THE FORCE DERIVATIVES OF A SHIP MOVING WITH FORWARD VELOCITY U .

A THESIS SUBMITTED TO THE UNIVERSITY OF MANCHESTER
FOR THE DEGREE OF DOCTOR OF PHILOSOPHY
IN THE FACULTY OF SCIENCE

By

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May 1992

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I would like to thank S.E.R.C. and B.M.T. Cortec. for their financial assistance, David Clarke for his suggestion as to the direction of the research, Eric Watson, and especially Professor Ursell for his guidance and patience throughout the three years of Post-Graduate research.

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Abstract

This problem is suggested by B.M.T. Cortec (formerly British Maritime Technology) who are investigating the problem of calculating the hydrodynamic forces and moments on a manoeuvring ship, and the variation of the hydrodynamic forces and moments over the ship's hull, moving with forward velocity U through a fluid. These are described by the hydrodynamic force derivatives.

We consider the ship held at a small angle of yaw α and assume that the ship's hull has a plane of symmetry which is vertical and that the free surface of the fluid can be neglected. Thus we consider uniform flow past a fixed closed body.

Consider the wake flow far behind the ship. The momentum change of the retarded fluid velocity in the wake gives rise to the drag force on the body. Thus the drag on the body is expressed in terms of a wake traverse.

The fluid flow at large distances from the body is to first order a uniform stream of velocity U. This suggests approximating the Navier-Stokes equations to obtain the linear Oseen's equations for fluid flow.

We consider the Lamb-Goldstein velocity representation for Oseen flow. (Lamb, Hydrodynamics 1932 art 342 .) Lamb considers two dimensional flow and three dimensional axisymmetric flow, and Goldstein extends the theory for general flow in three dimensions.(Proc.Royal Soc. 1931 a.)

The velocity representation is $\underline{u} = \underline{\nabla \phi} + \underline{w}$ where the potential ϕ is defined as $\phi = \frac{D \ln r}{2\pi \rho U} - (1/\rho U) \int_{-\infty}^{x} \left\{ p(x'y) + \frac{D \cos \theta'}{2\pi r'} \right\} dx'$ in two dimensional flow, but this does not in fact define ϕ in a shadow region behind the ship.

We find other interesting difficulties associated with this representation; thus, from this definition, it is unclear whether ϕ is continuous.

We consider the above difficulty and give the complete Fourier expansions for the velocity and pressure. We also find expressions for the drag, lift and moment on the body.

We consider Oseen's velocity and pressure representations for Oseen flow given in terms of a surface distribution of singularities called Oseenlets, and its equivalence in two dimensional flow to the Lamb-Goldstein velocity representation.

We consider the velocity in the far field laminar wake and the solutions for this flow given by Lagerstrom and Landau and Lifshitz.

Declaration

No portion of the work referred to in this Thesis has been submitted in support of an application for another degree or qualification of this or any other University or institution of learning.

Edmund Chadwick.

Chapter 1

Research on ship manoeuvrability.

1.1 Introduction

We deal with a problem in ship manoeuvrability suggested by Dr. D. Clarke of B.M.T. Cortec[1], Wallsend, Newcastle. This problem has arisen from research at B.M.T. into the manoeuvrability of large tankers and of ferry boats. The hull design of a ship affects its manoeuvrability; of particular importance is research into the relations between the hull design and the manoeuvring characteristerics of the ship. We discuss this research, and its applications at B.M.T., briefly below.

Large oil tankers cause great damage to the environment from spillage after collision, and their poor manoeuvrability causes them problems when docking. B.M.T. have recently developed a computer program for simulating the docking procedure. The computer screen gives the view as it would appear from the ship's bridge and the Captain of the ship is given the same manoeuvrability controls as those on the ship's bridge. The computer estimates the ship response to the Captain's controls from the force derivatives[2] data particular to the ship's hull design.

The design of ferry boats is increasingly governed by economic considerations;

[1]formally British Maritime Technology.
[2]definition given in the Chapter 1 appendix.

increasing the width of the boat increases its volume capacity. Such designs severely limit the manoeuvring ability of the boats. However, within the next few years government legislation will be introduced whereby every boat must satisfy certain manoeuvring criteria in order to be considered seaworthy. The manoeuvrability criteria are expressed in terms of the force derivatives of the boat.

We see that there is a need at B.M.T. to calculate the force derivatives, which determine the manoeuvring characteristics, of a ship accurately. There are full scale experimental tests which can be undertaken in order to calculate them. However these are expensive and difficult to perform. The important methods of estimating the force derivatives are scale model testing and theoretical approaches.

Model test results at B.M.T. are obtained either from experiments using a tow path or a rotating arm. In both cases the model is held in a fixed position. In the first case the model is moved at a constant velocity and its axis is held at a fixed angle to the forward velocity. In the second case the model is placed at the end of a large rotating arm with its axis held at a fixed angle to the velocity of the model. The results are obtained during the period of motion when the model's angular velocity is constant. The model is divided into sections and the force and moment on each section is calculated. There has been difficulty though in extracting accurate data from the experiments. The experiments are also time-consuming and costly.

The theoretical approach to ship manoeuvring is discussed next.

1.2 Outline of the research

We first consider the ship to be at a fixed angle of yaw α and the following assumptions[3] to be applicable for the motion:

- The ship's hull has a plane of symmetry and this plane is vertical.
- The fluid surface lies in the horizontal plane. This implies that the waves are short and that the height of the fluid raised at the bow is of order lower than the length and beam dimensions of the ship.

This means that the solution of the fluid flow is thus equivalent to the solution for fluid flow past the hull surface reflected about the mean free surface, and that the mean free surface is now composed of streamlines of the flow. (We may define the mean free surface as follows: Far from the ship disturbance, we expect the fluid to lie in a horizontal plane. We take the extension of this horizontal plane in the fluid region to be the mean free surface of the fluid. In two-dimensional flow, we expect the fluid to lie on a horizontal line far from the ship disturbance and we take the extension of this horizontal line in the fluid to be the mean free surface of the fluid. Thus if we consider the waves to be very short, the fluid surface approximates to the mean free surface of the fluid.)

We now formulate the problem mathematically below:

We consider a fluid infinite in extent with uniform velocity U flowing past a fixed closed body. This body is considered to have two planes of symmetry, perpendicular to each other, one plane parallel to the flow and the other plane at an angle α to the flow.

[3]discussed in the Chapter 1 appendix.

We usually consider the ship to be slender. This means that the length dimension of the body in the plane perpendicular to the two planes of symmetry is considered to be of lower order than the length dimension of the body in the planes of symmetry. We define the body length as the length l along the line of intersection of the two planes of symmetry of the body from the body bow to the body stern.

We would like to calculate the hydrodynamic drag force, the lift force and the moment on the body and the changes in these forces and moment along the body length.

We now consider my theoretical approach to the problem:

Consider the steady wake flow far behind the ship. The momentum change of the retarded fluid velocity in the wake gives rise to the drag force on the body. Thus the drag on the body is expressed in terms of a wake traverse.

The fluid flow at large distances from the body is to first order a uniform stream of velocity U. This suggests approximating the Navier-Stokes equations to obtain the linear Oseen's equations for fluid flow.

Oseen replaced the Navier-Stokes equations for a body fixed in a uniform stream

$$\rho(u_j \frac{\partial u_i}{\partial x_j}) = -\frac{\partial p}{\partial x_i} + \mu(\frac{\partial^2 u_i}{\partial x_j^2} + X_i)$$

by the linearised equations in which

$$u_1 \frac{\partial u_i}{\partial x_1} + u_2 \frac{\partial u_i}{\partial x_2} + u_3 \frac{\partial u_i}{\partial x_3}$$

is replaced by

$$U \frac{\partial u_i}{\partial x_1}$$

The inertial cartesian coordinate system (x_1, x_2, x_3) is used above where the uniform stream velocity U is taken to act in the x_1 direction.

This is expected to be correct near infinity, but causes difficulties near the body where boundary conditions need to be applied. (In principle this later difficulty might be overcome by matched asymptotic expansions, see Rosenhead p.187, Proudman and Pearson (1959), and Kaplun and Lagerstrom (1959) but we shall be mainly concerned with expansions at a great distance.)

We shall consider the Lamb-Goldstein velocity representation for Oseen flow. (Lamb, Hydrodynamics 1932 art 342.) Lamb uses this velocity representation in two dimensional flow and three dimensional axisymmetric flow, and Goldstein extends the theory for general flow in three dimensions. (Proc. Roy. Soc. 1931a.)

We shall consider the Lamb-Goldstein velocity representation for Oseen flow in two dimensions only, and in the discussion sections (7.0.3) and (7.0.4) we shall review Lamb's and Goldstein's treatments of steady Oseen flow in three dimensional flow.

The velocity representation is $\underline{u} = \underline{\nabla\phi} + \underline{w}$ where the potential ϕ is defined as $\phi = \frac{D\log r}{2\pi\rho U} - (1/\rho u)\int_{-\infty}^{x}\{p(x') + \frac{D\cos\theta'}{2\pi r'}\}dx'$, but this does not in fact define ϕ in the shadow region behind the ship.

We find other interesting difficulties associated with this representation; thus, from this definition, it is unclear whether ϕ is continuous.

We shall consider the above difficulty and give the complete Fourier expansions for the velocity and pressure for two dimensional flow. We shall also find expressions for the drag, lift and moment on the body. The velocity and pressure in the far field wake and the expressions for the drag, lift and moment agree with those given by Lagerstrom, Imai (1951) and Goldstein (1933) in their respcetive treatments of Oseen flow.

We consider Oseen's velocity and pressure representations for Oseen flow given in terms of a surface distribution of singularities called Oseenlets, and its equivalence to the Lamb-Goldstein velocity representation.

1.3 Brief review of papers written on ship manoeuvrability.

In aerodynamics, calculations for the drag and lift forces on an aerofoil in two dimensions, and a streamlined wing in three dimensions, in a uniform stream are well known. However there are many more difficulties associated with the calculations of forces on a ship. In aerodynamics, we deal with streamlined wings and using the Kutta-Joukowski condition we can determine the forces on the wing from the calculation of shed vorticity. However in ship motion we deal with bluff bodies. Separation occurs and vorticity is shed from unknown positions on the hull. This vorticity diffuses in a region behind the ship called the wake and is associated with the drag and the lift on the body. We see that in comparison with aerodynamic theory, the problem of ship manoeuvrability is complex and difficult to solve. However attempts at theoretical models have been made and some papers on manoeuvrability theory are listed below.

1.Clarke, D: (1972) A two-dimensional strip method for surface ship hull derivatives: comparison of theory with experiment on a segmented tanker model. (Journal of Mechanical Engineering Sciences, pp 53-61, paper 8)

2.Newman, J N: (1972) Some theories on ship manoeuvring. (J.M.E.S. pp 34-42, paper 6.)

3.Gadd, G E: (1984) A calculation method for forces on ships at small angles of yaw. (Royal Institution of Naval Architects. pp 257-267)

We shall review the work by Lamb and by Goldstein on Oseen flow which provides the stimulus for my approach to the ship manoeuvrability problem in the discussion sections (7.0.3) and (7.0.4) respectively. We next review the three papers listed above.

1.3.1 Papers using slender body inviscid theory.

Clarke's paper and Newman's paper use results derived in Lamb; Lamb considers a body moving in an infinite inviscid fluid. (Lamb, Hydrodynamics 1932 art 117 .) Lamb gives the equations of motion for the body (art 124) which involve the hydrodynamic forces, and moments, on the body and also terms including the velocity, and angular velocity, derivatives of the kinetic energy of the body.

Slender body theory is then used by both Clarke and Newman so that the forces and moments on the ship are found in terms of integrals along the body length.

However Clarke's method differs from Newman's:
Clarke first calculates the side force on a curve lying in an arbitrary plane perpendicular to the ship's principal axis. The total force on the body can then be found by summing this value over all the perpendicular 'strips' along the body length. This gives the total force as an integral over the body length.

Newman calculates the total side force from Lamb's relation. He then invokes the slender body assumption in order to find the force and moment as an integral over the body length.

Clarke's method agrees well with experimental results, which suggests the validty of his approach using an irrotational fluid strip theory method. However, the theory does not produce accurate results near the stern of the ship. Attempts to overcome this difficulty have been made and are discussed in the following subsections.

1.3.2 The equations of motion for a body moving in an infinite inviscid fluid.

(Lamb Hydrodynamics, 1932 ed, chapter 6.)

Lamb gives the equations of motion of the body in the coordinate reference frame moving with the body. The equations of motion for the fluid and body system must first be found. However, the fluid momentum cannot be determined; we cannot determine the fluid momentum as a Green's surface integral over the body boundary since this integral over a closed surface enclosing the body and tending to infinity is indeterminate. Lamb overcomes this problem by finding the equations of motion of the system in terms of the impulse of the system. Lamb finds that the impulse change of the system is the same as the momentum change of the body. Lamb defines the impulse of the system, after Lord Kelvin, as 'the properly adjusted impulsive wrench which when applied instantaneously to the body,when the system is at rest, counteracts the impulsive pressures due to the fluid on the surface of the body and generates the momentum of the body'. By considering the form of the energy of the system and the change in the energy and the impulse of the system over an infinitesimal time, the equations of motion of the system, and thus the body, are found.

1.3.3 Inviscid flow models incorporating shed vortex sheets.

We deal specifically here with the method described by Gadd. Uniform flow past a fixed ship at a fixed angle of yaw α in inviscid irrotational theory can be repesented by a Green's surface source and dipole distribution. Another representation would be a vortex distribution on the body surface. We find that for a slender ship the vortex distribution takes the form of longitudinal bound vortices which stop abruptly at the stern of the ship.

One method of representing the flow past the body in a more realistic way than that in slender body inviscid irrotational flow is proposed to be by continuing these longitudinal vortices into the fluid region behind the stern of the ship. This creates a region of vorticity, a wake, behind the ship. This is the proposition which Gadd uses for his inviscid flow model which incorporates shed vortex sheets. There is theoretical motivation for doing this since it is argued that this method is similar to applying a Kutta condition on a trailing edge, which is done in aerodynamics.

Gadd considers the wake to lie in the plane of symmetry of the hull and to have depth d, the ship depth. Longitudinal trailing vortex filaments are distributed over the wake. The strength of the trailing vortex filaments is equivalent to the strength of the bound vortex filaments at the stern of the ship at the same depth.

Gadd divides the hull and wake into panels and distributes sources and normal dipoles over the panels to satisfy the above conditions.

Hence Gadd's calculation for the side force on the ship considers also the effect of trailing vorticity in a wake region whose strength is obtained from the strength of the bound vorticity at the stern of the ship at the same depth.

This method is similar to the calculation of the lift L on a streamlined wing in aerodynamics where a trailing vortex sheet emanates from the trailing edge of the streamlined wing in uniform flow. Gadd's method is an attempt to apply a condition for the wake vorticity similar to the Kutta condition for aerodynamics streamline wing flow, although a ship is a bluff body.

(Gadd extends the method for the case of a ship moving through restricted water by using an iterative procedure: He distributes panels in this case over the bottom surface of the fluid as well as the hull surface and wake. He then distributes sources over the surface panels to counteract the normal velocity from the solution in unrestricted water on the fluid bottom surface. The affect of this distribution is to modify the flow at the hull, so a new solution is obtained for the distribution of singularities over the hull surface. The method is then repeated.

Gadd also considers allowing for the viscous effects of the fluid by considering the displacement thickness of the boundary layer. He does this by considering the iterative boundary layer calculation.

1.4 Ch1 Appendix: The force derivatives definition.

We consider a ship moving with a perturbed velocity $\underline{u}$ to a forward velocity U in the x direction . We also consider the rate of turn of the ship $\dot{\alpha}$. The hydrodynamic forces and moments are then assumed to be directly proportional to the perturbations u, v, $\dot{\alpha}$, and their derivatives $\dot{u}$, $\dot{v}$, $\ddot{\alpha}$. (Clarke, The application of manoeuvring criteria in hull design using linear theory. R.I.N.A. trans. 1983 pp 45-68.)

We consider the drag, the lift, and the moment acting on the ship. The force derivatives are defined as the rate of change of the drag, lift and moment, with respect to the perturbations u, v, $\dot{\alpha}$, $\dot{u}$, $\dot{v}$, and $\ddot{\alpha}$. Hence by the Taylor expansion they are the constants of proportionality in the linearised equation relating the hydrodynamic force with the perturbations. We note that in the experimental model tests the model was held in a fixed position. This means that the perturbations $\dot{u}$, $\dot{v}$, and $\ddot{\alpha}$, are zero. In the theoretical model we will use we will assume that the mean position of the model is fixed and a uniform stream U flows past it.

1.5 Ch1 Appendix: The basic assumptions made in ship manoeuvrability.

We consider the motion of the ship through the free surface of the sea; the ship is considered to be a rigid body and thus to have six degrees of freedom. However, the motion associated with some of the degrees of freedom is considered to be negligible. We now consider Naval Architecture terminology, the terminology used in describing ship motion.

1.5.0.1 Naval Architecture Terminology.

We assume that the ship's hull has a plane of symmetry and we consider the restricted motion where the plane of symmetry is kept vertical. We also assume that the fluid free surface lies in a horizontal plane. (This assumption is discussed later.) We consider the cartesian coordinate system (x, y, z) which is fixed in the ship and moves with the ship; we let the x axis lie along the intersection of the two planes and its direction to be from stern to bow. We let the z axis point vertically upwards. The following terminology is used for the forces and moments acting on the ship:

X is the force in the x direction called the surge force.
Y is the force in the y direction called the sway force.
Z is the force in the z direction called the heave force.
N is the moment in the z direction called the yaw moment.
The moment in the y direction is called the pitch moment.
The moment in the x direction is called the roll moment.

Diagram showing the forces and moments acting on a ship.

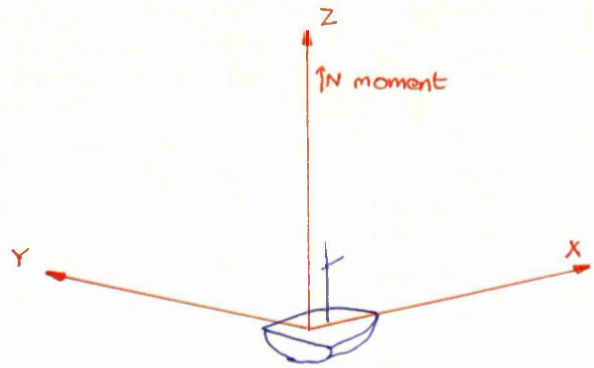
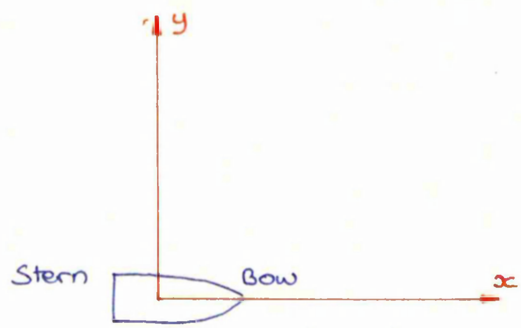


Diagram naming the positions on a ship.



Thus we assume the ship moves with surge, sway and yaw only. This is the usual assumption is made in the literature on ship manoeuvrability.

We will now make an assumption about the free surface of the fluid through which the ship moves.

1.5.0.2 The free surface assumption

We assume that the free surface acts as an undisturbed solid plane. However, the fluid surface will actually be raised at the bow. We estimate the height of the raised bow by considering Bernoulli's equation along a free surface streamline.

Bernoulli's equation gives:

$$\frac{1}{2}\underline{u}^2 + gz = \text{constant}$$

where $\underline{u}$ is the fluid velocity, g is the gravitational constant and z is the upwards vertical displacement. So near the bow we expect an increase in the height of the fluid of the order of $\frac{U^2}{2g}$.

We also consider the length scale of waves generated by this motion. We find below that the horizontal length scale is of order U^2/g. Therefore near the bow, the slope is not small. Thus we need an inner expansion for this region. However, for the moment we shall ignore this difficulty.

We consider the equation for the time independent surface waves around a body travelling with forward velocity U.

The wave equation is obtained by the appropriate linearisation for the kinematic surface boundary condition and the dynamic surface boundary condition.

If we consider the z displacement of the free surface to be $\zeta(x, y)$, then the kinematic boundary condition gives $\frac{D}{Dt}(\zeta - z) = 0$

The linearisation of this condition gives

$$U\frac{\partial \zeta}{\partial x} - \frac{\partial \phi}{\partial z} = 0$$

where $\underline{u}' = \underline{u} - (U, 0, 0) = \underline{\nabla\phi}$

Hence $\underline{u}'$ is the perturbed velocity of the motion.

The dynamic boundary equation is obtained from Bernoulli's equation of motion for the fluid. On the free surface, this gives the condition

$$\tfrac{1}{2}\rho\underline{u}^2 - \rho g\zeta = 0$$

The linearisation of this condition gives

$$\rho U\frac{\partial \phi}{\partial x} - \rho g\zeta = 0$$

Hence the appropriate wave condition is

$$U^2\frac{\partial^2 \phi}{\partial x^2} - g\frac{\partial \phi}{\partial z} = 0$$

If we assume the fluid surface acts as a solid plane, then the vertical velocity of the surface is negligible, or$\frac{\partial \phi}{\partial z} \to 0$

Hence we expect waves to be short, of order U^2/g. This is the short wave approximation.

Chapter 2

The derivation of the Oseen equations.

2.1 The derivation of Oseen's equations.

Since we are dealing with steady flow, we obtain the Navier-Stokes equations for steady incompressible flow:

$$\rho u_j^\dagger \frac{\partial u_i^\dagger}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \mu \nabla^2 u_i^\dagger \tag{2.1}$$

We can now obtain Oseen's equations from the Navier-Stokes equations for steady flow. (The Navier-Stokes equation derivation is given in the appendix.)

Oseen's approximation to the fluid flow is that the velocity perturbation $\underline{u}$ to the uniform stream U is small compared to the stream velocity U.

We let the uniform stream U be parallel to the x_1 axis. Thus the velocity $\underline{u}^\dagger$ is given by

$$(u_1^\dagger, u_2^\dagger, u_3^\dagger) = (U + u_1, u_2, u_3)$$

where the Oseen approximation is $|u_i| \ll U$.

Considering the Navier-Stokes equation, the term $u_j^\dagger \frac{\partial}{\partial x_j}$ is

$$U\left(\frac{\partial}{\partial x_1} + \frac{u_1 \partial}{U \partial x_1} + \frac{u_2 \partial}{U \partial x_2} + \frac{u_3 \partial}{U \partial x_3}\right)$$

Applying Oseen's approximation that $\frac{|u_i|}{U} \ll 1$, we obtain

$$u_j^\dagger \frac{\partial}{\partial x_j} = U \frac{\partial}{\partial x_1}$$

Thus:

$$U \frac{\partial \underline{u}}{\partial x_1} = -(1/\rho)\underline{\nabla p} + \nu \nabla^2 \underline{u}$$

Since the flow is incompressible, taking the divergence of the above equation, we obtain

$$\nabla^2 p = 0$$

The equations

$$U\frac{\partial \underline{u}}{\partial x_1} = -(1/\rho)\underline{\nabla p} + \nu\nabla^2\underline{u} \tag{2.2}$$

$$\underline{\nabla}.\underline{u} = 0 \tag{2.3}$$

are the Oseen equations for steady flow.

Change of cartesian coordinate notation. In the vector analysis considered to obtain the Navier-Stokes equations for steady incompressible flow, it was convenient to label the cartesian coordinate variables x_1, x_2 and x_3.

However, the preferred notation for this subject is to denote the cartesian coordinate variables by x, y and z. We therefore change the coordinate variables to x, y and z where $x = x_1$, $y = x_2$ and $z = x_3$.

We do not change the vector description. Hence we still describe $\underline{u}^\dagger$ and $\underline{u}$ as $(u_1^\dagger, u_2^\dagger, u_3^\dagger)$ and (u_1, u_2, u_3) respectively. The suffix 1 denotes the x direction, the suffix 2 the y direction and 3 the z direction.

2.2 Ch2 Appendix: The derivation of Oseen's equations.

Oseen's equations are an approximation to the Navier-Stokes equations. The Navier-Stokes equations describe the Newtonian fluid. They are found by considering the constitutive relations of the fluid.

Oseen flow deals with a certain type of fluid flow; we consider uniform fluid flow past a body of velocity U. Oseen flow is applicable in this case far from the body surface.

We first apply Oseen theory to the problem of uniform steady two dimensional fluid flow of velocity U past a body.

Oseen's approximation assumes that the velocity perturbations to the uniform stream are small. We denote the vector $\underline{u}^{\dagger}$ for the fluid velocity field, $\underline{u}$ for the perturbation velocity, and U for the uniform stream velocity. Thus Oseen's approximation depends upon the condition that $\mid \underline{u} \mid \ll U$.

In the region far from the body, the fluid velocity approaches the velocity of the uniform stream U. Hence $\mid \underline{u} \mid \ll U$ and Oseen's approximation is valid in this region. On the body surface, however, the condition $\underline{u}^{\dagger}.\underline{n} = 0$ must be satisfied, where $\underline{n}$ is the unit normal to the surface. If the unit normal vector direction is close to being parallel to the uniform stream direction, then $\mid \underline{u} \mid$ is of order U and Oseen's approximation is not applicable. Thus in the region near to the body we do not expect the approximation to be valid.

It is noted, however, that for a slender body whose axis is close to being parallel to the uniform stream, the unit normal vector direction is close to being perpendicular to the uniform stream direction. Hence Oseen flow is applicable in regions close to a slender body orientated in this way.

We will now obtain Oseen's equations from the Navier-Stokes equations for fluid flow. Navier-Stokes equations are found by considering the equations of motion for an element of fluid and by considering the stress and rate of strain relations of the fluid. We give a preliminary derivation of the equations of motion for an element of fluid and a preliminary discussion of the stress and rate of strain relations of the fluid.

Coordinate notation. We use suffix notation to label the axes and to represent vectors; thus the cartesian coordinate axes are labelled (x_1, x_2, x_3), and the vector $\underline{u}$ is represented by u_j where $j = 1$, 2 or 3. The uniform stream U is taken to be in the x_1 direction.

The equations of motion of an element of fluid. A Newtonian fluid is assumed to have continuous density and thus the motion to obey continuum mechanics. We consider an element of fluid of volume δV, density ρ and velocity $\underline{u}^\dagger$. A force $\underline{f}$ is exerted on the fluid element. Thus, from Newton's equations of motion, the force equals the rate of momentum change:

$$f_i = \frac{D}{Dt}(\rho \delta V u_i^\dagger)$$

The operator $\frac{D}{Dt}$ is the rate of change with time of the function calculated in the fluid element. (Since the fluid element is not stationary, this operator is different from the partial derivative operator $\frac{\partial}{\partial t}$.)

We now find the force f_i in terms of the stress tensor τ_{ij}

Stress. We invoke Cauchy's stress principle that the fluid has finite stress which is a function both of position within the fluid and of time. We consider the force on a region of fluid over part of the bounding surface ΔS. We let the unit vector $\underline{n}$ be normal to the surface and pointing away from the fluid. Then if $\underline{f}$ is the force on the fluid over the elemental area δS, we have from Cauchy that the vector $\overset{n}{\underline{T}}$ such that:

$$\overset{n}{\underline{T}} = \lim_{\Delta S \to 0} \frac{\underline{f}}{\Delta S}$$

The stress tensor. We now find the stress tensor of the fluid.

Let us consider a region of fluid enclosed by the surface S. The force on the fluid, $\underline{f}$, is such that

$$f_i = \int_S \overset{n}{\tau}_i ds \tag{2.4}$$

We consider this surface as its area tends to zero. Using Cauchy's stress principle,

$$|\lim_{S\to 0} f_i| \ll \max| \overset{n}{\tau}_i | \lim_{S\to 0} S = 0 \tag{2.5}$$

We use this result in considering the surface, S, of a tetrahedron. The tetrahedron has four corners positioned at $(0,0,0)$ $(a,0,0)$ $0,b,0)$ and $(0,0,c)$.

We let the area of triangle $0ab$ be A_1,

the area of triangle $0ac$ be A_2,

the area of triangle $0bc$ be A_3,

and the area of triangle abc be A_n.

From equation (2.4), we have that:

$$f_i - \overset{n}{\tau}_i A + \overset{-\hat{x}_1}{\tau_i} A_1 + \overset{-\hat{x}_2}{\tau_i} A_2 + \overset{-\hat{x}_3}{\tau_i} A_3 = 0$$

Applying equation (2.5), and using the fact that $\frac{A_i}{A_n} = n_i$, we obtain

$$\overset{n}{\tau}_i = \overset{-\hat{x}_j}{\tau_i} n_j$$

where the repeated suffix implies a summation over j.

For ease of notation we let $\overset{-\hat{x}_j}{\tau_i} = \tau_{ij}$. We see that τ_{ij} is independent of the normal vector $\underline{n}$. Once the components are known, any vector $\overset{n}{\tau}_i$ may be calculated. τ_{ij} is a tensor operator on the normal vector $\underline{n}$ and is called the stress tensor. In this notation,

$$\overset{n}{\tau}_i = \tau_{ij} n_j \tag{2.6}$$

Equation of motion of the fluid element in terms of the stress tensor. The equation of motion of the fluid element is found by considering equation (2.4):

$$f_i = \int_S \overset{n}{\tau}_i \, ds$$

Using equation (2.6) and applying the divergence theorem, we obtain

$$f_i = \int_S \overset{n}{\tau}_i \, ds = \int_S \tau_{ij} n_j ds = \int_V \frac{\partial \tau_{ij}}{\partial x_j} dV$$

Hence $f_i = \delta V \frac{\partial \tau_{ij}}{\partial x_j}$ and so the equation of motion becomes:

$$\frac{D(\rho u_i^\dagger)}{Dt} = \frac{\partial \tau_{ij}}{\partial x_j} \tag{2.7}$$

We now find the constitutive relations, the relations between the components of the stress tensor τ_{ij} and the rate of strain tensor e_{ij}, in order to obtain the Navier-Stokes equations of motion.

The constitutive relations for the fluid. Applying stresses on the surface of the fluid element makes the fluid element distort. Thus we expect a relation between the stress tensor field τ_{ij} and the rate of strain tensor field e_{ij}.

From Taylors expansion we express the change in velocity $\delta u_i^\dagger$ for a displacement δx_i as

$$\delta u_1^\dagger = (1/2)(e_{11}\delta x_1 + e_{12}\delta x_2 + e_{13}\delta x_3) + (1/2)(\omega_2 \delta x_3 - \omega_3 \delta x_2)$$

$$\delta u_2^\dagger = (1/2)(e_{21}\delta x_1 + e_{22}\delta x_2 + e_{23}\delta x_3) + (1/2)(\omega_3 \delta x_1 - \omega_1 \delta x_3)$$

$$\delta u_3^\dagger = (1/2)(e_{31}\delta x_1 + e_{32}\delta x_2 + e_{33}\delta x_3) + (1/2)(\omega_1 \delta x_2 - \omega_2 \delta x_1)$$

where $e_{ij} = \frac{\partial u_i^\dagger}{\partial x_j} + \frac{\partial u_j^\dagger}{\partial x_i}$ and $\underline{\omega} = \underline{\nabla} \times \underline{u}^\dagger$, $\underline{\omega} = (\omega_1, \omega_2, \omega_3)$

The terms in e_{ij} give the distortion of the fluid element, whereas the velocity change due to the terms in ω_i is $(1/2)(\underline{\omega} \times \underline{\delta r})$ which represents a rotation of the fluid element. [The term $\underline{\delta r}$ equals $(\delta x_1, \delta x_2, \delta x_3)$.]

The tensor components e_{ij} give the different rate of strains of the fluid element.

The simplest relation between the stress tensor field τ_{ij} and the rate of strain tensor field e_{ij} is linear:

$$\tau_{ij} = A_{ij} + B_{ij,mp}e_{mp} \tag{2.8}$$

In deriving Navier-Stokes equations we assume the relation takes the above linear form. Since the Navier-Stokes equations give accurate fluid flow descriptions, the above linear relation must be a property of the fluid.

We assume the fluid is isotropic. Thus if we rotate the coordinate axes we expect the form of the stress-rate of strain relations to remain the same. This implies that the tensors A_{ij} and $B_{ij,mp}$ are isotropic. Therefore

$$A_{ij} = -p\delta_{ij}$$

$$B_{ij,mp} = \lambda\delta_{ij,mp} + (\mu/2)(\delta_{im}\delta_{jp} + \delta_{ip}\delta_{jm}) + \nu(\delta_{im}\delta_{jp} - \delta_{ip}\delta_{jm})$$

Substituting these relations into equation (2.8) gives:

$$\tau_{ij} = -p\delta_{ij} + \lambda\delta_{ij}e_{mm} + \mu e_{ij}$$

Since the fluid is incompressible, we expect the rate of increase of a fluid element e_{mm} to be zero. This means that the dilatation of the rate of strain tensor is zero.

Hence we obtain the constitutive relation

$$\tau_{ij} = -p\delta_{ij} + \mu e_{ij} \tag{2.9}$$

where p is defined as $p = -(1/2)\tau_{kk}$ and may be called the pressure of the fluid.

We now substitute the constitutive relation into the equation of motion for a fluid element in order to obtain Navier-Stokes equations.

2.2.1 The Navier-Stokes equations.

The equation of motion of the fluid is given by equation (2.7) as

$$\frac{D(\rho u_i^\dagger)}{Dt} = \frac{\partial \tau_{ij}}{\partial x_j}$$

The constitutive relations for the fluid is given by equation (2.9) as

$$\tau_{ij} = -p\delta_{ij} + \mu(\frac{\partial u_i^\dagger}{\partial x_j} + \frac{\partial u_j^\dagger}{\partial x_i})$$

Thus

$$\frac{D(\rho u_i^\dagger)}{Dt} = -\frac{\partial p}{\partial x_i} + \mu(\nabla^2 u_i^\dagger + \frac{\partial^2 u_j^\dagger}{\partial x_i \partial x_j})$$

Since we are dealing with an incompressible fluid, $\frac{\partial u_j^\dagger}{\partial x_j} = 0$

by considering the infinitesimal changes in the function $f(x_1, x_2, x_3, t)$ operated on by the operator $\frac{D}{Dt}$, we find that

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u_i^\dagger \frac{\partial}{\partial x_i}$$

Since we are dealing with steady flow, we obtain the Navier-Stokes equations for steady incompressible flow:

$$\rho u_j^\dagger \frac{\partial u_i^\dagger}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \mu \nabla^2 u_i^\dagger \tag{2.10}$$

We can now obtain Oseen's equations from the Navier-Stokes equations for steady flow.

2.2.2 Oseen's equations

Oseen's approximation to the fluid flow is that the velocity perturbation $\underline{u}$ to the uniform stream U is small compared to the stream velocity U.

We let the uniform stream U be parallel to the x_1 axis. Thus the velocity $\underline{u}^\dagger$ is given by

$$(u_1^\dagger, u_2^\dagger, u_3^\dagger) = (U + u_1, u_2, u_3)$$

where the Oseen approximation is $|u_i| \ll U$.

Considering the Navier-Stokes equation, the term $u_j^\dagger \frac{\partial}{\partial x_j}$ is

$$U(\frac{\partial}{\partial x_1} + \frac{u_1 \partial}{U \partial x_1} + \frac{u_2 \partial}{U \partial x_2} + \frac{u_3 \partial}{U \partial x_3})$$

Applying Oseen's approximation that $\frac{|u_i|}{U} \ll 1$, we obtain

$$u_j^\dagger \frac{\partial}{\partial x_j} = U \frac{\partial}{\partial x_1}$$

Thus:

$$U \frac{\partial \underline{u}}{\partial x_1} = -(1/\rho)\underline{\nabla p} + \nu \nabla^2 \underline{u}$$

Since the flow is incompressible, taking the divergence of the above equation, we obtain

$$\nabla^2 p = 0$$

The equations

$$U \frac{\partial \underline{u}}{\partial x_1} = -(1/\rho)\underline{\nabla p} + \nu \nabla^2 \underline{u} \tag{2.11}$$

$$\nabla^2 p = 0 \tag{2.12}$$

are the Oseen equations for steady flow.

Change of cartesian coordinate notation. In the vector analysis considered to obtain the Navier-Stokes equations for steady incompressible flow, it was convenient to label the cartesian coordinate variables x_1, x_2 and x_3.

However, it is sometimes preferable to denote the cartesian coordinate variables by x, y and z where $x = x_1$, $y = x_2$ and $z = x_3$.

We do not change the vector description. Hence we still describe $\underline{u}^\dagger$ and $\underline{u}$ as $(u_1^\dagger, u_2^\dagger, u_3^\dagger)$ and (u_1, u_2, u_3) respectively. The suffix 1 denotes the x direction, the suffix 2 the y direction and 3 the z direction.

Chapter 3

The Lamb-Goldstein velocity representation in two dimensions.

We consider the steady two dimensional Oseen flow of an infinite fluid of uniform velocity U past a closed body. Thus we expect a wake region of rotational flow and outside this a region of nearly irrotational potential flow.

We shall find the complete expansions for the fluid velocity and pressure and verify that these expansions are compatible with the known theory. The complete expansion of the velocity and pressure is necessary for use in asymptotic matching of near field and far field flows. The complete expansions are found by extending the theory first given by Lamb and Goldstein. (See sections (7.0.3) and (7.0.4) respectively.) Some of the coefficients in the expansions can be expressed in terms of the drag, the lift, and the moment on the body due to the action of the fluid. These expansions are shown to be compatible with the far-field Laminar-wake theory given by Lagerstrom and the Oseen velocity representation of Oseen flow in two dimensions.

We first extend the theory given by Lamb and Goldstein. The Lamb-Goldstein method involves a decomposition of the fluid velocity.

In this chapter we obtain the complete expansions for the perturbation Oseen velocity $\underline{u}$ and the pressure p. We do this by considering the Lamb-Goldstein velocity representation of Oseen flow; the perturbation velocity $\underline{u}$ is represented as a summation of a velocity potential $\underline{\nabla\phi}$ and a rotational velocity $\underline{w}$. Lamb and Goldstein define the potential ϕ as $\frac{\partial\phi}{\partial x} = -\frac{p}{\rho U}$. (In section (3.1.3) we shall see that this implies that outside the wake region the perturbation velocity is nearly $\underline{u} = \underline{\nabla\phi}$.) Both Lamb and Goldstein do not properly define ϕ; we investigate this difficulty within this section.

We consider the rotational velocity $\underline{w}$ such that $\underline{w} = \underline{u} - \underline{\nabla\phi}$, and the streamfunction Ψ of the velocity $\underline{w}$. (Hence $\underline{w} = (\frac{\partial\Psi}{\partial y}, -\frac{\partial\Psi}{\partial x})$ since $\underline{\nabla}.\underline{w} = \underline{\nabla}.\underline{u} = 0$.)

Hence the Lamb-Goldstein velocity representation is a decomposition of the perturbation velocity $\underline{u}$ into a potential velocity $\underline{\nabla\phi}$ and a rotational velocity $\underline{w}$.

$$\underline{u} = \underline{\nabla\phi} + \underline{w}$$

We first find the Fourier expansions for ϕ and Ψ; we equate some of the coefficients in the expansion with the lift, drag and moment on the body. We next finally give the complete expansions for the perturbation Oseen velocity $\underline{u}$ and pressure p.

