mathcal{R}^3)$ have mutually disjoint supports contained in W and $\theta_k(\mathbf{x}_k) = 1$ for each $1 \leq k \leq d$. If $\{\mathbf{u}_{j,\epsilon}(s)\}_{1\leq j \leq d}$ is a base of $F(s)$ such that $\mathbf{u}_j(0) = \mathbf{u}_j$ and $\mathbf{u}(s) = \sum_{j=1}^{d} \alpha_{j,\epsilon}(s)\mathbf{u}_{j,\epsilon}(s)$, where $\alpha_j \in C^1$, we have

$$\begin{aligned} &\frac{d[\mathbf{u}(s).({}^t d\phi_s)^{-1}(\mathbf{n})]_{s=0}}{ds} \mid_{\partial\Omega}= \\ &\sum \dot{\alpha}_{j,\epsilon}(0)\mathbf{u}_j.\mathbf{n} \mid_{\partial\Omega} + \sum \alpha_{j,\epsilon}(0)[\dot{\mathbf{u}}_{j,\epsilon}.\mathbf{n} - \mathbf{u}_j.{}^t d\psi_\epsilon(\mathbf{n})] \mid_{\partial\boldsymbol{\Omega}} . \end{aligned} \tag{A.16}$$

It suffices to verify that, for ϵ sufficiently small,

$$\sum \beta_{j,\epsilon}^2 + \sum \alpha_{j,\epsilon}^2 = 1$$

and

$$\sum \beta_{j,\epsilon} v_j + \sum \alpha_{j,\epsilon}[p(\dot{\mathbf{u}}_{j,\epsilon}).\mathbf{n} - \mathbf{u}_j.{}^t d\psi_\epsilon.\mathbf{n}] \mid_{\partial\Omega}= 0$$

cannot both be true.

Clearly, $\lim_{\epsilon\to 0} \psi_\epsilon = 0$ in $H^2(\mathcal{R}^3)$, so, because of Lemma 19,

$$\lim_{\epsilon\to 0} p(\dot{\mathbf{u}}_{j,\epsilon}), \mathbf{n} \mid_{\partial\Omega}= 0$$

in $H^{\frac{3}{2}}(\partial\Omega)$. Since the vectors $\{v_j \mid_{\partial\Omega\setminus W}\}$ are linearly independent and

$$\sum \beta_{j,\epsilon} v_j \mid_{\partial\Omega\setminus W}= 0,$$

$\beta_{j,\epsilon} = 0$.

Therefore,

$$\lim_{\epsilon\to 0} \sum \alpha_{j,\epsilon}\mathbf{u}_j.{}^t d\psi_\epsilon(\mathbf{n}) \mid_{\partial\Omega}= 0 \tag{A.17}$$

in $H^{\frac{3}{2}}(\partial\Omega)$. So, for all k

$$0 = \lim_{\epsilon \to 0} \theta_k(\mathbf{x})\epsilon^{1+\nu} \cos\left(\frac{\mathbf{a}_k \mathbf{x}}{\epsilon}\right) \mathbf{a}_k . \sum \alpha_{j,\epsilon} \mathbf{u}_j(\mathbf{x})\mathbf{n}(\mathbf{x}_k).\mathbf{n}(\mathbf{x}) \mid_{\partial\Omega} \tag{A.18}$$

in $H^{\frac{3}{2}}(\partial\Omega)$.This implies that

$$\lim_{\epsilon \to 0} \mathbf{a}_k . \sum \alpha_{j,\epsilon} \mathbf{u}_j(\mathbf{x}_k) = 0. \tag{A.19}$$

It must then be the case that $\lim_{\epsilon \to 0} \alpha_{j,\epsilon} = 0$.

The spectrum of $A(\Omega_\phi)$ is continuous in ϕ and so G_n must be open.

The proof that G_n is dense in E.

Let $F^\phi_{[0,n]}$ be the vector space generated by the eigenvectors of $A(\Omega_\phi)$ associated with an eigenvalue in the range $[0, n]$. Let $\hat{F}^\phi_\lambda$ be the set of $\mathbf{u}.\mathbf{n} \mid_{\partial\Omega}$, where $\mathbf{u}$ runs through the space spanned by the eigenvectors of $A(\Omega_\phi)$ associated with λ. Let U be any open set of E.