Therefore we divide the chapter into the following subsections:

3.1 The definition of the potential ϕ.

3.2 The streamfunction Ψ of the velocity $\underline{w}$.

3.3 Symmetric flow.

3.4 Antisymmetric flow.

3.5 The complete expansions for the velocity and pressure.

3.1 The potential function ϕ.

We now give Oseen's uniform steady flow in two dimensions past a body boundary. (See section 2.1 equations (2.2) and (2.3).) In terms of the perturbed velocity $\underline{u}$, we have

$$U\frac{\partial \underline{u}}{\partial x} = -(1/\rho)\underline{\nabla p} + \nu\nabla^2\underline{u} \tag{3.1}$$

We follow the method first used by Lamb and Goldstein of decomposing the perturbation velocity potential in the form

$$\underline{u} = \underline{\nabla\phi} + \underline{w}$$

where

$$\frac{\partial \phi}{\partial x} = -\frac{p}{\rho U} \tag{3.2}$$

There are good reasons for using this decomposition especially when considering steady uniform flow past fixed bodies; for this type of flow, we expect regions of almost no vorticity within the fluid (and therefore potential flow), and we also expect regions of vorticity within the fluid (and therefore rotational flow). We call the region of rotational flow the wake. Thus outside the wake we expect potential flow, this decomposition is useful because we find that outside the wake the potential flow is given very nearly by $\underline{u} = \underline{\nabla\phi}$; the velocity is satisfied very nearly by the velocity potential $\underline{\nabla\phi}$ of equation (3.2). Therefore outside the wake the function $\underline{w}$ is effectively zero.

We will see in section (3.2.2) that substituting the velocity $\underline{u} = \underline{\nabla\phi} + \underline{w}$ into the Oseen equations we obtain a differential equation in $\underline{w}$ only such that

$$\frac{\partial \underline{w}}{\partial x} = \frac{\nu}{U}\nabla^2\underline{w} \tag{3.3}$$

and $\underline{\nabla}.\underline{w} = 0$

It is not clear, however, that the decomposition given by Lamb and Goldstein is complete. This is because the potential ϕ is not defined uniquely in equation (3.2):

In order to obtain a complete expansion for the potential ϕ, we must define ϕ uniquely. We must also make sure the definition of ϕ isn't a divergent intergral.

Thus I define the potential ϕ as

$$\phi = \frac{D \ln r}{2\pi\rho U} - \frac{1}{\rho U}\int_{-\infty}^{x}\left\{p(x', y) + \frac{D\cos\theta'}{2\pi r'}\right\} dx'$$

where $r'\cos\theta' = x'$, $r'\sin\theta' = y$ and $-\frac{D\cos\theta}{2\pi r}$ is the leading order term in the expansion of the pressure.

Hence

$$\frac{\partial\phi}{\partial x} = -\frac{p}{\rho U}$$

However, this doesn't define the potential ϕ everywhere in the fluid; there is a shadow region where ϕ is undefined. We shall continue ϕ analytically into this region. However, this means that we obtain a discontinuity line.

We must define ϕ uniquely in order to obtain a complete expansion for $\underline{u}$. It is especially important for our problem to know that we have the complete expansion and not a partial expansion since the solution for $\underline{u}$ may be used in an asymptotic matching. (Oseen flow is applicable in the region far from the body.)

The subsection subdivides into the following parts:

3.1.1 Definition of the potential ϕ.

3.1.2 The analytic continuation of ϕ into the shadow region showing the existence of a discontinuity line in ϕ.

3.1.3 The fluid motion outside the wake region.

3.1.1 The definition of the potential ϕ.

We now consider the Lamb-Goldstein velocity decomposition given by equations (3.2) in Oseen flow. The potential ϕ is first given which satisfies equation (3.2).

ϕ is defined as :

$$\phi = \frac{D \ln r}{2\pi\rho U} - (1/\rho U)\int_{-\infty}^{x}\left\{p(x', y) + \frac{D\cos\theta'}{2\pi r'}\right\}dx' \tag{3.4}$$

Hence $\frac{\partial\phi}{\partial x} = -(1/\rho U)p$ as given in equation (3.2).

The term $-\frac{D\cos\theta}{2\pi\rho U r}$ is included in the integrand so that the integral is not divergent.

The prsssure p satisfies the differential equation $\nabla^2 p = 0$ and since p is everywhere continuous and tends to zero at infinity, it has the form given by equation (3.23).

From section (3.5.2), we see that the leading order term of p is $-\frac{D\cos\theta}{2\pi\rho U r}$, but if this term was the integrand in equation (3.4), the resulting integral would be divergent.

However, all other terms in the expansion of the pressure give convergent integrals and so

$$\int_{-\infty}^{x}\left\{p(x', y) + \frac{D\cos\theta'}{2\pi r'}\right\}dx'$$

is a convergent integral.

3.1.2 The analytic continuation of ϕ into the shadow region giving a discontinuity line.

The definition of ϕ in equation (3.4) does not give ϕ everywhere in the fluid region where Oseen flow is valid. In this subsection we find the region where ϕ is undefined (called the shadow region) and give a method to continue ϕ analytically into this region. In order to do this, we must find the region of fluid where Oseen flow is valid.

Oseen flow is valid in the region where the Oseen approximation holds. This is the region where the magnitude of the perturbation velocity, $|\underline{u}|$, is much less than the uniform stream velocity U, so $|\underline{u}| \ll U$.

Far from the body the fluid tends to flow as the uniform stream and so Oseen flow is valid . Near the body the fluid velocity $u^\dagger$ must tend to zero since the fluid is assumed viscous, and so Oseen flow is invalid. (However, there are some Reynolds number flows where Oseen flow is a good approximation almost everywhere within the fluid. Also Oseen flow is valid in regions close to a slender body whose length axis is closely aligned to the uniform stream direction.)

For the problem of concern to us, we expect Oseen flow only in the region a distance at least R from body where R is much greater than the body dimension l, $R \gg l$. We now consider the position of the coordinates in order to define the shadow region of the fluid.

We position the coordinates such that the x-axis is parallel to the uniform stream direction and the body is located at $(-R, 0)$. Thus Oseen flow is valid in the region $(x+R)^2 + y^2 > R^2$, and the shadow region is where $(x+R)^2 + y^2 > R^2$, $x > -R$ and $y^2 < R^2$ are all satisfied. We draw the shadow region next.

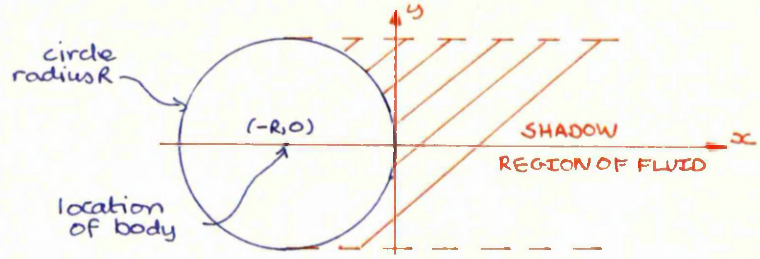
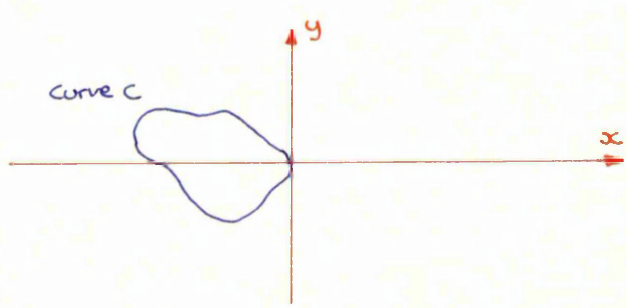


Figure 3.1: The shadow region of the fluid.

However, in the following analysis we consider a general closed curve C rather than the specific curve $(x+R)^2+y^2 = R^2$ as the boundary for Oseen flow validity in the fluid region. This has the added advantage for application to those flows where Oseen flow is a good approximation very close to the body. In these cases it may be a good approximation to take the closed curve C to be the body boundary.

We next consider the position of the boundary curve C in the coordinate frame. The boundary curve C intersects the x-axis at more than one point. The origin of the coordinate frame is placed at the point where the body curve crosses the x-axis for the greatest value of x. Thus the origin is located at the rear of the body. (When the axes are chosen in this way, the discontinuity line will lie along the line $y = 0$, $x > 0$.)

The position of the coordinate system in relation to the body is shown diagramatically below:



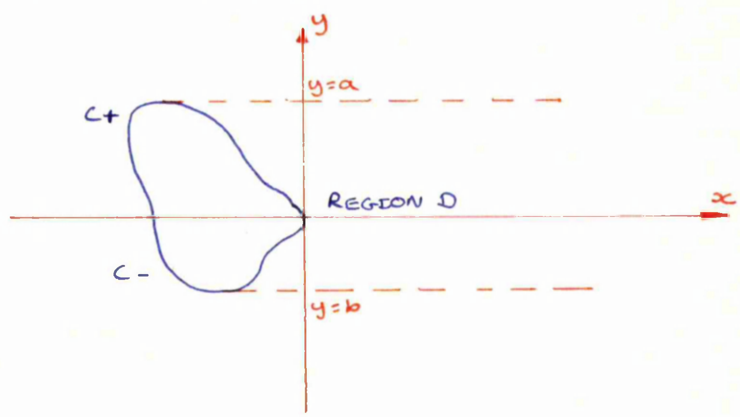
In order to describe the region of the fluid where ϕ is undefined by equation (3.2) we first define a region D. We represent the boundary curve C by $[x(q), y(q)]$ for some parameter q. We let the maximum value of $y(q)$ be a and the minimum value of $y(q)$ be b.

We consider the two points on the curve at $y = a$ and at $y = b$. (If $y(q)$ equals a at more than one point on the boundary curve, we consider the point whose x value is the least, and similarly when $y(q)$ equals b.) These two points split the curve C into two parts, the curve C_+ which passes around the front of the body and the curve C_- which passes around the rear of the body.

We define the region D as that bounded by the curve C_- and the two semi-infinite lines parallel to the x-axis starting from the points on the curve at $y = a$ and at $y = b$ and finishing at $x = \infty$.

The potential ϕ defined by equation (3.4) is an integration of the pressure over a horizontal integral path to the x-axis.

We show the region D in the diagram below:



We consider continuing analytically ϕ into the domain D. We divide the region D into two parts, the region D_+, where $y > 0$, and the region D_-, where $y < 0$.

We first consider the continuation of ϕ into the region D_+.

Outside the region D, we see that:

$$\frac{\partial^2 \phi}{\partial y^2} = -\frac{1}{\rho U}\frac{\partial^2}{\partial y^2}\left(\int_{-\infty}^{x} p(x', y)dx'\right) = -\frac{1}{\rho U}\int_{-\infty}^{x} \frac{\partial^2}{\partial y^2} p(x', y)dx'$$

Since $\nabla^2 p = 0$ from equation (), then

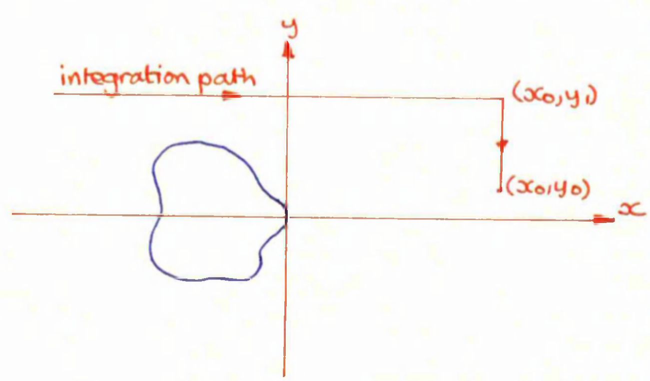
$$\frac{\partial^2 \phi}{\partial y^2} = \frac{1}{\rho U}\int_{-\infty}^{x} \frac{\partial^2}{\partial x'^2} p(x', y)dx' = \frac{1}{\rho U}\frac{\partial p}{\partial x} = \frac{1}{\rho U}\frac{\partial}{\partial x}\left(-\rho U \frac{\partial \phi}{\partial x}\right) = -\frac{\partial^2 \phi}{\partial x^2}$$

and so $\nabla^2 \phi = 0$ and

$$\frac{\partial^2 \phi}{\partial y^2} = \frac{1}{\rho U}\frac{\partial p}{\partial x}. \tag{3.5}$$

Since the pressure p is defined everywhere in the fluid, equation (3.5) will be used to continue ϕ into the region D_+. We consider a point (x_0, y_0) within the fluid region D_+.The potential ϕ is known at the point (x_0, y_1), where $y_1 > a$. Equation (3.5) gives us the change in the second partial derivative with respect to y of ϕ in the region outside the domain D. However by letting equation (3.5) hold within the region D_+ we can continue ϕ from the point (x_0, y_1) to the point x_0, y_0. Hence by letting equation (3.5) hold within the region D_+ we have a continuation of ϕ within this region and we are thus able to find a value for ϕ at the point (x_0, y_0).

We show the continuation used diagrammatically below:



We next give the definition of the potential ϕ in the region D_+ using this continuation. Using equation (3.5):

$$\frac{\partial^2\phi}{\partial y^2} = \frac{1}{\rho U}\frac{\partial p}{\partial x}$$

we can express $\phi(x_0, y_0)$ in terms of $\phi(x_0, y_1)$ and $\frac{\partial\phi}{\partial y}(x_0, y_1)$:

$$\frac{\partial^2\phi}{\partial y^2} = \frac{1}{\rho U}\frac{\partial p}{\partial x}$$

implies that, for $y > 0$:

$$\left[\ \frac{\partial\phi}{\partial y}\ \right]_y^{y_1} = \frac{1}{\rho U}\int_y^{y_1}\frac{\partial p}{\partial x}(x, y')dy' \tag{3.6}$$

Hence

$$\frac{\partial\phi}{\partial y}(x_0, y) = \frac{\partial\phi}{\partial y}(x_0, y_1) - \frac{1}{\rho U}\int_y^{y_1}\frac{\partial p}{\partial x}(x_0, y')dy'$$

and so

$$[\phi(x_0, y)]_{y_0}^{y_1} = (y_1 - y_0)\frac{\partial\phi}{\partial y}(x_0, y_1) - \frac{1}{\rho U}\int_{y_0}^{y_1}\int_y^{y_1}\frac{\partial p}{\partial x}(x_0, y')dy'dy \tag{3.7}$$

Equation (3.7) gives the potential $\phi(x_0, y_0)$, since we know $\phi(x_0, y_1)$, $\frac{\partial\phi}{\partial y}(x_0, y_1)$ and the fluid pressure p and thus we have defined ϕ in the fluid domain D_+. We now show that $\nabla^2\phi = 0$ in the fluid domain D_+.

Hence, we have found $\phi(x, y)$ within the fluid domain D_+. The potential ϕ satisfies

$$\frac{\partial^2\phi}{\partial y^2} = \frac{1}{\rho U}\frac{\partial p}{\partial x}$$

within the domain D_+ and also outside the domain D. Within this fluid region, $\frac{\partial}{\partial x}(\frac{\partial^2\phi}{\partial y^2}) = \frac{1}{\rho U}\frac{\partial^2 p}{\partial x^2} = -\frac{1}{\rho U}\frac{\partial^2 p}{\partial y^2}$ since $\nabla^2 p = 0$ from equation (2.3).

Hence $\frac{\partial^2}{\partial y^2}(\frac{\partial\phi}{\partial x} + \frac{p}{\rho U}) = 0$

This gives us $\frac{\partial\phi}{\partial x} + (1/\rho U)p = Ayf(x) + Bg(x)$

This equation holds for $\phi(x,y)$ in the region $y > 0$. However, we know that for $y > a$, equation (3.2) holds which implies that $\frac{\partial \phi}{\partial x} + (1/\rho U)p = 0$.

Hence $A = B = 0$ and $\frac{\partial \phi}{\partial x} + (1/\rho U)p = 0$ in the region $y > 0$.

Thus substituting the above equation into equation (3.5), we obtain $\nabla^2 \phi = 0$ in the region $y > 0$.

Hence we have found that in the region D_+, ϕ may be continued so that ϕ is single valued and

$$\frac{\partial \phi}{\partial x} + (1/\rho U)p = 0 \tag{3.8}$$

$$\nabla^2 \phi = 0 \tag{3.9}$$

Simiarly, ϕ can be defined in the region D_- by following the same method and we find that the equations (3.8) and (3.9) also hold in this region. Hence the equations (3.8) and (3.9) hold everywhere within the fluid.

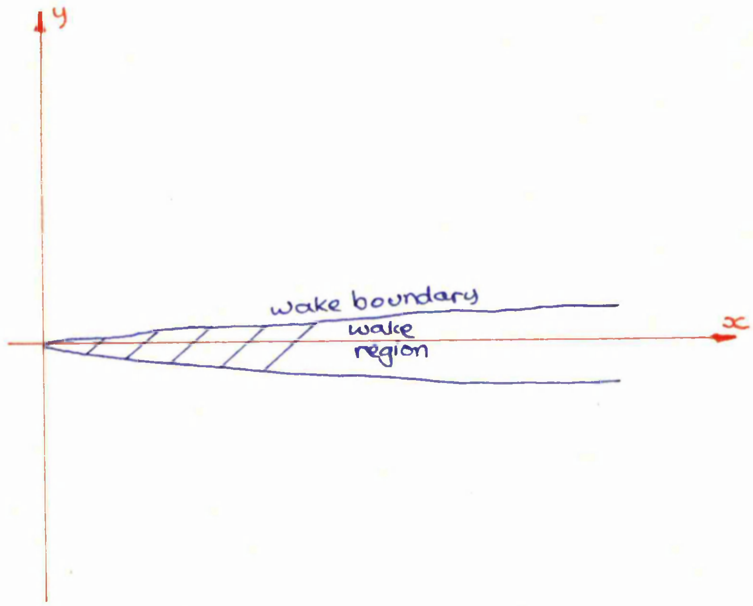
Since the pressure is defined everywhere within the fluid and is assumed to be continuous and single valued, then the function $\frac{\partial \phi}{\partial x}$ is continuous across $y = 0$, and $x > 0$, the boundary line between the domain D_+ and the domain D_-.

However, it may be that ϕ and $\frac{\partial \phi}{\partial y}$ are discontinuous across $y = 0$ and $x > 0$ and this will be investigated in the next subsection.

3.1.3 The fluid motion outside the wake region.

We consider the fluid motion past the body. We expect a region of vorticity (the wake region) to occur at the rear of the body.

We draw a diagram of the flow:



Applying Bernoulli's equation to the region of irrotational flow, we obtain

$$p + (1/2)\rho \underline{u}.\underline{u} = p|_{x=-\infty} + (1/2)\rho U^2$$

along a streamline, where we take $p|_{x=-\infty} = 0$, (see equation (2.4)).

Outside the wake region, since we are dealing with a region of nearly irrotational flow, we may consider the perturbation velocity given by $\underline{u} = \nabla\Phi$, where $\Phi(x, y)$ is some velocity potential.

Hence Bernoulli's equation becomes

$p + \rho U \frac{\partial \Phi}{\partial x} = 0$ to first order. (We consider the perturbation flow such that $|\underline{u}| \ll U$.)

From equation (2.2), we have

$$p + \rho U \frac{\partial \phi}{\partial x} = 0$$

Hence $\Phi - \phi = f(y)$ and so $\frac{\partial \Phi}{\partial y} = \frac{\partial \phi}{\partial y} + f'(y)$.

However, from equation (3.4), we have $\frac{\partial \phi}{\partial y} \to 0$ as $x \to -\infty$.

Since $\underline{u} = \underline{\nabla \Phi}$ is the perturbation velocity to the uniform stream at $x = -\infty$, we also expect that $\frac{\partial \Phi}{\partial y} \to 0$ as $x \to -\infty$.

Hence

$\underline{u} = \underline{\nabla \Phi} = \underline{\nabla \phi}$ in the irrotational flow domain of the fluid.

This gives us the boundary condition that $\underline{w} \to \underline{0}$ outside the wake. We now finish this subsection by looking at the form of the function $\underline{w}$.

3.2 The stream function Ψ.

The streamfunction is a measure of the flux across a line joining two points in the fluid (see Lamb Hydrodynamics art 59 and chapter 2 appendix 2.3.1). Thus if extra fluid is not injected into the fluid stream, (if there are no sources of fluid), then the streamfunction is defined at every point within the fluid.

Therefore we may define a streamfunction Ψ_u of the velocity perturbation $\underline{u}$ within the whole fluid. (Fluid does not flow across the body boundary and so there is no net flux out of the body boundary.)

However, if we consider the potential flow $\underline{\nabla\phi}$, there is a discontinuity line along $y = 0$, $x > 0$, and a region where Oseen flow is invalid within $(x + R)^2 + y^2 < R^2$.

The discontinuity line is equivalent to a line of multipoles, some of which may be sources, and there may be a net flux out (outflow of fluid) from the region $(x + R)^2 + y^2 < R^2$.

Thus the streamfunction of the potential flow $\underline{\nabla\phi}$, Ψ_ϕ, is ill defined due to the presence of sources within the fluid. However, if we define a cut along $y = 0$, $x > 0$ then Ψ_ϕ is defined uniquely everywhere, with the possibility of discontinuities in Ψ_ϕ and its derivatives on the infinite half line $y = 0$, $x > 0$.

Hence there are possibly discontinuities in $\Psi = \Psi_u - \Psi_\phi$ and its derivatives along the infinite half line $y = 0$, $x > 0$. We now investigate the discontinuities in Ψ and its derivatives.

3.2.1 The discontinuities in Ψ and its derivatives.

We find which derivatives of Ψ are continuous and which may have discontinuities on the infinite half line $x > 0$, $y = 0$.

Since the pressure p equals $-\rho U \frac{\partial \phi}{\partial x}$, and the potential satisfies Laplace's equation $\frac{\partial^2 \phi}{\partial y^2} = -\frac{\partial^2 \phi}{\partial x^2}$, it is possible to have discontinuities in ϕ and $\frac{\partial \phi}{\partial y}$ along the infinite half line $x > 0$, $y = 0$, but all other derivatives of ϕ are continuous; this follows from the assumption that the pressure and all its derivatives are continuous everywhere in the fluid.

We also assume that the velocity and its derivatives are continuous , and since

$$\underline{u} = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}\right) + \left(\frac{\partial \Psi}{\partial y}, -\frac{\partial \Psi}{\partial x}\right)$$

then it is only possible that there are discontinuities in Ψ and $\frac{\partial \Psi}{\partial x}$; all other derivatives of Ψ are continuous.

The discontinuity in $\frac{\partial \Psi}{\partial x}$ on the infinite half line $x > 0$, $y = 0$ gives a discontinuity term Ψ_0 in Ψ which is obtained by integration with respect to x.

However, we also obtain a term Ψ_1 giving a discontinuity in Ψ on the infinite half line giving a velocity field $(\frac{\partial \Psi_1}{\partial y}, -\frac{\partial \Psi_1}{\partial x})$ which is continuous within the fluid; this is obtained from the discontinuity in the streamfunction ψ_ϕ due to the outflux of the velocity potential $\underline{\nabla \phi}$ from the circular contour $(x - R)^2 + y^2 = R^2$. In section (3.3.2), we see that this outflux is related to the term $A \log r$ in the Fourier expansion of ϕ from the origin. (The velocity obtained from the source potential $A \log r$ is continuous and so the discontinuity term Ψ_1 must also have continuous derivatives.)

Thus the discontinuity in Ψ is $\Psi_0 + \Psi_1$ and the discontinuity in $\frac{\partial \Psi}{\partial x}$ is $\frac{\partial \Psi_0}{\partial x}$. We next find the functions Ψ_0 and Ψ_1 and so we first obtain the differential equation satisfied by Ψ.

3.2.2 The differential equation satisfied by the stream function Ψ.

When we substitute the equations (3.2) into the Oseen equation (2.2) we obtain a differential equation in $\underline{w}$:

Equations (3.2) are:

$$\underline{u} = \underline{\nabla\phi} + \underline{w}$$
$$\frac{\partial\phi}{\partial x} = -(1/\rho U)p$$

Equation (2.2) is:

$$U\frac{\partial\underline{u}}{\partial x} = -(1/\rho)\underline{\nabla p} + \nu(\nabla^2)\underline{u}$$

Hence we obtain

$$U\frac{\partial}{\partial x}(\underline{\nabla\phi} + \underline{w}) = -(1/\rho)\underline{\nabla p} + \nu\nabla^2(\underline{\nabla\phi} + \underline{w})$$

From equations (3.8) and (3.9) $\frac{\partial\phi}{\partial x} = -(1/\rho U)p$ and $\nabla^2\phi = 0$ hold everywhere within the fluid.

Therefore we obtain the equation

$$U\frac{\partial\underline{w}}{\partial x} = \nu\nabla^2\underline{w} \tag{3.10}$$

with the boundary condition that $\underline{w} = \underline{0}$ outside the wake.

We introduce a stream function Ψ since $\frac{\partial w_1}{\partial x} + \frac{\partial w_2}{\partial y} = 0$.

We write

$$\underline{w} = \left(\frac{\partial\Psi}{\partial y}, -\frac{\partial\Psi}{\partial x}\right)$$

Letting $\frac{U}{\nu} = 2k$, we obtain the equation

$$\left(\frac{\partial}{\partial y}, -\frac{\partial}{\partial x}\right)\left(\nabla^2 - 2k\frac{\partial}{\partial x}\right)\Psi = 0$$

Thus $(\nabla^2 - 2k\frac{\partial}{\partial x})\Psi = \text{const} = E$ and letting $\Psi_E = \Psi + \frac{Ex}{2k}$, we obtain

$$\left(\nabla^2 - 2k\frac{\partial}{\partial x}\right)\Psi_E = 0 \tag{3.11}$$

We now let $\Psi = e^{kx}F$.

Substituting into equation (3.11) we thus find that F satisfies the modified Helmholtz equation

$$(\nabla^2 - k^2)F = 0 \tag{3.12}$$

with separation of variable solutions $K_n(kr)\sin(n\theta + \gamma_{nK})$ and $I_n(kr)\sin(n\theta + \gamma_{nI})$ (We will find the expansion for F in symmetric flow in section (3.3.4) and in anti-symmetric flow in section (3.4.4) using Fourier's theorem given in appendix A.) The terms $I_n(kr)$ are not suitable since we must satisfy the condition that $(\frac{\partial\Psi}{\partial y}, -\frac{\partial\Psi}{\partial x}) \to \underline{0}$ in the far field.

From appendix (3.8.1) a function ϕ satisfying Laplace's equation obtained from a line distribution of normal dipoles of strength $S(\rho)$ along the line $l(\rho)$ parameterized by ρ has a discontinuity across the line at ρ of value $2\pi S(\rho)$.

The function $-K_0(kr)$ satisfies the Helmholtz equation and as $r \to 0$ then $-K_0(kr) \to \ln r$, the Laplace source. Hence using the argument in appendix (3.8.1), we can show that a function F satisfying the modified Helmholtz equation obtained from a line distribution of (Helmholtz) dipoles of strength $S(\rho)$ along the line $l(\rho)$ has a discontinuity across the line at ρ of value $2\pi S(\rho)$. The (Helmholtz) dipole is derived from the (Helmholtz) source $-K_0(kr)$.

Since $F = e^{-kx}\Psi$, there is a discontinuity line in F along $y = 0$, $x > 0$ and so for the function

$$F(x) = -\int_0^\infty S(\xi)\frac{\partial}{\partial y}K_0[k\{(x-\xi)^2 + y^2\}^{1/2}]d\xi$$

which satisfies the modified Helmholtz equation , then

$$[F(x)]_{y\to 0_-}^{y\to 0_+} = e^{-kx}\,[\Psi]_{y\to 0_-}^{y\to 0_+} = 2\pi S(x)$$

We use this equation to find the functions $\Psi_0(x)$ and $\Psi_1(x)$ which give the discontinuities in Ψ and $\frac{\partial \Psi}{\partial x}$. These discontinuities both occur in symmetric flow which is discussed next.

3.3 Symmetric flow.

We define two particular flows $\underline{u}_S$ and $\underline{u}_A$ which we call the perturbation velocity to symmetric flow and antisymmetric flow respectively, and we show that the general perturbation velocity $\underline{u}$ is given by $\underline{u} = \underline{u}_S + \underline{u}_A$

The perturbation velocity components u_i are split into two parts symmetric u_{iS} and antisymmetric u_{iA} about $y = 0$:

$$u_i = u_{iS} + u_{iA} \ , \quad u_{iS}(x,y) = u_{iS}(x,-y) \ , \quad u_{iA}(x,y) = -u_{iA}(x,-y) \ , \quad i = 1,2$$

The incompressibility condition gives

$$\frac{\partial}{\partial x}u_{1S}(x,y) + \frac{\partial}{\partial x}u_{1A}(x,y) + \frac{\partial}{\partial y}u_{2S}(x,y) + \frac{\partial}{\partial y}u_{2A}(x,y) = 0$$

Substituting in the above equations we obtain

$$\frac{\partial}{\partial x}u_{1S}(x,-y) - \frac{\partial}{\partial x}u_{1A}(x,-y) + \frac{\partial}{\partial y}u_{2S}(x,-y) - \frac{\partial}{\partial y}u_{2A}(x,-y) = 0$$

Making the variable change from y to $-y$, gives

$$\frac{\partial}{\partial x}u_{1S}(x,y) - \frac{\partial}{\partial x}u_{1A}(x,y) - \frac{\partial}{\partial y}u_{2S}(x,y) + \frac{\partial}{\partial y}u_{2A}(x,y) = 0$$

Thus if we define $\underline{u}_S + (u_{1S}, u_{2A})$ and $\underline{u}_A = (u_{1A}, u_{2S})$ then $\underline{\nabla}.\underline{u}_S = 0$, $\underline{\nabla}.\underline{u}_A = 0$.

Since the boundary condition $\underline{u}^{\dagger}.\underline{n}$ is linear, where $\underline{u}^{\dagger}$ is the fluid velocity, we solve for the two flows $\underline{u}_S$ and $\underline{u}_A$ separately and the perturbation velocity is given by $\underline{u} = \underline{u}_S + \underline{u}_A$.

$\underline{u}_S$ and $\underline{u}_A$ are defined as the perturbation velocities for symmetric flow and antisymmetric flow respectively. Thus in symmetric flow the axis $y = 0$ is a streamline, and the flow for $y < 0$ is a reflection of the flow for $y > 0$ about the line $y = 0$. We now consider the properties of the pressure p, potential ϕ and the streamfunction Ψ of $\underline{w}$ which is Ψ_s in symmetric flow and Ψ_a in antisymmetric flow.

The pressure p, potential ϕ and streamfunction ψ in symmetric flow and antisymmetric flow. From Oseen's equations (3.2) and (3.3), we have

$$U\frac{\partial \underline{u}}{\partial x} = -\frac{1}{\rho}\underline{\nabla p} + (\nu\nabla^2)\underline{u}\,, \quad \nabla^2 p = 0$$

In symmetric flow, we see that $p(x,y) - p(x,-y) =$ constant $= 0$ since p is assumed continuous everywhere. Similarly in antisymmetric flow $p(x,y) = -p(x,-y)$.

The potential ϕ is defined as

$$\phi(x,y) = \int_{-\infty}^{x} p(x',y)dx'$$

Hence in symmetric flow the potential ϕ is symmetric about the line $y = 0$ and in antisymmetric flow the potential ϕ is antisymmetric.

The streamfunction is defined as $\Psi = \Psi_u - \Psi_\phi$ where

$$\underline{u} = \left(\frac{\partial \Psi_u}{\partial y}, -\frac{\partial \Psi_u}{\partial x}\right) \; and \; \underline{\nabla\phi} = \left(\frac{\partial \Psi_\phi}{\partial y}, -\frac{\partial \Psi_\phi}{\partial x}\right)$$

In symmetric flow Ψ_u and Ψ_ϕ are antisymmetric and so Ψ is antisymmetric about the line $y = 0$.

Similarly, in antisymmetric flow Ψ is symmetric about the line $y = 0$.

We now consider the discontinuities in ϕ, Ψ and their derivatives for symmetric flow.

In this section, we find the discontinuities in ϕ and Ψ for symmetric Oseen flow, and thus find the complete Fourier expansions of ϕ and Ψ.

This section is divided into the following parts:

3.3.1 The discontinuity in symmetric flow in ϕ.

3.3.2 The expansion for ϕ in symmetric flow.

3.3.3 The discontinuity in Ψ for symmetric flow.

3.3.4 The expansion of Ψ in symmetric flow.

3.3.1 The discontinuity in ϕ for symmetric flow.

In symmetric flow, ϕ must be symmetric about $y = 0$ and so ϕ must be continuous across $y = 0$.

Since $\frac{\partial \phi}{\partial x} = -(1/\rho U)p$, it is continuous everywhere within the fluid.

However, we may expect a discontinuity in $\frac{\partial \phi}{\partial y}$ across the line $y = 0$, $x > 0$.

Across $y = 0$, we know that $\frac{\partial}{\partial y}\frac{\partial \phi}{\partial x} = -(1/\rho U)\frac{\partial p}{\partial y}$ is continuous and thus $\frac{\partial}{\partial x}\frac{\partial \phi}{\partial y}$ must therefore also be continuous. So:

$$\frac{\partial}{\partial x}\left.\frac{\partial \phi}{\partial y}\right|_{y=o+} = \frac{\partial}{\partial x}\left.\frac{\partial \phi}{\partial y}\right|_{y=0-}$$

This implies that:

$$\frac{\partial}{\partial x}\left\{\left.\frac{\partial \phi}{\partial y}\right|_{y=0+} - \left.\frac{\partial \phi}{\partial y}\right|_{y=0-}\right\} = 0$$

and therefore

$$\left.\frac{\partial \phi}{\partial y}\right|_{y=0+} - \left.\frac{\partial \phi}{\partial y}\right|_{y=0-} = \text{constant} = B \tag{3.13}$$

We find the potential ϕ_0 which gives the required discontinuity. Consider first the potential of a line of sources from $x = 0$ to $x = A$. This potential is $Re[\phi_A]$ where:

$$\begin{aligned}
\phi_A &= \int_0^A \log(z-\zeta)d\zeta \\
&= [-(z-\zeta)\log(z-\zeta) + (z-\zeta)]_0^A + F(A) \\
&= \{z\log z - z - (z-A)\log(z-A) + z - A\} + F(A)
\end{aligned}$$

We want the potential ϕ_A to be finite as $A \to \infty$. Thus we choose $F(A)$ such that as $A \to \infty$ then $\phi_A(z) \not\to \infty$ for all z.

We let $\phi_{OA} = \lim_{A\to\infty} \phi_A$

Choosing $f_A(z) = A + (z-A)\log(Ae^{i\pi}) + z$ we obtain

$$
\begin{aligned}
\phi_{OA}(z) &= z\log z - z - (z-A)\log(z-A) + z - A + A \\
&\quad +(z-A)\log(Ae^{i\pi}) + z \\
&= z\log z - (z-A)\log(1-z/A) + z
\end{aligned}
$$

As $A \to \infty$,

$$
\begin{aligned}
z\log(1-z/A) &\to 0 \text{ and} \\
A\log(1-z/A) &\to A[-z/A - (1/2)(z/A)^2 - (1/3)(z/A)^3 - .....] \\
&\to -z
\end{aligned}
$$

Thus $\lim_{A\to\infty}\{\phi_{OA}(z)\} = z\log z$.

We have found a potential $Re\{\phi_{OA}(z)\}$ which has the correct discontinuity of equation (3.13) and satisfies Laplaces equation.

However, this function isn't symmetric about $y = 0$. Hence the symmetric potential ϕ_0 which satisfies the correct discontinuity is given by

$$
\phi_0 = -\frac{B}{2\pi} Re\{z\log(ze^{-i\pi})\} \tag{3.14}
$$

for $0 \le \theta \le 2\pi$.

We now verify that the potential $Re\{\phi_0(z)\}$ gives the correct type of discontinuity.

We consider the complex potential $\Phi_0 = \frac{B}{2\pi} z\log ze^{-i\pi}$. Taking the derivative,

$$
\frac{d\Phi_0}{dz} = -\frac{B}{2\pi}(1 + \log r + i(\theta - \pi)) = \frac{\partial\phi_0}{\partial x} - i\frac{\partial\phi_0}{\partial y}
$$

So we have that $\frac{\partial\phi_0}{\partial y} = \frac{B\theta - \pi}{2\pi}$

When $\theta = 0$, $\frac{\partial\phi_0}{\partial y} = 0$ and when $\theta = 2\pi$, $\frac{\partial\phi_0}{\partial y} = B$.

The potential $\phi_0 = Re\{\Phi_0\}$ has the discontinuity given by equation (3.13).

Also $\frac{\partial \phi_0}{\partial x} = -\frac{B}{2\pi}(1 + \ln r)$.

However, from equation (3.2), $p_0 = -\rho U \frac{\partial \phi_0}{\partial x}$, where p_0 is the pressure term associated to the potential term ϕ_0.

Therefore $p_0 = \frac{B}{2\pi}(1 + \ln r)$.

However, the pressure tends to zero as $r \to 0$, and so $B = 0$.

3.3.2 The expansion of ϕ in symmetric flow.

The potential ϕ is represented by $\phi = \phi_0 + \phi_1$, where ϕ_1 is continuous everywhere.

We now consider a new origin at $(-R, 0)$ with new cartesian coordinates (x^*, y^*). Thus $(x^*, y^*) = (x + R, y)$ and we let the polar coordinates from this new origin be (r^*, θ^*) and so $r^{*2} = (x + R)^2 + y^2$. Hence from section (3.1.2) we see that Oseen flow holds in the region $r^* > R$.

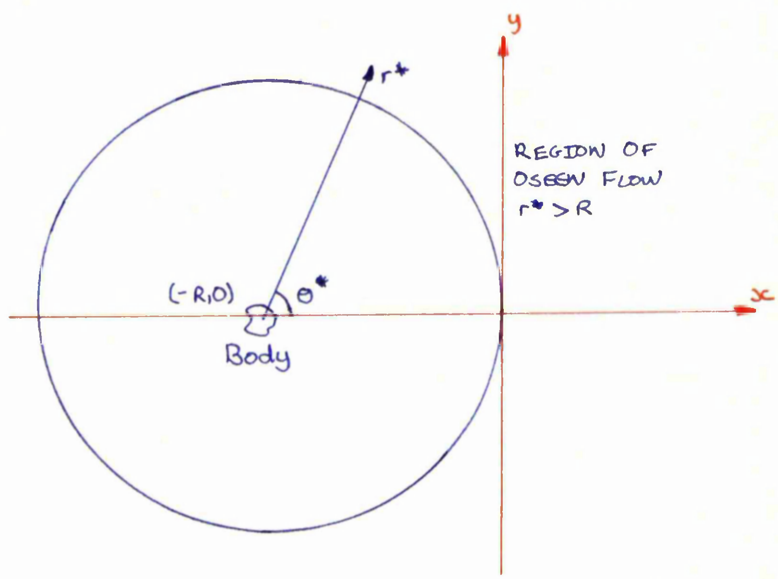


Figure 3.2: Diagram showing the region of Oseen flow with origin centered at the body.