Let U_1 be the open subset of U defined as

$$U_1 = \{\phi \in U; \dim F^\phi_{[0,n]} \text{ is minimal } \}.$$

Let U_2 be the open subset of U_1 in which the number of distinct eigenvalues is maximal. In U_2 the eigenvalues are then stable. Finally, let U_3 be the open subset of U_2 consisting of elements, ϕ, for which

$$\sum_{\lambda \in [0,n]} \dim \hat{F}^\phi_\lambda$$

is maximal.

Then according to Lemma 20,

$$\sum_{\lambda \in [0,n]} \dim \hat{F}^\phi_\lambda = \dim F^\phi_{[0,n]}$$

for $\phi \in U_3$. Thus, $U_3 \subset G_n$ and G_n is dense.

Appendix B

Some proofs from Chapter 4.

Let

$$\psi(\mathbf{x}) = \psi_1(\mathbf{x}) + \psi_2(\mathbf{x}),$$

where

$$\psi_1 = \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} a^{n+1} \alpha_{mn} \cos m\phi \left(\frac{(n-m)!}{(n+m)!} \right)^{1/2} \frac{P_n^m(\cos\theta)}{r^{n+1}},$$

and

$$\psi_2 = \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} a^{n+1} \alpha_{mn} \frac{(-1)^{m+n+1}}{((n+m)!(n-m)!)^{1/2}} \times \int_0^\infty \tfrac{k+K}{k-K} k^n \exp(-k(y+2/\epsilon)) J_m(kR) dk.$$

Suppose that $\Im K \neq 0$ and that a is smaller than $1/(2\epsilon)$.

The α_{mn}'s are constants and

$$\sum_{m=0}^{\infty} \sum_{n=m}^{\infty} |\alpha_{mn}|^2$$

exists.

We want to show that

$$(1 + r^2)^{-1/2}\psi(\mathbf{x}) \in L^2(\Omega^a), \tag{B.1}$$

$$\nabla\psi(\mathbf{x}) \in (L^2(\Omega^a))^3 \tag{B.2}$$

and

$$\psi(\mathbf{x})|_{FS} \in L^2(FS). \tag{B.3}$$

B.1 Proof of (B.1)

Evidently,

$$|\psi(\mathbf{x})|^2 \leq 2|\psi_1(\mathbf{x})|^2 + 2|\psi_2(\mathbf{x})|^2.$$

Clearly,

$$\int_{\Omega^a} (1 + r^2)^{-1}|\psi_1(\mathbf{x})|^2 dV$$

exists if and only if

$$\int_{\Omega^a} r^{-2}|\psi_1(\mathbf{x})|^2 dV$$

exists.

This last term is less than

$$\int_{r=a}^{\infty}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} drd\theta d\phi \sin\theta \left|\sum_{m=0}^{\infty}\sum_{n=m}^{\infty} a^{n+1}\alpha_{mn}\cos m\phi \left(\frac{(n-m)!}{(n+m)!}\right)^{1/2} \frac{P_n^m(\cos\theta)}{r^{n+1}}\right|^2.$$

Therefore,

$$\begin{aligned}\int_{\Omega^a} r^{-2}|\psi_1(\mathbf{x})|^2 dV &< \textstyle\sum_{m=0}^{\infty}\sum_{n=m}^{\infty} \frac{\pi a^{2n+2}|\alpha_{mn}|^2}{n+1/2} C_m \int_a^{\infty} \frac{dr}{r^{2n+2}} \\ &= \textstyle\sum_{m=0}^{\infty}\sum_{n=m}^{\infty} \frac{2\pi|\alpha_{mn}|^2}{(2n+1)^2} C_m a,\end{aligned}$$

where

$$C_m = \begin{cases} 2 & \text{if } m = 0 \\ 1 & \text{otherwise} \end{cases}$$

This last sum exists. The exchange of the order of summation and integration that has taken place above is justifiable because each term in the sum is non-negative.

Now split the $\psi_2(\mathbf{x})$ term in two as

$$\psi_2(\mathbf{x}) = a\alpha_{00}B_{00}(\mathbf{x}) + \sum_{m=0}^{\infty} \sum_{\substack{n=m \\ n \neq 0}}^{\infty} a^{n+1}\alpha_{mn}B_{mn}(\mathbf{x}),$$

where

$$B_{mn}(\mathbf{x}) = \frac{(-1)^{m+n+1}}{((n+m)!(n-m)!)^{1/2}} \int_0^\infty \frac{k+K}{k-K} k^n \exp(-k(y+2/\epsilon))J_m(kR)dk.$$

Using the well known identity:

$$\frac{P_n^m(\cos\theta')}{r'^{n+1}} = \frac{1}{(n-m)!} \int_0^\infty k^n \exp(-k(y+2/\epsilon))J_m(kR)dk,$$

we have

$$B_{00}(\mathbf{x}) = \frac{1}{r'} + \frac{2}{K}\frac{P_1(\cos\theta')}{r'^2} + \ldots + \frac{2}{K^p}p!\frac{P_p(\cos\theta')}{r'^{p+1}}$$
$$-\tfrac{2}{K^p}\int_0^\infty \tfrac{k^{p+1}}{k-K} \exp(-k(y+2/\epsilon))J_0(kR)dk,$$

where

$$r' = \sqrt{R^2 + (y+2/\epsilon)^2}$$

and

$$\theta' = \arctan\frac{R}{y+2/\epsilon}.$$

Therefore,

$$\int_{S_c} |B_{00}(\mathbf{x})|^2 dS = 2\pi$$
$$-4\Re \int_{S_c} dS \tfrac{1}{c}\left\{\int_0^\infty \tfrac{k}{k-K}\exp(-k(y+2/\epsilon))J_0(kR)dk\right\}$$
$$+4\int_{S_c} dS \left|\int_0^\infty \tfrac{k}{k-K}\exp(-k(y+2/\epsilon))J_0(kR)dk\right|^2,$$

where S_c is the hemisphere with radius c centred on the image point ($r' = 0$) and lying below $y = -2/\epsilon$. Denote by $R(c)$ the difference

$$\int_{S_c} |B_{00}(\mathbf{x})|^2 dS - 2\pi.$$

It is easy to see that $R(c)$ decays to zero as c approaches infinity. So 2π is the leading order term in the asymptotic expansion of

$$\int_{S_c} |B_{00}(\mathbf{x})|^2 dS.$$

Clearly, then,

$$\int_{\Omega^a} r'^{-2} |a\alpha_{00} B_{00}(\mathbf{x})|^2 dV \leq \int_{1/\epsilon}^{\infty} c^{-2} a^2 |\alpha_{00}|^2 \int_{S_c} |B_{00}(\mathbf{x})|^2 dS dc.$$

This exists because

$$2\pi \int_{1/\epsilon}^{\infty} c^{-2} \int_{S_c} |B_{00}(\mathbf{x})|^2 dS dc$$

exists. Therefore,

$$\int_{\Omega^a} (1 + r^2)^{-1} |B_{00}(\mathbf{x})|^2 dV$$

exists.

We now aim to show that

$$\sum_{m=0}^{\infty} \sum_{\substack{n = m \\ n \neq 0}}^{\infty} a^{n+1} \alpha_{mn} B_{mn}(\mathbf{x})$$

belongs to $L^2(\Omega_a)$. This will imply that

$$\int_{\Omega^a} (1 + r^{-2}) \left| \sum_{m=0}^{\infty} \sum_{n=m;n\neq 0}^{\infty} a^{n+1} \alpha_{mn} B_{mn}(\mathbf{x}) \right|^2 dV$$

exists.

Let

$$C_{mn}(y, p, q) = \mathcal{F}(B_{mn}(-1/\epsilon, R, \phi)) \exp(-\rho y),$$

where

$$\mathcal{F}(B_{mn}(-1/\epsilon, R, \phi))$$

denotes the Fourier transform of $B_{mn}(-1/\epsilon, R, \phi)$ when considered as a function in $\mathcal{R}^2$ and

$$\rho = \sqrt{p^2 + q^2}.$$

Let

$$\phi' = \arctan \frac{q}{p}.$$

It is easy to show that

$$C_{mn}(y,p,q) = 2\pi \frac{(-1)^{n/2+m+1}}{((n+m)!(n-m)!)^{1/2}} \frac{\rho + iK}{\rho - K} \rho^{n-1} \exp(-\rho(y + 2/\epsilon)) \cos m\phi'$$

when n is even and

$$C_{mn}(p,q) = 2\pi \frac{(-1)^{n/2+m+1/2}}{((n+m)!(n-m)!)^{1/2}} \frac{i\rho - K}{\rho - K} \rho^{n-1} \exp(-\rho(y + 2/\epsilon)) \cos m\phi'$$