We now consider Fourier's theorem . Although it would seem natural to apply Fourier's theorem for the variables (r^*, θ^*) we find that because the discontinuity line is along $x > 0$, $y = 0$ it is more useful to apply Fourier's theorem for the variables (r, θ). Fourier's theorem for Laplace's equation is given in appendix (A). Since ϕ is symmetric, we expect the potential ϕ to be of the form

$$\phi = \phi_0 + \sum_{n=1}^{n=\infty} \left(A'_n r^n + A_n r^{-n}\right) \cos n\theta + \frac{D}{2\pi\rho U} \log r$$

where A'_n, A_n and D are constants.

However, applying the condition that $\frac{\partial \phi}{\partial x} = -(1/\rho U)p$, where the pressure p is bounded at infinity, the above equation is considerably simplified.

For $n \geq 2$, the terms A'_n in the potential expansion give pressure terms of order at least

$$p = -\rho U \frac{\partial \phi}{\partial x} = O(r)$$

Thus $A'_n = 0$ for $n \geq 2$.

Therefore, as $r \to \infty$, we obtain

$$\frac{\partial \phi}{\partial x} = -(\frac{1}{\rho U})p = A'_1$$

However, from section (3.1.1) we take the pressure $p(x = -\infty, y) = 0$. Thus $A'_1 = 0$ and

$$\phi = \phi_0 + \sum_{n=1}^{n=\infty} \frac{A_n}{r^n} \cos n\theta + \frac{D}{2\pi\rho U} \log r \tag{3.15}$$

3.3.3 The discontinuity in Ψ for symmetric flow.

There is a discontinuity term Ψ_0 in Ψ which is calculated from the discontinuity in the velocity $\underline{\nabla\phi_0}$ and also a discontinuity term Ψ_1 in Ψ due to the application of the conservation of momentum law.

Therefore the discontinuity in $\Psi = \Psi_0 + \Psi_1$.

The discontinuity in the velocity along the infinite half line $x > 0$, $y = 0$ is given by equation (3.13)

$$\left.\frac{\partial\phi_0}{\partial y}\right|_{y=0+} - \left.\frac{\partial\phi_0}{\partial y}\right|_{y=0_-}$$

where

$$\phi_0 = -\frac{B}{2\pi}Re\{z\log(ze^{i\pi})\}$$

From section (3.3.1), we show that the constant $B = 0$ and so $\phi_0 \equiv 0$ and $\Psi_0 \equiv 0$.

We may also show this result by following the calculation described below. The full details of the method is given in the apppendix section (3.9).

The calculation is divided into four main parts:

1. The evaluation of the discontinuity term Ψ_0 in the stream function Ψ.
2. The integral representation of the function $F_0 = e^{-kx}\Psi_0$.
3. The evaluation of the integral in the far wake, particularly as $r \to \infty$, $\theta \to 0$.
4. The evaluation of the velocity term $\underline{u}_0 = \underline{\nabla\Phi_0} + \underline{\nabla\Psi_0}$ as $r \to \infty$, $\theta \to 0$.

Thus the above method calculates the velocity term $\underline{u}_0$ in the far wake close to the discontinuity line $y = 0$, $x > 0$.

We find that for $B \neq 0$ we obtain the result $\underline{u}_0 \to \infty$ in this region, which implies that $\underline{u} \to \infty$. Thus the condition of a uniform stream at infinity is violated and so we must have $B = 0$.

The first part of this section is to find the discontinuity in the stream function Ψ.

3.3.3.1 The discontinuity Ψ_1.

We now consider the discontinuity term Ψ_1. There is a discontinuity term Ψ_1 in Ψ which is related to the term $\frac{D}{2\pi\rho U}\log r$ in the expansion of ϕ given by equation (3.15). We will find this relation below. We introduce a cut along the infinite half line $y = 0$, $x > 0$, and thus the discontinuity in Ψ_1 is given by

$$\Psi_1(x > 0, y = 0_+) - \Psi_1(x > 0, y = 0_-) = -\int_0^{2\pi} \frac{\partial \Psi_1}{\partial \theta} d\theta$$

We consider the streamfunction Ψ_u of the velocity $\underline{u}$

$$\Psi_u = \Psi_\phi + \Psi$$

where Ψ is the streamfunction of $\underline{w}$ and Ψ_ϕ is the streamfunction of $\underline{\nabla\phi}$.(See section (3.1.2).)

Since there is no flux out of any closed contour around the body, then

$$\int_0^{2\pi} \frac{\partial \Psi_u}{\partial \theta} d\theta = 0$$

Therefore we obtain the relation

$$\int_0^{2\pi} \frac{\partial \Psi}{\partial \theta} d\theta = -\int_0^{2\pi} \frac{\partial \Psi_\phi}{\partial \theta} d\theta$$

However,

$$\frac{1}{r}\frac{\partial \Psi_\phi}{\partial \theta} = \underline{\nabla\Phi_\theta}.\underline{n_\theta} = \frac{\partial \Psi_\phi}{\partial y}\cos\theta - \frac{\partial \Psi_\phi}{\partial x}\sin\theta = \frac{\partial \phi}{\partial x}\cos\theta + \frac{\partial \phi}{\partial y}\sin\theta = \underline{\nabla\phi}.\underline{n_r} = \frac{\partial \phi}{\partial r}$$

where $\underline{n_\theta}$ is the unit normal in the θ direction and $\underline{n_r}$ is the unit normal in the r direction.

Thus

$$\Psi_1(x > 0, y = 0_+) - \Psi_1(x > 0, y = 0_-) = \int_0^{2\pi} \frac{\partial \phi}{\partial r} r\, d\theta$$

We refer to the expansion of ϕ given in section (3.3.2) equation (3.15).All the terms in the expansion except for the term $\frac{D}{2\pi\rho U}\log r$ give no contribution to the integral $\int_0^{2\pi}\frac{\partial\phi}{\partial r}rd\theta$.

Therefore

$$\int_0^{2\pi}\frac{\partial\phi}{\partial r}rd\theta = \int_0^{2\pi}\frac{\partial}{\partial r}(\frac{D}{2\pi\rho U}\log r)rd\theta = \frac{D}{\rho U}$$

Hence the discontinuity in Ψ_1 is

$$\Psi_1(x>0, y=0_+) - \Psi_1(x>0, y=0_-) = \frac{D}{\rho U}$$

3.3.3.2 The integral representation of the function Ψ_1.

Refering to appendix (3.3.1), the potential $F(x)$ due to a Helmholtz dipole line along $y = 0$, $x > 0$ of strength $S(x)$ is

$$F(x, y) = -\int_0^\infty S(\xi)\frac{\partial}{\partial y}K_0[k\{(x-\xi)^2 + y^2\}^{1/2}]d\xi$$

and has a discontinuity across the half line $y = 0$, $x > 0$ such that

$$[F(x)]_{y \to 0_-}^{y \to 0_+} = 2\pi S(x)$$

(For small r, $K_0(kr) \to -\log r$ and $\frac{\partial}{\partial y}(\log r)$ is a dipole orientated along the y-axis.)

We consider the equation $\Psi_1 = e^{kx}F_1$. Then $F_1(x, y)$ satisfies the modified Helmholtz equation and has a discontinuity across $y = 0$, $x > 0$ such that

$$[F_1(x)]_{y \to 0_-}^{y \to 0_+} = \frac{D}{\rho U}Ce^{-kx}$$

Therefore

$$F_1(x, y) = -\int_0^\infty \frac{D}{\rho U}e^{-k\xi}\frac{\partial}{\partial y}K_0[k\{(x-\xi)^2 + y^2\}^{1/2}]d\xi \tag{3.16}$$

We now consider the expansion of Ψ for symmetric flow.

3.3.4 The expansion of Ψ for symmetric flow.

We first find the differential equation satisfied by the velocity $\underline{w}$ of the Lamb-Goldstein velocity decomposition $\underline{u} = \underline{\nabla\phi} + \underline{w}$ where the potential ϕ is defined as $\phi = \frac{D\log r}{2\pi\rho U} - \frac{1}{\rho U}\int_{-\infty}^{x}\{p(x',y) + \frac{D\cos\theta'}{2\pi r'}dx'\}$. This is found by substituting the velocity decomposition into Oseen's equations for steady two dimensional flow given by

$$U\frac{\partial \underline{u}}{\partial x} = -\frac{1}{\rho}\underline{\nabla p} + (\nu\nabla^2)\underline{u}$$

Substituting $\underline{u} = \underline{\nabla\phi} + \underline{w}$ into the above equation gives

$$U\frac{\partial \underline{w}}{\partial x} = \nu\nabla^2\underline{w}$$

since

$$(\nabla^2)\underline{\nabla\phi} = \frac{\partial^2}{\partial x_i\partial x_i}\left(\frac{\partial\phi}{\partial x_j}\right) = \frac{\partial}{\partial x_j}\left(\frac{\partial^2\phi}{\partial x_i\partial x_i}\right) = \nabla(\nabla^2\phi) = 0$$

and $\frac{\partial\phi}{\partial x} = -\frac{1}{\rho U}p$.

The streamfunction Ψ of the velocity $\underline{w}$ gives

$\underline{w} = \left(\frac{\partial\Psi}{\partial y}, -\frac{\partial\Psi}{\partial x}\right)$ and so

$$\left(\nabla^2 - \frac{U}{\nu}\frac{\partial}{\partial x}\right)\Psi = E \quad \rightarrow \quad \left(\nabla^2 - \frac{U}{\nu}\frac{\partial}{\partial x}\right)\Psi_E = 0$$

where $\Psi = -\frac{\nu E x}{U} + \Psi_E$ and E is a constant.

Splitting the function Ψ_E into an antisymmetric part Ψ_{Es} and a symmetric part Ψ_{Ea}, we thus obtain

$\Psi_s = \Psi_{Es}$ and $\Psi_a = -\frac{\nu E x}{U} + \Psi_{Ea}$ where Ψ_s and Ψ_a are the streamfunctions in symmetric and antisymmetric flow respectively.

In symmetric flow, the streamfunction $\Psi = \Psi_s$ is antisymmetric about the line $y = 0$ and so the term $-\frac{\nu E x}{U}$ must be zero which implies that $E = 0$ in symmetric flow. Hence in symmetric flow

$$\left(\nabla^2 - \frac{U}{\nu}\frac{\partial}{\partial x}\right)\Psi_s = 0$$

In antisymmetric flow,

$$\left(\nabla^2 - \frac{U}{\nu}\frac{\partial}{\partial x}\right)\Psi_{Ea} = 0 \;\; and \;\; \Psi_a = -\frac{\nu E x}{U} + \Psi_{Ea} \tag{3.17}$$

We now give the expansion for Ψ in symmetric flow. Following the method in section (3.2.2), we consider the function $F_s = e^{-kx}\Psi_s$ which gives

$$(\nabla^2 - k^2)F_s = 0 \;\text{ where }\; 2k = \frac{U}{\nu}$$

Applying Fourier's theorem from appendix B.2, since $F_s - F_0 - F_1$ is continuous everywhere and is an antisymmetric function about $y = 0$, then

$$F_s - F_0 - F_1 = \sum_{n=1}^{\infty} f_n(r)\sin n\theta$$

where

$$f_n(r) = \frac{1}{\pi}\int_0^{2\pi}[F_s - F_0 - F_1]\sin\theta d\theta$$

Applying the operator

$$\frac{1}{r}\left\{\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)\right\}$$

to the above equation, and since $(\nabla^2 - k^2)F_s = 0$, $(\nabla^2 - k^2)F_0 = 0$ and $(\nabla^2 - k^2)F_1 = 0$, then we obtain the equation

$$r^2 f_n(r) + r f_n'(r) - [k^2r^2 + n^2]f_n(r) = 0$$

which has solutions $I_n(kr)$ and $K_n(kr)$, of the modified Bessel equation. Thus $F_s - F_0 - F_1$ is expressed in the form

$$F_s - F_0 - F_1 = \sum_{n=1}^{\infty}\{b_n K_n(kr) + b'_n I_n(kr)\}\sin n\theta$$

where b_n and b'_n are constants.

The streamfunction in symmetric flow is

$$\begin{aligned}\Psi_s &= e^{kx}F_s \\ &= \Psi_0 + \Psi_1 + e^{k(r\cos\theta)}\sum_{n=1}^{n=\infty}\{b_n K_n(kr) + b'_n I_n(kr)\}\sin n\theta\end{aligned}$$

However, we must satisfy the boundary condition at infinity that

$$\underline{w}_s = \left(\frac{\partial\Psi_s}{\partial y}, -\frac{\partial\Psi_s}{\partial x}\right) \rightarrow 0 \quad \text{as} \quad r \rightarrow \infty$$

This is satisfied by the terms $K_n(kr) \rightarrow \sqrt{\frac{\pi}{2kr}}e^{-kr}$ as $r \rightarrow \infty$. (From Abramowitz and Stegun pg 378 9.7.2.)

But,

$$I_n(kr) \rightarrow \frac{e^{kr}}{\sqrt{2\pi kr}}\left\{1 - (\frac{4n^2-1}{8kr}) + ...\right\}$$

Thus $b'_n = 0$. (From Abramowitz and Stegun pg 378 9.7.1.)

Thus the streamfunction in symmetric flow is

$$\Psi_s = \Psi_0 + \Psi_1 + e^{k(r\cos\theta)}\sum_{n=1}^{\infty} b_n K_n(kr)\sin n\theta \tag{3.18}$$

3.4 Antisymmetric flow.

In antisymmetric flow, the potential ϕ and the pressure p are antisymmetric about $y = 0$ and the streamfunction Ψ is symmetric about $y = 0$.

Thus in antisymmetric flow we have a discontinuity in ϕ along $y = 0$, $x > 0$. This is considered next.

3.4.1 The discontinuity in ϕ for antisymmetric flow.

In antisymmetric flow, there is a discontinuity $\frac{L(x)}{\rho U}$ in ϕ along $y = 0$, $x > 0$. Thus

$$\lim_{y \to 0_+ \; x>0} [\phi(x, y) - \phi(x, -y)] = \frac{L(x)}{\rho U}$$

However, we know that the pressure $p = -\frac{1}{\rho U}\frac{\partial \phi}{\partial x}$ is continuous along $y = 0$, $x > 0$. Thus

$$\begin{aligned} \frac{1}{\rho U}\frac{d}{dx}L(x) &= \frac{\partial}{\partial x}\{\lim_{y\to 0_+} \phi(x,y) - \phi(x,-y)\} = \lim_{y \to 0_+}\left\{\frac{\partial}{\partial x}\phi(x,y) - \frac{\partial}{\partial x}\phi(x,-y)\right\} \\ &= -\rho U\{\lim_{y\to 0_+} p(x,y) - p(x,-y)\} = 0 \end{aligned}$$

Thus $\frac{L(x)}{\rho U}$ is a constant $\frac{L}{\rho U}$.

From appendix section (3.3.1) the potential due to a line of dipoles of strength $\frac{L(x)}{2\pi\rho U}$ along the line $y = 0$, $x > 0$ has a discontinuity $\frac{L(x)}{\rho U}$ along this line.

Thus we express the term ϕ_1 giving a discontinuity in ϕ as

$$\phi_1 = \lim_{x_1 \to \infty} \int_0^{x_1} \frac{L(x)}{2\pi\rho U}\frac{\partial}{\partial y}\{\log[(x-\zeta)^2 + y^2]^{1/2} d\zeta + A(x_1)\}$$

where $A(x_1)$ is chosen so that as $x_1 \to \infty$, ϕ_1 tends to a limit.

This integral is solved more easily by using the complex variable $z = x + iy$. Thus ϕ_1 has the same discontinuity as $Re\{\Phi_1\}$ where

$$\begin{aligned}\Phi_1 &= \lim_{x_1\to\infty}\left\{\frac{L}{2\pi\rho U}\int_0^{x_1}\frac{\partial}{\partial y}\log(z-\xi)d\xi + A(x_1)\right\} \qquad (3.19)\\ &= \frac{L}{2\pi\rho U}\lim_{x_1\to\infty}\left\{-\int_0^{x_1} id[\log(z-\xi)] + A(x_1)\right\}\end{aligned}$$

since $\frac{\partial}{\partial y}\log(z-\xi)d\xi = i\frac{d\xi}{z-\xi} = -id(z-\xi)$. Therefore

$$\Phi_1 = -\frac{L}{2\pi\rho U}\lim_{x_1\to\infty}\{i\log(z-x_1) - i\log z + A(x_1)\}$$

and if we choose $A(x_1) = -i\log(x_1 e^{i\pi})$, then

$$\begin{aligned}\Phi_1 &= -\frac{L}{2\pi\rho U}\lim_{x_1\to\infty}\{i\log(1-\frac{z}{x_1}) - i\log z\}\\ &= i\frac{L}{2\pi\rho U}\log z\end{aligned}$$

In polar coordinates, $i\log z = i\{\log r + i\theta\} = i\log r - \theta$.

Thus

$$Re\{\Phi_1\} = -\frac{L\theta}{2\pi\rho U}$$

The function ϕ_1 is antisymmetric about $y = 0$ and therefore

$$\phi_1 = -\frac{L(\theta-\pi)}{2\pi\rho U} \qquad (3.20)$$

[We now check that ϕ_1 and its derivatives have the correct discontinuity along $y = 0$, $x > 0$:

$$\phi_1|_{\theta=0} - \phi_1|_{\theta=2\pi} = -\frac{L}{2\pi\rho U}(0-2\pi) = \frac{L}{\rho U}$$

and

$$\frac{\partial \phi_1}{\partial x} = -\frac{L}{2\pi\rho U}\frac{\partial \theta}{\partial x} = -\frac{L}{2\pi\rho U}\frac{\sin\theta}{r}$$

which is continuous along $y = 0$, $x > 0$.]

3.4.2 The expansion of ϕ in antisymmetric flow.

In antisymmetric flow, $\phi_a - \phi_1$ is continuous and we give an expansion for this function by applying Fourier's theorem. The potential ϕ satisfies Laplace's equation and from appendix (A) we see that

$$\phi_a - \phi_1 = \sum_{n=1}^{n+\infty} \left(\frac{B_n}{r^n} + B'_n r^n \right) \sin n\theta$$

But the terms in the expansion for B'_n where $n \geq 2$ give terms in the pressure at least of order r as $r \to \infty$:

B'_n gives $p = -\frac{1}{\rho U} \frac{\partial \phi}{\partial x} = O(r^{n-1})$. Thus $B'_n = 0$ for $n \geq 2$ for the pressure condition at infinity to be satisfied.

The potential term involving the coefficient B'_1 is $B'_1 r \sin\theta = B'_1 y$. However, since $\phi = -\rho U \int_{-\infty}^{x} p(x', y) dx'$, then $\frac{\partial}{\partial y}\phi(-\infty, y) = 0$. Thus $B'_1 = 0$.

Therefore the expansion for the potential ϕ in antisymmetric flow is

$$\phi_a = -\frac{L(\theta - \pi)}{2\pi \rho U} + \sum_{n=1}^{\infty} B_n \frac{\sin n\theta}{r^n} \qquad (3.21)$$

3.4.3 The discontinuity in Ψ for antisymmetric flow.

In antisymmetric flow, the streamfunction ψ is symmetric. Thus we may expect a discontinuity in the streamfunction derivative $\frac{\partial \psi}{\partial y}$, but no discontinuity in the streamfunction ψ.

However,

$$\underline{u} = \underline{\nabla \phi} + \underline{w}$$

and so

$$w_1 = \frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y}$$

Since $p = -\frac{1}{\rho U}\frac{\partial \phi}{\partial x}$, and p, $\underline{u}$, and their derivatives are assumed everywhere continuous, then $\frac{\partial \psi}{\partial y}$ is continuous everywhere.

Thus in antisymmetric flow, we expect no discontinuities in the streamfunction Ψ and its derivatives.

3.4.4 The expansion of Ψ for antisymmetric flow.

In antisymmetric flow $\Psi = \Psi_a$ and from equation (3.11) we see that

$$(\nabla^2 - \frac{U}{\nu}\frac{\partial}{\partial x})\Psi_a = E \quad \rightarrow \quad (\nabla^2 - \frac{U}{\nu}\frac{\partial}{\partial x})\Psi_{Ea} = 0$$

where $\Psi_a = \Psi_{Ea} - \frac{Ex\nu}{U}$.

Letting $\frac{U}{\nu} = 2k$, and by following the method in section (3.2.2) this gives

$$(\nabla^2 - k^2)F_a = 0 \text{ where } F_a = e^{-kx}\Psi_{Ea}$$

Applying Fourier's theorem from appendix (A), since F_a is continuous everywhere and symmetric about $y = 0$, then

$$F_a = \sum_{n=1}^{\infty} g_n(r)\cos n\theta$$

where

$$g_n(r) = \frac{1}{\pi}\int_0^{2\pi} F_a(r,\theta)\cos n\theta d\theta$$

Applying the operator

$$\frac{1}{r}\left\{\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)\right\}$$

to the above equation, and using the operator $(\nabla^2 - k^2)F_a = 0$, gives

$$r^2 g_n(r) + r g_n'(r) - [k^2r^2 + n^2]g_n(r) = 0$$

which has solutions $I_n(kr)$ and $K_n(kr)$, of the modified Bessel equation.

Thus F_a is expressed in the form

$$F_a = \sum_{n=0}^{\infty}\{a_n K_n(kr) + a_n' I_n(kr)\}\cos n\theta$$

where a_n and a_n' are constants. The streamfunction in antisymmetric flow is

$$
\begin{aligned}
\Psi_a &= e^{kx}F_a - \frac{Ex}{2k} \\
&= e^{k(r\cos\theta)} \sum_{n=0}^{n=\infty} \{a_n K_n(kr) + a_n\prime I_n(kr)\} \cos n\theta\} - \frac{E}{2k}(r\cos\theta)
\end{aligned}
$$

However, we must satisfy the boundary condition at infinity that

$$
\underline{w}_a = \left(\frac{\partial \Psi_a}{\partial y}, -\frac{\partial \Psi_a}{\partial x}\right) \rightarrow 0 \text{ as } r \rightarrow \infty
$$

This is satisfied by the terms $K_n(kr)$ since $K_n(kr) \rightarrow \sqrt{\frac{\pi}{2kr}}e^{-kr}$ as $r \rightarrow \infty$. (From Abramowitz and Stegun pg 378 9.7.2.)

But,

$$
I_n(kr) \rightarrow \frac{e^{kr}}{\sqrt{2\pi kr}} \left\{1 - (\frac{4n^2-1}{8kr}) + ...\right\}
$$

Thus $a_n' = 0$. (From Abramowitz and Stegun pg 378 9.7.1.)

This gives at infinity

$$
\underline{w}_a = \left(0, -\frac{E}{2k}\right)
$$

Therefore, $E = 0$ in order that the boundary condition at infinity is satisfied.

Thus the streamfunction in antisymmetric flow is

$$
\Psi_a = e^{k(r\cos\theta)} \sum_{n=1}^{\infty} a_n K_n(kr) \cos n\theta \tag{3.22}
$$

3.5 The complete expansions for ϕ, Ψ, velocity and pressure.

3.5.1 The expansion of the pressure p.

We consider the Fourier expansion for p. The pressure satisfies Laplace's equation and we apply Fourier's theorem (see appendix (A)) to give an expansion for the pressure p. The pressure is taken to tend to zero at infinity, (see section (3.1.1)) and so we obtain p as an expansion in the form

$$p(r,\theta) = \sum_{1}^{\infty} \left\{ (C_n r^{-n} \cos n\theta + D_n r^{-n}) \sin n\theta \right\} \tag{3.23}$$

3.5.2 Expansion of the potential ϕ.

We combine the expansions for the potential in symmetric flow, ϕ_s, equation (3.15) and antisymmetric flow, ϕ_a, (equation (3.21)) to obtain the general expansion for the potential ϕ.

$$\phi(r,\theta) = \frac{D}{2\pi\rho U}\log r - \frac{L(\theta-\pi)}{2\pi\rho U} + \sum_1^\infty (A_n r^{-n})\cos n\theta + (B_n r^{-n})\sin n\theta \qquad (3.24)$$

From equation (3.2), the pressure p and potential ϕ are related by

$$p = -\rho U \frac{\partial \phi}{\partial x}$$

Differentiating the above expansion with respect to x term by term we equate the coefficients in the pressure and potential expansions.

$$\begin{aligned} p &= -\frac{D}{2\pi}\frac{\cos\theta}{r} - \frac{L}{2\pi}\frac{\sin\theta}{r} + \rho U \sum_{n=1}^{\infty}\left\{ A_n n \frac{\cos(n+1)\theta}{r^{(n+1)}} + B_n n \frac{\sin(n+1)\theta}{r^{(n+1)}} \right\} \\ &= -\frac{D}{2\pi}\frac{\cos\theta}{r} - \frac{L}{2\pi}\frac{\sin\theta}{r} + \rho U \sum_{n=2}^{\infty}\left\{ (n-1)A_{n-1}\frac{\cos n\theta}{r^n} + B_{n-1}(n-1)\frac{\sin n\theta}{r^n} \right\} \end{aligned}$$

Therefore $C_1 = -\frac{D}{2\pi}$, $D_1 = -\frac{L}{2\pi}$, and for $n \geq 2$ $C_n = \rho U(n-1)A_{n-1}$, $D_n = \rho U(n-1)B_{n-1}$.

3.5.3 The expansion of the function Ψ.

We combine the expansions for the function Ψ in symmetric flow, Ψ_s, and antisymmetric flow, Ψ_a, to obtain the expansion for Ψ in general flow

$$\Psi(r,\theta) = \Psi_1 + e^{kx}\left\{\sum_{n=1}^{\infty} a_n K_n(kr)\cos n\theta + \sum_{n=1}^{\infty} b_n K_n(kr)\sin n\theta\right\} \tag{3.25}$$

where

$$\Psi_1 = -e^{kx}\int_0^{\infty} \frac{D}{2\pi\rho U} e^{-k\zeta} \frac{\partial}{\partial y} K_0[k\{(x-\zeta)^2 + y^2\}^{1/2}]d\zeta$$

3.5.4 The expansion for the velocity.

The perturbation velocity is given by

$$\underline{u} = \underline{\nabla \phi} + \underline{w}$$

We first consider the expansion of $\underline{\nabla \phi}$. From equation (3.24), we differentiate term by term to obtain

$$\underline{\nabla \phi} = \left(\sum_{n=1}^{\infty} \frac{1}{r^n} \{E_n \cos n\theta + F_n \sin n\theta\}, \sum_{n=1}^{\infty} \frac{1}{r^n} \{G_n \cos n\theta + H_n \sin n\theta\} \right) \qquad (3.26)$$

where E_n, F_n, G_n and H_n are constants related to the constants D, L, A_n and B_n.

We next consider the expansion of $\underline{w}$. We find the expansion of $w_1 = \frac{\partial \Psi}{\partial y}$ by differentiating w.r.t. y every term in the expansion of Ψ from equation (3.25).

We first consider the term $\frac{\partial}{\partial y} (K_n(kr) \cos n\theta)$.

By differentiating we obtain

$$\begin{aligned}
& \frac{\partial}{\partial y} (K_n(kr) \cos n\theta) \\
= \;& kK_n'(kr) \sin\theta \cos n\theta - \frac{n}{r} K_n(kr) \sin n\theta \cos\theta \\
= \;& 1/2 \left\{ kK_n'(kr) - \frac{n}{r} K_n(kr) \right\} \sin(n+1)\theta \\
- \;& 1/2 \left\{ kK_n'(kr) + \frac{n}{r} K_n(kr) \right\} \sin(n-1)\theta \\
= \;& (1/2) k K_{n+1}(kr) \sin(n+1)\theta - (1/2) k K_{n-1}(kr) \sin(n-1)\theta
\end{aligned}$$

since from Abramowitz and Stegun equation (9.6.28), (pg 376)

$kK_n'(kr) - \frac{n}{r} K_n(kr) = kK_{n+1}(kr)$ and $kK_n'(kr) + \frac{n}{r} K_n(kr) = kK_{n-1}(kr)$.

Similarly,

$$\frac{\partial}{\partial y} (K_n(kr) \sin n\theta) = (1/2) k K_{n-1}(kr) \cos(n-1)\theta - (1/2) k K_{n+1}(kr) \cos(n+1)\theta$$

Thus w_1 has the form

$$w_1 = \frac{\partial \Psi_1}{\partial y} + e^{kx}\left\{\sum_{n=0}^{\infty} c'_n K_n(kr)\cos n\theta + \sum_{n=1}^{\infty} d'_n K_n(kr)\sin n\theta\right\}$$

where c'_n and d'_n are constants. We now consider the function $\frac{\partial \Psi_1}{\partial y}$.

From equation (3.16),

$$\Psi_1 = e^{kx}\left\{-\int_0^{\infty} \frac{D}{2\pi\rho U} e^{-k\xi} \frac{\partial}{\partial y} K_0(kr_\xi)d\xi\right\}$$

where $r_\xi = [(x-\xi)^2 + y^2]^{1/2}$.

We change the variable of integration to $p = \xi - x$ to obtain

$$\Psi_1 = -\frac{D}{2\pi\rho U}\int_{-x}^{\infty} e^{-kp}\frac{\partial}{\partial y}K_0(kr_p)dp$$

where $r_p = [p^2 + y^2]^{1/2}$.

Differentiating w.r.t. y, we obtain

$$\frac{\partial \Psi_1}{\partial y} = -\frac{D}{2\pi\rho U}\int_{-x}^{\infty} e^{-kp}\frac{\partial^2}{\partial y^2}K_0(kr_p)dp$$

Using the condition that $(\nabla^2 - k^2)K_0(kr) = 0$, then

$$\frac{\partial \Psi_1}{\partial y} = -\frac{D}{2\pi\rho U}\int_{-x}^{\infty} e^{-kp}\left\{k^2 K_0(kr_p) - \frac{\partial^2}{\partial p^2}K_0(kr_p)\right\}dp$$

We now consider the integral

$$\int_{-x}^{\infty} e^{-kp}\frac{\partial^2}{\partial p^2}K_0(kr_p)dp$$

Integrating by parts, we obtain

$$\left[e^{-kp}\frac{\partial}{\partial p}K_0(kr_p)\right]_{-x}^{\infty} + k\int_{-x}^{\infty} e^{-kp}\frac{\partial}{\partial p}K_0(kr_p)dp$$

$$= e^{kx}\frac{\partial}{\partial x}K_0(kr) + k\left\{\left[e^{-kp}K_0(kr_p)\right]_{-x}^{\infty} + k\int_{-x}^{\infty} e^{-kp}K_0(kr_p)dp\right\}$$

$$= e^{kx}\frac{\partial}{\partial x}K_0(kr) - ke^{kx}K_0(kr) + k^2\int_{-x}^{\infty} e^{-kp}K_0(kr_p)dp$$

Therefore

$$\frac{\partial}{\partial y}\Psi_1 = \frac{D}{2\pi\rho U}e^{kx}\left\{\frac{\partial}{\partial x}K_0(kr) - kK_0(kr)\right\} \tag{3.27}$$

and w_1 has the expansion

$$w_1 = e^{kx}\left\{c_0K_0(kr) + \sum_{n=1}^{\infty} c_nK_n(kr)\cos n\theta + \sum_{n=1}^{\infty} d_nK_n(kr)\sin n\theta\right\} \tag{3.28}$$

where c_n and d_n are constants.

We now consider the expansion of w_2. Following the same method for finding the expansion of w_1, we differentiate term by term w.r.t. x the terms in the expansion for Ψ, and similarly we obtain

$$w_2 = -\frac{\partial \Psi}{\partial x} = -\frac{\partial \Psi_1}{\partial x} + \sum_{n=1}^{\infty} e'_n K_n(kr)\cos n\theta + \sum_{n=1}^{\infty} f'_n K_n(kr)\sin\theta$$

where e'_n and f'_n are constants.

We now evaluate $\frac{\partial \Psi_1}{\partial x}$

From equation (3.16),

$$\Psi_1 = -\frac{D}{2\pi\rho U}e^{kx}\int_0^{\infty} e^{-k\xi}\frac{\partial}{\partial y}K_0(kr_\xi)d\xi$$

where $r_{xi} = [(x-\xi)^2 + y^2]^{1/2}$. Differentiating w.r.t. x

$$\frac{\partial \Psi_1}{\partial x} = k\Psi_1 - \frac{D}{2\pi\rho U}e^{kx}\int_0^{\infty} e^{-k\xi}\frac{\partial}{\partial y}\frac{\partial}{\partial x}K_0(kr_\xi)d\xi$$

We consider the function

$$\frac{\partial}{\partial x}K_0(kr_\xi)$$

where $r_\xi = [(x-\xi)^2 + y^2]^{1/2}$. We change from the variable x to the variable $p = \xi - x$, and so

$$\frac{\partial}{\partial x}K_0(kr_\xi) = \frac{dp}{dx}\frac{\partial}{\partial p}K_0(kr_p) = -\frac{\partial}{\partial p}K_0(kr_p)$$

where $r_p = [p^2 + y^2]^{1/2}$.

Therefore

$$\frac{\partial \Psi_1}{\partial x} = k\Psi_1 + \frac{D}{2\pi\rho U}\int_0^{\infty} e^{-kp}\frac{\partial}{\partial p}\left\{\frac{\partial}{\partial y}K_0(kr_p)\right\}d\xi(p)$$

The variable $\xi(p) = p + x$, and so changing the variable of integration and then differentiating by parts we obtain

$$
\begin{aligned}
\frac{\partial \Psi_1}{\partial x} &= k\Psi_1 + \frac{D}{2\pi\rho U}\int_{-x}^{\infty} e^{-kp}\frac{\partial}{\partial p}\left\{\frac{\partial}{\partial y}K_0(kr_p)\right\}dp \\
&= k\Psi_1 + \frac{D}{2\pi\rho U}\left\{\left[e^{-kp}\frac{\partial}{\partial y}K_0(kr_p)\right]_{-x}^{\infty} + k\int_{-x}^{\infty} e^{-kp}\frac{\partial}{\partial y}K_0(kr_p)dp\right\} \\
&= k\Psi_1 - \frac{D}{2\pi\rho U}e^{kx}\frac{\partial}{\partial y}K_0(kr) - k\Psi_1 \\
&= -\frac{D}{2\pi\rho U}e^{kx}\frac{\partial}{\partial y}K_0(kr) \qquad (3.29)
\end{aligned}
$$

Thus w_2 has the expansion

$$
w_2 = e^{kx}\left\{\sum_{n=1}^{\infty} e_n K_n(kr)\cos n\theta + \sum_{n=1}^{\infty} f_n K_n(kr)\sin n\theta\right\} \qquad (3.30)
$$

We expected the expansions for w_1 and w_2 to be of the form given by equations (3.28) and (3.30) respectively since w_1 and w_2 are continuous functions which tend to zero as $r \to \infty$ and which satisfy the modified Helmholtz equation.

$$
(\nabla^2 - 2k\frac{\partial}{\partial x})(w_1, w_2) = (0, 0)
$$

The coefficients e_n and f_n of equation (3.30) are related to the coefficints c_n and d_n of equation (3.28) since the continuity equation $\underline{\nabla}.\underline{w} = 0$ is satisfied by the velocity $\underline{w}$.

3.6 The form of the solution in the far field wake.

In the far field,

$$\phi \sim \frac{D}{2\pi\rho U}\log r - \frac{L(\theta-\pi)}{2\pi\rho U}$$

and

$$\Psi \sim \Psi_1 + s\sqrt{\frac{\pi}{2kr}}e^{k(x-r)}$$

since from Abramowitz and Stegun equation (9.7.2) (pg 378),the asymptotic expansion of $K_n(kr)$ is

$$K_n(kr) \sim \sqrt{\frac{\pi}{2kr}}e^{-kr}\left(1 + \frac{4n^2-1}{8kr} + ...\right)$$

and $s = \sum_{n=1}^{\infty} a_n$, where the coefficients a_n are given in section (3.4.3).

Thus in the far field wake,

$$\Psi \sim \Psi_1 + s\sqrt{\frac{\pi}{2kx}}e^{\frac{-ky^2}{2x}}$$

From equation (3.27), we have that

$$\frac{\partial \Psi_1}{\partial y} = \frac{D}{2\pi\rho U}e^{kx}\left\{\frac{\partial}{\partial x}K_0(kr) - kK_0(kr)\right\} = \frac{D}{2\pi\rho U}e^{kx}\left\{-kK_1(kr)\cos\theta - kK_0(kr)\right\}$$

and so in the far field wake

$$\frac{\partial \Psi_1}{\partial y} \sim -\frac{D}{\pi\rho U}k\sqrt{\frac{\pi}{2kx}}e^{\frac{-ky^2}{2x}} \tag{3.31}$$

From equation (3.2.9), we have that

$$\frac{\partial \Psi_1}{\partial x} = \frac{D}{2\pi\rho U}e^{kx}\frac{\partial}{\partial y}K_0(kr) = -\frac{D}{2\pi\rho U}ke^{kx}K_1(kr)\sin\theta$$

and so in the far field wake

$$\frac{\partial \Psi_1}{\partial x} = -\frac{D}{2\pi\rho U}\sqrt{\frac{k\pi}{2}}\frac{y}{x\sqrt{x}}e^{-\frac{ky^2}{2x}} \tag{3.32}$$

The second order terms for the far field wake velocity are due to the stream function term in antisymmetric flow

$$\Psi \sim s\sqrt{\frac{\pi}{2kr}}e^{k(x-r)} = s\sqrt{\frac{\pi}{2kx}}e^{-\frac{ky^2}{2x}}$$

This gives

$$w_{2a} = -\frac{\partial \Psi_a}{\partial x} = -\frac{\partial}{\partial x}\left\{s\sqrt{\frac{\pi}{2kx}}e^{-\frac{ky^2}{2x}}\right\} = -s\sqrt{\frac{\pi}{2kx}}e^{-\frac{ky^2}{2x}}\left\{-\frac{1}{2x}+\frac{ky^2}{2x^2}\right\}$$

and

$$w_{1a} = \frac{\partial \Psi_a}{\partial y} = \frac{\partial}{\partial y}\left\{s\sqrt{\frac{\pi}{2kx}}e^{-\frac{ky^2}{2x}}\right\} = s\sqrt{\frac{\pi}{2kx}}\left\{-\frac{ky}{x}\right\}e^{-\frac{ky^2}{2x}}$$

Thus in the far field wake,

$$\underline{w} \sim \left(-\frac{D}{\pi\rho U}k\sqrt{\frac{\pi}{2kx}}e^{-k\frac{y^2}{2x}} - \frac{sy}{x}\sqrt{\frac{\pi k}{2x}}e^{-\frac{ky^2}{2x}}, \frac{D}{2\pi\rho U}\sqrt{\frac{k\pi}{2}}\frac{y}{x\sqrt{x}}e^{-\frac{ky^2}{2x}} + s\sqrt{\frac{\pi}{2kx}}e^{-\frac{ky^2}{2x}}\left\{\frac{1}{2x}-\frac{k}{2}\right.\right. \tag{3.33}$$

This is an important result; we note that the order of the function w_2 is $\frac{1}{x\sqrt{x}}e^{-\frac{ky^2}{2x}}$ in the far field. we shall see later that this implies that there is no contribution to the lift on the body due to a wake traverse.

3.7 Ch3 Appendix: The stream function ψ.

We can define a streamfunction ψ if the integral

$$\oint_C \underline{u}.\underline{n}dl = \int_S \underline{\nabla}.\underline{u}ds = 0$$

where C is a closed curve and S is the area within the closed curve C.