when n is odd. We now use Parseval's equality to show that

$$\int_{-1/\epsilon}^{\infty} \int_{\mathcal{R}^2} |B_{mn}(\mathbf{x})|^2 dV = \int_{-1/\epsilon}^{\infty} \int_{\mathcal{R}^2} |C_{mn}(y,p,q)|^2 dV$$

and, thus

$$\begin{aligned}
\int_{\Omega^a} |B_{mn}(\mathbf{x})|^2 dV \quad &\leq \tfrac{M}{(n+m)!(n-m)!} \int_{-1/\epsilon}^{\infty} dy \int_0^{\infty} d\rho \int_0^{2\pi} d\phi \left|\tfrac{\rho+K}{\rho-K}\right|^2 \\
&\quad \times \rho^{2n-1} \exp(-2\rho(y + 2/\epsilon)) \\
&= \tfrac{2\pi M}{(n+m)!(n-m)!} \int_0^{\infty} d\rho \left|\tfrac{\rho+K}{\rho-K}\right|^2 \\
&\quad \times \rho^{2n-1} \int_{-1/\epsilon}^{\infty} dy \exp(-2\rho(y + 2/\epsilon)) \\
&= \tfrac{\pi M}{(n+m)!(n-m)!} \int_0^{\infty} d\rho \left|\tfrac{\rho+K}{\rho-K}\right|^2 \\
&\quad \times \rho^{2n-2} \exp(-2\rho/\epsilon) \\
&\leq \tfrac{C}{(n+m)!(n-m)!} \int_0^{\infty} d\rho \rho^{2n-2} \exp(-2\rho/\epsilon) \\
&= C \tfrac{(2n-2)!}{(n+m)!(n-m)!} (\epsilon/2)^{2n-1},
\end{aligned}$$

where M and C are constants independent of n and m.

We now use the fact that if the sum

$$\sum_{n=1}^{\infty} 2^n |a_n|^2$$

exists then

$$|\sum_{n=1}^{\infty} a_n|^2 \leq \sum_{n=1}^{\infty} 2^n |a_n|^2$$

to see that

$$\int_{\Omega^a} \left| \sum_{m=0}^{\infty} \sum_{n=m;n\neq 0}^{\infty} a^{n+1}\alpha_{mn} B_{mn}(y, R, \phi) \right|^2 dV$$
$$\leq \textstyle\sum_{m=0}^{\infty} \sum_{n=m;n\neq 0}^{\infty} 2^{m+n} a^{2n+2} |\alpha_{mn}|^2 \int_{\Omega^a} |B_{mn}(y, R, \phi)|^2 dV$$
$$\leq Ca^3 \textstyle\sum_{m=0}^{\infty} \sum_{n=m;n\neq 0}^{\infty} 2^{m+n} |\alpha_{mn}|^2 \frac{(2n-2)!}{(n+m)!(n-m)!} (a\epsilon/2)^{2n-1}$$
$$\leq 2Ca^3 \textstyle\sum_{m=0}^{\infty} \sum_{n=m;n\neq 0}^{\infty} |\alpha_{mn}|^2 \frac{(2n-2)!}{(n+m)!(n-m)!} (a\epsilon)^{2n-1}.$$

It is true that

$$\frac{(2n-2)!}{(n+m)!(n-m)!} \leq \frac{(2n-2)!}{n!n!}$$
$$= \tfrac{2n(2n-2)(2n-4)\ldots 2}{n!} \times \tfrac{(2n-1)(2n-3)(2n-5)\ldots 1}{n!} \times \tfrac{1}{2n(2n-1)}$$
$$< \tfrac{2^{2n}}{2n(2n-1)}.$$

Therefore,

$$\int_{\Omega^a} \left| \sum_{m=0}^{\infty} \sum_{n=m;n\neq 0}^{\infty} a^{n+1}\alpha_{mn} B_{mn}(y, R, \phi) \right|^2 dV$$

exists if

$$\sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \frac{(2a\epsilon)^{2n-1}}{n^2} |\alpha_{mn}|^2$$

exists. If $a < 1/(2\epsilon)$ the result is true.

B.2 Proof of (B.2)

$$\nabla\psi_1(\mathbf{x}) = \textstyle\sum_{m=0}^{\infty} \sum_{n=m}^{\infty} a^{n+1}\alpha_{mn} \left(\frac{(n-m)!}{(n+m)!}\right)^{1/2}$$

$$\times \left\{ \tfrac{-(n+1)}{r^{n+2}} P_n^m(\cos\theta)\cos m\phi\, \mathbf{e}_r \right.$$
$$- \tfrac{\sin\theta}{r^{n+2}} P_n^{m\prime}(\cos\theta)\cos m\phi\, \mathbf{e}_\theta$$
$$\left. - \tfrac{m}{r^{n+2}\sin\theta} P_n^m(\cos\theta)\sin m\phi\, \mathbf{e}_\phi \right\}$$

Therefore,

$$\int_{\Omega^a} |\nabla\psi_1(\mathbf{x})|^2 dV \leq \sum_{m=0}^{\infty}\sum_{n=m}^{\infty}\int_a^\infty \left(\frac{a}{r}\right)^{2n+2} |\alpha_{mn}|^2 \left(\frac{(n+1)^2}{n+1/2} + m\right)\pi C_m dr \quad \text{(B.4)}$$
$$+ \textstyle\int_a^\infty \int_{-1}^1 r^2 \sum_{m=0}^\infty \left| \sum_{n=m}^\infty \frac{a^{n+1}}{r^{n+2}} \alpha_{mn} P_n^{m\prime}(c) \left(\frac{(n-m)!}{(n+m)!}\right)^{1/2} \sqrt{1-c^2} \right|^2 \pi C_m dc dr$$
$$+T, \qquad ,$$

where

$$T \leq \int_{\Omega_2^a} \sum_{m=0}^\infty \left| \sum_{n=m}^\infty \frac{a^{n+1}}{r^{n+2}} \alpha_{mn} P_n^{m\prime}(c)(\cos\theta) \sqrt{\frac{(n-m)!}{(n+m)!}} \sin\theta\cos m\phi \right|^2 dV,$$

where Ω_2^a is the region between the surfaces of the spheres centred on the origin and of radii a and $2a$. Let

$$h = \sum_{m=0}^\infty \sum_{n=m}^\infty \alpha_{mn} \cos m\phi \sqrt{\frac{(n-m)!}{(n+m)!}} P_n^m(\cos\theta)$$

and let

$$g = \sum_{m=0}^\infty \sum_{n=m}^\infty \frac{1}{2^{n+1}} \alpha_{mn} \cos m\phi \sqrt{\frac{(n-m)!}{(n+m)!}} P_n^m(\cos\theta).$$

Lemma 11 implies that h and g belong to $H^{\frac{1}{2}}(\Sigma_a)$ and $H^{\frac{1}{2}}(\Sigma_{2a})$ respectively. By the theory of Lions and Magenes [19, Chapter 2], there exists a unique function in $H^1(\Omega_2^a)$ satisfying Laplace's equation and whose traces on the boundaries are h and g. Clearly this function is the restriction of ψ_1 to Ω_2^a. Therefore, $\nabla\psi_1$ belongs to $L^2(\Omega_2^a)$ and so T is finite.

It is easy to see that the first term on the right hand side of equation (B.4) is finite. The second term is bounded by

$$M \sum_{m=0}^\infty \sum_{n=m}^\infty \int_{2a}^\infty dr \int_{-1}^1 dc |\alpha_{mn}|^2 \left(\frac{2a}{r}\right)^{2n+2} \frac{(n-m)!}{(n+m)!}(1-c^2)\left(P_n^{m\prime}(c)\right)^2,$$

where M is some constant. Define

$$I = \int_{-1}^{1} \left(P_n^{m\prime}(c) \right)^2 (1-c^2) dc.$$

Integration by parts and the equation satisfied by Legendre functions give

$$I = n(n+1) \int_{1}^{1} \left(P_n^m(c) \right)^2 dc - m^2 \int_{-1}^{1} \frac{\left(P_n^m(c) \right)^2}{1-c^2} dc.$$

Thus

$$I = \frac{2n(n+1)(n+m)!}{(2n+1)(n-m)!} - m\frac{(n+m)!}{(n-m)!}.$$

So

$$\int_{\Omega^a} |\nabla \psi_1(\mathbf{x})|^2 dV$$

exists if

$$\sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \int_{2a}^{\infty} dr \frac{(2a)^{2n}}{r^{2n+2}} |\alpha_{mn}|^2 \left(\frac{2n(n+1)}{2n+1} - m \right)$$

exists. This last term equals

$$\frac{1}{2a} \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} |\alpha_{mn}|^2 \left(\frac{2n(n+1)}{(2n+1)^2} - \frac{m}{2n+1} \right)$$

and, therefore, exists.