For our problem, taking a closed contour enclosing the body , $\oint_C \underline{u}.\underline{n}dl$ is non zero. Thus in order to define the streamfunction ψ we consider a simply connected region of fluid by introducing the half line cut along $y = 0$, $x > 0$.

Thus in the region defined in this way,

$$\int_S \underline{\nabla}.\underline{u}ds = 0$$

due to mass conservation and so

$$\int_S \underline{\nabla}.\underline{u}ds = \oint_C \underline{u}.\underline{n}dl = 0$$

Thus we may define the single valued function ψ such that

$$\psi(p) = \int_{p_0}^{p} (u_1 n_1 + u_2 n_2)dl$$

where n is the unit normal from left to right for increasing positive dl and p_0 is a reference point and p is a general point.

Thus

$$\delta\psi = \frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}dy = u_1 dy - u_2 dx$$

and so

$$(u_1, u_2) = \left(\frac{\partial\psi}{\partial y}, -\frac{\partial\psi}{\partial x}\right)$$

3.8 Ch3 Appendix: The discontinuities due to source and dipole lines.

In this section, we look at the discontinuities in the potential function and its derivatives which satisfies Laplace's equation due to source and dipole lines.

We first consider the discontinuity in the potential due to a dipole line.

3.8.1 The discontinuity due to a dipole line.

We next consider the discontinuity due to a particular line of dipoles; we consider the dipole line along the axis $y = 0$, the dipoles orientated in the positive y direction and having strength $\eta(x)$.

Thus the potential due to such a line of dipoles is given by

$$I(x,y) = \int_{-\infty}^{\infty} \frac{\eta(\zeta)y}{(x-\zeta)^2 + y^2} d\zeta$$

To find the discontinuity, we consider this integral for $y = \epsilon \to 0$.

We break the integral into three parts:

$$I = \int_{-\infty}^{x-\delta} \frac{\eta(\zeta)y}{(x-\zeta)^2 + y^2} d\zeta + \int_{x-\delta}^{x+\delta} \frac{\eta(\zeta)y}{(x-\zeta)^2 + y^2} d\zeta + \int_{x+\delta}^{\infty} \frac{\eta(\zeta)y}{(x-\zeta)^2 + y^2} d\zeta$$

where $y = \epsilon \to 0$ and $\delta > 0$ is small, but $\delta \gg \epsilon$.

Thus

$$\int_{-\infty}^{x-\delta} \frac{\eta(\zeta)\epsilon}{(x-\zeta)^2 + \epsilon^2} d\zeta \to \epsilon \int_{-\infty}^{x-\delta} \frac{\eta(\zeta)}{(x-\zeta)^2} d\zeta = O(\epsilon)$$

Therefore,

$$I = \int_{x-\delta}^{x+\delta} \frac{\eta(x+\zeta-x)y}{(x-\zeta)^2 + y^2} + O(\epsilon)$$

where $|\zeta - x| < \delta \to 0$

and so $\eta(x + \zeta - x) = \eta(x) + (\zeta - x)\eta'(x) + ...$

Hence

$$I = \eta(x) \int_{x-\delta}^{x+\delta} \frac{y}{(x-\zeta)^2 + y^2} d\zeta + O(\delta)$$

where $|\zeta - x| = O(\delta)$.

Thus,

$$I = \eta(x) \left[\tan^{-1} \left(\frac{x - \zeta}{y} \right) \right]_{x-\delta}^{x+\delta} + O(\delta)$$

As $y = \epsilon \to 0$, $\frac{\delta}{\epsilon} \to \infty$

and so for $y \to 0_+$,

$$I \to \eta(x)\{\frac{\pi}{2} - (-\frac{\pi}{2})\} = \pi\eta(x)$$

For $y \to 0_-$,

$$I \to \eta(x)\{-\frac{\pi}{2} - \frac{\pi}{2}\} = -\pi\eta(x)$$

Thus the discontinuity in I across the dipole line is

$$I(x,y)|_{y\to 0_+} - I(x,y)|_{y \to 0_-} = 2\pi\eta(x)$$

We now consider the discontinuity due to a source line.

3.8.2 The discontinuity due to a source line.

We consider the discontinuty due to a particular line of sources; we consider the source line along $y = 0$ and having strength $\eta(x)$.

The potential due to such a line of sources is given by

$$J(x,y) = \int_{-\infty}^{\infty} \eta(\zeta) \log\{(x-\zeta)^2 + y^2\}^{\frac{1}{2}} d\zeta$$

Thus $\frac{\partial}{\partial y} J(x,y) = I(x,y)$ where

$$I(x,y) = \int_{-\infty}^{\infty} \frac{\eta(\zeta) y}{\{(x-\zeta)^2 + y^2\}^{\frac{1}{2}}} d\zeta$$

Following the method for finding the discontinuity due to a dipole line, we thus find

$$I(x,y)|_{y\to 0_+} - I(x,y)|_{y\to 0_-} = \left.\frac{\partial}{\partial y} J(x,y)\right|_{y\to 0_+} - \left.\frac{\partial}{\partial y} J(x,y)\right|_{y\to 0_-} = 2\pi\eta(x)$$

3.9 Ch3 Appendix: Calculation to show $B = 0$ and thus the discontinuity in the potential in the symmetric flow case is zero.

A brief description of the calculation is given below.

The calculation is divided into four main parts:

1. The evaluation of the discontinuity term Ψ_0 in the stream function Ψ.
2. The integral representation of the function $F_0 = e^{-kx}\Psi_0$.
3. The evaluation of the integral in the far wake, particularly as $r \to \infty$, $\theta \to 0$.
4. The evaluation of the velocity term $\underline{u}_0 = \underline{\nabla\Phi_0} + \underline{\nabla\Psi_0}$ as $r \to \infty$, $\theta \to 0$.

Thus the above method calculates the velocity term $\underline{u}_0$ in the far wake close to the discontinuity line $y = 0$, $x > 0$.

We find that for $B \neq 0$ we obtain the result $\underline{u}_0 \to \infty$ in this region, which implies that $\underline{u} \to \infty$. Thus the condition of a uniform stream at infinity is violated and so we must have $B = 0$.

The first part of this section is to find the discontinuity in the stream function Ψ.

If $B \neq 0$ then Ψ and its derivatives cannot all be continuous. We consider the possibility of discontinuities in Ψ and its derivatives.

Since u is the fluid velocity, it is continuous everywhere. Across $y = 0$, $\frac{\partial\phi}{\partial x}$ is continuous, but $\frac{\partial\phi}{\partial y}$ has a discontinuity of value B.

Across $y = 0$, $\frac{\partial\Psi}{\partial y}$ is continuous and $w_2 = -\frac{\partial\Psi}{\partial x}$ has a discontinuity of value $-B$.

Thus the discontinuity in Ψ is Bx. We next construct a potential Ψ_0 having this discontinuity and satisfying $(\nabla^2 - 2k\frac{\partial}{\partial x})\Psi = 0$

The function $F_0 = e^{-kx}\Psi_0$

Letting $\Psi_0 = e^{kx}F_0(x, y)$, we obtain

$$e^{kx}(\nabla^2 - k^2)F_0 = 0$$

where F_0 has a discontinuity across $y = 0$, $x > 0$ of value Bxe^{-kx}.

In polar coordinates, F_0 satisfies the equation

$$\left[\frac{1}{r}\left\{\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}) + \frac{\partial}{\partial \theta}(\frac{1}{r}\frac{\partial}{\partial \theta})\right\} - k^2\right] F_0 = 0$$

An important solution to this equation is the source solution $K_0(kr)$, a solution of the modified Bessel's equation of order zero. This solution occurs for F_0 independant of θ and so the equation reduces to

$$\left\{k^2r^2\frac{d^2}{d(kr)} + kr\frac{d}{d(kr)} - (kr)^2\right\} F_0 = 0$$

This solution is important because it behaves like a source of fluid: As $r \to 0$, we find that $K_0(kr) \to -\log r$, so the flux out from a singularity is

$$\int_0^{2\pi} \frac{\partial}{\partial r}(-\log r) r d\theta = -2\pi$$

Thus the solution $K_0(kr)$ gives a constant flux out of value -2π.

The discontinuity due to a dipole line: We next consider the discontinuity due to a particular line of dipoles; we consider the dipole line along the axis $y = 0$, the dipoles orientated in the positive y direction and having strength $\eta(x)$. Thus the potential due to such a line of dipoles is given by

$$I(x, y) = \int_{-\infty}^{\infty} \frac{\eta(\zeta)y}{(x - \zeta)^2 + y^2} d\zeta$$

From the previous appendix section, we see that the discontinuity in I across the dipole line is

$$I(x, y)|_{y \to 0_+} - I(x, y)|_{y \to 0_-} = 2\pi\eta(x)$$

3.9.1 The integral representation of the function F_0.

As $r \to 0$, the function $K_0(kr) \to -\log r$,

$$\Rightarrow \frac{\partial}{\partial y}\{K_0(kr)\} \to -\frac{\frac{\partial r}{\partial y}}{r} = -\frac{y}{r^2}$$

Thus the function $F_0(x, y)$ satisfies the modified Helmholtz equation and has a discontinuity across $y = 0$ of $\eta(x)$, where

$$F_0(x, y) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \eta(\zeta)\frac{\partial}{\partial y}\{K_0(kr)\}d\zeta$$

The line of dipoles along $y = 0$, $x > 0$ of strength Bxe^{-kx} gives F_0 as

$$F_0(x, y) = -\frac{B}{2\pi}\int_0^{\infty} \xi e^{-k\xi}\frac{\partial}{\partial y}K_0(k\{[x-\xi]^2 + y^2\})d\xi$$

The integral representation of $K_0(kr)$.

From Abramowitz and Stegun p376, 9.6.24 , we represent $K_0(kr)$ by the integral

$$K_0(kr) = \int_0^{\infty} e^{-kr\cosh\mu^*}d\mu^* = \int_{-\infty}^{\infty} e^{-kr\cosh\mu^*}\frac{d\mu^*}{2} \tag{3.34}$$

where $r^2 = (x - \mu^*)^2 + y^2$.

We put the variables x and y in a combination easier to manipulate than in the variable r. We do this by raising the line of integration in the μ^* plane by $i\theta$, since the function $e^{-kr\cosh(\mu^*-i\theta)}$ is analytic everywhere and so there are no poles of this integral in the region

$$0 \le Im\{(\mu^*)\} \le \theta \quad -\infty \le Re\{\mu^*\} \le \infty$$

in the μ^* plane.

This procedure is valid only in the case for $x > 0$, otherwise the integral is divergent.

Thus the variable of integration is changed in equation (3.34) to $\mu^* = \mu - i\theta$ and we obtain

$$K_0(kr) = \int_{-\infty+i\theta}^{\infty+i\theta} e^{-kr\cosh(\mu-i\theta)}\frac{d\mu}{2} = \int_{-\infty}^{\infty} e^{-kr\cosh(\mu-i\theta)}\frac{d\mu}{2}$$

However, since $re^{i\theta} = x + iy$, we obtain

$$K_0(kr) = \int_{-\infty}^{\infty} \exp(-kx\cosh\mu + iky\sinh\mu)\frac{d\mu}{2}$$

for $x \geq 0$.

Similarly, we may lower the line of integration by $i\theta$, which gives a convergent integral only in the case $x < 0$, to obtain

$$K_0(kr) = \int_{-\infty}^{\infty} \exp(kx\cosh\mu + iky\sinh\mu)\frac{d\mu}{2}$$

Hence we can represent $K_0(kr)$ by an integral valid for all x by

$$K_0(kr) = \int_{-\infty}^{\infty} \exp(-k|x|\cosh\mu + iky\sinh\mu)\frac{d\mu}{2}$$

where $r^2 = x^2 + y^2$.

We now consider $x < 0$ only. Then $|x - \xi| = |\xi - x| = |\xi + |x|| = \xi + |x|$ for $\xi > 0$.

Thus,

$$\begin{aligned} F_0 &= -\frac{B}{2\pi}\frac{\partial}{\partial y}\int_0^{\infty} \xi e^{-k\xi}d\xi \int_{-\infty}^{\infty} \exp(-k(|x| + \xi)\cosh\mu + iky\sinh\mu)\frac{d\mu}{2} \\ &= -\frac{B}{2\pi}\frac{\partial}{\partial y}\int_{-\infty}^{\infty} \frac{d\mu}{2}\exp(-k(|x|\cosh\mu + iky\sinh\mu)\left[\int_0^{\infty} \xi e^{-k\xi(1+\cosh\mu)}d\xi\right] \end{aligned}$$

Integrating by parts gives

$$\int_0^{\infty} \xi e^{-k\xi(1+\cosh\mu)}d\mu = \frac{1}{k^2(1+\cosh\mu)^2}$$

So

$$F_0 = -\frac{B}{2\pi}\frac{\partial}{\partial y}\int_{-\infty}^{\infty}\frac{d\mu}{2}\exp(-k|x|\cosh\mu + iky\sinh\mu)\frac{1}{k^2(1+\cosh\mu)^2}$$

$$= -\frac{Bi}{4\pi k}\int_{-\infty}^{\infty} d\mu \exp(-k|x|\cosh\mu + iky\sinh\mu)\frac{\sinh\mu}{(1+\cosh\mu)^2} \qquad (3.35)$$

We make the substitution $|x| = -x = r\cos\beta$, $y = r\sin\beta$.

This is valid in the range $|\beta| \leq \frac{\pi}{2}$.

Thus,

$$\begin{aligned} -k|x|\cosh\mu + iky\sinh\mu &= -kr(\cosh\mu\cos\beta - i\sinh\mu\sin\beta) \\ &= -kr\cosh(\mu - i\beta) \end{aligned}$$

Thus

$$F_0(r,\beta) = -\frac{iB}{4\pi k}\int_{-\infty}^{\infty} d\mu \exp(-kr\cosh(\mu - i\beta))\frac{\sinh\mu}{(1+\cosh\mu)^2} \text{ where } |\beta| \leq \frac{\pi}{2}$$

There is a pole when $\cosh \mu = -1$, and $\mu = \pm i\pi$.

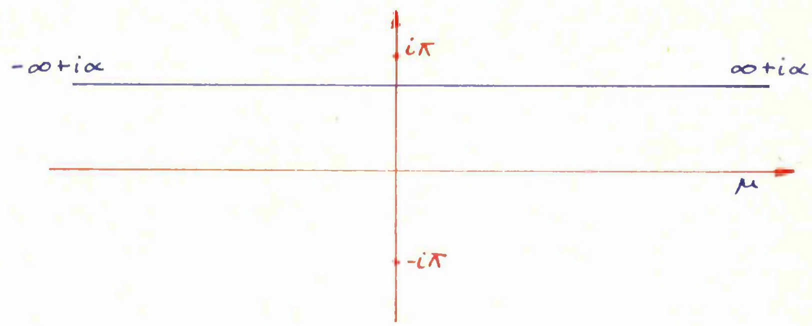


Figure 3.3: The μ-plane.

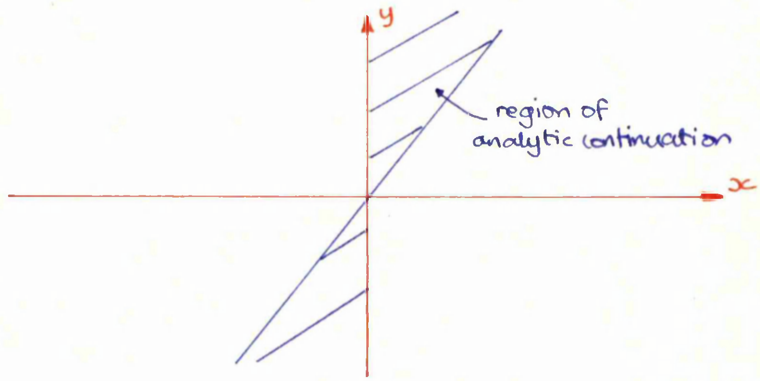


Figure 3.4: The (r, β)-plane.

Now, we consider an analytic continuation into the region $x > 0$. For this purpose, we move the contour of integration in the μ-plane.

By Cauchy, if there is no pole of a function $f(x, y)$ in the region considered, then $\oint_C f(x, y)dl = 0$, where C is a closed curve surrounding the region considered.

Thus, for

$$f(r, \beta) = \exp\left\{-kr\cosh(\mu - i\beta)\frac{\sinh \mu}{(1 + \cosh \mu)^2}\right\}$$

we have that

$$\oint_C f(r, \beta)d\mu \quad = \quad \int_{-\infty}^{\infty} f(r, \beta)d\mu - \int_{-\infty+i\alpha}^{\infty+i\alpha} f(r, \beta)d\mu$$

$$+\lim_{X\to\infty}\left\{\int_{X}^{X+i\alpha} f(r,\beta)d\mu - \int_{-X}^{-X+i\alpha} f(r,\beta)d\mu\right\}$$

Where the closed curve C is the rectangle with corners positioned in the μ plane at $-X$, $-X+i\alpha$, X and $X+i\alpha$. We move along the curve C in the anticlockwise direction.

But

$$\lim_{X\to\infty}\left|\int_{X}^{X+i\alpha} f(r,\beta)d\mu\right|^2 = \lim_{r\to\infty}\left\{\alpha^2|f(r,\beta)|^2\right\} \longrightarrow 0$$

Thus we write

$$F_0(r,\beta) = -\frac{Bi}{4\pi k}\int_{-\infty+i\alpha}^{\infty+i\alpha} d\mu \exp\left\{-kr\cosh(\mu-i\beta)\right\}\frac{\sinh\mu}{(1+\cosh\mu)^2} \tag{3.36}$$

where we must have $|\alpha|<\pi$ in order to avoid the poles at $\mu=\pm i\pi$.

Checking the integral has the right discontinuity across $y = 0$, $x > 0$.

This is an analytic continuation of $F_0(r,\beta)$ since by making the variable change $\mu' + i\alpha = \mu$, we have

$$\exp\{-kr\cosh(\mu - i\beta)\} = \exp\{-kr\cosh(\mu' - i(\beta-\alpha))\}$$

and thus the integrand is valid in the new region $|\beta - \alpha| < \frac{\pi}{2}$. Thus the integrand gives an expression for $F_0(r,\beta)$ in the range $-\frac{\pi}{2} + \alpha < \beta < \frac{\pi}{2} + \alpha$.

Using this analytic continuation, we can thus find $F_0(r,\beta)$ everywhere by raising the line of integration except along the line $y = 0$, $x > 0$.

To find the value of $F_0(r,\beta)$, for $\beta = \pi$, we integrate along the line from $\mu = -\infty + i\pi$ to $\mu = \infty + i\pi$ and just under the pole at $\mu = i\pi$.

Thus we have, for $\beta = \pi$,

$$F_0(r,\beta) = -\frac{Bi}{4\pi k}\, ⨍_{-\infty+i\pi}^{\infty+i\pi} d\mu \exp(kr\cosh\mu)\frac{\sinh\mu}{(1+\cosh\mu)^2}$$

since when $\beta = \pi$, then $\cosh(\mu - i\beta) = \cosh(\mu - i\pi) = -\cosh\mu$.

Similarly for $\beta = -\pi$,

$$F_0(r,\beta) = -\frac{Bi}{4\pi k}\, ⨍_{-\infty-i\pi}^{\infty-i\pi} d\mu \exp(kr\cosh\mu)\frac{\sinh\mu}{(1+\cosh\mu)^2}$$

Raising the line of integration by $2i\pi$ by making the variable substitution $\mu' = \mu + 2i\pi$, we obtain

$$F_0(r,\beta) = -\frac{Bi}{4\pi k}\, ⨍_{-\infty+i\pi}^{\infty+i\pi} d\mu' \exp(kr\cosh\mu')\frac{\sinh\mu'}{(1+\cosh\mu')^2}$$

The discontinuity across $y = 0$, $x > 0$, is $F_0(r,\beta=\pi) - F_0(r,\beta=-\pi)$ and from the above equations, we see that

$$F(r,\beta=-\pi) - F(r,\beta=\pi) = -\frac{Bi}{4\pi k}\oint_C d\mu \exp(kr\cosh\mu)\frac{\sinh\mu}{(1+\cosh\mu)^2}$$

$$= 2\pi i \times (\text{ the residue at } \mu = i\pi)$$

where $\beta = \pi - \theta$

and the closed curve C includes the pole at $\mu = i\pi$.

The residue at $\mu = i\pi$.

We find the residue at $\mu = i\pi$ by making the transformation $\mu = \eta + i\pi$.

Thus

$$\frac{\sinh \mu}{(1+\cosh \mu)^2} = -\frac{\sinh \eta}{(\cosh \eta - 1)^2} = -(1/2)\frac{\cosh \frac{\eta}{2}}{\sinh^3 \frac{\eta}{2}}$$

and also

$$\exp(kr\cosh\mu) = \exp(-kr\cosh\eta) = \exp(-kr)\exp(-2kr\sinh^2\frac{\eta}{2})$$

Thus

$$\exp(kr\cosh\mu)\frac{\sinh\mu}{(1+\cosh\mu)^2}$$

$$= \frac{\exp(-kr)(1-2kr\sinh^2\eta/2+...)(-1/2)(1+(1/2)\eta^2/4+...)}{\eta^3/8(1+(1/6)\eta^2/4+...)^3}$$

$$= -\frac{4}{\eta^3}\frac{(1+(1/2)\eta^2/4+...)}{(1+(1/2)\eta^2/4+...)}\exp(-kr)(1-2kr\sinh^2\eta/2+...)$$

$$= -\frac{4}{\eta^3}(1+0(\eta^4))\exp(-kr)(1-\frac{kr\eta^2}{2})$$

Thus we see that the residue is $2kr\exp(-kr)$

Thus

$$F_0(r,\beta=-\pi) - F_0(r,\beta=\pi) = -\frac{Bi}{4\pi k}2\pi i 2kre^{-kr} = -Bre^{-kr} = -Bxe^{-kx}$$

Hence $F_0(r,\beta=\pi) - F_0(r,\beta=-\pi) = Bxe^{-kx}$.

Thus the discontinuity in Ψ from $y = 0_+$ to $y = 0_-$ $x > 0$ is Bx, which gives a discontinuity in w_2 of $-B$ where $\underline{w} = (w_1, w_2)$.

Since $\underline{u} = \underline{\nabla \phi} + \underline{w}$ and $\underline{u}$ is continuous everywhere, then there must be a discontinuity in $\frac{\partial \phi}{\partial y}$ of B, across $y = 0$, $x > 0$ which is consistent with equation (3.13).

3.9.1.1 The evaluation of the integral in the far wake.

We evaluate the integral function $F_0(r, \beta)$ for large r and for $\beta \rightarrow \pi$. The method of steepest descents cannot be used since the saddle point of the integral and the pole of the integral coalesce as $\beta \rightarrow \pi$.

We find that the leading term in the uniform asymptotic expansion is a Fresnel integral.The value of the Fresnel integral is then given for $r \rightarrow \infty$, $\beta \rightarrow \pi$.

From equation (3.36),

$$F_0(r, \beta) = -\frac{iB}{4\pi k}\int_{-\infty+i\alpha}^{\infty+i\alpha} d\mu \exp\{-kr\cosh(\mu - i\beta)\}\frac{\sinh\mu}{(1+\cosh\mu)^2}$$

In order to find the value of $F_0(r, \beta)$ at $\beta = \pi$, we must choose $\alpha = \pi$. However, there is a pole at $\cosh\mu = -1$ which is $\mu = i\pi$.

The analytic continuation for $F_0(r, \beta)$ in the range $\frac{\pi}{2} \leq \beta \leq \pi$, where $F(r, \beta)$ is in the integral form above, is made by raising the line of integration , thus increasing the value of α from $\alpha = 0$.

Thus the integral path moves under the pole at $\mu = i\pi$. Hence we consider the integral

$$F_0(r, \beta) = -\frac{iB}{4\pi k}\fint_{-\infty+i\pi}^{\infty+i\pi} d\mu \exp\{-kr\cosh(\mu - i\beta)\}\frac{\sinh\mu}{(1+\cosh\mu)^2}$$

We change the variable of integration so that the integral path is along the real axis. We let $\mu = \nu + i\pi$, which gives

$$F_0(r, \gamma) = -\frac{iB}{4\pi k}\fint_{-\infty}^{\infty} d\nu \exp\{-kr\cosh(\nu + i\gamma)\}\frac{\sinh\nu}{(\cosh\nu - 1)^2}d\nu \tag{3.37}$$

where $\gamma = \pi - \beta$. Thus as $\beta \rightarrow \pi$, then $\gamma \rightarrow 0$.

We find the stationary point of the above equation. This is where

$$\frac{d}{d\nu}(\cosh(\nu + i\gamma)) = 0\ ,\ \Rightarrow\ \ \nu = -i\gamma$$

is the stationary point.

We change the variable of integration from ν to v and also the path of integration so that the stationary point is at $v = 0$ and the path of integration is along the real v axis, going through the stationary point.

We let

$$v = 2\sinh\frac{(\nu + i\gamma)}{2} \tag{3.38}$$

and so $\cosh(\nu + i\gamma) - 1 = (1/2)v^2$.

Thus

$$F_0(r,\gamma) = -\frac{iBe^{-kr}}{4\pi k}\int_{-\infty}^{\infty}\exp(-\frac{krv^2}{2})\left\{\frac{\sinh\nu}{(\cosh\nu - 1)^2}\frac{d\nu}{dv}\right\}dv \tag{3.39}$$

Thus the pole in the integrand is now at the point v_0, where

$$\cosh(i\nu) - 1 = 2\sinh^2(1/2)(i\gamma) = (1/2)v_0^2$$

From equation (3.38), when $\nu = 0$, $v = 2\sinh(\frac{i\gamma}{2}) = 2i\sin\frac{\gamma}{2}$.

In the v plane we have the integral path going through the stationary point at $v = 0$ and along the real axis, with the pole of the integral at $v = v_0 = 2i\sin\frac{\gamma}{2}$.

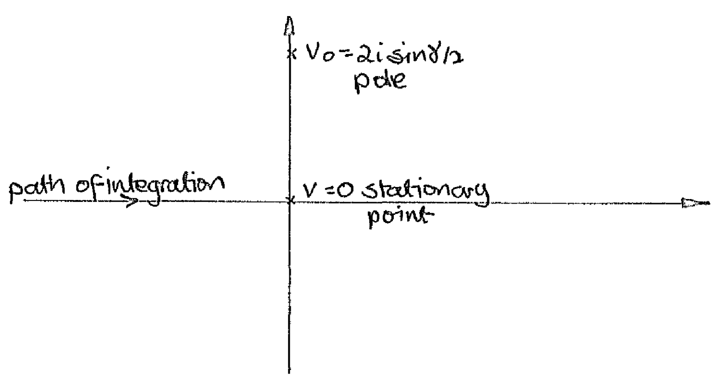


Figure 3.5: The integral path in the v-plane

We would like to apply the method of steepest descents to the integral. However, we are given v_0 is small so the stationary point and the pole are close together.

However, we can expand the non-exponential function inside the integral in powers of $(v - v_0)$. We obtain terms of negative order, giving Fresnel integrals to leading order in the expansion. The pole is of order 3 and the leading order term is of order $(v - v_0)^{-3}$.

We consider the bracketed function inside the integral of equation (3.39). This is

$$\left\{ \frac{\sinh \nu}{(\cosh \nu - 1)^2} \frac{d\nu}{dv} \right\}$$

We consider what happens to this function as $\nu \longrightarrow 0$.

$$\text{As } \nu \longrightarrow 0, \qquad \sinh \nu \longrightarrow \nu + \tfrac{\nu^3}{3!} + \ldots$$
$$\cosh \nu \longrightarrow 1 + \tfrac{\nu^2}{2!} + \ldots$$

Hence,

$$\frac{\sinh \nu}{(\cosh \nu - 1)^2} \rightarrow \frac{\nu}{[\nu^2]^2} = \frac{1}{\nu^3} \tag{3.40}$$

From equation (3.38), we have that $v = 2 \sinh \frac{(\nu + i\gamma)}{2}$, and so differentiating w.r.t v, we obtain

$$1 = 2 \cosh \left(\frac{\nu + i\gamma}{2} \right) (1/2) \frac{d\nu}{dv}$$

so as $\nu \longrightarrow 0$, $\frac{d\nu}{dv} \longrightarrow \frac{1}{\cos \gamma / 2}$ and so

$$\nu \longrightarrow \frac{(v - v_0)}{\cos \gamma / 2} \tag{3.41}$$

and

$$\left\{ \frac{\sinh \nu}{(\cosh \nu - 1)^2} \frac{d\nu}{dv} \right\} \longrightarrow \frac{1}{\cos \frac{\gamma}{2}} \left(\frac{1}{\nu^3} \right) \tag{3.42}$$

Since $\frac{\sinh \nu}{(\cosh \nu - 1)^2} \frac{d\nu}{dv}$ is analytic everywhere except $\nu = 0$, we expand this function in a Laurent series. From equation (3.40), the lowest power of ν in the series must be -3.

Therefore,

$$\left\{\frac{\sinh\nu}{(\cosh\nu-1)^2}\frac{d\nu}{dv}\right\} = \frac{A_{-3}(\gamma)}{(v-v_0)^3} + \frac{A_{-2}(\gamma)}{(v-v_o)^2} + \frac{A_{-1}(\gamma)}{(v-v_0)} + \sum_0^\infty D_m(\gamma)v^m \qquad (3.43)$$

We can find the first term in the expansion since, from equation (3.42) and the first term on the right hand side of equation (3.43), we have that:

$$\frac{1}{\cos\gamma/2}\left(\frac{1}{\nu^3}\right) = A_{-3}\frac{(\cos\gamma/2)^3}{(\cos\gamma/2)(v-v_0)3} = \frac{A_{-3}(\gamma)}{(v-v_0)^3} \qquad (3.44)$$

Thus $A_{-3}(\gamma) = \cos^2\gamma/2$ in order that equation (3.40) holds as $\nu \to 0$.

We can find $A_{-1}(\gamma)$ by integrating the function $\left\{\frac{\sinh\nu}{(\cosh\nu-1)^2}\frac{d\nu}{dv}\right\}$ round a closed contour enclosing v_0.

Also A_{-2} may be calculated by letting $v = 0$ on both sides of equation (3.43).

However, it is more convenient to use a different method in order to express $F_0(r,\beta)$ in terms of an integral of lowest order -1; we express the integral in a different way so that the pole inside the integral is of order one. This is done by considering the following:

Consider the function

$$\frac{\sinh\nu}{(\cosh\nu-1)^2}$$

We have

$$\frac{\sinh\nu}{(\cosh\nu-1)^2} = -\frac{d}{d\nu}\left(\frac{1}{\cosh\nu-1}\right)$$

and

$$\frac{1}{\cosh\nu-1} = \frac{1}{2\sinh^2\nu/2} = -\frac{d}{d\nu}\left(\coth\frac{\nu}{2}\right)$$

Thus

$$\frac{\sinh\nu}{(\cosh\nu-1)^2} = \frac{d^2}{d\nu^2}\left(\frac{\cosh\nu+1}{\sinh\nu}\right)$$

Considering equation (3.37) we have

$$F_0(r,\gamma) = -\frac{iB}{4\pi k} \fint_{-\infty}^{\infty} d\nu \exp[-kr\cosh(\nu + i\gamma)] \frac{\sinh\nu}{(\cosh\nu - 1)} d\nu$$

Thus

$$F_0(r,\gamma) = -\frac{iB}{4\pi k} \fint_{-\infty}^{\infty} d\nu \exp(-kr\cosh(\nu + i\gamma)) \frac{d^2}{d\nu^2} \left[\frac{\cosh\nu + 1}{\sinh\nu}\right]$$

Since $[\exp(-kr\cosh(\nu + i\gamma))]_{\nu=\pm\infty} = 0$, integration by parts gives:

$$F_0(r,\gamma) = -\frac{iB}{4\pi k} \fint_{-\infty}^{\infty} d\nu \frac{d^2}{d\nu^2} \{\exp(-kr\cosh(\nu + i\gamma))\} \left(\frac{\cosh\nu + 1}{\sinh\nu}\right)$$

If we consider a function $f(\nu + i\gamma)$, then $\frac{\partial^2}{\partial\nu^2} = f''(\nu + i\gamma)$ and $\frac{\partial^2 f}{\partial\gamma^2} = i^2 f''(\nu + i\gamma)$, which gives $\frac{\partial^2 f}{\partial\nu^2} = -\frac{\partial^2 f}{\partial y^2}$. Thus we write

$$F_0(r,\gamma) = \frac{iB}{4\pi k} \frac{d^2}{d\gamma^2} \fint_{-\infty}^{\infty} d\nu \exp(-kr\cosh(\nu + i\gamma)) \frac{\cosh\nu + 1}{\sinh\nu}$$

Following the same variable and integration path changes used in obtaining equation (3.39), we now obtain

$$F_0(r,\gamma) = \frac{iB}{4\pi k} e^{-kr} \frac{d^2}{d\gamma^2} \int_{-\infty}^{\infty} \exp(\frac{-krv^2}{2}) \left\{\frac{\cosh\ nu + 1}{\sinh\nu} \frac{d\nu}{dv}\right\} dv$$

where $\cosh(\nu + i\gamma) - 1 = \frac{v^2}{2}$.

Thus the pole of the integrand is at $v_0 = 2i\sin 1/2\gamma$, and the saddle point is at $v = 0$.

3.9.1.2 The order of the pole.

We find the order of the pole by looking at the way in which

$$\left(\frac{\cosh\nu+1}{\sinh\nu}\right)\frac{d\nu}{dv}$$

behaves as $\nu \to 0$.

We use the same method that was used to find the way in which

$$\frac{\sinh\nu}{(\cosh\nu-1)^2}\frac{d\nu}{dv}$$

behaved as $\nu \to 0$.

As $\nu \to 0$,

$$\frac{\cosh\nu+1}{\sinh\nu} \to \frac{1+\{1+\nu^2/2+\ldots\}}{\nu+\nu^3/3+\ldots} \to \frac{2}{\nu}$$

From equation (3.41) we have $\frac{d\nu}{dv} = \frac{1}{\cos\gamma/2}$ as $\nu \to 0$.

Thus as $\nu \to 0$,

$$\left(\frac{\cosh\nu+1}{\sinh\nu}\right)\frac{d\nu}{dv} \to \frac{2}{\cos\gamma/2}\left(\frac{1}{\nu}\right)$$

From equation (3.41), we see that as $\nu \to 0$, $v - v_0 \to \nu\cos\gamma/2$.

Thus as $\nu \to 0$, $v \to v_0$,

$$\left(\frac{\cosh\nu+1}{\sinh\nu}\right)\frac{d\nu}{dv} \to \frac{2}{\cos\gamma/2}\frac{\cos\gamma/2}{v-v_0} = \frac{2}{v-v_0}$$

Since the function is analytic except at $\nu = 0$, it may be expressed as a Laurent expansion. However, all the coefficients of terms of lower order than -1 must be zero. Hence this function has the expansion

$$\left(\frac{\cosh\nu+1}{\sinh\nu}\right)\frac{d\nu}{dv} = \frac{A_{-1}(\gamma)}{v-v_0} + A_0(\gamma) + \sum_1^\infty A_m(\gamma)v^m \qquad (3.45)$$

where $A_{-1} = 2$.

Substituting this expression back into the integral, we obtain an asymptotic expansion for the integral, with the first term being a Fresnel integral.

$$\begin{aligned} &\int_{-\infty}^{\infty} \exp\{-kr\cosh(\nu + i\gamma)\}\frac{\cosh\nu + 1}{\sinh\nu}d\nu \\ \sim \quad & e^{-kr}\int_{-\infty}^{\infty} \exp\{-\frac{krv^2}{2}\}\frac{A_{-1}}{v - v_0}dv + e^{-kr}\sum_{v=0}^{v=M} A_m \int_{-\infty}^{\infty} v^m \exp(-\frac{krv^2}{2})dv \end{aligned} \tag{3.46}$$

where M is chosen according to the degree of approximation wanted.

We find the first two terms in the series, A_{-1} and A_0.

The terms A_{-1} and A_0 .

To find A_{-1}, we integrate round $\nu = 0$, $v = v_0$ for the function $\frac{\cosh\nu+1}{\sinh\nu}$.

Hence,

$$\oint \frac{\cosh\nu + 1}{\sinh\nu}d\nu = A_{-1}\int \frac{dv}{v - v_0}$$

Thus

$$\oint \frac{\cosh\nu + 1}{\sinh\nu}d\nu = \oint \frac{(2 + \nu/2! + \nu/4! + ...)}{(\nu + \nu^3/3! + ...)}d\nu = 4\pi i$$

Therefore $4\pi i = A_{-1}.2\pi i$ and $A_{-1} = 2$.

To find A_0, we put $v = 0$, which implies that $\nu = -i\gamma$, and substituting into the expansion of equation (3.45), we obtain

$$-\frac{\cosh i\gamma + 1}{\sinh i\gamma}\left.\frac{d\nu}{dv}\right|_{\nu=i\gamma} = \frac{A_{-1}}{v_0} + A_0$$

From equation (3.38) $v = 2\sinh\{1/2(\nu + i\gamma)\} \Rightarrow \frac{d\nu}{dv} = 1$ for $\nu = -i\gamma$.

Thus,

$$A_0 = -\frac{2}{2i\sin\gamma/2} + \frac{\cos\gamma + 1}{i\sin\gamma}$$

But

$$\frac{1 + \cos\gamma}{\sin\gamma} = \frac{1 + (2\cos^2\gamma/2 - 1)}{2\sin\gamma/2\cos\gamma/2} = \frac{cos\frac{\gamma}{2}}{\sin\frac{\gamma}{2}}$$

and so

$$A_0 = \frac{-1 + (1 - 2\sin^2\gamma/4)}{2i\sin\gamma/4\cos\gamma/4} = i\tan\gamma/4$$

Hence

$$A_{-1} = 2, \quad A_0 = i\tan\gamma/4 \tag{3.47}$$

Approximations to the integral for certain fluid regions.

From equations (3.46) and (3.47) we therefore have, taking the first two terms in the expansion:

$$F_0(r,\theta) \sim \frac{iBe^{-kr}}{4\pi k}\frac{d^2}{d\theta^2}\left\{\int_{-\infty}^{\infty} \exp(-1/2krv^2)\frac{2}{v-v_0}dv + \int_{-\infty}^{\infty} i\tan(\theta/4)\exp(-1/2krv^2)dv\right\}$$

We consider the first integral in the above equation which is of the form:

$$I = \int_{-\infty}^{\infty} \frac{\exp(\frac{-krv^2}{2})}{v-v_0}dv$$

where $v_0 = 2i\sin\theta/2$ from equation (3.38).

Changing the variable of integration to $t = \sqrt{1/2kr}\,u$ gives

$$I = \int_{-\infty}^{\infty} \frac{e^{-t^2}}{t-t_0}dt$$

where $t_0 = \sqrt{1/2kr}\,u_0 = i\sqrt{2kr}\sin\theta/2$

We find the approximations to this integral, and the corresponding fluid regions, for $t_0 \to i\infty$ and $t_0 \to 0$.

For large r, if t_0 is constant $\sqrt{r}\theta$ is constant. If $\sqrt{r}theta$ is of order one then we are in the wake flow and as $\sqrt{r}\theta \to 0$, we move outside the wake to the fluid region of no vorticity. Thus the wake boundary is parameterised by a curve $\sqrt{r}\theta =$ constant.