$$\begin{aligned} \nabla \psi_2(\mathbf{x}) \qquad &= \textstyle\sum_{m=0}^{\infty} \sum_{n=m}^{\infty} a^{n+1} \alpha_{mn} \frac{(-1)^{m+n+1}}{((n+m)!(n-m)!)^{1/2}} \\ &\times \left\{ \textstyle\int_0^\infty \frac{k+K}{k-K} k^{n+1} \exp(-k(y+2/\epsilon)) J_m{}'(kR) dk \cos m\phi \, \mathbf{e}_R \right. \\ &- \textstyle\int_0^\infty \frac{k+K}{k-K} k^{n+1} \exp(-k(y+2/\epsilon)) J_m(kR) dk \cos m\phi \, \mathbf{e}_y \\ &\left. - \textstyle\frac{1}{R} \int_0^\infty \frac{k+K}{k-K} k^{n} \exp(-k(y+2/\epsilon)) J_m(kR) m dk \sin m\phi \, \mathbf{e}_\phi \right\}. \end{aligned}$$

Rewrite this as

$$\nabla \psi_2(\mathbf{x}) \qquad = \textstyle\sum_{m=0}^{\infty} \sum_{n=m}^{\infty} a^{n+1} \alpha_{mn} \frac{(-1)^{m+n+1}}{((n+m)!(n-m)!)^{1/2}}$$

$$\times \left\{ \int_0^\infty \tfrac{k+K}{k-K} k^{n+1} \exp(-k(y+2/\epsilon)) \right.$$
$$\left\{ \tfrac{1}{2}(J_{m-1}(kR) - J_{m+1}(kR)) \cos m\phi \, \mathbf{e}_R \right.$$
$$-J_m(kR) dk \cos m\phi \, \mathbf{e}_y$$
$$\left. \left. -\tfrac{1}{2}(J_{m+1}(kR) + J_{m-1}(kR)) \sin m\phi \, \mathbf{e}_\phi \right\} dk \right\}.$$

It is clear from the previous section that

$$\nabla \psi_2(\mathbf{x}) \in (L^2(\Omega_a))$$

if we bear in mind that

$$J_{-1}(z) = -J_1(z).$$

B.3 Proof of (B3)

$$\psi(\mathbf{x})|_{FS} = \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} a^{n+1} \alpha_{mn} (-1)^{m+n+1}$$
$$\times \tfrac{2K}{((n+m)!(n-m)!)^{1/2}} \int_0^\infty \tfrac{k^n}{k-K} \exp(-2k/\epsilon) J_m(kR) dk$$

(Recall that

$$\frac{P_n^m(\cos\theta)}{r^{n+1}} = (-1)^{m+n} \frac{P_n^m(\cos\theta')}{r'^{n+1}}$$

on FS.)

The Fourier transform of $\psi(\mathbf{x})|_{FS}$ is

$$\chi(p,q) = 4\pi \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} a^{n+1} \alpha_{mn} (-1)^{m+n+1} A_m$$
$$\tfrac{K}{((n+m)!(n-m)!)^{1/2}} \tfrac{k^n}{k-K} \exp(-2k/\epsilon) \cos m\phi',$$

where

$$A_m = \begin{cases} (-1)^{m/2} & \text{if } m \text{ is even} \\ (-1)^{(m+1)/2} & \text{if } m \text{ is odd} \end{cases}$$

$$k = \sqrt{p^2 + q^2}$$

and

$$\phi' = \arctan \frac{q}{p}.$$

So, $\psi(\mathbf{x})|_{FS} \in L^2(FS)$ if and only if $\chi \in L^2(\mathcal{R}^2)$. We, once again, use the result

$$|\sum_{n=1}^{\infty} a_n|^2 \leq \sum_{n=1}^{\infty} 2^n |a_n|^2$$

to see that the Fourier transform of $\psi(\mathbf{x})$ belongs to $L^2(\mathcal{R}^2)$ if

$$\sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \frac{(2a^2)^n}{(n+m)!(n-m)!} \left(\int_0^{\infty} dk \frac{k^{2n+1}}{|k-K|^2} \exp(-4k/\epsilon) \right) |\alpha_{mn}|^2$$

exists. For very large n each term in the sum is close to

$$\frac{(2n-1)!}{(n+m)!(n-m)!} (\epsilon a/2)^{2n} |\alpha_{mn}|^2.$$

This last term is bounded by

$$M \frac{1}{n} (\epsilon a)^{2n} |\alpha_{mn}|^2,$$

where M is a constant. As a is smaller than $1/\epsilon$, the series converges and we are done.