As $\sqrt{r}\theta \to 0$, we approach the line $y = 0$, $x > 0$ which is the discontinuity line.

We now consider the integral

$$I = \int_{-\infty}^{\infty} \frac{e^{-t^2}}{t-t_0}dt$$

We first consider the integral for $t_0 \to i\infty$. We expect the main contributions to this integral will come near $t = 0$ and $t = t_0$. However, for large t_0 the integrand is exponentially small.

Thus,

$$I \sim \int_{-M}^{M} \frac{e^{-t^2}}{t - t_0} dt \sim \frac{1}{t_0} \int_{-M}^{M} e^{-t^2} dt \sim \frac{1}{t_0} \int_{-\infty}^{\infty} e^{-t^2} dt = \frac{\sqrt{\pi}}{t_0}$$

$$\left(\left[\int_{-\infty}^{\infty} e^{-t^2} dt \right]^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx dy = \int_0^{2\pi} \int_0^{\infty} r e^{-r^2} dr d\theta \right)$$

We now give the first term in the expansion of $F_0(r,\theta)$ for $t_0 \to i\infty$

$$\begin{aligned} F_0(r,\theta) &\sim \frac{iBe^{-kr}}{2\pi k}\frac{d^2}{d\theta^2}\left(\frac{\sqrt{\pi}}{i\sqrt{2kr}\sin\theta/2}\right) \\ &\sim \frac{Be^{-kr}}{2k\sqrt{2\pi kr}}\frac{d^2}{d\theta^2}\left(\frac{1}{\sin\theta/2}\right) \end{aligned} \tag{3.48}$$

But

$$\frac{d^2}{d\theta^2}\left(\frac{1}{\sin\theta/2}\right) = \frac{d}{d\theta}\left(\frac{-(1/2)\cos\theta/2}{\sin^2\theta/2}\right) = \frac{1/4\sin^2\theta/2 + 1/2\cos^2\theta/2}{\sin^3\theta/2}$$

and so

$$F_0(r,\theta) \sim \frac{Be^{-kr}}{2k\sqrt{2\pi kr}}\left(\frac{1/4\sin^2\theta/2 + 1/2\cos^2\theta/2}{\sin^3\theta/2}\right) \tag{3.49}$$

We now consider the integral I for $t_0 \to 0$. We consider the summation representation of the integral given in Abramowitz and Stegun (p297 eq 7.1.4, 7.1.8) The function $w(z)$ is defined as

$$w(z) = \frac{i}{\pi}\int_{-\infty}^{\infty}\frac{e^{-t^2}}{z-t}dt$$

which is represented by the summation

$$w(z) = \sum_{n=0}^{\infty}\frac{(iz)^n}{\Gamma(\frac{n}{2}+1)}$$

where Γ is the gamma or factorial function.

Therefore, by comparison,

$$I = \int_{-\infty}^{\infty}\frac{e^{-t^2}}{t-t_0}dt$$

has the summation

$$I = \pi i\sum_{n=0}^{\infty}\frac{(it_0)^n}{\Gamma(n/2+1)}$$

Considering small t_0, the first term in the expansion will be of highest order. Thus, for small t_0, we obtain the approximation

$$I \sim \pi i \left(\frac{1}{\Gamma(1)} + \frac{it_0}{\Gamma(3/2)} - \frac{t_0^2}{\theta(2)} \right)$$

This gives

$$F_0(r,\theta) \sim \frac{iBe^{-kr}}{2\pi k} \frac{d^2}{d\theta^2} \left\{ \pi i (1 + \frac{2it_0}{\sqrt{\pi}} - t_0^2 + ...) \right\} \tag{3.50}$$

where $t_0 = i\sqrt{2kr} \sin \theta/2$

The expansion of $\sin \theta/2$ is

$\sin \theta/2 = \theta/2 - \frac{(\theta/2)^3}{3!} + \frac{\theta/2^5}{5!} + ...$ and so $t_0 = i\sqrt{2kr}(\theta/2 - \frac{\theta^3}{48} + ...)$.

Therefore, the second derivative of with respect to θ gives the term:

$$\frac{d^2}{d\theta^2} t_0 = -\frac{i\sqrt{2kr}\,\theta}{8} + O(\theta^2)$$

The second derivative of t_0^2 gives a term independant of θ to first order:

$$\frac{d^2}{d\theta^2} t_0^2 = -2kr \frac{d^2}{d\theta^2} \left(\frac{\theta}{2} - \frac{\theta^3}{48} + ... \right)^2 = -2kr \frac{d^2}{d\theta^2} \left(\frac{\theta^2}{4} - \frac{\theta^4}{48} + O(\theta^6) \right) = kr + O(\theta^2)$$

The second derivative of t_0^n for $n > 2$ will have a first order term θ^{n-2}, and so all these terms will be at the greatest of order θ.

Hence we have

$$\frac{d^2}{d\theta^2} \left\{ \pi i (1 + \frac{2it_0}{\sqrt{\pi}} - t_0^2 + ...) \right\} = \{-\pi i\}\{-kr\} + O(\theta) = i\pi kr + O(\theta)$$

Thus substituting this expression into equation (3.50) we obtain

$$F_0(r,\theta) \sim \frac{iBe^{-kr}}{2\pi k} i\pi kr = -\frac{B}{2} e^{-kr}$$

For $t_0 \to 0$ and $\theta \to 2\pi$, for small α where $-\alpha + \theta = 2\pi$, then $\sin\theta/2 = \sin 1/2(2\pi + \alpha) = -sin\alpha/2$

Since $t_0 = i\sqrt{2kr}\sin\theta/2$, then following the same method we obtain for $\theta \to 2\pi$,

$$F_0(r,\theta) \sim -\frac{iBe^{-kr}}{2\pi k} i\pi kr = \frac{B}{2}e^{-kr}$$

This result is also evident from the fact that in symmetric flow the function Ψ and thus $F_0(r,\theta)$ must be antisymmetric about $y = 0$

We now find the vector $\underline{w}$ for the fluid regions considered above. From section (3.2.2)

$$\underline{u} = \underline{\nabla\phi} + \underline{w}, \quad \underline{w} = \left(\frac{\partial\Psi}{\partial y}, -\frac{\partial\Psi}{\partial x}\right), \quad \Psi_0 = e^{kx}F_0(x,y)$$

Therefore, for $t_0 \to \infty$

$$\Psi_0 = \frac{Be^{-kr(r-x)}}{2k\sqrt{2\pi kr}}\left(\frac{1/4\sin^2\theta/2 + 1/2\cos^2\theta/2}{\sin^3\theta/2}\right)$$

Thus,

$$\frac{\partial\Psi_0}{\partial\theta} = \frac{Be^{-kr(1-\cos\theta)}}{2k\sqrt{2kr}}\left\{-kr\sin\theta\left(\frac{1/4\sin^2\theta/2+1/2\cos^2\theta/2}{\sin^3\theta/2}\right) - \left(\frac{5/8\sin^3\theta/2\cos\theta/2+3/4\cos^3\theta/2}{\sin^4\theta/2}\right)\right\}$$

and

$$\frac{\partial\Psi_0}{\partial r} = \frac{-Be^{-kr(1-\cos\theta)}}{2k\sqrt{2kr}}\left(k + \frac{1}{2r}\right)$$

Where

$$\frac{\partial\Psi_0}{\partial x} = \frac{\partial\Psi_0}{\partial x}\cos\theta - \frac{1}{r}\frac{\partial\Psi_0}{\partial\theta}\sin\theta$$

and

$$\frac{\partial \Psi_0}{\partial y} = \frac{\partial \Psi_0}{\partial r} \sin\theta + \frac{1}{r} \frac{\partial \Psi_0}{\partial \theta} \cos\theta$$

Thus

$$|\underline{w}| = O(\frac{e^{-kr}}{\sqrt{r}})$$

and so as $t_0 \rightarrow i\infty$, $r \rightarrow \infty$ and so

$$\frac{\partial \Psi_0}{\partial x} \rightarrow 0\,, \quad \frac{\partial \Psi_0}{\partial y} \rightarrow 0$$

Next we consider the case for $t_0 \to 0$.

Following the same method, we obtain for $\theta \to 0$

$$\Psi_0 = -\frac{B}{2} r e^{-kr(1-\cos\theta)}$$

Thus

$$\frac{\partial \Psi_0}{\partial r} = -\frac{B}{2} e^{-kr(1-\cos\theta)} \{-kr(1-\cos\theta)+1\}$$

and

$$\frac{\partial \Psi_0}{\partial \theta} = -\frac{B}{2} e^{-kr(1-\cos\theta)} (-kr\sin\theta)$$

As $t_0 \to 0$,

$$\frac{\partial r}{\partial x} \to 1\,, \quad \frac{\partial r}{\partial y} \to 0\,, \quad \frac{\partial \theta}{\partial x} \to 0\,, \quad \frac{\partial \theta}{\partial y} \to \frac{1}{r}$$

Thus

$$\frac{\partial \Psi_0}{\partial x} \to -\frac{B}{2}\,, \quad \frac{\partial \Psi_0}{\partial y} \to 0$$

Similarly, for $t_0 \to 0$, $\theta \to 2\pi$ we have

$$\frac{\partial \Psi_0}{\partial x} \to \frac{B}{2}\,, \quad \frac{\partial \Psi_0}{\partial y} \to 0$$

Therefore there is a discontinuity in $\frac{\partial \Psi_0}{\partial x}$ across $y = 0$, $x > 0$ of value $-B$:

$$\left.\frac{\partial \Psi_0}{\partial x}\right|_{y \to 0_+} - \left.\frac{\partial \Psi_0}{\partial x}\right|_{y \to 0_-} = -B$$

Since we are dealing with symmetric flow, we find that for $\underline{u} = (u_1, u_2)$, we have $u_2 = 0$ along $y = 0$.

The velocity is

$$\underline{u} = \underline{\nabla\phi} + \left(\frac{\partial \Psi}{\partial y}, -\frac{\partial \Psi}{\partial x}\right)$$

and so we have

$$\frac{\partial \phi_0}{\partial y} - \frac{\partial \Psi_0}{\partial x} = 0$$

Therefore we have a discontinuity across the line $y = 0$, $x > 0$ of

$$\left.\frac{\partial \phi_0}{\partial y}\right|_{y \to 0_+} - \left.\frac{\partial \phi_0}{\partial y}\right|_{y \to 0_-} = B$$

Referring to equation (3.13), we see that this is the correct discontinuity in ϕ_0.

3.9.1.3 The velocity far from the body.

As $r \to \infty$, $t_0 \to i\infty$,

$$\frac{\partial \Psi_0}{\partial x} \to 0\,, \quad \frac{\partial \Psi_0}{\partial y} \to 0$$

Letting $\underline{w}_0$ be the term in $\underline{w}$ corresponding to Ψ_0, we thus have that $\underline{w}_0 \to 0$.

As $r \to \infty$ and $t_0 \to 0$,

$\underline{w}_0 \to (0, B/2)$ for $\theta \to 0$ and

$\underline{w}_0 \to (0, -B/2)$ for $\theta \to 2\pi$.

In order to calculate $\underline{u}$ we must also find $\underline{\nabla \phi_0}$ for large r.

From equation (3.14), we have

$$\phi_0 = -\frac{B}{2\pi} Re\{z \log z - zi\pi\}$$

Thus,

$$\frac{\partial \phi_0}{\partial x} = -\frac{B}{2\pi}(1 + \log r)\,, \quad \frac{\partial \phi_0}{\partial y} = \frac{B}{2\pi}(\theta - \pi)$$

Thus we see that as $r \to \infty$ and $t \to i\infty$ then

$$\underline{u} \to \left(\infty, \frac{B}{2\pi}(\theta - \pi)\right)$$

and as $r \to \infty$, $t_0 \to 0$ then

$$\underline{u} \to (\infty, 0)$$

Hence the condition that the velocity is a uniform stream of velocity $(U, 0, 0)$ at infinity is violated. Thus the constant B in the discontinuity potential ϕ_0 must be zero. This means that there is no discontinuity in the potential ϕ in the symmetric flow case.

Chapter 4

The drag, lift and moment on the body.

4.1 The drag force, lift force and moment on the body in two dimensional flow.

In this chapter, we express the force and moment on the body as integrals over a closed contour far away from the body in the far field but enclosing the body. We substitute for the functions ϕ and Ψ in the far field into the closed contour integrals and thus find coefficients in the Fourier expansions of the functions ϕ and Ψ related to the drag, lift and moment on the body.

We find that the coefficient L in the expansion of ϕ is the lift on the body and the coefficient D in the expansion of Ψ is the drag on the body. Thus the drag is expressed in terms of a wake traverse but the lift is not. It is important here to note the result given in Landau and Lifshitz for three dimensional steady far field flow past a body where both the lift and the drag are expressed in terms of wake traverses.

We first find the force on the body as an integral over a closed contour enclosing the body. We consider the force $\underline{F}$ on the body due to the fluid. We express the force as an integral expression over a closed contour C enclosing the body. The force is

$$F_i = \int_C p_{ij} n_j dl$$

where C is the body contour and $\underline{n}$ is the outward pointing normal to the body surface.

We consider the closed contour C' within the fluid and enclosing the body contour C.

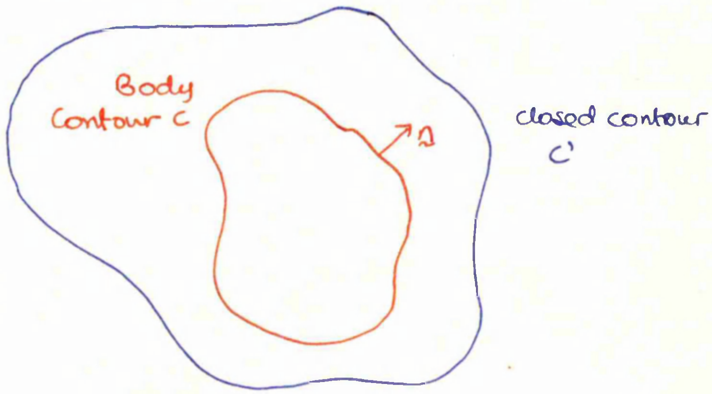


Figure 4.1: The closed contours C' and C.

The vector $\underline{n}$ is the outward pointing normal to the contour C' and so by the divergence theorem

$$\int_{C'} p_{ij}n_j dl + \int_C p_{ij}(-n_j)dl = \int\int_A \frac{\partial p_{ij}}{\partial x_j} dA$$

where A is the fluid area between the closed contours C and C'.

Therefore

$$F_i = \int_C p_{ij}n_j dl = \int_{C'} p_{ij}n_j dl - \int\int_A \frac{\partial p_{ij}}{\partial x_j} dA$$

From the momentum equation,

$$\frac{D}{Dt}(\rho u_i^\dagger) = \frac{\partial p_{ij}}{\partial x_j}$$

(For an elemental volume of fluid, $\frac{D}{Dt}\int\int_{\delta A} \rho u_i^\dagger dA = \delta F_i = \int_{\delta C} p_{ij}n_j dl$.)

Thus

$$\int\int_A \frac{\partial p_{ij}}{\partial x_j} dA = \int\int_A \rho \frac{Du_i^\dagger}{Dt} dA = \int\int_A \rho u_j^\dagger \frac{\partial u_i^\dagger}{\partial x_j} dA = \int\int_A \frac{\partial}{\partial x_j}(\rho u_i^\dagger u_j^\dagger) dA$$

since $\underline{\nabla}.\underline{u}^\dagger = 0$.

By the divergence theorem,

$$\int\int_A \frac{\partial}{\partial x_j}(\rho u_i^\dagger u_j^\dagger)dA = \int_{C'} \rho u_i^\dagger u_j^\dagger n_j dl - \int_C \rho u_i^\dagger u_j^\dagger n_j dl$$

However, the fluid velocity on the body contour is zero and so we obtain the force equation

$$F_i = \int_{C'} (p_{ij} - \rho u_i^\dagger u_j^\dagger) n_j dl \qquad (4.1)$$

where $p_{ij} = -p\delta_{ij} + \mu e_{ij}$. (See equation (2.9).)

We consider the contribution to the force F_i from the tensor e_{ij}. We thus first consider the integral

$$\int_{C'} \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) n_j dl$$

We decompose the fluid velocity in the form $\underline{u} = \underline{\nabla\phi} + \underline{w}$ and first consider the contribution to the integral

$$\int_{C'} 2\mu \left(\frac{\partial^2 \phi}{\partial x_i \partial x_j}\right) n_j dl$$

From equation (3.26), the velocity potential is given in the form

$$\underline{\nabla\phi} = \sum_{n=1}^{\infty} \left(E_n \frac{\cos n\theta}{r^n} + F_n \frac{\sin n\theta}{r^n}, G_n \frac{\cos n\theta}{r^n} + H_n \frac{\sin n\theta}{r^n}\right)$$

We consider the contour C' to be the circumference of a circle centred at the origin and of radius r. Thus the function $\frac{\partial^2\phi}{\partial x_i \partial x_j}$ will be of highest order $\frac{1}{r^2}$ in the far field.

Since the contour length is of order r, then the contribution to the integral from $\underline{\nabla\phi}$ is of order $\frac{1}{r}$ and tends to zero as $r \to \infty$.

We next consider the contribution to the integral

$$\int_{C'} \mu \left(\frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i} \right) n_j dl$$

From equation (3.33), it is seen that the velocity $\underline{w}$ is exponentially small outside the wake region. Therefore the integral approximates to

$$\int_{-\infty}^{\infty} \mu \left(\frac{\partial w_i}{\partial x_1} + \frac{\partial w_1}{\partial x_i} \right) dx_2$$

for large r.

The velocity component w_i and its derivatives with respect to x_1 and x_2 are of the form $f(x_1, x_2)e^{-\frac{kx_2^2}{2x_1}}$.

In order that an integral of the form

$$\int_{-\infty}^{\infty} f(x_1, x_2)e^{-\frac{kx_2^2}{2x_1}} dx_2$$

is non zero as $x_1 \to \infty$, then $f(x_1, x_2)$ must be at least of order $\frac{1}{\sqrt{x_1}}$, since

$$\int_{-\infty}^{\infty} \frac{e^{-\frac{kx_2^2}{2x_1}}}{\sqrt{x_1}} dx_2 = \int_{-\infty}^{\infty} e^{-p^2} dp \sqrt{\frac{2}{k}} = \sqrt{\frac{2\pi}{k}}$$

From equation (3.33), in the far field $\underline{w}$ is of order

$$\underline{w} = \left(\frac{e^{-\frac{kx_2^2}{2x_1}}}{\sqrt{x_1}}, \frac{x_2}{x_1\sqrt{x_1}} e^{-\frac{kx_2^2}{2x_1}} \right)$$

Thus $\frac{\partial w_i}{\partial x_j}$ is of greatest order $\frac{e^{-\frac{kx_2^2}{2x_1}}}{x_1\sqrt{x_1}}$ and so

$$\int_{-\infty}^{\infty} \mu \left(\frac{\partial w_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) n_j dl \to 0$$

as $r \to \infty$.

We now refer to equation (4.1) and consider the other terms in the integrand of the integral for the force F_i.

$u_i^\dagger$ is the fluid velocity and $u_i^\dagger = U_i + u_i$ where u_i is the perturbed fluid velocity and $U_i = U$ for $i = 1$ and 0 for $i = 2$.

We substitute $u_i^\dagger = U_i + u_i$ into equation (4.1). From equation (3.24) in the far field $|\underline{\nabla\phi}|^2 = O(1/r^2)$ and from equations (3.28) and (3.30) in the far field $|\underline{w}|^2 = O(\frac{e^{-2k(r-x)}}{r})$. Thus using the method in section (5.3.1) we see that

$$\int_0^{2\pi} |\underline{w}|^2 r d\theta = O(\frac{1}{\sqrt{kr}} \int_{-\infty}^{\infty} e^{-t^2} dt) \to 0$$

Therefore $\int_0^{2\pi} u_i u_j n_j r d\theta \to 0$ as $r \to \infty$.

The pressure $p = -\rho U \frac{\partial \phi}{\partial x}$ and so

$$F_i = \int_0^{2\pi} \left\{ \rho U \frac{\partial \phi}{\partial x} n_j - \rho(u_i U_j + u_j U_i) n_j \right\} r d\theta$$

The velocity flux out of a closed contour is zero and so $\int_0^{2\pi} u_j n_j r d\theta = 0$. Therefore

$$F_i = \int_0^{2\pi} \left\{ \rho U \frac{\partial \phi}{\partial x} n_i - \rho U u_i n_1 \right\} r d\theta \qquad (4.2)$$

We now calculate the drag and the lift on the body.

4.1.1 The drag on the body.

We substitute expressions for ϕ and Ψ into the integral exprression for the drag, and find coefficients in the expansion of Ψ related to the drag.

From equation (4.2), the drag on the body is given by

$$\text{Drag} = F_1 = \int_0^{2\pi} \left\{ -\rho U \frac{\partial \Psi}{\partial y} n_1 \right\} r d\theta$$

The function w_1 is zero outside the wake region due to the exponential factor $e^{-kr(1-\cos\theta)}$ which tends to zero rapidly except in the wake region where θ is small such that

$r(1 - \cos\theta) = \frac{r\theta^2}{2}$ is of order one.

Thus w_1 contributes to the drag integral only in the wake region where $n_1 = \cos\theta \sim 1$, and $rd\theta = dy$.

Within the wake region, from equation (3.33),

$$w_1 \sim -\frac{D}{\pi \rho U} k \sqrt{\frac{\pi}{2kx}} e^{-\frac{ky^2}{2x}}$$

and so

$$\text{Drag} = \frac{D}{\pi} k \sqrt{\frac{\pi}{2kx}} \int_{-\infty}^{\infty} e^{-\frac{ky^2}{2x}} dy$$

We change the variable of integration to $q = \sqrt{\frac{k}{2x}} y$ and so

$$\begin{aligned} \text{Drag} &= \frac{D}{\pi} k \sqrt{\frac{\pi}{2kx}} \sqrt{\frac{2x}{k}} \int_{-\infty}^{\infty} e^{-q^2} dq \\ &= \frac{D}{\pi} k \frac{\pi}{k} = D \end{aligned}$$

This result is expected since

$$\text{Drag} = \int_0^{2\pi} \left\{ -\rho U \frac{\partial \Psi}{\partial y} n_1 \right\} r d\theta$$

where $\Psi \sim \Psi_1 + s\sqrt{\frac{\pi}{2kr}} e^{-kr(1-\cos\theta)}$ in the far field from section (3.6). Therefore

$$\begin{aligned}
\text{Drag} &= \int_0^{2\pi} -\rho U \frac{\partial \Psi_1}{\partial y} \cos\theta r d\theta \\
&\sim -\left\{ \int_{x\to\infty,\ y=0}^{y=\infty} + \int_{x\to\infty,\ y=-\infty}^{y=0} \right\} \rho U \frac{\partial \Psi}{\partial y} dy \\
&\sim \int_0^{2\pi} -\rho U \frac{1}{r} \frac{\partial \Psi}{\partial \theta} r d\theta \\
&= -\rho U [\Psi]_{y\to 0_+}^{y\to 0_-} \\
&= -\rho U \left(-\frac{D}{\rho U} \right) \\
\text{Drag} &= D \qquad (4.3)
\end{aligned}$$

Thus we see that the drag is expressed as a wake traverse.

4.1.2 The lift on the body.

We substitute expressions for ϕ and Ψ into the integral expression for the lift and find the coefficient in the expansion of ϕ related to the lift. From equation (4.2), the lift on the body is given by

$$\text{Lift} = F_2 = \int_0^{2\pi} \left\{ \rho U \frac{\partial \phi}{\partial x} n_2 - \rho U u_2 n_1 \right\} r d\theta$$

We consider first the integral

$$\int_0^{2\pi} \left\{ \rho U \frac{\partial \phi}{\partial x} \sin\theta r d\theta - \rho U \frac{\partial \phi}{\partial y} \cos\theta \right\} r d\theta$$

However, $\frac{\partial \phi}{\partial \theta} = \frac{\partial x}{\partial \theta}\frac{\partial \phi}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial \phi}{\partial y} = -\frac{\partial \phi}{\partial x} r \sin\theta + \frac{\partial \phi}{\partial y} r \cos\theta$, and so the integral becomes

$$-\rho U \int_0^{2\pi} \frac{\partial \phi}{\partial \theta} d\theta = -\rho U [\phi]_0^{2\pi}$$

The discontinuity term in ϕ is $-\frac{L\theta}{2\pi\rho U}$, and so the integral has value

$$-\rho U \left(-\frac{L}{2\pi \rho U} \right) [\theta]_0^{2\pi} = L$$

We consider next the integral

$$\int_0^{2\pi} \rho U \frac{\partial \Psi}{\partial x} \cos\theta r d\theta$$

From equation (3.33), in the far field wake

$$w_2 \sim \frac{1}{x} \left\{ \sqrt{\frac{\pi}{2kx}} e^{-\frac{ky^2}{2x}} \right\}$$

and so using the same method for evaluating the drag in the previous section (4.1.1), we find that this integral will be of order $\frac{1}{x}$, and so tends to zero as $r \to \infty$. Therefore the expression for the lift is

$$\text{Lift} = L \tag{4.4}$$

Thus we see that the lift is not expressed as a wake traverse; the lift is associated with the velocity potential $\underline{\nabla\phi}$ and not the wake velocity $\underline{w}$.

It is important to compare this result with that obtained in the Landau and Lifshitz book Fluid Mechanics for three dimensional steady flow past a body in the far field. They express both the drag and the lift in terms of wake traverses. Thus the approach given by Landau and Lifshitz cannot be used for far field flow past a body in two dimensional flow since the premise that the lift is expressed in terms of a wake traverse is invalid. We shall consider the Landau and Lifshitz approach in section (6.1.2).

We next consider the integral expression for the moment on the body.

4.1.3 The moment on the body.

We similarly express the moment on the body about axes through the origin as an integral over a closed contour enclosing the body.

We substitute into this integral expression the functions ϕ and Ψ in the far field and evaluate the resulting integral. We split the integral and evaluate each part. There are many resulting integrals to be evaluated and the procedure although straightforward is long.

We evaluate the integral expression for the contribution $M^{(1)}$ to the moment M such that the integral is evaluated using expressions for the Oseen velocity $\underline{u}$ and Oseen pressure p. We see that $M^{(1)}$ does not give all the terms in the moment expression because higher order pressure and velocity terms are involved in its evaluation. In order to find these extra terms in the moment, it is necessary to consider the second and third approximations to the Oseen equations. (See Filon (1928) and Imai (1951).)

We consider $M^{(1)}$ only, and find that it depends upon coefficients in the expansion of both ϕ and Ψ such that

$$-M^{(1)} = -2\rho U\pi B_1 + \frac{LD}{2\pi\rho U^2} - 2\frac{\rho U s\pi}{k} - \frac{2\nu L}{U}$$

where B_1 and s are constants given in equations (3.24) and (3.33) respectively. These terms are also obtained by Filon and Imai.

We first express the moment on the body as an integral over the closed contour enclosing the body.

The vector cross product $\underline{r} \times \underline{F}$, where $\underline{r} = (x_1, x_2)$ and $\underline{F} = (F_1, F_2)$ is expressed in tensor notation as

$$[\underline{r} \times \underline{F}]_i = -\epsilon_{ijk} x_j F_k$$

where $i, j, k = 1, 2, 3$ and $x_3 = F_3 = 0$.

The moment on the body $\underline{M}$ is

$$-M_i = \int_C \epsilon_{ijk} x_j p_{kq} n_q dl \quad \text{as} \quad \delta F_k = p_{kq} n_q \delta l$$

By the divergence theorem,

$$\left(\int_{C'} - \int_C\right) \epsilon_{ijk} x_j p_{kq} n_q dl = \int\int_A \frac{\partial}{\partial x_q}(\epsilon_{ijk} x_j p_{kq}) dA$$

Hence

$$-M_i = \int_{C'} \epsilon_{ijk} x_j p_{kq} n_q dl - \int\int_A \frac{\partial}{\partial x_q}(\epsilon_{ijk} x_j p_{kq}) dA$$

We consider the vector $\epsilon_{ijk} p_{kj}$. Since the pressure tensor p_{kj} is symmetric, this vector is $(0,0)$.

Therefore,

$$\begin{aligned}
\int\int_A \frac{\partial}{\partial x_q}(\epsilon_{ijk} x_j p_{kq}) dA &= \int\int_A \left\{ \epsilon_{ijk} \delta_{jq} p_{kq} + \epsilon_{ijk} x_j \frac{\partial p_{kq}}{\partial x_q} \right\} dA \\
&= \int\int_A \epsilon_{ijk} x_j \frac{\partial p_{kq}}{\partial x_q} dA \\
&= \int\int_A \epsilon_{ijk} x_j \frac{\partial}{\partial x_q}(\rho u_k^\dagger u_q^\dagger) dA \\
&= \int\int_A \frac{\partial}{\partial x_q}(\epsilon_{ijk} x_j \rho u_k^\dagger u_q^\dagger) dA
\end{aligned}$$

using $\epsilon_{ijk} \rho u_k^\dagger u_j^\dagger = 0_i$.

Therefore

$$-M_i = \int_{C'} \epsilon_{ijk} x_j p_{kq} n_q dl - \int\int_A \frac{\partial}{\partial x_q}(\epsilon_{ijk} x_j \rho u_k^\dagger u_q^\dagger) dA$$

Applying the divergence theorem,

$$\int\int_A \frac{\partial}{\partial x_q}(\epsilon_{ijk}x_j\rho u_k^\dagger u_q^\dagger)dA = \left(\int_{C'} - \int_C\right)\epsilon_{ijk}x_j\rho u_k^\dagger u_q^\dagger n_q dl$$

However, the fluid velocity is zero on the body contour C and so

$$-M_i = \int_{C'} \epsilon_{ijk}x_j(p_{kq} - \rho u_k^\dagger u_q^\dagger)n_q dl \qquad (4.5)$$

Evaluating the contribution to the moment on the body from the velocity potential We next substitute the functions ϕ and Ψ in the far field into the integral over the closed contour enclosing the body. We evaluate the resulting integrals. Although this procedure is straightforward, it is long. We first consider the equation (4.4)

$$-M_i = \int_{C'} \epsilon_{ijk} x_j (p_{kq} - \rho u_k^\dagger u_q^\dagger) n_q dl$$

The velocity $u_k^\dagger u_q^\dagger$ is

$$u_k^\dagger u_q^\dagger = (U_k + u_k)(U_q + u_q) = U_k U_q + U_k u_q + u_k U_q + u_k u_q$$

where $U_k = U$ for $k = 1$ and $U_k = 0$ for $k = 2$.

We substitute the above expression for $u_k^\dagger u_q^\dagger$ into equation (4.5) for M_i and thus find $M^{(1)}$, the contribution to the moment satisfying Oseen's first approximation (the Oseen equations). We also substitute for p_{ij}, $p_{ij} = -p\delta_{ij} + \mu e_{ij}$, into equation (4.5). Thus we express the $M_i^{(1)}$ as the summation of six integrals over the contour C' which we determine individually. Thus we consider six integrals a_i, b_i, c_i, d_i, e_i, and f_i over the contour C' such that

$$\begin{aligned}
a_i &= -\int_{C'} \epsilon_{ijk} x_j p \delta_{kq} n_q dl \\
b_i &= -\int_{C'} \epsilon_{ijk} x_j \rho U_k U_q n_q dl \\
c_i &= -\int_{C'} \epsilon_{ijk} x_j \rho U_k u_q n_q dl \\
d_i &= -\int_{C'} \epsilon_{ijk} x_j \rho u_k U_q n_q dl \\
e_i &= -\int_{C'} \epsilon_{ijk} x_j \rho u_k u_q n_q dl \\
f_i &= -\int_{C'} \epsilon_{ijk} x_j \mu e_{kq} n_q dl
\end{aligned}$$

We first consider the integral expression for a_i:

$$
\begin{aligned}
a_i &= -\int_{C'} \epsilon_{ijk} x_j p \delta_{kq} n_q dl \\
&= -\int_{C'} \epsilon_{ijk} x_j p n_k dl
\end{aligned}
$$

Since $x_3 = n_3 = 0$ then $a_1 = a_2 = 0$. a_3 is such that

$$
\begin{aligned}
a_3 &= -\int_{C'} \{\epsilon_{321} x_2 n_1 p + \epsilon_{312} x_1 n_2 p\} dl \\
&= \int_{C'} p(x_2 n_1 - x_1 n_2) dl
\end{aligned}
$$

We take the contour C' over the circle circumference radius $r \to \infty$ and so

$$a_3 = \int_0^{2\pi} p(x_2 n_1 - x_1 n_2) r d\theta$$

$$\int_0^{2\pi} p(x_2 n_1 - x_1 n_2) r d\theta$$

However, $rn_1 = x_1$ and $rn_2 = x_2$.

Therefore

$$a_3 = 0$$

We next consider the integral b_i such that

$$b_i = -\int_{C'} \epsilon_{ijk} x_j \rho U_k U_q n_q dl$$

However $b_1 = b_2 = 0$ because $x_3 = n_3 = 0$. b_3 is such that

$$b_3 = -\rho \int_{C'} \{\epsilon_{321} x_2 U_1 (U_1 n_1 + U_2 n_2) + \epsilon_{312} x_1 U_2 (U_1 n_1 + U_2 n_2)\} dl$$

U_i is such that $U_1 = U$ and $U_2 = 0$, so

$$b_3 = \rho U^2 \int_0^{2\pi} x_2 x_1 d\theta$$

The function $x_2 x_1$ equals $(1/2) r^2 \sin 2\theta$ and is antisymmetric in the variable θ and so

$$b_3 = 0$$

[We consider the single valued function $f(\theta)$ antisymmetric in θ. Thus $f(\theta) = -f(-\theta) = -f(2\pi - \theta)$. Therefore

$$\begin{aligned} \int_0^{2\pi} f(\theta) d\theta &= \int_0^{2\pi} f(\theta) d\theta - \int_\pi^{2\pi} f(2\pi - \theta) d\theta \\ &= \int_0^{\pi} f(\theta) d\theta - \int_\pi^{0} f(\alpha)[-d\alpha] \\ &= 0 \ \ ] \end{aligned}$$

We next consider the integral c_i such that

$$c_i = -\rho \int_{C'} \epsilon_{ijk} x_j U_k u_q n_q dl$$

$c_1 = c_2 = 0$ because $x_3 = n_3 = 0$ and c_3 is such that

$$c_3 = -\rho \int_{C'} \{\epsilon_{321} x_2 U(u_1 n_1 + u_2 n_2) dl$$

since $U_1 = U$ and $U_2 = 0$. Hence

$$c_3 = \rho U \int_{C'} x_2(u_1 n_1 + u_2 n_2) dl$$

We take the contour C' over the circle circumference radius $r \to \infty$, and so

$$c_3 = \rho U \int_0^{2\pi} x_2(u_1 n_1 + u_2 n_2) r d\theta$$

We first consider the contribution to this integral from the velocity term $\underline{\nabla \phi}$. We name this contribution c_3^ϕ. We shall later consider the contribution to this integral from the velocity term $\underline{w}$. We substitute the expansion of $\underline{\nabla \phi}$ into the above integral equation. (The expansion of $\underline{\nabla \phi}$ is obtained from the expansion of ϕ given in section (3.5.2) .) As $r \to \infty$, all the terms in the expansion of ϕ give integral contributions for c_3^ϕ which tend to zero except for the term $B_1 \frac{\sin\theta}{r}$.

Therefore, from appendix section (4.2), we obtain

$$c_3^\phi = \rho U \left(-\frac{\pi B_1}{2} - \frac{\pi B_1}{2} \right) = -\rho U \pi B_1$$

The contribution to the integral expression for c_3 from the velocity term $\underline{w} = (\frac{\partial\psi}{\partial y}, -\frac{\partial\psi}{\partial x})$ is considered later. (We name this contribution c_3^ψ and therefore $c_3 = c_3^\phi + c_3^\psi$.)

We next consider the integral d_i such that

$$d_i = -\rho \int_{C'} \epsilon_{ijk} x_j u_k U_q n_q dl$$

$d_1 = d_2 = 0$ because $x_3 = n_3 = 0$ and d_3 is such that

$$d_3 = -\rho \int_{C'} \{\epsilon_{321} x_2 u_1 U n_1 + \epsilon_{312} x_1 u_2 U n_1\} dl$$

since $U_1 = U$ and $U_2 = 0$. Therefore

$$d_3 = \rho U \int_{C'} (x_2 u_1 n_1 - x_1 u_2 n_1) dl$$

We take the contour C' over the circle circumference radius $r \to \infty$ and so

$$d_3 = \rho U \int_0^{2\pi} (x_2 u_1 n_1 - x_1 u_2 n_1) r d\theta$$

We consider the contribution to this integral from the velocity term $\underline{\nabla\phi}$. We call this contribution d_3^ϕ.

We follow the method for calculating c_3^ϕ and similarly find that as $r \to \infty$ all the terms in the expansion of ϕ give integral contributions for d_3^ϕ which tend to zero except for the term $\phi = B_1 \frac{\sin\theta}{r}$. [See section (3.5.2).]

Therefore, from appendix (4.2), we obtain

$$d_3^\phi = \rho U(-\frac{\pi B_1}{2} - \frac{\pi B_1}{2}) = -\rho U \pi B_1$$

We next consider the integral e_i such that

$$e_i = -\rho \int_{C'} \epsilon_{ijk} x_j u_k u_q n_q dl$$

$e_1 = e_2 = 0$ because $x_3 = n_3 = 0$ and e_3 is such that

$$e_3 = -\rho \int_{C'} \{\epsilon_{312} x_1 u_2 (u_1 n_1 + u_2 n_2) + \epsilon_{321} x_2 u_1 (u_1 n_1 + u_2 n_2)\} dl$$

We take the contour C' along the circle circumference radius $r \to \infty$ and so

$$e_3 = \rho \int_{C'} \{x_2 u_1 (u_1 n_1 + u_2 n_2) - x_1 u_2 (u_1 n_1 + u_2 n_2)\} r d\theta$$

We consider the contribution to this integral from the velocity term $\underline{\nabla \phi}$. We name this contribution e_3^ϕ. We substitute the expansion of $\underline{\nabla \phi}$ into the above integral equation. The expansion of $\underline{\nabla \phi}$ is obtained from the expansion of ϕ given in section (3.5.2). We find that as $r \to \infty$, all the terms in the expansion of ϕ give integral contributions for e_3^ϕ which tend to zero except for the first two terms:

$$\begin{aligned}
\phi &= -\frac{L}{2\pi\rho U}(\theta - \pi) + \frac{D}{2\pi\rho U}\ln r + \ldots \\
\underline{\nabla \phi} &= -\frac{L}{2\pi\rho U}\left(-\frac{\sin\theta}{r}, \frac{\cos\theta}{r}\right) + \frac{D}{2\pi\rho U}\left(\frac{\cos\theta}{r}, \frac{\sin\theta}{r}\right) + \ldots
\end{aligned}$$

Substituting the velocity term $\underline{\nabla \phi}$ into the integral expression for e_3, we obtain:

$$e_3^\phi = \frac{LD}{8\pi\rho U^2} + \frac{LD}{8\pi\rho U^2} + \frac{LD}{8\pi\rho U^2} + \frac{LD}{8\pi\rho U^2}$$

where

$$\rho \int_0^{2\pi} x_2 u_1^2 n_1 r d\theta = \frac{LD}{8\pi\rho U^2} \quad e.t.c.$$

[See appendix section (4.2).]