Appendix C

Sobolev Spaces

This appendix is intended as a brief overview of the theory of Sobolev spaces. We will concentrate only on those results that are directly relevant to our task.

Let us suppose that Ω is an open subset of $\mathcal{R}^3$. Denote by $\mathcal{D}(\Omega)$ the space of infinitely smooth functions whoose support is completely contained in Ω. A distribution on Ω is defined as any member of the dual space of $\mathcal{D}(\Omega)$; i. e. for any distribution, g, and any member of $\mathcal{D}(\Omega)$, f, the integral

$$\int_\Omega gf dV \text{ exists.}$$

The derivative up to any order of a distribution can be defined via the equation

$$\int_\Omega (D^\alpha g) f dV = (-1)^{|\alpha|} \int_\Omega g (D^\alpha f) dV,$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, each α_i is a non-negative integer,

$$\mid \alpha \mid = \sum_{i=1}^{3} \alpha_i,$$

and

$$D \equiv \frac{\partial^{|\alpha|}}{\partial x^{\alpha_1} \partial y^{\alpha_2} \partial z^{\alpha_3}}.$$

Armed with this definition of the distributional derivative, we can go on to define the Sobolev spaces $H^s(\Omega)$.

Firstly, if s is a non-negative integer, then $g \in H^s(\Omega)$ if and only if

$$\sum_{|\alpha|=s} \int_\Omega \mid D^\alpha g \mid^2 dV \text{ exists.}$$

The $H^s(\Omega)$ norm of g is then defined to be

$$\| g \|_{H^s(\Omega)} \equiv \left(\sum_{|\alpha|=0}^{s} \int_\Omega \mid D^\alpha g \mid^2 dV \right)^{\frac{1}{2}}. \tag{C.1}$$

If s is non-negative, but not an integer, and if we write $s = m + t$, where m is an integer and $0 < t < 1$, then the $H^s(\Omega)$ norm of g is defined to be

$$\| g \|_{H^s(\Omega)} \equiv \left(\| g \|^2_{H^m(\Omega)} + \sum_{|\alpha|=m} \int \int_{\Omega \otimes \Omega} \frac{\mid D^\alpha g(\mathbf{x}) - D^\alpha g(\mathbf{y}) \mid^2}{\mid \mathbf{x} - \mathbf{y} \mid^{3+t/2}} d^3\mathbf{x} d^3\mathbf{y} \right)^{1/2}. \tag{C.2}$$

Of course, $g \in H^s(\Omega)$ if and only if the right hand side of (C.2) exists. If $s = 0$ then it is usual to write $H^s(\Omega)$ as $L^2(\Omega)$.

If $s < 0$, then we cannot simply define $H^s(\Omega)$ as the dual space of $H^{-s}(\Omega)$ because this is not a space of distributions. Instead, the space $H_0^{-s}(\Omega)$ is defined as the closure of $\mathcal{D}(\Omega)$ with respect to the norm defined in (C.1) or (C.2) (depending, of course, on whether s is an integer or not). $H^s(\Omega)$ is then defined as the dual of $H_0^{-s}(\Omega)$. It is known that this space consists of distributions.

If $0 \leq s < t$, then $H^t(\Omega) \subset H^s(\Omega)$. This is an obvious consequence of (C.1) and (C.2). Furthermore, the imbedding is compact. Suppose, now, that $s < t \leq 0$. Let g be any member of $H^t(\Omega)$. The duality product between g and any element, f, of $H_0^{-t}(\Omega)$ is

$$< g, f >= \int_\Omega g\bar{f} dV, \tag{C.3}$$

where $\overline{f}$ denotes the complex conjugate of f. Clearly, (C.3) will also be true for any element of $H_0^{-s}(\Omega)$ since $H_0^{-s}(\Omega)$ is contained in $H_0^{-t}(\Omega)$. Therefore, g belongs to $H^s(\Omega)$. Again, the imbedding is known to be compact.

Similar spaces can be defined on surfaces in $\mathcal{R}^3$. They are normed in a similar way to the Sobolev spaces that are defined on subsets of $\mathcal{R}^3$ of non-zero measure. They have identical compactness and imbedding properties as before and if the surface, $\partial\Omega$, is closed, then $H^{-s}(\partial\Omega)$ is the dual space of $H^s(\partial\Omega)$ regardless of whether s is positive or negative.

If $\partial\Omega$ is the boundary of Ω, then $H^s(\partial\Omega)$, for $s \geq \frac{1}{2}$, consists entirely of traces of distributions in $H^{s+\frac{1}{2}}(\Omega)$. The trace of a distribution is analogous to the boundary value of a function. Furthermore, for every $g \in H^s(\partial\Omega)$ there exists an element, g^*, of $H^{s+\frac{1}{2}}(\Omega)$ whose trace on $\partial\Omega$ equals g. g^* is called a *lifting* of g. It can be shown that this lifting operation is continuous; i. e. there exists a positive constant K, independent of g and g^* such that

$$\| g^* \|_{H^{s+\frac{1}{2}}(\Omega)} \leq K \| g \|_{H^s(\partial\Omega)} .$$

In addition to the spaces that have already been defined, we shall need the space $H^1_{loc}(\Omega)$. This is the completion of functions with compact support in Ω with respect to the H^1 norm. This means that the restriction to any compact subset, Ω', of Ω of any distribution in $H^1_{loc}(\Omega)$ is in $H^1(\Omega')$.

Appendix D

Uniqueness Proof

In this appendix we prove that any function, $\mathbf{u}$, satisfying

$$(\lambda + \mu)\nabla\nabla.\mathbf{u} + \mu\nabla^2\mathbf{u} + \rho\omega^2\mathbf{u} = \mathbf{0}$$

in any exterior domain Ω_e, vanishing at infinity, and having homogeneous Dirichlet or Neumann boundary conditions, vanishes identically when the Lamé constants satisfy

$$\Re\left(\lambda + \frac{2}{3}\mu\right) > 0,$$

$$\Re\mu > 0,$$

$$\Im\left(\lambda + \frac{2}{3}\mu\right) < 0$$

and

$$\Im\mu < 0.$$

It is well known that $\mathbf{u}$ may be written as

$$\mathbf{u} = \nabla\Phi + \nabla \times \Psi,$$

where

$$\nabla^2\Phi + \frac{\rho\omega^2}{\lambda + 2\mu}\Phi = 0$$

and

$$\nabla^2\Psi + \frac{\rho\omega^2}{\mu}\Psi = \mathbf{0}.$$

We require that $\mathbf{u}$ vanish at infinity; therefore, Φ and Ψ cannot be exponentially increasing functions. They must, therefore, be exponentially decreasing. Whence, $\mathbf{u}$ is exponentially decreasing. Therefore,

$$0 = \lim_{R\to\infty} \int_{\Sigma_R} \overline{\mathbf{u}}.\sigma(\mathbf{u}).\mathbf{n}dS,$$

where Σ_R denotes the surface of a sphere of radius R, $\mathbf{n}$ is an outward pointing normal to Σ_R and $\sigma(\mathbf{u})$ is the stress tensor. From the divergence theorem we have

$$\int_{\Omega_e} \nabla.(\overline{\mathbf{u}}.\sigma(\mathbf{u}).\mathbf{n})dV = 0,$$

where the homogeneous boundary condition has been taken into account. Thus

$$\int_{\Omega_e} \nabla\overline{\mathbf{u}} : \sigma(\mathbf{u})dV = \rho\omega^2 \int_{\Omega_e} \mathbf{u}.\overline{\mathbf{u}}dV.$$

Taking the imaginary part of this equation yields

$$\Im(\lambda + \frac{2}{3}\mu)|e_{kk}|^2 + 2\Im\mu(e_{ij} - \frac{1}{3}e_{kk}\delta_{ij})(\overline{e_{ij}} - \frac{1}{3}\overline{e_{kk}}\delta_{ij}) = 0,$$

where the summation convention is employed and

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial n_j} + \frac{\partial u_j}{\partial n_i}\right).$$

The conditions on the Lamé constants imply that $\mathbf{e}$ vanishes. Thus $\mathbf{u}$ is constant. The condition that it must vanish at infinity implies it vanishes identically.

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