Thus

$$e_e^\phi = \frac{LD}{2\pi\rho U^2}$$

Therefore the moment on the body due to the velocity $\underline{\nabla\phi}$ is $M_3^{(1)\phi}$ such that

$$\begin{aligned} -M_3^{(1)\phi} &= c_3^\phi + d_3^\phi + e_3^\phi \\ &= -\rho U\pi B_1 - \rho U\pi B_1 + \frac{LD}{2\pi\rho U^2} \\ &= -2\rho U\pi B_1 + \frac{LD}{2\pi\rho U^2} \end{aligned}$$

We now consider the contributions to the moment due to the velocity $\underline{w}$.

In the far field, the function $\underline{w}$ exponentially falls to zero outside the wake region. Inside the wake region, from equation (3.33), $\underline{w}$ is

$$\underline{w} = \left(-\frac{D}{\pi\rho U} k\sqrt{\frac{\pi}{2kx_1}} e^{-\frac{kx_2^2}{2x_1}}, -\frac{D}{2\pi\rho U}\frac{x_2}{x_1}\sqrt{\frac{k\pi}{2x_1}} e^{-\frac{kx_2^2}{2x_1}} \right)$$

The terms in the moment integral which include the functions w_1 and w_2 will only give non zero integral contributions to the moment if they are at least of order

$$\frac{e^{-\frac{kx_2^2}{2x_1}}}{\sqrt{x_1}}$$

We see this by considering the order of the term in the drag integral in section (4.1.1).

By considering c_i, d_i, e_i we see that the highest order terms in the moment integral which include the functions w_1 and w_2 are

$\rho U(x_2 w_1 n_1)$, $\rho U(x_1 w_2 n_1)$ and $\rho(x_1 w_2 w_1 n_1)$ where in the far field wake $n_1 \to 1$.

We obtain a non zero integral contribution to the moment from the terms in the moment integral $\rho U(x_2 w_1 n_1)$ and $\rho U(x_1 w_2 n_1)$ which come from the integrals c_3 and d_3 respectively.

The contribution from the integral c_3 is

$$\rho U \int_0^{2\pi} x_2 w_1 n_1 r d\theta$$

which is

$$\rho U \int_{-\infty}^{\infty} x_2 \left(-\frac{s x_2}{x_1}\right) \sqrt{\frac{k\pi}{2x_1}} e^{-\frac{kx_2^2}{2x_1}} dx_2$$

If we change the variable of integration to $q = \sqrt{\frac{k}{2x_1}} x_2$ then the integral becomes

$$-\rho U s \sqrt{\frac{k\pi}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{x_1}} \frac{2p^2}{k} e^{-p^2} dp \sqrt{\frac{2x_1}{k}} = -\frac{\rho U s \pi}{k}$$

The contributions from the integral d_3 are

$$-\rho U \int_0^{2\pi} x_1 w_2 n_1 r d\theta$$

and

$$\rho U \int_0^{2\pi} x_2 w_1 n_1 r d\theta$$

The first integral gives the contribution

$$-\rho U \int_{-\infty}^{\infty} x_1 s \sqrt{\frac{\pi}{2kx_1}} e^{-\frac{kx_2^2}{2x_1}} \left\{\frac{1}{2x_1} - \frac{kx_2^2}{2x_1^2}\right\} dx_2$$

If we change the variable of integration to $q = \sqrt{\frac{k}{2x_1}} x_2$, then the integral becomes

$$-\rho U s \sqrt{\frac{\pi}{2k}} \int_{-\infty}^{\infty} \frac{e^{-q^2}}{\sqrt{x_1}} \left\{\frac{1}{2} - q^2\right\} \sqrt{\frac{2x_1}{k}} dq =$$

The second integral is identical to the itegral contribution c_3 and therefore gives the contribution

$$-\frac{\rho U s \pi}{k}$$

Thus the contribution to the moment from the integrals c_i, d_i and e_i involving the velocity $\underline{w}$ is

$$-\frac{3\rho U s\pi}{2k} \tag{4.6}$$

We finally consider the integral f_i such that

$$f_i = -\mu \int_{C'} \epsilon_{ijk} x_j \left(\frac{\partial u_k}{\partial x_q} + \frac{\partial u_q}{\partial x_k} \right) n_q dl$$

Therefore $f_1 = f_2 = 0$ and

$$\begin{aligned} f_3 &= - \quad \mu \int_0^{2\pi} \left\{ x_1^2 \left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2} \right) - x_2^2 \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) + 2x_1x_2 \left(\frac{\partial u_2}{\partial x_2} - \frac{\partial u_1}{\partial x_1} \right) \right\} d\theta \\ &= \quad -\mu \int_0^{2\pi} \left\{ (x_1^2 - x_2^2) \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) - 2x_1x_2 2 \frac{\partial u_1}{\partial x_1} \right\} d\theta \\ &= \quad -\mu \int_0^{2\pi} \left\{ 2r^2 \cos 2\theta \frac{\partial^2 \phi}{\partial x_1 \partial x_2} - 2r^2 \sin 2\theta \frac{\partial^2 \phi}{\partial x_1^2} \right\} d\theta \end{aligned}$$

where C' is the contour circumference of the circle radius r centred at the origin.

We first consider the contribution to this integral from the velocity potential $\underline{\nabla \phi}$.

By inspection of the expansion for the velocity potential given in section (3.26), we see that the terms in the expansion for $n > 1$ will give integral contributions to f_3 at the greatest of order $\frac{1}{r}$. Thus there is no contribution to the integral from these terms. Therefore we consider the first two terms in the expansion for ϕ

$$\phi = -\frac{D}{2\pi\rho U} \log r - \frac{L\theta}{2\pi\rho U}$$

We find $\frac{\partial^2 \phi}{\partial x_1^2}$ and $\frac{\partial^2 \phi}{\partial x_1 \partial x_2}$.

$$-2\pi\rho U \frac{\partial \phi}{\partial x_1} = D\frac{x_1}{r^2} - L\frac{x_2}{r^2} \quad \text{and} \quad -2\pi\rho U \frac{\partial \phi}{\partial x_2} = D\frac{x_2}{r^2} + L\frac{x_1}{r^2}$$

and so

$$\begin{aligned} -2\pi\rho U \frac{\partial^2 \phi}{\partial x_1^2} &= D\left(\frac{1}{r^2} - \frac{2x_1^2}{r^4} \right) + 2L\frac{x_1x_2}{r^4} \\ &= -D\frac{\cos 2\theta}{r^2} + L\frac{\sin 2\theta}{r^2} \end{aligned}$$

and

$$
\begin{aligned}
-2\pi\rho U \frac{\partial^2 \phi}{\partial x_1 \partial x_2} &= -2D\frac{x_1 x_2}{r^4} - L\left(\frac{1}{r^2} - \frac{2x_2^2}{r^4}\right) \\
&= -D\frac{\sin 2\theta}{r^2} - L\frac{\cos 2\theta}{r^4}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
2\pi\rho U f_3 &= 2\mu \int_0^{2\pi} \{\cos 2\theta\,(-D\sin 2\theta - L\cos 2\theta) - \sin 2\theta\,(-D\cos 2\theta + L\sin 2\theta)\}\, d\theta \\
&= 2\mu \int_0^{2\pi} -L(\cos^2 2\theta + \sin^2 2\theta) d\theta \\
&= -4\pi\mu L
\end{aligned}
$$

We next consider contributions to the integral f_3 from the velocity $\underline{w}$. Since the velocity $\underline{w}$ is exponentially small outside the wake, then

$$\begin{aligned} f_3 &= -\mu \int_{-\infty}^{\infty} (\epsilon_{312} x_1 e_{21} + \epsilon_{321} x_2 e_{11}) dx_2 \\ &= -\int_{-\infty}^{\infty} \left\{ x_1 \left(\frac{\partial w_2}{\partial x_1} + \frac{\partial w_1}{\partial x_2} \right) - 2x_2 \frac{\partial w_1}{\partial x_1} \right\} dx_2 \end{aligned}$$

From the symmetry of the above equation, we will only obtain terms contributing to the integral which are antisymmetric in x_2 for the function w_1 and symmetric in x_2 for the function w_2.

Refering to section (3.6), the leading order term in the streamfunction for antisymmetric flow in the far field is

$$\Psi_a = s\sqrt{\frac{\pi}{2kx_1}} e^{-\frac{kx_2^2}{2x_1}}$$

which give the leading order terms in the velocity far field for antisymmetric flow:

$$\underline{w_a} = \left(-s\frac{x_2}{x_1}\sqrt{\frac{k\pi}{2x_1}} e^{-\frac{kx_2^2}{2x_1}}, \frac{s}{x_1}\sqrt{\frac{\pi}{2kx_1}} e^{-\frac{kx_2^2}{2x_1}} \left\{ \frac{1}{2} - \frac{kx_2^2}{2x_1} \right\} \right)$$

given in equation (3.33).

By inspection of the integral f_3, the only term in the integrand contributing to the integral is $\mu x_1 \frac{\partial w_1}{\partial x_2}$, the other terms give contributions which tend to zero as $x_1 \to \infty$.

Thus

$$f_3 = -\mu \int_{-\infty}^{\infty} x_1 \frac{\partial w_1}{\partial x_2} dx_2$$

and

$$
\begin{aligned}
\frac{\partial w_{1a}}{\partial x_2} &= -s\frac{x_2}{x_1}\sqrt{\frac{k\pi}{2x_1}}e^{-\frac{kx_2^2}{2x_1}}\left(\frac{1}{x_2}-\frac{kx_2}{x_1}\right) \\
&= -\frac{s}{x_1}\sqrt{\frac{k\pi}{2x_1}}e^{-\frac{kx_2^2}{2x_1}}\left(1-\frac{kx_2^2}{x_1}\right)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
f_3 &= s\mu\sqrt{\frac{k\pi}{2}}\int_{-\infty}^{\infty}\frac{e^{-\frac{kx_2^2}{2x_1}}}{\sqrt{x_1}}\left\{1-\frac{kx_2^2}{x_1}\right\}dx_2 \\
&= s\mu\sqrt{\frac{k\pi}{2}}\int_{-\infty}^{\infty}\frac{e^{-p^2}}{\sqrt{x_1}}\left\{1-2p^2\right\}\sqrt{\frac{2x_1}{k}}dp \\
&= s\mu\sqrt{\pi}\int_{-\infty}^{\infty}e^{-p^2}\{1-2p^2\}dp \\
&= 0
\end{aligned}
$$

Thus the contribution to the moment from the term in the integrand involving e_{ij} is zero.

Hence

$$
\int_{C'}\epsilon_{ijk}\mu e_{kq}n_q dl = 0
$$

Alternatively, we obtain this result by expressing w_1 and w_2 in terms of the streamfunction Ψ of the velocity $\underline{w}$.

This gives

$$
\begin{aligned}
f_3 &= -\mu\int_{-\infty}^{\infty}\left\{x_1\left(-\frac{\partial^2\Psi}{\partial x_1^2}+\frac{\partial^2\Psi}{\partial x_2^2}\right)-2x_2\frac{\partial^2\Psi}{\partial x_1\partial x_2}\right\}dx_2 \\
&= -\mu\int_{-\infty}^{\infty}\left(-x_1\frac{\partial^2\Psi}{\partial x_1^2}+2\frac{\partial\Psi}{\partial x_1}\right)dx_2
\end{aligned}
$$

by integrating by parts. We then find expressions for $\frac{df_3}{dx_1}$ and $\frac{d^2f_3}{dx_1^2}$ and find that

$$\frac{d^2f_3}{dx_1^2} - 4k\frac{df_3}{dx_1} + 4k^2f_3 = 0$$

and so $f_3 = (A + Bx_1)e^{2kx_1}$. However, f_3 cannot be exponentially large and so $A = B = 0$ and therefore $f_3 = 0$.

We now investigate further approximations to the Oseen solution and the effect of these terms on the lift, drag and moment.

4.1.4 Further approximations to the Oseen solution.

We consider further approximations to the Oseen solution for steady two dimensional flow. We consider flow such that the velocity perturbations to a uniform stream U are small and so represent the fluid velocity $u^\dagger$ and pressure $p^\dagger$ in the forms

$$\underline{u^\dagger} = \underline{u_0} + \underline{u_1} + \underline{u_2} + \underline{u_3} + ... \;, \quad p^\dagger = p_0 + p_1 + p_2 + p_3 + ...$$

where $\underline{u_0} = Ui$, and $\underline{u_1} = \underline{u}$, the Oseen velocity.

The Navier-Stokes equations are

$$\left\{(\underline{u_0} + \underline{u_1} + \underline{u_2} + ...).\underline{\nabla}\right\}(\underline{u_0}+\underline{u_1}+\underline{u_2}+...) = -\frac{1}{\rho}\nabla(p_0+p_1+p_2+...) = \nu(\nabla^2)(\underline{u_0}+\underline{u_1}+\underline{u_2}+....)$$

and the continuity equation $\underline{\nabla}.\underline{u^\dagger} = 0$ gives $\underline{\nabla}.\underline{u_0} = 0$, $\underline{\nabla}.\underline{u_1} = 0$......

Thus we linearise the Navier-Stokes equations for flow perturbation to a uniform stream U.

The term of highest order in the linearisation gives

$$(\underline{u_0}.\underline{\nabla})\underline{u_0} = -\frac{1}{\rho}\underline{\nabla p_0} + \nu(\nabla^2)\underline{u_0}$$

which gives $P_0 =$ constant.

The first approximation gives

$$(\underline{u_0}.\underline{\nabla})\underline{u_1} + (\underline{u_1}.\underline{\nabla})\underline{u_0} = -\frac{1}{\rho}\underline{\nabla p_1} + \nu(\nabla^2)\underline{u_1}$$

which are the Oseen equations

$$U\frac{\partial \underline{u}}{\partial x_1} = -\frac{1}{\rho}\underline{\nabla p} + \nu(\nabla^2)\underline{u}$$

where $\underline{u_1} = u$ and $p_1 = p$.

The second approximation to the Oseen solution gives

$$(\underline{u_0}.\underline{\nabla})\underline{u_2} + (\underline{u_1}.\underline{\nabla})\underline{u_1} + (\underline{u_2}.\underline{\nabla})\underline{u_0} = -\frac{1}{\rho}\underline{\nabla p_2} + \nu(\nabla^2)\underline{u_2}$$

and so

$$(U\frac{\partial}{\partial x_1})\underline{u_2} + (\underline{u_1}.\underline{\nabla})\underline{u_1} = -\frac{1}{\rho}\underline{\nabla p_2} + \nu(\nabla^2)\underline{u_2}$$

Similarly, the third approximation gives

$$U\frac{\partial \underline{u_3}}{\partial x_1} + (\underline{u_1}.\underline{\nabla})\underline{u_2} + (\underline{u_2}.\underline{\nabla})\underline{u_1} = -\frac{1}{\rho}\underline{\nabla p_3} + \nu(\nabla^2)\underline{u_3}$$

Filon (1928 Phil. Trans. Roy. Soc. 227 93-135) considers the possibility of contributions to the lift, drag and moment from the second approximation to the Oseen solution.

Imai (1951 Proc. Roy. Soc. A 208 487-516) finds the second and third approximations to the Oseen solution in the far field wake and calculates the contribution to the lift, drag and moment. He finds these contributions vanish conditionally and not because of their order.

In fact, there is a contribution to the moment from the second approximation to the Oseen solution which is of order $\log r$, (Filon pg 131), but Imai shows this term is balanced by an identical contribution but of opposite sign from the third approximation to the Oseen solution (see also Lamimar Boundary Layers, Rosenhead pg 197 ch4.8).

We have therefore not found all the terms in the moment expression, only those derived from the Oseen first approximation solution, the Oseen equations. The contributions to the moment from the second and third Oseen approximations are given by Filon (1928) and Imai (1951).

Collecting together all the contributions, we obtain

$$M_3^{(1)} = -2\rho U \pi B_1 + \frac{LD}{2\pi\rho U^2} - 2\frac{\rho U s \pi}{k} - \frac{2\nu L}{U} \tag{4.7}$$

Thus we see that the moment on the body involves coefficients from the expansions of both ϕ and Ψ.

We now investigate the moment calculation for three dimensional flow where the Oseen approximation is valid.

4.2 Ch4.1 Appendix: Evaluation of the terms in the moment integral expression.

We evaluate the integral expression

$$c_3^\phi = \rho U \int_0^{2\pi} x_2 \left(\frac{\partial \phi}{\partial x_1} n_1 + \frac{\partial \phi}{\partial x_2} n_2 \right) r d\theta$$

We split the above integral expression into two parts, and we first consider the first part of the integral expression

$$\int_0^{2\pi} x_2 \frac{\partial \phi}{\partial x_1} \cos\theta r d\theta$$

We substitute the potential expansion of ϕ into this integral. The terms in the expansion of ϕ which are symmetric about $y = 0$ give no contribution to the integral since $\int_0^{2\pi} f(\theta) d\theta = 0$ for $f(\theta) = -f(-\theta)$. We consider the first term $-\frac{L(\theta-\pi)}{2\pi\rho U}$ in the expansion of the antisymmetric potential. This gives a contribution to the integral of

$$-\int_0^{2\pi} x_2 \frac{\partial}{\partial x_1} \left(\frac{L(\theta - \pi)}{2\pi\rho U} \right) \cos\theta r d\theta = -\frac{L}{2\pi\rho U} \int_0^{2\pi} r \sin^2\theta \cos\theta d\theta = 0$$

We next consider the second term $\frac{B_1 \sin\theta}{r}$ in the expansion of the antisymmetric potential. This gives a contribution to the integral of

$$\int_0^{2\pi} r \sin\theta \left(-2B_1 \frac{\sin\theta\cos\theta}{r^2} \right) \cos\theta r d\theta$$

since $\frac{\partial}{\partial x_1}\left(\frac{\sin\theta}{r}\right) = -2\frac{\sin\theta\cos\theta}{r^2}$.

Thus the term $\frac{B_1 \sin\theta}{r}$ gives a contribution to the integral of

$$-\frac{B_1}{2} \int_0^{2\pi} \sin^2 2\theta d\theta = -\frac{B_1}{4} \int_0^{2\pi} (1 - \cos 4\theta) d\theta = -\frac{\pi B_1}{2}$$

The other terms in the antisymmetric expansion of ϕ are

$$\frac{B_n \sin n\theta}{r^n}$$

where $n \geq 2$. For $n \geq 2$, the integral contributions are of order $\frac{1}{r^{(n-1)}}$ and thus tend to zero as $r \to \infty$.

We next consider the other part of the integral expression for c_3^ϕ.

$$\int_0^{2\pi} x_2 \frac{\partial \phi}{\partial x_2} \sin\theta r d\theta \tag{4.8}$$

We follow the same method for calculating the first term in the integral expression $\int_0^{2\pi} x_2 \frac{\partial\phi}{\partial x_1} \cos n\theta r d\theta$.

We find the symmetric terms in the expansion of ϕ give no contribution to the integral of equation (4.6).

We therefore only consider the terms in the antisymmetric expansion of ϕ, the first term being $-\frac{L}{2\pi\rho U}\frac{(\theta-\pi)}{2\pi}$. This term gives a contribution to the integral of

$$\int_0^{2\pi} x_2 \left(\frac{L}{2\pi\rho U}\frac{\cos\theta}{r}\right) \sin\theta r d\theta$$

which gives

$$\frac{Lr}{2\pi\rho U}\left[\frac{\sin^3\theta}{3}\right]_0^{2\pi} = 0$$

We now consider the next term in the antisymmetric expansion of ϕ which is $\frac{B_1 \sin\theta}{r}$. The integral contribution from this term is

$$\int_0^{2\pi} r^2 \sin^2\theta \left(\frac{B_1 \cos 2\theta}{r^2}\right) d\theta = -\frac{B_1 \pi}{2}$$

We now consider the integral d_3^ϕ

$$d_3^\phi = \rho U \int_0^{2\pi} \left(x_2 \frac{\partial \phi}{\partial x_1} n_1 - x_1 \frac{\partial \phi}{\partial x_2} n_1 \right) r d\theta$$

We split this integral into two parts and we first consider the part of the integral

$$\int_0^{2\pi} x_2 \frac{\partial \phi}{\partial x_1} \cos\theta d\theta$$

This integral has already been evaluated in this section and has value

$$-\frac{\pi B_1}{2}$$

(See the start of this appendix section.)

We next consider the part of the integral

$$-\int_0^{2\pi} x_1 \frac{\partial \phi}{\partial x_2} \cos\theta r d\theta$$

The symmetric terms in the expansion of ϕ give no contribution to this integral. Therefore we consider the antisymmetric expansion of ϕ only.

We first consider the term $-\frac{L}{2\pi\rho U}(\theta - \pi)$ in the antisymmetric expansion of ϕ. This gives a contribution to the integral of

$$\begin{aligned}
& \int_0^{2\pi} x_i \frac{L}{2\pi\rho U} \frac{\cos\theta}{r} \cos\theta r d\theta \\
= & \frac{Lr}{2\pi\rho U} \int_0^{2\pi} (1 - \sin^2\theta) \cos\theta d\theta \\
= & \frac{Lr}{2\pi\rho U} [-\sin\theta - 1/3 \sin^3\theta]_0^{2\pi} \\
= & \ 0
\end{aligned}$$

We next consider the term $\frac{B_1 \sin\theta}{r}$.

This term gives a contribution to the integral of

$$
\begin{aligned}
& -\int_0^{2\pi} r\cos\theta \frac{B_1\cos 2\theta}{r} d\theta \\
= & -B_1 \int_0^{2\pi} \{1/2\cos 2\theta(1+\cos 2\theta\} d\theta \\
= & -\frac{B_1}{2} \int_0^{2\pi} \{\cos 2\theta + 1/2(1+\cos 4\theta)\} d\theta \\
= & -\frac{B_1 \pi}{2}
\end{aligned}
$$

We now consider the contribution to the moment from the integral

$$e_3 = \rho \int_0^{2\pi} \{x_2 u_1(u_1 n_1 + u_2 n_2) - x_1 u_2(u_1 n_1 + u_2 n_2)\} r d\theta$$

We consider the contribution to this integral from the velocity potential $\underline{\nabla \phi}$. As $r \to \infty$, only the first two terms in the expansion for ϕ give contributions to the integral which do not tend to zero. The first two terms in the expansion of ϕ are

$$\phi = -\frac{L}{2\pi\rho U}(\theta - \pi) + \frac{D}{2\pi\rho U} \ln r$$

which give the terms in the velocity potential

$$\underline{\nabla \phi} = -\frac{L}{2\pi\rho U}\left(-\frac{\sin\theta}{r}, \frac{\cos\theta}{r}\right) + \frac{D}{2\pi\rho U}\left(\frac{\cos\theta}{r}, \frac{\sin\theta}{r}\right)$$

We first consider the integral

$$\rho \int_0^{2\pi} x_2 u_1^2 n_1 r d\theta$$

If u_1^2 is a symmetric function in θ, then the integral is zero. Hence we look for antisymmetric combinations of the function

$$u_1^2 = -\left\{\frac{L}{2\pi\rho U}\left(-\frac{\sin\theta}{r}\right) + \frac{D}{2\pi\rho U}\left(\frac{\cos\theta}{r}\right)\right\}\left\{-\frac{L}{2\pi\rho U}\left(-\frac{\sin\theta}{r}\right) + \frac{D}{2\pi\rho U}\left(\frac{\cos\theta}{r}\right)\right\}$$

The antisymmetric combinations of the above function give

$$2 \times \left(-\frac{L}{2\pi\rho U}\right)\left(\frac{D}{2\pi\rho U}\right)\left(-\frac{\sin\theta}{r}\right)\left(\frac{\cos\theta}{r}\right) = \frac{LD}{2\pi\rho^2 U^2}\frac{\sin\theta\cos\theta}{r^2}$$

Therefore the integral $\rho \int_0^{2\pi} x_2 u_1^2 n_1 r d\theta$ becomes

$$\begin{aligned}
& \rho \int_0^{2\pi} r \sin\theta \left(\frac{LD}{22\pi^2\rho^2 U^2} \frac{\sin\theta\cos\theta}{r^2} \right) \cos\theta r d\theta \\
= \; & \frac{LD}{2\pi^2 \rho U^2} \int_0^{2\pi} 1/4 \sin^2 2\theta d\theta \\
= \; & \frac{LD}{8\pi^2 \rho U^2} \int_0^{2\pi} 1/2(1 - \cos 4\theta) d\theta \\
= \; & \frac{LD}{8\pi \rho U^2}
\end{aligned}$$

We next consider the integral

$$\rho \int_0^{2\pi} \{x_2 u_1 u_2 n_2\} r d\theta$$

If the function $u_1 u_2$ is antisymmetric, the integral is zero. Hence we consider combinations of the function

$$u_1 u_2 = -\left(\frac{L}{2\pi\rho U}\left\{-\frac{\sin\theta}{r}\right\} + \frac{D}{2\pi\rho U}\left\{\frac{\cos\theta}{r}\right\}\right)\left(-\frac{L}{2\pi\rho U}\left\{\frac{\cos\theta}{r}\right\} + \frac{D}{2\pi\rho U}\left\{\frac{\sin\theta}{r}\right\}\right)$$

which are symmetric in θ.

The symmetric combinations give

$$\begin{aligned} & \frac{LD}{4\pi^2\rho^2U^2}\frac{\sin^2\theta}{r^2} - \frac{LD}{4\pi^2\rho^2U^2}\frac{\cos^2\theta}{r^2} \\ = \quad & -\frac{LD}{4\pi^2\rho^2U^2}\frac{\cos 2\theta}{r^2} \end{aligned}$$

Therefore the integral becomes

$$\begin{aligned} & -\frac{LD}{4\pi^2\rho U^2}\int_0^{2\pi} r\sin\theta\frac{\cos 2\theta}{r^2}\sin\theta r d\theta \\ = \quad & -\frac{LD}{8\pi^2\rho U^2}\int_0^{2\pi}\{\cos 2\theta - 1/2(1+\cos 4\theta)\}d\theta \\ = \quad & \frac{LD}{8\pi\rho U^2} \end{aligned}$$

We next consider the integral

$$-\rho \int_0^{2\pi} x_1 u_2 u_1 n_1 r d\theta$$

We obtain a contribution to this integral from the symmetric part of the function $u_1 u_2$ which is

$$-\frac{LD}{4\pi^2\rho^2U^2}\frac{\cos 2\theta}{r^2}$$

Hence the integral becomes

$$\begin{aligned}
& \frac{LD}{4\pi^2\rho U^2}\int_0^{2\pi} r\cos\theta\frac{\cos 2\theta}{r^2}\cos\theta r d\theta \\
= & \frac{LD}{8\pi^2\rho U^2}\int_0^{2\pi}\{\cos 2\theta 1/2(1+\cos 4\theta)\}d\theta \\
= & \frac{LD}{8\pi\rho U^2}
\end{aligned}$$

We finally consider the integral

$$-\rho \int_0^{2\pi} x_1 u_2 u_2 n_2 r d\theta$$

If the function $u_2 u_2$ is symmetric in θ then the integral is zero. Thus we look for combinations of the function

$$u_2 u_2 = \left(-\frac{L}{2\pi\rho U}\frac{\cos\theta}{r} + \frac{D}{2\pi\rho U}\frac{\sin\theta}{r}\right)\left(-\frac{L}{2\pi\rho U}\frac{\cos\theta}{r} + \frac{D}{2\pi\rho U}\frac{\sin\theta}{r}\right)$$

which are antisymmetric about θ, which give

$$-\frac{LD}{4\pi^2\rho^2U^2}\frac{\sin 2\theta}{r^2}$$

Therefore the integral becomes

$$\begin{aligned}
& \frac{LD}{4\pi^2\rho U^2}\int_0^{2\pi} r\cos\theta\frac{\sin 2\theta}{r^2}\sin\theta r d\theta \\
= & \frac{LD}{8\pi^2\rho U^2}\int_0^{2\pi}\sin^2 2\theta d\theta \\
= & \frac{LD}{8\pi\rho U^2}
\end{aligned}$$

4.3 The drag force, lift force and moment on the body in three dimensional flow.

From the Oseen representation of Oseen flow, we show that we can expand the perturbation velocity $\underline{u}$ and pressure p in terms of the fundamental solutions $\underline{u}^{(i)}$ and $p^{(i)}$ where $i = 1, 2, 3$. We obtain

$$u_k = A_i u_k^{(i)} + B_{il} \frac{\partial u_k^{(i)}}{\partial x_l} + ...$$

and

$$p = A_i p^{(i)} + B_{il} \frac{\partial p^{(i)}}{\partial x_l} + ...$$

where

$$A_i = \int \int_{S_0} \left\{ -p\delta_{ij} + \mu \frac{\partial u_i}{\partial x_j} - \rho U u_i \delta_{j1} \right\} n_j dS$$

and

$$B_{il} = \int \int_{S_0} \left\{ x_l (p\delta_{ij} - \mu \frac{\partial u_i}{\partial x_j} + \rho U u_i \delta_{j1}) - \rho U x_i u_j \delta_{l1} + \mu u_i \delta_{jl} \right\} n_j dS$$

Further, we find these coefficients are related to the force and moment on the body.

We expand the force F_i and the moment M_i on the body in the form

$$F_i = F_i^{(I)} + F_i^{(II)} + ... \quad M_i = M_i^{(I)} + M_i^{(II)} + ...$$

where $F_i^{(N)}$ is the force due to the velocity and pressure terms of the Nth order.

We find

$$F_i = F_i^{(I)} = A_i$$

and

$$M_i = M_i^{(I)} + M_i^{(II)} + M_i^{(III)} = -\epsilon_{ijk}B_{jk} + M_i^{(II)} + M_i^{(III)}$$

Thus we obtain expansions for the velocity and pressure, the coefficients of the terms in the expansions being related to the force and moment.

However, the nine derivatives of the fundamental solutions are not linearly independent. We thus consider a different representation of this expansion where the terms are linearly independent. We also consider the expansion of the velocity and pressure from the Lamb-Goldstein velocity decomposition.

We consider

(4.3.1). The Oseen representation of Oseen flow in three dimensions.

(4.3.2). Obtaining the velocity and pressure expansions in terms of the fundamental solutions and their derivatives.

(4.3.3). Showing the relations between the coefficients in this expansion and the force and moment on the body.

(4.3.4). Representing the velocity and pressure in an expansion which has linearly independant terms.

4.3.1 The Oseen representation of Oseen's equations in three dimensions.

(This method is identical to obtaining the Oseen representation of Oseen flow in two dimensions, which is covered in detail in chapter five.)

We consider the Oseen equations with body force $-\underline{f}(\underline{x})$ per unit volume,

$$\rho U \frac{\partial \underline{u}}{\partial x_1} = -\underline{\nabla} p + \mu \nabla^2 \underline{u} - \underline{f}\,, \quad \underline{\nabla}.\underline{u} = 0$$

Let $\underline{u}^{(a)}(\underline{x})$, $p^{(a)}(\underline{x})$ and $\underline{u}^{(b)}(\underline{x})$, $p^{(b)}(\underline{x})$ be solutions with negative body forces $\underline{f}^{(a)}(\underline{x})$, $\underline{f}^{(b)}(\underline{x})$ respectively. Then

$$\begin{aligned}
\frac{\partial}{\partial y_1}\left\{\rho U u_i^{(a)}(\underline{y}) u_i^{(b)}(\underline{x}-\underline{y})\right\} &= -\frac{\partial}{\partial y_i}\left\{p^{(a)}(\underline{y}) u_i^{(b)}(\underline{x}-\underline{y}) + u_i^{(a)}(\underline{y}) p^{(b)}(\underline{x}-\underline{y})\right\} \\
&\quad + \mu \frac{\partial}{\partial y_j}\left\{\frac{\partial u_i^{(a)}(\underline{y})}{\partial y_j} u_i^{(b)}(\underline{x}-\underline{y}) - u_i^{(a)}(\underline{y}) \frac{\partial u_i^{(b)}(\underline{x}-\underline{y})}{\partial y_j}\right\} \\
&\quad - f_i^{(a)}(\underline{y}) u_i^{(b)}(\underline{x}-\underline{y}) + u_i^{(a)}(\underline{y}) f_i^{(b)}(\underline{x}-\underline{y})
\end{aligned}$$

Hence we let the integral I be

$$\begin{aligned}
I &= \int\!\!\int_S \left\{p^{(a)} u_j^{(b)} + u_j^{(a)} p^{(b)} + \mu\left(u_i^{(a)} \frac{\partial u_i^{(b)}}{\partial y_j} - \frac{\partial u_i^{(a)}}{\partial y_j} u_i^{(b)}\right) + \rho U u_i^{(a)} u_i^{(b)} \delta_{j1}\right\} n_j dS_y \\
&= \int\!\!\int\!\!\int_V \left(u_i^{(a)} f_i^{(b)} - f_i^{(a)} u_i^{(b)}\right) dV_y
\end{aligned}$$

where n_j is the outward normal to S, the boundary surface of V.

We take V to be bounded externally by the large sphere S_R given by $|\underline{y}-\underline{x}| = R$, internally by the small sphere S_δ given by $|\underline{y}-\underline{x}| = \delta$ and by the surface S_0 lying between S_δ and S_R.

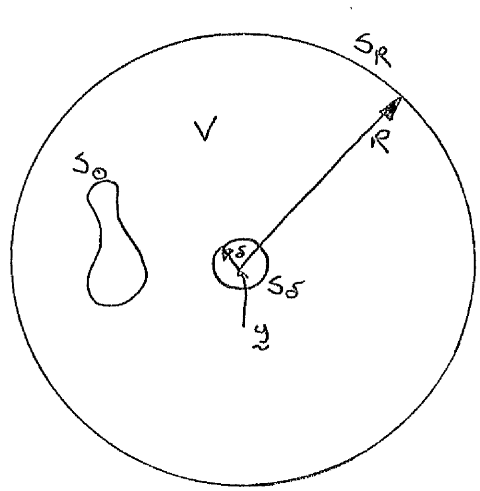


Figure 4.2: The surface bounding the volume V.

The Oseen fundamental solutions are defined as

$$u_i^{(k)}(\underline{x}) = \mu^{-1} w_{ki}(\underline{x}) \ , \quad p^{(k)}(\underline{x}) = q_k(\underline{x})$$

where

$$w_{ki}(\underline{x}) = \frac{\partial^2 E}{\partial x_k \partial x_i} - \nabla^2 E \delta_{ki} \ , \quad q_k = \frac{\partial}{\partial x_k}\left(\frac{1}{4\pi r}\right)$$

$$E(\underline{x}) = \frac{1}{8\pi k}\int_0^{k(r-x_1)} \frac{1-e^{-t}}{t} dt \ , \quad r = |\underline{x}| \ , \quad k = \frac{U}{2\nu}$$

(See Oseen Hydrodynamik.)

First we put $\underline{z} = \underline{x} - \underline{y}$ into the integral I and take

$$u_j^{(b)}(\underline{z}) = \mu^{-1} w_{kj}(\underline{z}) \ , \ \ p^{(b)}(\underline{z}) = q_k(\underline{z}) \ , \ \ \underline{f}^{(b)}(\underline{z}) = \underline{0}$$

When $\delta \to 0$ and $R \to \infty$, we get, for any $\underline{x}$ outside S_0

$$\begin{aligned} u_k^{(a)}(\underline{x}) \ \ = \ \ & -\int\int_{S_0} \left\{ p^{(a)}(\underline{y})\mu^{-1} w_{kj}(\underline{x}-\underline{y}) + \underline{u}_j^{(a)}(\underline{y}) q_k(\underline{x}-\underline{y}) + u_i^{(a)}(\underline{y}) \frac{\partial}{\partial y_j} w_{ki}(\underline{x}-\underline{y}) \right. \\ & \left. - \frac{\partial u_i^{(a)}}{\partial y_j} w_{ki}(\underline{x}-\underline{y}) + 2k u_i^{(a)}(\underline{y}) w_{ki}(\underline{x}-\underline{y}) \delta_{j1} \right\} n_j dS_y \\ & + \mu^{-1} \int\int\int_{V_0} f_i^{(a)}(\underline{y}) w_{ki}(\underline{x}-\underline{y}) dV_y \end{aligned}$$

where n_j is now outwards from S_0 and hence inwards to V_0, the space exterior to S_0. In particular, if there is no internal boundary surface S_0 we have

$$u_k^{(a)}(\underline{x}) = \mu^{-1} \int\int\int_{-\infty}^{\infty} f_i^{(a)}(\underline{y}) w_{ki}(\underline{x}-\underline{y}) dV_y = \mu^{-1} w_{ki} * f_i^{(a)}$$

Similarly, we find the pressure p in terms of an integral distribution over the surface S_0. We put $u_j^{(b)}(\underline{z}) = q_j(\underline{z})$, $p^{(b)}(\underline{z}) = -\rho U q_1(\underline{z})$ into the integral I. When $\delta \to 0$ and $R \to \infty$ we get, for any $\underline{x}$ outside S_0,

$$
\begin{aligned}
p^{(a)}(\underline{x}) \;=\; & -\int\int_{S_0} \Big\{ p^{(a)}(\underline{y}) q_j(\underline{x}-\underline{y}) - \rho U u_j^{(a)}(\underline{y}) q_1(\underline{x}-\underline{y}) + \mu u_i^{(a)}(\underline{y}) \frac{\partial q_i}{\partial y_j}(\underline{x}-\underline{y}) \\
& -\mu \frac{\partial u_i^{(a)}}{\partial y_j}(\underline{y}) q_i(\underline{x}-\underline{y}) + \rho U u_i^{(a)}(\underline{y}) q_i(\underline{x}-\underline{y}) \delta_{j1} \Big\} n_j dS_y \\
& + \int\int\int_{V_0} f_i^{(a)}(\underline{y}) q_i(\underline{x}-\underline{y}) dV_y
\end{aligned}
$$

where again n_j is outwards from S_0, and if there is no external boundary S_0, then

$$p^{(a)}(\underline{x}) = \int\int\int_{-\infty}^{\infty} f_i^{(a)}(\underline{y}) q_i(\underline{x}-\underline{y}) dV_y = q_i * f_i^{(a)}$$

4.3.2 The velocity and pressure expansions in terms of fundamental solutions.

From Oseen Hydrodynamik, the Oseen representation of Oseen flow in three dimensions is

$$\begin{aligned} u_k(\underline{x}) &= -\int\int_{S_0}\left\{p(\underline{y})\mu^{-1}w_{kj}(\underline{x}-\underline{y})+u_j(\underline{y})q_k(\underline{x}-\underline{y})+u_i(\underline{y})\frac{\partial}{\partial y_j}w_{ki}(\underline{x}-\underline{y})\right. \\ &\quad \left. -\frac{\partial u_i}{\partial y_j}w_{ki}(\underline{x}-\underline{y})+2ku_i(\underline{y})w_{ki}(\underline{x}-\underline{y})\delta_{j1}\right\}n_j dS \end{aligned} \tag{4.9}$$

where $\underline{x}$ is the position vector of a general point outside the closed surface S_0 and $\underline{y}$ is the position vector parameterising the closed surface S_0.

$u_j^{(k)} = \mu^{-1}w_{kj}$ and $p^{(k)} = q_k$ are the fundamental velocity and pressure solutions respectively. These are given in Oseen Hydrodynamik as

$$q_k = \frac{\partial}{\partial x_k}\left(\frac{1}{4\pi r}\right)$$

$$w_{ki} = \frac{\partial^2 E}{\partial x_k \partial x_i} - \nabla^2 E\delta_{ki}$$

where

$$E = \frac{1}{8\pi k}\int_0^{k(r-x_1)}\frac{1-e^{-t}}{t}dt$$

We consider the point $\underline{x}$ far enough from the surface S_0 such that $|\underline{y}| < |\underline{x}|$ for all $\underline{y}$ on S_0.

Hence we can expand the fundamental solutions as a Taylor series:

$$w_{ki}(\underline{x}-\underline{y}) = w_{ki}(\underline{x}) - y_i\frac{\partial}{\partial x_j}w_{ki}(\underline{x}) + (1/2)y_j y_l \frac{\partial^2}{\partial x_j \partial x_l}w_{ki}(\underline{x}) + ...$$

$$q_k(\underline{x}-\underline{y}) = q_k(\underline{x}) - y_j\frac{\partial}{\partial x_j}q_k(\underline{x}) + (1/2)y_j y_l \frac{\partial^2}{\partial x_j \partial x_l}q_k(\underline{x}) + ...$$

Substituting into equation (4.9), and writing $w_{ki} = w_{ki}(\underline{x})$, we obtain

$$\begin{aligned} u_k(\underline{x}) &= -\int\int_{S_0}\left\{p(\underline{y})\mu^{-1}\left(w_{kj} - y_l\frac{\partial w_{kj}}{\partial x_l} + ...\right) + u_j(\underline{y})\left(q_k - y_l\frac{\partial q_k}{\partial x_l} + ...\right)\right. \\ &\quad + u_i(\underline{y})\left(-\frac{\partial w_{ki}}{\partial x_j} + y_l\frac{\partial^2 w_{ki}}{\partial x_j \partial x_l} - ...\right) - \frac{\partial}{\partial y_j}u_i(\underline{y})\left(w_{ki} - y_l\frac{\partial w_{ki}}{\partial x_l} + ...\right) \\ &\quad \left. + 2ku_i(\underline{y})\left(w_{ki} - y_l\frac{\partial w_{ki}}{\partial x_l} + ...\right)\delta_{j1}\right\} n_j dS \end{aligned}$$

Here $\frac{\partial q_k}{\partial x_l} = \nabla^2 w_{kl} - 2k\frac{\partial w_{kl}}{\partial x_1}$, and since there is no fluid outflow from S_0, then $\int\int_{S_0} u_j(\underline{y})n_j dS = 0$. Hence we can write

$$\begin{aligned} u_k(\underline{x}) &= w_{ki}(\underline{x})\int\int_{S_0}\left\{-\mu^{-1}p\delta_{ij} + \frac{\partial u_i}{\partial y_j} - 2ku_i\delta_{j1}\right\} n_j dS \\ &\quad + \frac{\partial w_{ki}(\underline{x})}{\partial x_l}\int\int_{S_0}\left\{\mu^{-1}p y_l \delta_{ij} - 2ku_j y_i \delta_{l1} + u_i\delta_{jl} - \frac{\partial u_i(\underline{y})}{\partial y_j}y_l + 2ku_i y_l \delta_{j1}\right\} n_j dS \\ &\quad + ... \\ &= \mu^{-1}\left\{A_i w_{ki} + B_{il}\frac{\partial w_{ki}}{\partial x_l} + ...\right\} \end{aligned}$$

where

$$A_i = \int\int_{S_0}\left\{-p\delta_{ij} + \mu\frac{\partial u_i}{\partial y_j} - \rho U u_i \delta_{j1}\right\} n_j dS \tag{4.10}$$

$$B_{il} = \int\int_{S_0}\left\{y_l\left(p\delta_{ij} - \mu\frac{\partial u_i}{\partial y_j} + \rho U u_i \delta_{j1}\right) - \rho U y_i u_j \delta_{l1} + \mu u_i \delta_{jl}\right\} n_j dS \tag{4.11}$$

We note that in this expansion the nine derivatives of the fundamental velocity $\frac{\partial w_{ki}}{\partial x_l}$ are not linearly independent. This will be discussed later in section (4.3.4).

Similarly, from Oseen's Hydrodynamik, the Oseen pressure in Oseen flow can be expressed as

$$
\begin{aligned}
p(\underline{x}) \;=\; & -\int\int_{S_0} \left\{ p(\underline{y}) \left(q_j - y_k \frac{\partial q_j}{\partial x_k} + \ldots \right) - \rho U u_j \left(q_1 - y_k \frac{\partial q_1}{\partial x_k} + \ldots \right) \right. \\
& + \mu u_i(\underline{y}) \left(-\frac{\partial q_i}{\partial x_j} + y_k \frac{\partial^2 q_i}{\partial x_k \partial x_j} - \ldots \right) - \mu \frac{\partial u_i}{\partial y_j}(\underline{y}) \left(q_i - y_k \frac{\partial q_i}{\partial x_k} + \ldots \right) \\
& \left. + \rho U u_i(\underline{y}) \left(q_i - y_k \frac{\partial q_i}{\partial x_k} + \ldots \right) \delta j1 \right\} n_j dS
\end{aligned}
$$

This gives

$$
p(\underline{x}) = A_i q_i(\underline{x}) + B_{ik} \frac{\partial q_i}{\partial x_k}(\underline{x}) + \ldots. \tag{4.12}
$$

where $\frac{\partial q_i}{\partial x_k} = \frac{\partial q_k}{\partial x_i}$.

Thus we have found the velocity and pressure expansions in terms of the fundamental solutions and their derivatives. We now consider the relations between the coefficients in these expansions and the force and moment on the body.

4.3.3 Coefficients in the expansion in terms of the force and moment.

From section (4.1.1), following the same argument in three dimensions, the force on the body can be expressed as an integral over a closed surface S_0 enclosing the body

$$F_i = \int\int_{S_0} \left(-p\delta_{ij} + \mu \frac{\partial u_i^\dagger}{\partial x_j} - \rho u_i^\dagger u_j^\dagger \right) n_j dS$$

Considering the Oseen linearisation valid on and outside S_0,

$$u_i^\dagger = U\delta_{i1} + u_i^{(I)} + u_i^{(II)} + ... \quad p = p^{(I)} + p^{(II)} + ...$$

and substituting these expressions into the above integral, gives us a corresponding expansion for the force F_i such that

$$F_i = F_i^{(I)} + F_i^{(II)} + ...$$

where

$$F_i^{(I)} = \int\int_{S_0} \left\{ -p^{(I)}\delta_{ij} + \mu \frac{\partial u_i^{(I)}}{\partial x_j} - \rho U \left(\delta_{i1} u_j^{(I)} + \delta_{j1} u_i^{(I)} \right) \right\} n_j dS \tag{4.13}$$

and

$$F_i^{(II)} = \int\int_{S_0} \left\{ -p^{(II)}\delta_{ij} + \mu \frac{\partial u_i^{(II)}}{\partial x_j} - \rho U \left(\delta_{i1} u_j^{(II)} + \delta_{j1} u_i^{(II)} \right) - \rho u_i^{(I)} u_j^{(I)} \right\} n_j dS$$

From equation (4.10) we thus see that

$$F_i^{(I)} = A_i \tag{4.14}$$

The moment is

$$
\begin{aligned}
M_i &= \int\int_S \epsilon_{ijk} x_j p_{kl} n_l dS \\
&= \epsilon_{ijk} \left\{ \int\int_{S_0} x_j p_{kl} n_l dS - \int\int\int_V \frac{\partial}{\partial x_l}(x_j p_{kl})\, dV \right\} \\
&= \epsilon_{ijk} \left[\int\int_{S_0} x_j \left\{ -p\delta_{kl} + \mu \left(\frac{\partial u_l}{\partial x_k} + \frac{\partial u_k}{\partial x_l} \right) \right\} n_l dS - \int\int\int_V \left(\delta_{jl} p_{kl} + x_j \rho u_l^\dagger \frac{\partial u_k}{\partial x_l} \right) dV \right] \\
&= \epsilon_{ijk} \left[\int\int_{S_0} x_j \left(-p\delta_{kl} + \mu \frac{\partial u_k}{\partial x_l} \right) n_l dS \right. \\
&\quad \left. - \int\int\int_V \left\{ x_j \rho u_l \frac{\partial u_k}{\partial u_l} - p\delta_{jk} + \mu \frac{\partial u_k}{\partial x_j} \right\} dV \right] \\
&= \epsilon_{ijk} \left[\int\int_{S_0} x_j \left(-p\delta_{kl} + \mu \frac{\partial u_k}{\partial x_l} \right) n_l dS \right. \\
&\quad \left. - \int\int\int_V \left\{ \frac{\partial}{\partial x_l} \left(\rho x_j u_k^\dagger u_l^\dagger + \mu u_k \delta_{jl} \right) - \delta_{jl} \rho u_k^\dagger u_l^\dagger \right\} dV \right] \\
&= \epsilon_{ijk} \left[\int\int_{S_0} x_j \left(-p\delta_{kl} + \mu \frac{\partial u_k}{\partial x_l} \right) n_l dS \right. \\
&\quad \left. - \left(\int\int_{S_0} - \int\int_S \right) \left(\rho x_j u_k^\dagger u_l^\dagger + \mu u_k \delta_{jl} \right) n_l dS + \int\int\int_V \rho u_j^\dagger u_k^\dagger dV \right] \\
&= \epsilon_{ijk} \int\int_{S_0} \left\{ x_j \left(-p\delta_{kl} + \mu \frac{\partial u_k}{\partial x_l} - \rho u_k^\dagger u_l^\dagger \right) - \mu u_k \delta_{jl} \right\} n_l dS
\end{aligned}
$$

Thus we obtain the expression for the moment

$$M_i = \epsilon_{ijk} \int\int_{S_0} \left\{ x_j \left(-p\delta_{kl} + \mu \frac{\partial u_k}{\partial x_l} - \rho u_k u_l \right) - \mu u_k \delta_{jl} \right\} n_l dS$$

Substituting the velocity and pressure linearisations into the above integral, we obtain

$$M_i = M_i^{(I)} + M_i^{(II)} + ...$$

where

$$M_i^{(I)} = \epsilon_{ijk} \int\int_{S_0} \left\{ x_j \left(-p^{(I)}\delta_{kl} + \mu \frac{\partial u_k^{(I)}}{\partial x_l} - \rho U \left(\delta_{k1} u_l^{(I)} + \delta_{l1} u_k^{(I)} \right) \right) - \mu u_k^{(I)} \delta_{jl} \right\} n_l dS \tag{4.15}$$

$$\begin{aligned} M_i^{(II)} &= \epsilon_{ijk} \int\int_{S_0} \left\{ x_j \left(-p^{(II)}\delta_{kl} + \mu \frac{\partial u_k^{(II)}}{\partial x_l} - \rho U \left(\delta_{k1} u_l^{(II)} + \delta_{l1} u_k^{(II)} \right) - \rho u_k^{(I)} u_l^{(I)} \right) \right. \\ & \left. -\mu u_k^{(II)} \delta_{jl} \right\} n_l dS \end{aligned}$$

Since $\epsilon_{ijk} = -\epsilon_{ikj}$, we can write

$$\begin{aligned} M_i^{(I)} &= \epsilon_{ijk} \int\int_{S_0} \left\{ x_j \left(-p^{(I)}\delta_{kl} + \mu \frac{\partial u_k^{(I)}}{\partial x_l} - \rho U \delta_{l1} u_k^{(I)} \right) + \rho U x_k u_l^{(I)} \delta_{j1} - \mu u_k^{(I)} \delta_{jl} \right\} n_l dS \\ & -\epsilon_{ijk} B_{kj} = \epsilon_{ijk} B_{jk} \end{aligned} \tag{4.16}$$

Hence the force and the moment are related to the coefficients of the velocity and pressure expansions by equations (4.14) and (4.16).

We can also derive these results by considering the surface S_0 as a large sphere S_R; we substitute the expansions for $u_k^{(I)}$ and $p^{(I)}$ in the integrals for $F_i^{(I)}$ and $M_i^{(II)}$ and evaluate the integrals as $R = |\underline{x}| \to \infty$.

We now consider which terms in the expansion contribute to the respective integrals:

The order of the fundamental velocity on the sphere radius R is $\frac{1}{R}$ inside the wake and $\frac{1}{R^2}$ outside the wake.

In the wake x_2, x_3 $= \mathrm{O}(\sqrt{R})$ and the derivative of the velocity w.r.t x_2 or x_3 reduces the order by $\sqrt{R}$

Hence inside the wake the order of the fundamental velocity first derivative is at greatest $\mathrm{O}(\frac{1}{R\sqrt{R}})$. Outside the wake , the order of the fundamental velocity first derivative is $\mathrm{O}(\frac{1}{R^3})$.

The order of the fundamental velocity second derivative is $\mathrm{O}(\frac{1}{R^2})$ inside the wake and $\mathrm{O}(\frac{1}{R^4})$ outside the wake.

Similarly, the order of the fundamental pressure is $\frac{1}{R^2}$, fundamental pressure first derivative $\frac{1}{R^3}$ and fundamental pressure second derivative $\frac{1}{R^4}$.

The area of the surface S_R outside the wake is $\mathrm{O}(R^2)$ and inside the wake $\mathrm{O}(R)$, since x_2, x_3, are of order $\frac{1}{\sqrt{R}}$ inside the wake.

Thus the contribution to the force $F_i^{(I)}$ from the first derivative terms $\frac{\partial \underline{u}^{(n)}}{\partial x_m}$, $\frac{\partial p^{(n)}}{\partial x_m}$ are $\mathrm{O}(\frac{1}{R\sqrt{R}}.R)$ inside the wake and $\mathrm{O}(\frac{1}{R^3}.R^2)$ outside the wake. The contributions from higher derivatives are even smaller, so we need only consider the contributions to $F_i^{(I)}$ from $\underline{u}^{(n)}$ and $p^{(n)}$.

Similarly, for $M_i^{(I)}$, the contributions from the second derivatives $\frac{\partial^2 \underline{u}^{(n)}}{\partial x_m \partial x_q}$, $\frac{\partial^2 p^{(n)}}{\partial x_m \partial x_q}$ are $O\left(R.\frac{1}{R^2}.R\right)$ inside the wake and $O\left(R.\frac{1}{R^4}.R^2\right)$ outside the wake. The term possibly giving a contribution to the moment is

$$\epsilon_{i1k} \int \int_{S_0} x_1(-\rho U) \frac{\partial^2 u_k^{(n)}}{\partial x_m \partial x_p} dx_2 dx_3$$

where k, m and p are 2 or 3.

However, integration with respect to x_m gives values of $\frac{\partial u_k^{(n)}}{\partial x_p}$ outside the wake, which are of $O(\frac{1}{R^3})$, and so the contribution to the moment is $O(R.\frac{1}{R^3}.\sqrt{R})$.

Thus in the case of $M_i^{(I)}$, we have to consider $\underline{u}^{(n)}$, $p^{(n)}$ and $\frac{\partial \underline{u}^{(n)}}{\partial x_m}$, $\frac{\partial p^{(n)}}{\partial x_m}$ only.

The force integral is given as an integral over the surface S_0, enclosing the body S, on and outside which the Oseen approximation is valid, by equation (4.13).

$$F_i^{(I)} = \int\int_{S_0} \left\{ -p^{(I)}\delta_{ij} + \mu \frac{\partial u_i^{(I)}}{\partial x_j} - \rho U \left(\delta_{i1} u_j^{(I)} + \delta_{j1} u_i^{(I)} \right) \right\} n_j dS$$

and

$$u_i^{(I)} = A_m u_i^{(m)} + B_{mn} \frac{\partial u_i^{(m)}}{\partial x_n} + ...$$

$$p^{(I)} = A_m p^{(m)} + B_{mn} \frac{\partial p^{(m)}}{\partial x_n} + ...$$

We consider the integral

$$f_i^{(m)} = \int\int_{S_0} \left\{ -p^{(m)}\delta_{ij} + \mu \frac{\partial u_i^{(m)}}{\partial x_j} - \rho U \left(\delta_{i1} u_j^{(m)} + \delta_{j1} u_i^{(m)} \right) \right\} n_j dS$$

and so $F_i^{(I)} = A_m f_i^{(m)}$.

The functions $p^{(m)}$ and $u_i^{(m)}$ are the three fundamental pressures and velocities respectively. Thus this integral is singular at the origin. We consider the surface of a small sphere S_δ whose centre is located at the origin. Applying the divergence theorem to the above integral, we obtain

$$\begin{aligned} f_i^{(m)} &= \int\int_{S_\delta} \left\{ -p^{(m)}\delta_{ij}\mu \frac{\partial u_i^{(m)}}{\partial x_j} - \rho U \left(\delta_{i1} u_j^{(m)} + \delta_{j1} u_i^{(m)} \right) \right\} n_j dS \\ &\quad + \int\int\int_V \frac{\partial}{\partial x_j} \left\{ -p^{(m)}\delta_{ij} + \mu \frac{\partial u_i^{(m)}}{\partial x_j} - \rho U \left(\delta_{i1} u_j^{(m)} + \delta_{j1} u_i^{(m)} \right) \right\} dV \end{aligned}$$

But

$$
\begin{aligned}
\frac{\partial}{\partial x_j} & \left\{-p^{(m)}\delta_{ij} + \mu\frac{\partial u_i^{(m)}}{\partial x_j} - \rho U\left(\delta_{i1}u_j^{(m)} + \delta_{j1}u_i^{(m)}\right)\right\} dV \\
&= -\frac{\partial p^{(m)}}{\partial x_i} + \mu\frac{\partial^2 u_i^{(m)}}{\partial x_j \partial x_j} - \rho U\frac{\partial u_i^{(m)}}{\partial x_1} \\
&= 0
\end{aligned}
$$

from the Oseen equations.

Thus

$$
f_i^{(m)} = \int\int_{S_\delta} \left\{-p^{(m)}\delta_{ij} + \mu\frac{\partial u_i^{(m)}}{\partial x_j} - \rho U\left(\delta_{i1}u_j^{(m)} + \delta_{j1}u_i^{(m)}\right)\right\} n_j dS \tag{4.17}
$$

From equation (4.15), we obtain the expression for the moment

$$M_i^{(I)} = \epsilon_{ijk} \int\int_{S_0} \left\{ x_j \left(-p\delta_{kl}^{(I)} + \mu \frac{\partial u_k^{(I)}}{\partial x_l} - \rho U \delta_{k1} u_l^{(I)} - \rho U \delta_{l1} u_k^{(I)} \right) - \mu u_k^{(I)} \right\} n_l dS$$

Substituting in the expansions for $u_k^{(I)}$ and $p^{(I)}$, we obtain

$$\begin{aligned} M_i^{(I)} &= \epsilon_{ijk} A_m \int\int_{S_0} \left\{ x_j \left(-p^{(m)} \delta_{kl} + \mu \frac{\partial u_k^{(m)}}{\partial x_l} - \rho U \delta_{l1} u_k^{(m)} \right) \right. \\ &\quad \left. + \rho U x_k u_l^{(m)} \delta_{j1} - \mu u_k^{(m)} \delta_{jl} \right\} n_l dS \\ &\quad + \epsilon_{ijk} B_{mn} \int\int_{S_0} \left\{ x_j \left(-\frac{\partial p^{(m)}}{\partial x_n} \delta_{kl} + \mu \frac{\partial^2 u_k^{(m)}}{\partial x_l \partial x_n} - \rho U \delta_{l1} \frac{\partial u_k^{(m)}}{\partial x_n} \right) \right. \\ &\quad \left. + \rho U x_k \frac{\partial u_l^{(I)}}{\partial x_n} \delta_{j1} - \mu \frac{\partial u_k^{(m)}}{\partial x_n} \delta_{jl} \right\} n_l dS \\ &\quad + \ldots . \\ &= A_m J_i^{(m)} + B_{mn} L_i^{(m,n)} \end{aligned}$$

where $J_i^{(m)}$ and $L_i^{(m,n)}$ are given by the two integral expressions above.

Thus,

$$J_i^{(m)} = \epsilon_{ijk} \int\int_{S_0} \left\{ x_j \left(-p^{(m)}\delta_{kl} + \mu \frac{\partial u_k^{(m)}}{\partial x_l} - \rho U \left(\delta_{k1} u_l^{(m)} + \delta_{l1} u_k^{(m)} \right) \right) - \mu u_k^{(m)} \delta_{jl} \right\} n_l dS$$

$$= \epsilon_{ijk} \left[\int\int_{S_\delta} \left\{ x_j \left(-p^{(m)}\delta_{kl} + \mu \frac{\partial u_k^{(m)}}{\partial x_l} - \rho U \left(\delta_{k1} u_l^{(m)} + \delta_{l1} u_k^{(m)} \right) \right) - \mu u_k^{(m)} \delta_{jl} \right\} n_l dS \right.$$

$$+ \int\int\int_V \frac{\partial}{\partial x_l} \left\{ x_j \left(-p^{(m)}\delta_{kl} + \mu \frac{\partial u_k^{(m)}}{\partial x_l} - \rho U \left(\delta_{k1} u_l^{(m)} + \delta_{l1} u_k^{(m)} \right) \right) \right.$$

$$\left. \left. -\mu u_k^{(m)} \delta_{jl} \right\} dV \right]$$

However,

$$\epsilon_{ijk} \quad \frac{\partial}{\partial x_l} \left[x_j \left\{ -p^{(m)}\delta_{kl} + \mu \frac{\partial u_k^{(m)}}{\partial x_l} - \rho U \left(\delta_{k1} u_l^{(m)} + \delta_{l1} u_k^{(m)} \right) - \mu u_k^{(m)} \delta_{jl} \right\} \right]$$

$$= \epsilon_{ijk} \left[\delta_{jl} \left\{ -p^{(m)}\delta_{kl} + \mu \frac{\partial u_k^{(m)}}{\partial x_l} - \rho U \left(\delta_{k1} u_l^{(m)} + \delta_{l1} u_k^{(m)} \right) \right\} \right.$$

$$\left. + x_j \left\{ \frac{\partial}{\partial x_1} \left(-p^{(m)} + \mu \frac{\partial u_k^{(m)}}{\partial x_l} \right) - \rho U \delta_{k1} \frac{\partial u_l^{(m)}}{\partial x_l} - \rho U \delta_{l1} \frac{\partial u_k^{(m)}}{\partial x_l} \right\} - \mu u_k^{(m)} \delta_{jl} \right]$$

$$= \epsilon_{ijk} \left[-p^{(m)}\delta_{jk} + \mu \left(\frac{\partial u_k^{(m)}}{\partial x_l} - \frac{\partial u_k^{(m)}}{\partial x_j} \right) - \rho U \left(\delta_{k1} u_j^{(m)} + \delta_{j1} u_k^{(m)} \right) \right.$$

$$\left. + x_j \left\{ \rho U \frac{\partial u_k^{(m)}}{\partial x_1} - \rho U \frac{\partial u_k^{(m)}}{\partial x_1} \right\} \right]$$

$$= 0$$

Hence

$$J_i^{(m)} = \epsilon_{ijk} \int\int_{S_\delta} \left\{ x_j \left(-p^{(m)}\delta_{kl} + \mu \frac{\partial u_k^{(m)}}{\partial x_l} - \rho U \left(\delta_{k1} u_l^{(m)} + \delta_{l1} u_k^{(m)} \right) \right) - \mu u_k^{(m)} \delta_{jl} \right\} n_l dS \tag{4.18}$$

We now obtain the expansion of the fundamental solutions as $r = |\underline{x}| \to 0$.

We have

$$E = \int_0^{k(r-x_1)} \frac{1-e^{-t}}{8\pi kt} dt$$

This gives the expansion

$$E = \frac{1}{8\pi} \sum_{n=0}^{\infty} \frac{(-k)^n (r-x_1)^{n+1}}{(n+1)!(n+1)}$$

since $e^{-t} = \sum_{n=0}^{n=\infty} \frac{(-t)^n}{n!}$

Thus for small r,

$$E = \frac{1}{8\pi}(r - x_1) + O(r^2) \quad w_{kn} = \frac{\partial^2 E}{\partial x_k \partial x_n} - \nabla^2 E \delta_{kn}$$

and hence

$$w_{kn} = -\frac{1}{8\pi}\left(\frac{1}{r}\delta_{kn} + \frac{x_k x_n}{r^3}\right) + O(1)$$

$$\frac{\partial w_{kn}}{\partial x_m} = \frac{1}{8\pi}\left(\frac{x_m}{r^3}\delta_{kn} - \frac{\delta_{km}x_n + \delta_{mn}x_k}{r^3} + \frac{3x_k x_m x_n}{r^5}\right) + O(r^{-1})$$

where $u_n^{(k)} = \mu^{-1} w_{kn}$ and $p^{(k)} = q_k = -\frac{1}{4\pi}\frac{x_k}{r^2}$

We now evaluate the integrals $f_i^{(n)}$, $J_i^{(n)}$ and $L_i^{(m,n)}$ over the surface S_δ.

$$
\begin{aligned}
f_i^{(n)} &= \int\int_{S_\delta} \left\{ -p^{(n)}\delta_{ij} + \frac{\partial w_{in}}{\partial x_j} + O\left(\frac{1}{r}\right) \right\} \frac{x_j}{r} dS \\
&= \int\int_{S_\delta} \left\{ \frac{x_n}{4\pi r^3}\frac{x_i}{r} + \frac{1}{8\pi}\left(\frac{\delta_{in}}{r^2} - \frac{2x_i x_n}{r^4} + \frac{3x_i x_n}{r^4} \right) \right\} dS + O(\delta) \\
&= \frac{1}{8\pi}\int\int_{S_\delta} \left(\frac{\delta_{in}}{r^2} + \frac{3x_i x_n}{r^4} \right) dS + O(\delta) \\
&= \frac{1}{8\pi}\left\{ 4\pi\delta_{in} + 3.\frac{4}{3}\pi\delta_{in} \right\} + O(\delta) \\
&= \delta_{in}
\end{aligned}
$$

Thus

$$
\begin{aligned}
F_i^{(I)} &= A_n f_i^{(n)} = A_n \delta_{in} \\
&= A_i
\end{aligned}
$$

The moment expression is given by

$$M_i^{(I)} = A_m J_i^{(m)} + B_{mn} L_i^{(m,n)} + ...$$

where

$$J_i^{(m)} = \epsilon_{ijk} \int\int_{S_\delta} \left\{ x_j \left(-p^{(m)}\delta_{kl} + \mu \frac{\partial u_k^{(m)}}{\partial x_l} - \rho U \left(\delta_{k1} u_l^{(m)} + \delta_{l1} u_k^{(m)} \right) \right) - \mu u_k^{(m)} \delta_{jl} \right\} n_l dS$$

However, the contribution from $\underline{u}^{(n)}$ and $p^{(n)}$ is O(δ) and $\frac{\partial u_k^{(m)}}{\partial x_l}$ is O($\frac{1}{\delta^2}$). Therefore $J_i^{(m)} = 0$.

We now calculate $L_i^{(m,n)}$.

The contribution to the moment from the term $\frac{\partial p^{(n)}}{\partial x_m}$ is

$$\epsilon_{ijk} \int\int_{S_\delta} x_j \frac{\partial p^{(n)}}{\partial x_m} \delta_{kl} \frac{x_l}{r} dS = 0$$

since $\epsilon_{ijk} x_j x_k = 0$ from symmetry.

The contribution from the term $-\mu \frac{\partial u_k^{(n)}}{\partial x_m} \delta_{jl}$ is

$$\begin{aligned}
\epsilon_{ijk} \quad & \int\int_{S_\delta} \left(-\mu \frac{\partial u_k^{(m)}}{\partial x_n} \delta_{jl} \right) n_l dS \\
= \quad & \frac{\epsilon_{ijk}}{8\pi} \int\int_{S_\delta} \left(-\frac{x_n}{r^3}\delta_{km} - 3\frac{x_k x_m x_n}{r^5} + \frac{x_k \delta_{mn} + x_m \delta_{kn}}{r^3} \right) \delta_{jl} n_l dS \\
= \quad & \frac{\epsilon_{ijk}}{8\pi} \int\int_{S_\delta} \left(-\frac{x_n \delta_{km} n_j}{r^3} - 3\frac{x_k x_m x_n}{r^5} n_j + \frac{x_k \delta_{mn}}{r^3} n_j + \frac{x_m \delta_{kn}}{r^3} n_j \right) dS \\
= \quad & 0
\end{aligned}$$

Hence,

$$
\begin{aligned}
L_i^{(m,n)} &= \epsilon_{ijk} \int\int_{S_\delta} \left\{ x_j \frac{\partial^2 w_{kn}}{\partial x_l \partial x_m} - \delta_{jl} \frac{\partial w_{kn}}{\partial x_m} + O(r^{-1}) \right\} \frac{x_l}{r} dS \\
&= \frac{\epsilon_{ijk}}{8\pi} \int\int_{S_\delta} \left\{ \frac{x_j x_l}{r} \left[\left(\frac{\delta_{lm}}{r^3} - \frac{3x_l x_m}{r^5} \right) \delta_{kn} - \frac{\delta_{km}\delta_{ln} + \delta_{mn}\delta_{kl}}{r^3} \right.\right. \\
&\quad \left. +3\frac{(\delta_{km}x_n + \delta_{mn}x_k)x_l}{r^5} + 3\frac{(\delta_{kl}x_m x_n + \delta_{lm}x_k x_n + \delta_{ln}x_k x_m)}{r^5} - 15\frac{(x_k x_l x_m x_n)}{r^7} \right] \\
&\quad \left. -\frac{x_j}{r} \left(\frac{x_m}{r^3}\delta_{kn} - \frac{\delta_{km}x_n + \delta_{mn}x_k}{r^3} + 3\frac{x_k x_m x_n}{r^5} \right) \right\} dS + O(\delta) \\
&= \frac{\epsilon_{ijk}}{8\pi} \int\int_{S_\delta} x_j \left\{ -\frac{2x_m}{r^4}\delta_{kn} - \frac{\delta_{km}x_n + \delta_{mn}x_k}{r^4} + 3\frac{\delta_{km}x_n + \delta_{mn}x_k}{r^4} + 9\frac{x_k x_m x_n}{r^6} \right. \\
&\quad \left. -15\frac{x_k x_m x_n}{r^6} - \frac{x_m}{r^4}\delta_{kn} + \frac{\delta_{km}x_n + \delta_{mn}x_k}{r^4} - 3\frac{x_k x_m x_n}{r^6} \right\} dS + O(\delta) \\
&= \frac{3\epsilon_{ijk}}{8\pi} \int\int_{S_\delta} \left\{ \frac{-\delta_{kn}x_j x_m + \delta_{km}x_j x_n + \delta_{mn}x_j x_k}{r^4} - 3\frac{x_j x_k x_m x_n}{r^6} \right\} dS + O(\delta) \\
&= \frac{3\epsilon_{ijk}}{8\pi} \int\int_{S_\delta} \left(\frac{-\delta_{kn}x_j x_m + \delta_{km}x_j x_n}{r^4} \right) dS + O(\delta) \\
&= \frac{3\epsilon_{ijk}}{8\pi} \left(-\delta_{kn}\frac{4}{3}\pi\delta_{jm} + \delta_{km}\frac{4}{3}\pi\delta_{jn} \right) + O(\delta) \\
&= -\frac{1}{2}\epsilon_{imn} + \frac{1}{2}\epsilon_{inm} + O(\delta) \\
&= \epsilon_{inm}
\end{aligned}
$$

Thus

$$
\begin{aligned}
M_i^{(I)} &= B_{mn} L_i^{(m,n)} \\
&= -\epsilon_{imn} B_{mn}
\end{aligned}
$$

4.3.4 The velocity and pressure expansion with linearly independent terms.

However, the expansions given for the velocity and pressure are not linearly independent, since

$$\frac{\partial u_k^{(n)}}{\partial x_n} = \frac{\partial u_n^{(k)}}{\partial x_n} = 0\ , \quad \frac{\partial p^{(n)}}{\partial x_n} = \frac{\partial^2}{\partial x_n \partial x_n}\left(\frac{1}{4\pi r}\right) = 0$$

Thus of the nine solutions $\frac{\partial \underline{u}^{(n)}}{\partial x_m}$, $\frac{\partial p^{(n)}}{\partial x_m}$, only eight are independent. We now give eight linearly independent solutions for the velocity and pressure.

The equations $\nabla^2 p^{(I)} = 0$ has five independent solutions of the form $B_{nm}\frac{\partial p^{(n)}}{\partial x_m}$.

With $x_1 = r\cos\theta$, $x_2 = r\sin\theta\cos\psi$, $x_3 = r\sin\theta\sin\psi$, these may be taken, with appropriate velocity fields, to be

$$\begin{aligned}
p^{(I)} &= \frac{1}{4\pi r^3}P_2(\cos\theta) = \frac{1}{2}\frac{\partial p^{(1)}}{\partial x_1}\ , \quad \underline{u}^{(I)} = \frac{1}{2}\frac{\partial \underline{u}^{(1)}}{\partial x_1} \\
p^{(I)} &= \frac{1}{4\pi r^3}P_2'(\cos\theta)\sin\theta\cos\psi = \frac{\partial p^{(1)}}{\partial x_2} = \frac{\partial p^{(2)}}{\partial x_1}\ , \quad \underline{u}^{(I)} = \frac{1}{2}\left(\frac{\partial \underline{u}^{(1)}}{\partial x_2} + \frac{\partial \underline{u}^{(2)}}{\partial x_1}\right) \\
p^{(I)} &= \frac{1}{4\pi r^3}P_2'(\cos\theta)\sin\theta\sin\psi = \frac{\partial p^{(1)}}{\partial x_3} = \frac{\partial p^{(3)}}{\partial x_1}\ , \quad \underline{u}^{(I)} = \frac{1}{2}\left(\frac{\partial \underline{u}^{(1)}}{\partial x_3} + \frac{\partial \underline{u}^{(3)}}{\partial x_1}\right) \\
p^{(I)} &= \frac{1}{4\pi r^3}P_2''(\cos\theta)\sin^2\theta\cos 2\psi = \frac{\partial p^{(2)}}{\partial x_2} - \frac{\partial p^{(3)}}{\partial x_3}\ , \quad \underline{u}^{(I)} = \frac{\partial \underline{u}^{(2)}}{\partial x_2} - \frac{\partial \underline{u}^{(3)}}{\partial x_3} \\
p^{(I)} &= \frac{1}{4\pi r^3}P_2''(\cos\theta)\sin^2\theta\sin 2\psi = 2\frac{\partial p^{(2)}}{\partial x_3} = 2\frac{\partial p^{(3)}}{\partial x_2}\ , \quad \underline{u}^{(I)} = \frac{\partial \underline{u}^{(2)}}{\partial x_3} + \frac{\partial \underline{u}^{(3)}}{\partial x_2}
\end{aligned}$$

Each of these solutions gives $M_i^{(I)} = 0$.

The remaining three solutions of the Oseen equations have $p^{(I)} = 0$ and $\underline{u}^{(I)}$ antisymmetrized to give

$$\begin{aligned}
p^{(I)} &= 0\,, \quad \underline{u}^{(I)} = \mathrm{curl}\underline{u}^{(1)} = \frac{\partial \underline{u}^{(2)}}{\partial x_3} - \frac{\partial \underline{u}^{(3)}}{\partial x_2}\,, \quad M_i^{(I)} = 2\delta_{i1} \\
p^{(I)} &= 0\,, \quad \underline{u}^{(I)} = \mathrm{curl}\underline{u}^{(2)} = \frac{\partial \underline{u}^{(3)}}{\partial x_1} - \frac{\partial \underline{u}^{(1)}}{\partial x_3}\,, \quad M_i^{(I)} = 2\delta_{i2} \\
p^{(I)} &= 0\,, \quad \underline{u}^{(I)} = \mathrm{curl}\underline{u}^{(3)} = \frac{\partial \underline{u}^{(1)}}{\partial x_2} - \frac{\partial \underline{u}^{(2)}}{\partial x_1}\,, \quad M_i^{(I)} = 2\delta_{i3}
\end{aligned}$$

Chapter 5

The Oseen velocity representation in two dimensions.

We consider in this section Oseen's representation in two dimensional steady flow of the velocity $\underline{u}$ and the pressure p satisfying Oseen's equations.

Oseen expresses both the fluid velocity and also the pressure at a point q as an integration of a velocity and pressure distribution over a closed contour C enclosing the body but excluding the point q.

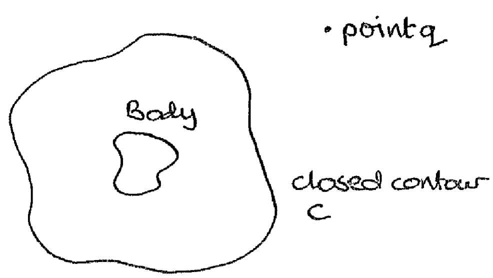


Figure 5.1: The position of the contour C enclosing the body.

Oseen obtains the results

$$u_k = -\frac{1}{4\pi\nu}\int_C \left\{ t_{jk}\left(\nu\frac{\partial u_j}{\partial n} - \frac{1}{\rho}pn_j\right) - u_j\left(\nu\frac{\partial t_{jk}}{\partial n} - \frac{1}{\rho}\tau_k n_j\right) - Uu_j t_{jk} n_1 \right\} dl$$

and

$$p = \frac{\rho}{2\pi}\int_C \left\{ \tilde{u}_j\left(\nu\frac{\partial u_j}{\partial n} - \frac{1}{\rho}pn_j\right) - u_j\left(\nu\frac{\partial \tilde{u}_j}{\partial n} - \frac{1}{\rho}\tilde{p}n_j\right) - Uu_j\tilde{u}_j n_1 \right\} dl$$

for functions t_{jk} and τ_k and particular solutions of $\tilde{u}_j$ and $\tilde{p}$ all defined later.

This section is concerned with Oseen's method in two dimensions given in Oseen and Lagerstrom to obtain the particular velocity and pressure distributions over the closed contour C which give the velocity and pressure at the point q.

We first consider the Oseen velocity expression and next give a brief description of how we obtain it.

We shall define a function f such that the integration of f over the closed contour L not enclosing the body is zero. The function f involves the velocity, pressure, adjoint veloctу $\tilde{u}$ and adjoint pressure $\tilde{p}$ where the adjoin t velocity and pressure satisfy Oseen's adjoint equation. The definition of f is given later.

The closed contour L consists of three curves; the closed contour C enclosing the body, the closed contour of a circle circumference centre at the point q radius $\mathcal{R} \to 0$ and a closed contour of a circle circumference centred on the body and enclosing the body and the point q of radius $R \to \infty$.

We shall first consider two particular solutions of $\tilde{u}_j$ and $\tilde{p}$, t_{jk} and τ_k respectively such that the fluid velocity at the point q is expressed as an integration of a fluid and pressure distribution (the function f) over the closed contour C.

Oseen finds that these two particular solutions are

$$k\underline{\underline{t}} = \begin{pmatrix} \frac{\partial g(\mathcal{R})}{\partial x} & \frac{\partial g(\mathcal{R})}{\partial y} \\ \frac{\partial g(\mathcal{R})}{\partial y} & -\frac{\partial g(\mathcal{R})}{\partial x} \end{pmatrix} + 2kK_0(k\mathcal{R})e^{k(x-x_0)} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\underline{\tau} = 2\rho\nu\nabla(\log\mathcal{R})$$

where $g(\mathcal{R}) = \log\mathcal{R} + K_0(k\mathcal{R})e^{k(x-x_c)}$, $\mathcal{R} = [(x-x_c)^2 + (y-y_c)^2]^{1/2}$ and $k = \frac{U}{2\nu}$. (x_c, y_c) is a point on the curve.

We now give fully the method to obtain the Oseen velocity representation. We first define the function f.

We first consider the function f given below which contains the perturbation velocity $\underline{u}$ and the pressure p (satisfying Oseen's equations), and the adjoint velocity $\tilde{\underline{u}}$ and the adjoint pressure $\tilde{p}$ (which satisfying the adjoint to Oseen's equations). Hence

$$U\frac{\partial \underline{u}}{\partial x} = -\frac{1}{\rho}\underline{\nabla p} + \nu\nabla^2\underline{u} \quad \underline{\nabla}.\underline{u} = 0 \tag{5.1}$$

and

$$U\frac{\partial \tilde{\underline{u}}}{\partial x} = \frac{1}{\rho}\underline{\nabla \tilde{p}} - \nu\nabla^2\tilde{\underline{u}} \quad \underline{\nabla}.\tilde{\underline{u}} = 0 \tag{5.2}$$

Oseen considers the function f along the closed contour C whose outward pointing normal is $\underline{n}$ such that, in vector notation,

$$f(x, y; \underline{n}) = \tilde{u}_j\left(\nu\frac{\partial u_j}{\partial n} - \frac{1}{\rho}pn_j\right) - u_j\left(\nu\frac{\partial u_j}{\partial n} - \frac{1}{\rho}\tilde{p}n_j\right) - Uu_j\tilde{u}_jn_1$$

The integration I of the function $f(x, y; \underline{n})$ over the closed contour L is zero. This is shown by applying the divergence theorem.

$$I = \int_L \{\tilde{u}_j(\nu\frac{\partial u_j}{\partial n} - \frac{1}{\rho}pn_j) - u_j(\nu\frac{\partial u_j}{\partial n} - \frac{1}{\rho}\tilde{p}n_j) - Uu_j\tilde{u}_jn_1\}dl$$

where dl is a length element of the contour.

Applying the divergence theorem,

$$I_L = \int\int_S \left\{ \frac{\partial}{\partial x_j}\left(\nu\tilde{u}_i\frac{\partial u_i}{\partial x_j} - \frac{1}{\rho}p\tilde{u}_j - \nu u_i\frac{\partial \tilde{u}_i}{\partial x_j} + \frac{1}{\rho}\tilde{p}u_j\right) - \frac{\partial}{\partial x_1}(Uu_j\tilde{u}_j)\right\} ds \qquad (5.3)$$

where S is the area enclosed by the contour L and ds is an element of area.

(The divergence theorem states that for a vector function $h_i(x_1, x_2)$ defined in a region S bounded by a curve L, having outward normal $\underline{n}$ to the closed curve, then $\int_C h_i(x_1, x_2)n_idl = \int\int_S \frac{\partial h_i}{\partial x_i}(x_1, x_2)ds$.

We note that $Uu_j\tilde{u}_jn_1 = h_in_i$ where $\underline{h} = (Uu_j\tilde{u}_j, 0, 0)$)

From equations (5.1) and (5.2), $\frac{\partial u_i}{\partial x_i} = 0$ and $\frac{\partial \tilde{u}_i}{\partial x_i} = 0$. Hence the bracketed expression inside the surface integral for I_L of equation (5.3) is

$$\nu\tilde{u}_i\frac{\partial^2 u_i}{\partial x_j\partial x_j} - \frac{1}{\rho}\tilde{u}_i\frac{\partial p}{\partial x_j} - \nu u_i\frac{\partial \tilde{u}_i}{\partial x_j\partial x_j} + \frac{1}{\rho}u_j\frac{\partial \tilde{p}}{\partial x_j} - Uu_j\frac{\partial \tilde{u}_j}{\partial x_1} - U\tilde{u}_j\frac{\partial u_j}{\partial x_1}$$

which is equivalent to

$$\tilde{u}_i\left\{\nu\nabla^2 u_i - \frac{1}{\rho}\frac{\partial p}{\partial x_i} - U\frac{\partial u_i}{\partial x_1}\right\} + u_i\left\{-\nu\nabla^2\tilde{u}_i + \frac{1}{\rho}\frac{\partial \tilde{p}}{\partial x_i} - U\frac{\partial \tilde{u}_i}{\partial x_1}\right\} \qquad (5.4)$$

From equations (5.1) and (5.2), this expression is zero. Therefore the area integral I_L is zero, and so

$$I_L = \int_L \{\tilde{u}_j(\nu\frac{\partial u_j}{\partial n} - \frac{1}{\rho}pn_j) - u_j(\nu\frac{\partial \tilde{u}_j}{\partial n} - \frac{1}{\rho}\tilde{p}n_j) - Uu_j\tilde{u}_jn_1\}dl = 0 \qquad (5.5)$$

We now consider the closed contour L not enclosing the body and consisting of three curves: the closed contour C enclosing the body, a closed contour of a circle circumference centre at the point q radius $\mathcal{R} \rightarrow 0$, and a closed contour of circle circumference centred on the body and enclosing the body and the point q of radius $R \rightarrow \infty$. The closed contour L is shown diagrammatically below:

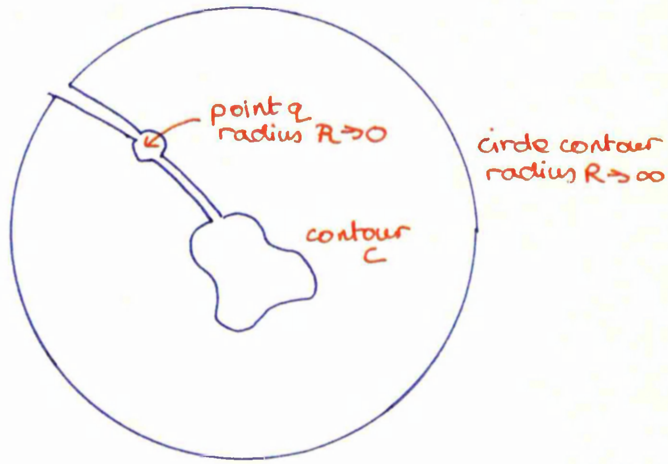


Figure 5.2: The closed contour L.

We apply Green's integral theorem for particular values of $\tilde{u}_j$ and $\tilde{p}$.

We first consider Oseen's representation in two dimensional steady flow of the velocity $\underline{u}$.

We consider two particular solutions of $\tilde{u}_j$ and $\tilde{p}$, t_{jk} and τ_k respectively and which are singular at the point q. (Where $k = 1, 2$.) Thus

$$I_k = \int_L \{t_{jk}\left(\nu\frac{\partial u_j}{\partial n} - \frac{1}{\rho}pn_j\right) - u_j\left(\nu\frac{\partial t_{jk}}{\partial n} - \frac{1}{\rho}\tau_k n_j\right) - Uu_jt_{jk}n_1\}dl \qquad (5.6)$$

We consider t_{jk} and τ_k such that the contribution to I_k around the point q is some multiple ot the velocity u_k. The point q is at position (q_1, q_2) in the cartesian coordinate reference frame.

Oseen gives the solution (Hydrodynamik, section 4.5 pg. 37)

$$2k\underline{\underline{t}} = \begin{pmatrix} \frac{\partial g(\mathcal{R})}{\partial x_1} & \frac{\partial g(\mathcal{R})}{\partial x_2} \\ \frac{\partial g(\mathcal{R})}{\partial x_2} & -\frac{\partial g(\mathcal{R})}{\partial x_1} \end{pmatrix} + 2kK_0(kr)e^{-k(x_1-x_0)} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{5.7}$$

where

$$g(\mathcal{R}) = \ln \mathcal{R} + K_0(k\mathcal{R})e^{-k(x_1-q_1)} \quad k = \frac{U}{2\nu} \quad \mathcal{R} = [(x_1-q_1)^2 + (x_2-q_2)^2]^{1/2} \tag{5.8}$$

and

$$\underline{\tau(\mathcal{R})} = -2\rho\nu\nabla(\log 1/\mathcal{R}) = 2\rho\nu\nabla(\log \mathcal{R}) \tag{5.9}$$

We now show the solution satisfies the equation

$$U\frac{\partial t_{jk}}{\partial x_1} = \frac{1}{\rho}\frac{\partial \tau_k}{\partial x_j} - \nu\nabla^2 t_{jk} \tag{5.10}$$

We first consider the case for $j = 1$, $k = 1$.

Hence

$$\nu\nabla^2 t_{11} + U\frac{\partial t_{11}}{\partial x_1} - \frac{1}{\rho}\frac{\partial \tau_1}{\partial x_1} = 0$$

Substituting the functions t_{11} and τ_1 from equations (5.7) and (5.9) into the above equation, and using the fact that $\nabla^2 \log r = 0$, we obtain

$$\begin{aligned}
& \nu\nabla^2\frac{\partial}{\partial x_1}(K_0(k)e^{-k(x_1-q_1)}) + U\frac{\partial^2}{\partial x_1^2}(\log\mathcal{R} + K_0(k\mathcal{R})e^{-k(x_1-q_1)}) \\
+ & \ \nu\nabla^2(2kK_0(k\mathcal{R})e^{-k(x_1-q_1)}) + U\frac{\partial}{\partial x_1}(2kK_0(k\mathcal{R})e^{-k(x_1-q_1)}) - \frac{k}{\rho}(2\rho\nu)\frac{\partial}{\partial x_1}(\frac{\partial}{\partial x_1}\log\mathcal{R}) \\
= & \ 0
\end{aligned}$$

The left hand side of the equation is identically zero since

$$(\nabla^2 + 2k)\{K_0(kr)e^{-x_1}\} = 0$$

We next consider equation (5.10) for $j = 1$, $k = 2$:

$$U\frac{\partial t_{12}}{\partial x_1} = \frac{1}{\rho}\frac{\partial \tau_2}{\partial x_1} - \nu\nabla^2 t_{12}$$

We substitute into this equation expressions for t_{12} and τ_2 from equations (5.7) and (5.9). Since $\nabla^2 \log r = 0$, we obtain

$$\begin{aligned} & U\tfrac{\partial^2}{\partial x_1 \partial x_2}\log\mathcal{R} + U\tfrac{\partial}{\partial x_2}\{\tfrac{\partial}{\partial x_1}K_0(k\mathcal{R})e^{-k(x_1-q_1)}\} \\ = \quad & U\tfrac{\partial^2}{\partial x_1 \partial x_2}\log\mathcal{R} - \nu\tfrac{\partial}{\partial x_2}\{\nabla^2 K_0(k\mathcal{R})e^{-k(x_1-q_1)}\} \end{aligned}$$

The L.H.S and R.H.S of the equation are identically equal since $(\nabla^2 + 2k\frac{\partial}{\partial x_1})K_0(kr)e^{-kx_1} = 0$.

We next consider $j = 2$, $k = 1$. This gives the same equation as above since $t_{12} = t_{21}$ and $\frac{\partial \tau_2}{\partial x_1} = \frac{\partial \tau_1}{\partial x_2}$.

We finally consider the equation (5.10) for $j = 2$, $k = 2$:

$$U\frac{\partial t_{22}}{\partial x_1} = \frac{1}{\rho}\frac{\partial \tau_2}{\partial x_2} - \nu\nabla^2 t_{22}$$

We substitute into this equation expressions for t_{22} and τ_2 from equations (5.7) and (5.9). Since $\nabla^2 \log r = 0$, we obtain

$$-U\frac{\partial}{\partial x_1}\left(\frac{\partial}{\partial x_1}\{\log\mathcal{R} + K_0(k\mathcal{R})e^{-k(x_1-q_1)}\}\right) = \nu\frac{\partial}{\partial x_1}\left(\nabla^2\{K_0(k\mathcal{R})e^{-k(x_1-q_1)}\}\right)$$

and the L.H.S and R.H.S of the above equation are identically equal.

We must also show that the solution satisfies the equation

$$\frac{\partial t_{jk}}{\partial x_j} = 0 \tag{5.11}$$

We first consider equation (5.11) for $k = 1$ and substitute into it expressions for t_{11} and t_{21} from equation (5.7)

$$\frac{\partial}{\partial x_1}\left\{\frac{\partial g(\mathcal{R})}{\partial x_1} + 2kK_0(k\mathcal{R})e^{-k(x_1-q_1)}\right\} + \frac{\partial}{\partial x_2}\left\{\frac{\partial g(\mathcal{R})}{\partial x_2}\right\} = 0$$

which gives

$$\nabla^2(log\mathcal{R}) + (\nabla^2 + 2k)K_0(k\mathcal{R})e^{-k(x_1-q_1)} = 0$$

The L.H.S. of this equation is identically zero.

We finally consider equation (5.11) for $k = 2$ and substitute into it expressions for t_{12} and t_{22} from equation (5.7). We thus obtain

$$\frac{\partial^2 g(\mathcal{R})}{\partial x_1 \partial x_2} + \frac{\partial^2[-g(\mathcal{R})]}{\partial x_2 \partial x_1} = 0$$

and the L.H.S of this equation is identically zero.

5.0.5 Evaluation of the contour integral around the point q.

We now evaluate the part of the contour integral L around the point q. We take the contour along the circle circumference radius $\mathcal{R}$ centred at the point q and consider the value of the integral as $\mathcal{R} \to 0$.

Hence we evaluate the integral expression

$$\lim_{\mathcal{R}\to 0}\left\{\int_0^{2\pi}\left[t_{jk}(\nu\frac{\partial u_j}{\partial \mathcal{R}} - \frac{1}{\rho}pn_j) - u_j(\nu\frac{\partial t_{jk}}{\partial \mathcal{R}} - \frac{1}{\rho}p_k n_j) - Uu_j t_{jk} n_1\right]\mathcal{R}d\alpha\right\} \tag{5.12}$$

We consider the function $t_{jk}(\mathcal{R})$ as $\mathcal{R} \to 0$.

From equation (5.8),

$$\frac{\partial g}{\partial x_1} = \frac{x_1}{\mathcal{R}^2} - kK_0(k\mathcal{R})e^{-k(x_1-q_1)} + \frac{x_1}{\mathcal{R}}e^{-k(x_1-q_1)}\frac{\partial K_0(k\mathcal{R})}{\partial \mathcal{R}}$$

As $\mathcal{R} \to 0$, $K_0(k\mathcal{R}) \to -\log\mathcal{R}$ and $e^{-k(x_1-q_1)} \to 1$. Therefore

$$\frac{\partial g}{\partial x_1} \to k\log\mathcal{R}$$

Also, from equation (5.8),

$$\frac{\partial g}{\partial x_2} \to \frac{x_2}{r^2} + \frac{x_2}{\mathcal{R}}\frac{\partial K_0(k\mathcal{R})}{\partial \mathcal{R}}e^{-k(x_1-q_1)}$$

Therefore as $\mathcal{R} \to 0$, $\frac{\partial g}{\partial x_2} \to 0$.

Thus as $\mathcal{R} \to 0$,

$$\begin{aligned} k\underline{\underline{t}} &\to \begin{pmatrix} k\log\mathcal{R} & 0 \\ 0 & -k\log\mathcal{R} \end{pmatrix} + 2k(-\log\mathcal{R})\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ &\to -\log\mathcal{R}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

Hence

$$t_{ij} \to -\delta_{ij}\log\mathcal{R} \tag{5.13}$$

We evaluate the part of the integral expression of equation (5.12)

$$\lim_{r\to 0}\left\{\int_0^{2\pi}\frac{1}{\rho}\tau_k n_j u_j \mathcal{R} d\alpha\right\} \tag{5.14}$$

From equation (5.9), $\tau_k = 2\rho\nu\frac{\partial}{\partial x_k}\log\mathcal{R} = 2\rho\nu\frac{x_k}{\mathcal{R}^2} = 2\rho\nu\frac{n_k}{\mathcal{R}}$

Thus the integral is

$$\lim_{\mathcal{R}\to 0}\left\{2\nu\int_0^{2\pi} n_k n_j u_j d\alpha\right\}$$

As $\mathcal{R}\to 0$, we expand u_j in a Taylor series around $\mathcal{R}=0$. Expressing the function u_j in polar coordinates $(\mathcal{R},\alpha)$ about an origin at the point q, then:

$$\begin{aligned} u_j(\mathcal{R},\alpha) &= u_j(0,\alpha)+\mathcal{R}\frac{\partial}{\partial\mathcal{R}}u_j(0,\alpha)+\frac{r^2}{2!}\frac{\partial^2}{\partial\mathcal{R}^2}u_j(0,\alpha)+\ldots \\ &= u_j\big|_q+\mathcal{R}\frac{\partial u_j}{\partial\mathcal{R}}\bigg|_q+\frac{\mathcal{R}^2}{2!}\frac{\partial^2 u_j}{\partial\mathcal{R}^2}\bigg|_q+\ldots. \end{aligned} \tag{5.15}$$

Where $u_j\big|_q$ is the value of u_j at the point q. Therefore this expansion is independant of the variable α, and so the integral approximates to

$$\lim_{\mathcal{R}\to 0}\left\{2\nu\left(u_j\big|_q+\mathcal{R}\,\frac{\partial u_j}{\partial\mathcal{R}}\bigg|_q+\ldots\right)\int_0^{2\pi} n_k n_j d\alpha\right\}$$

However, $\int_0^{2\pi} n_k n_j d\alpha = \delta_{ij}\pi$ since $\underline{n}=(\cos\alpha,\sin\alpha)$.

Therefore the integral of equation (5.14) approximates to

$$2\nu\pi u_k(q_1,q_2)$$

as $\mathcal{R}\to 0$.

We next evaluate part of the integral expression of equation (5.12)

$$\lim_{\mathcal{R}\to 0}\left\{-\nu\int_0^{2\pi}u_j\frac{\partial t_{jk}}{\partial\mathcal{R}}\mathcal{R}d\alpha\right\} \tag{5.16}$$

However, from equation (5.13), as $\mathcal{R}\to 0$, $\frac{\partial t_{jk}}{\partial\mathcal{R}}\to\frac{\delta_{jk}}{\mathcal{R}}$.

Thus the integral is

$$\lim_{\mathcal{R}\to 0}\left\{\nu\int_0^{2\pi}u_k d\alpha\right\} = 2\pi\nu u_k(q_1,q_2)$$

We finally evaluate the other integral contributions to equation (5.12). As $\mathcal{R}\to 0$, $t_{jk}\to -\delta_{jk}\log\mathcal{R}$.

Hence

$$\int_0^{2\pi} t_{jk}a_j r_q d\alpha = -a_k 2\pi\mathcal{R}\log\mathcal{R}\to 0$$

for some constant a_j. Therefore the other integral contributions to equation (5.12) are zero, and so

$$\begin{aligned}&\lim_{\mathcal{R}\to 0}\left[\int_0^{2\pi}\left\{t_{jk}(\nu\frac{\partial u_j}{\partial\mathcal{R}}-\frac{1}{\rho}pn_j)-u_j(\nu\frac{\partial t_{jk}}{\partial\mathcal{R}}-\frac{1}{\rho}\tau_k n_j)-Uu_jt_{jk}n_1\right]\mathcal{R}d\alpha\right\}\\ =\ & 4\pi\nu u_k\end{aligned}$$

We consider the closed contour L consisting of three curves: the closed contour C enclosing the body, a closed contour over a circle circumference centre at the point q radius $\mathcal{R} \to 0$, and a closed contour over the circle circumference enclosing the body radius $R \to \infty$, as shown in figure (5.2).

We consider the integral I_k given by equation (5.6). We will prove later in section (5.3.1) that this integral over the circle circumference enclosing the body radius $R \to \infty$ is zero. Hence

$$u_k(q_1, q_2) = -\frac{1}{4\pi\nu} \int_C \left\{ t_{jk}(\nu \frac{\partial u_j}{\partial n} - \frac{1}{\rho} p n_j) - u_j(\nu \frac{\partial t_{jk}}{\partial n} - \frac{1}{\rho} \tau_k n_j) - U u_j t_{jk} n_1 \right\} dl$$

where $t_{jk}(x_1, x_2; q_1, q_2)$ and $\tau_k(x_1, x_2; q_1, q_2)$.

However, the variables x_1 and x_2 in the above functions are constrained to lie on the curve C. Therefore we rename the variables such that

$t_{jk}(x_1^c, x_2^c; q_1, q_2)$ and $\tau_k(x_1^c, x_2^c; q_1, q_2)$.

where (x_1^c, x_2^c) are the points that lie on the curve C.

The point q is chosen arbitrarily, and so we could choose the point q to be any point lying outside the closed curve C.

Therefore we rename the point (q_1, q_2) and consider the general point (x_1, x_2). Therefore we consider the functions

$t_{jk}(x_1, x_2; x_1^c, x_2^c)$ and $\tau_k(x_1, x_2; x_1^c, x_2^c)$.

Thus, changing from the cartesian vector notation (x_1, x_2) to the usual cartesian notation (x, y), then

$$u_k(x,y) = \frac{1}{4\pi\nu} \int_C t_{jk}(x,y;x_c,y_c)\left(\nu\frac{\partial}{\partial n}u_j(x_c,y_c) - \frac{1}{\rho}p(x_c,y_c)n_j\right) \tag{5.17}$$

$$- \quad u_j(x_c,y_c)\left(\nu\frac{\partial}{\partial n}t_{jk}(x,y;x_c,y_c) - \frac{1}{\rho}\tau_k(x_c,y_c)n_j\right) \tag{5.18}$$

$$- \quad Uu_j(x_c,y_c)t_{jk}(x,y;x_c,y_c)n_1 dl \tag{5.19}$$

where $x_1 = x$, $x_2 = y$, $x_1^c = x_c$ and $x_2^c = y_c$.

Thus, from equation (5.7),

$$k\underline{\underline{t}} = \begin{pmatrix} \frac{\partial g(\mathcal{R})}{\partial x} & \frac{\partial g(\mathcal{R})}{\partial y} \\ \frac{\partial g(\mathcal{R})}{\partial y} & -\frac{\partial g(\mathcal{R})}{\partial x} \end{pmatrix} + 2kK_0(k\mathcal{R})e^{k(x-x_c)}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{5.20}$$

$$\underline{\tau} = 2\rho\nu\nabla(\log\mathcal{R}) \tag{5.21}$$

where $g(\mathcal{R}) = \log\mathcal{R} + K_0(k\mathcal{R})e^{k(x-x_c)}$, $\mathcal{R} = [(x-x_c)^2 + (y-y_c)^2]^{1/2}$ and $k = \frac{U}{2\nu}$.

We see this is equivalent to the velocity representation of Lagerstrom. Lagerstrom considers the superposition of fundamental solutions within the area A bounding the closed curve C. Applying the divergence theorem to the integral of equation (5.6) over the closed curve C and using the result in equation (5.4), we obtain the relation

$$\int_C \left\{ t_{jk}(\nu \frac{\partial u_j}{\partial n} - \frac{1}{\rho} p n_j) - u_j(\nu \frac{\partial t_{jk}}{\partial n} - \frac{1}{\rho}\tau_k n_j) - U u_j t_{jk} n_1 \right\} dl$$

$$= \int\int_A \left\{ t_{jk}(\nu \nabla^2 u_j - \frac{1}{\rho}\frac{\partial p}{\partial x_j} - U\frac{\partial u_j}{\partial x_1}) + u_j(-\nu + \nabla^2 t_{jk} + \frac{1}{\rho}\frac{\partial \tau_k}{\partial x_j} - U\frac{\partial t_{jk}}{\partial x_1} \right\} dA$$

Lagerstrom considers a distribution of points Q within the area A such that

$$U\frac{\partial \underline{u}}{\partial x} + \frac{1}{\rho}\underline{\nabla p} - \nu\nabla^2 \underline{u} = \begin{cases} \underline{f}(Q) \text{ at the points } Q \\ 0 \text{ not at } Q \end{cases} \tag{5.22}$$

t_{jk} and p_k are singular at the general point (x, y) outside the area A, and so, in cartesian vector subscript notation

$$\int\int_A u_j(-\nu\nabla^2 t_{jk} + \frac{1}{\rho}\frac{\partial \tau_k}{\partial x_j} - U\frac{\partial t_{jk}}{\partial x_1})dA$$

$$\rightarrow \quad u_j\left(-\nu\nabla^2 t_{jk} + \frac{1}{\rho}\frac{\partial \tau_k}{\partial x_j} - U\frac{\partial t_{jk}}{\partial x_1}\right) dQ$$

$$\rightarrow \quad 0$$

Therefore

$$u_k = \frac{1}{4\pi\nu}\int\int_A t_{jk} f_j(Q) dQ \tag{5.23}$$

which is the result obtained in Lagerstrom.

5.1 The integral representation of the pressure p.

We consider particular solutions of the adjoint velocity $\tilde{u}_j$ and pressure $\tilde{p}$ such that the pressure at the point q is expressed as an integration of a fluid and pressure distribution (the function f) over the closed contour C. These solutions are $\underline{\tilde{u}} = \underline{\nabla(\log \mathcal{R})}$ and $\tilde{p} = \rho U \frac{\partial}{\partial x}(\log \mathcal{R})$ such that the pressure p is

$$p = \frac{\rho}{2\pi} \int_C \left\{ \tilde{u}_j \left(\nu \frac{\partial u_j}{\partial n} - \frac{1}{\rho} p n_j \right) - u_j \left(\nu \frac{\partial \tilde{u}_j}{\partial n} - \frac{1}{\rho} \tilde{p} n_j \right) - U u_j \tilde{u}_j n_1 \right\} dl$$

We consider equation (5.5) for the integral I:

$$I = \int_L \{ \tilde{u}_j (\nu \frac{\partial u_j}{\partial n} - \frac{1}{\rho} p n_j) - u_j (\nu \frac{\partial u_j}{\partial n} - \frac{1}{\rho} \tilde{p} n_j) - U u_j \tilde{u}_j n_1 \} dl \tag{5.24}$$

and we choose $\tilde{u}_j$ and $\tilde{p}$ such that

$$\tilde{u}_j = \frac{\partial}{\partial x_j}(\log \mathcal{R}) \ , \quad \tilde{p} = \rho U \frac{\partial}{\partial x_1}(\log \mathcal{R})$$

We now show that the above expressions for $\tilde{u}_j$ and $\tilde{p}$ satisfy the Oseen adjoint equations

$$U \frac{\partial \tilde{u}_j}{\partial x_1} = \frac{1}{\rho} \frac{\partial \tilde{p}}{\partial x_j} - \nu \nabla^2 \tilde{u}_j \ \text{ and } \ \frac{\partial \tilde{u}_j}{\partial x_j} = 0$$

by substituting $\tilde{u}_j$ and $\tilde{p}$ into the Oseen adjoint equations. This gives

$$U \frac{\partial}{\partial x_1} \frac{\partial}{\partial x_j} \log \mathcal{R} = \frac{1}{\rho} (\rho U) \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_1} \log \mathcal{R} - \nu \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) \frac{\partial}{\partial x_j} \log \mathcal{R}$$

and

$$\frac{\partial}{\partial x_1} \frac{\partial}{\partial x_1} (\log \mathcal{R}) + \frac{\partial}{\partial x_2} \frac{\partial}{\partial x_2} (\log \mathcal{R}) = 0$$

The L.H.S. and the R.H.S. of the above equations are identically equal since $\nabla^2(\log\mathcal{R}) = 0$.

The contour L comprises of the closed contour C enclosing the body, the circle circumference of radius $\mathcal{R} \rightarrow 0$ centred at the point q, and the circle circumference radius $R \rightarrow \infty$ centred on the body and enclosing the contour C and the point q. (See figure (5.2).)

We first consider the above integral over the part of the contour L along the circle circumference radius $\mathcal{R} \rightarrow 0$ centred at the point q. This is

$$\lim_{\mathcal{R}\rightarrow 0}\left\{\int_0^{2\pi}\{\tilde{u}_j(\nu\frac{\partial u_j}{\partial\mathcal{R}} - \frac{1}{\rho}pn_j) - u_j(\nu\frac{\partial\tilde{u}_j}{\partial\mathcal{R}} - \frac{1}{\rho}\tilde{p}n_j) - Uu_j\tilde{u}_jn_1\}\mathcal{R}d\alpha\right\} \qquad (5.25)$$

For the above equation, $\frac{\partial}{\partial x_j}(\log\mathcal{R}) = \frac{n_j}{\mathcal{R}}$ and $\frac{\partial}{\partial x_1}(\log\mathcal{R}) = \frac{n_1}{\mathcal{R}}$ since $\underline{n} = (\cos\alpha, \sin\alpha)$.

We first consider the integral

$$\lim_{\mathcal{R}\rightarrow 0}\left\{\int_0^{2\pi}\tilde{u}_j\left\{\nu\frac{\partial u_j}{\partial\mathcal{R}} - \frac{1}{\rho}pn_j\right\}\mathcal{R}d\alpha\right\}$$

This equals

$$\begin{aligned} & \lim_{\mathcal{R}\rightarrow 0}\int_0^{2\pi}\nu\frac{\partial u_j}{\partial\mathcal{R}} - \frac{1}{\rho}pn_jn_jd\alpha \\ = \ & \lim_{\mathcal{R}\rightarrow 0}\nu\frac{\partial u_j}{\partial\mathcal{R}}\int_0^{2\pi}n_jd\alpha - \frac{p}{\rho}\int_0^{2\pi}n_jn_jd\alpha \end{aligned}$$

The velocity and pressure and their derivatives are continuous at the point q and so we find their Taylor expansion in $\mathcal{R}$ as $\mathcal{R} \rightarrow 0$ and bring them outside the integral. (See equation (5.15).) Therefore

$$\lim_{\mathcal{R}\rightarrow 0}\left\{\int_0^{2\pi}\tilde{u}_j\left(\nu\frac{\partial u_j}{\partial\mathcal{R}} - \frac{1}{\rho}pn_j\right)\mathcal{R}d\alpha\right\} = -\frac{2\pi}{\rho}p$$

We next consider the integral

$$\lim_{\mathcal{R}\to 0}\left\{\int_0^{2\pi} -u_j\left(\nu\frac{\partial \tilde{u}_j}{\partial \mathcal{R}} - \frac{1}{\rho}\tilde{p}n_j\right)\mathcal{R}d\alpha\right\}$$

$\tilde{u}_j = \frac{n_j}{\mathcal{R}}$, and so $\frac{\partial \tilde{u}_j}{\partial \mathcal{R}} = -\frac{n_j}{\mathcal{R}^2}$.

Thus this integral is

$$\begin{aligned} &\lim_{\mathcal{R}\to 0}\left\{\frac{\nu u_j}{\mathcal{R}}\int_0^{2\pi} n_j d\alpha + U u_j\int_0^{2\pi} n_1 n_j d\alpha\right\} \\ = \;& U u_1 \pi \end{aligned}$$

We finally consider the integral

$$\lim_{\mathcal{R}\to 0}\left\{\int_0^{2\pi} -U u_j \tilde{u}_j n_1 \mathcal{R} d\alpha\right\}$$

This equals

$$\lim_{\mathcal{R}\to 0}\left\{-U u_j\int_0^{2\pi}\frac{n_j}{\mathcal{R}} n_1 \mathcal{R} d\alpha\right\} = -U u_j \pi$$

Therefore,

$$p(q_1,q_2) = -\frac{\rho}{2\pi}\lim_{\mathcal{R}\to 0}\left\{\int_0^{2\pi}\{\tilde{u}_j\left(\nu\frac{\partial u_j}{\partial \mathcal{R}} - \frac{1}{\rho}p n_j\right) - u_j\left(\nu\frac{\partial \tilde{u}_j}{\partial \mathcal{R}} - \frac{1}{\rho}\tilde{p}n_j\right) - U u_j \tilde{u}_j n_1\} d\alpha\right\} \tag{5.26}$$

We take the contour L along the closed contour C enclosing the body, the circle circumference radius $\mathcal{R} \to 0$ centred at the point q, and the circle circumference radius $R \to \infty$ centred on the body enclosing both the contour C and the point q.

We show in the next section, (5.3.2), that there is no contribution to the integral I of equation (5.22) from the circle circumference radius $R \to \infty$ centred on the body and enclosing the contour C and point q.

Thus,

$$p(q_1, q_2) = \frac{\rho}{2\pi} \int_C \{\tilde{u}_j(\nu \frac{\partial u_j}{\partial n} - \frac{1}{\rho} p n_j) - u_j(\nu \frac{\partial \tilde{u}_j}{\partial n} - \frac{1}{\rho} \tilde{p} n_j) - U u_j \tilde{u}_j n_1\} dl$$

where the functions $\tilde{u}_j$, $\tilde{p}$ and u_j are evaluated on the curve C. We let the general point on the curve C be (x_1^c, x_2^c), and so in the above integral equation $\tilde{u}_j(x_1^c, x_2^c; q_1, q_2)$, $\tilde{p}(x_1^c, x_2^c; q_1, q_2)$ and $u_j(x_1^c, x_2^c)$.

The point q at position (q_1, q_2) was arbitrarily chosen except for the only constraint that it lied outside the closed curve C. Thus we consider a general point at (x_1, x_2) lying outside the closed curve C and therefore obtain an integral representation for the pressure $p(x_1, x_2)$.

We express the equation in the more familiar cartesian coordinate notation (x, y) where $x = x_1$ and $y = x_2$.

Hence the Oseen integral representation for the pressure is

$$p(x, y) = \frac{\rho}{2\pi} \int_C \left\{ \tilde{u}_j \left(\nu \frac{\partial u_j}{\partial n} - \frac{1}{\rho} p n_j \right) - u_j \left(\nu \frac{\partial \tilde{u}_j}{\partial n} - \frac{1}{\rho} \tilde{p} n_j \right) - U u_j \tilde{u}_j n_1 \right\} dl \quad (5.27)$$

where $\underline{\tilde{u}} = \underline{\nabla(\log \mathcal{R})}$ and $\tilde{p} = \rho U \frac{\partial}{\partial x}(\log \mathcal{R})$.

$\mathcal{R}^2 = (x - x_c)^2 + (y - y_c)^2$ where (x_c, y_c) is a point on the curve C.

5.2 The expansion for the velocity and pressure calculated from the Oseen velocity and pressure representations.

5.2.1 The expansion for the velocity from the Oseen velocity representation.

We shall now show that the expansion for the Oseen velocity representation is equivalent to the expansion for the velocity obtained from the Lamb-Goldstein velocity representation. We consider the tensor function t_{jk}. From equation (5.7), we see that it involves summation of the functions $\log \mathcal{R}$, $K_0(k\mathcal{R})e^{kx}$ and their derivatives.

We also consider the vector function τ_k. From equation (5.21), we see that it involves derivatives of the function $\log \mathcal{R}$.

The function $\log \mathcal{R}$ satisfies Laplace's equation and the function $K_0(k\mathcal{R})$ satisfies the modified Helmholtz equation.

Therefore the velocity u_k involves the summation of functions which solve either Laplace's equation or the modified Helmholtz equation. Thus the velocity is represented in the form

$$\underline{u} = \underline{\nabla \phi} + \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}\right)$$

where

$$\nabla^2 \phi = 0 \quad \text{and} \quad (\nabla^2 - 2k)\frac{\partial}{\partial x}\psi = 0$$

(This is the Lamb-Goldstein velocity representation of two dimensional steady Oseen flow. (See section (3.1).))

We now find the highest order terms in the potential expansion from Oseen's integral representation of the velocity. By inspection of equation (5.18), the highest order terms are $\frac{\partial \phi}{\partial x} = \frac{\partial}{\partial x}(\log r)$, $\frac{\partial \phi}{\partial x} = \frac{\partial}{\partial y}(\log r)$ and similar equations for $\frac{\partial \phi}{\partial y}$.

These terms can only come from either the function $\log r$ or θ.

Thus the potential expansion for ϕ is

$$\phi = \frac{D}{2\pi\rho U}\log r - \frac{L(\theta-\pi)}{2\pi\rho U} + \text{ lower order terms}$$

Hence we see that we obtain an expansion for $\underline{u}$ from the Oseen velocity representation which is equivalent to the expansion for the velocity obtained from the Lamb-Goldstein velocity representation.

5.2.2 The expansion for the pressure from the Oseen pressure integral representation.

Similarly, by inspection of equation (5.19), we see tht the pressure p satisfies Laplace's equation with terms in the expansion of highest order $\frac{\partial}{\partial x}(\log r)$ and $\frac{\partial}{\partial y}(\log r)$.

Therefore, referring to equation (3.24), we see that the expansion derived from the Oseen pressure representation and the Lamb-Goldstein pressure expansion are equivalent.

5.3 The evaluation of the integral I for the velocity representation and pressure representation over the circular contour circumference radius R.

5.3.1 The evaluation of the integral I for the velocity representation over the circular contour circumference radius R.

We represent the velocity as $\underline{u} = \underline{\nabla\phi} + \underline{w}$.

As $R \to \infty$ then $|\underline{\nabla\phi}|$ is of highest order $(\frac{1}{R^2})$.

We first consider the contribution to the integral from the velocity potential only. Each term in the integrand is of order $\frac{1}{R^2}$ and hence this gives no contribution to the integral.

We next consider the contribution to the integral from the function $\underline{w}$ only.

As $R \to \infty$, $K_n(kR) \to \frac{e^{-kR}}{\sqrt{kR}}$

Therefore, the functions w_1 and w_2 are of the form $\frac{g(\Theta)e^{-kR(1-\cos\Theta)}}{\sqrt{kR}}$ for some function $g(\Theta)$.

where the polar coordinates (R, Θ) have origin centred on the body.

As $R \to \infty$, this function is exponentially small and so will give no contribution to the integral, except in the region for small Θ.

For small Θ, the function tends to $\frac{g(0)e^{-kR\frac{\Theta^2}{2}}}{\sqrt{kR}}$

Therefore in this region, the highest order terms in the integrand are of the form $\frac{e^{-kR\Theta^2}}{kR}$. Thus the integral contribution from the function $\underline{w}$ is of order

$$\int_{-\Theta}^{\Theta} \frac{e^{-kR\beta^2}}{kR} R d\beta$$

where $\Theta \to 0$ as $R \to \infty$.

Making the variable change $\beta = \sqrt{\frac{2}{R}}t$, then the integral contribution is of order

$$\sqrt{\frac{2}{R}}\int_{-\sqrt{\frac{2}{R}}T}^{\sqrt{\frac{2}{R}}T} e^{-t^2}\,dt$$

which tends to zero as $R \to \infty$.

Hence the integral tends to zero as $R \to \infty$.

5.3.2 The evaluation of the integral I for the pressure representation over the circular contour circumference radius R.

From equation (5.22), the integral I for the pressure representation over the circular contour circumference radius R is

$$I = \int_{\theta=0}^{\theta=2\pi} \left\{ \tilde{u}_j \left(\nu \frac{\partial u_j}{\partial n} - \frac{1}{\rho} p n_j \right) - u_j \left(\nu \frac{\partial u_j}{\partial n} - \frac{1}{\rho} \tilde{p} n_j \right) - U u_j \tilde{u}_j n_1 \right\} R d\theta$$

From equation (5.25), we see that $\tilde{u}_j$ and $\tilde{p}$ are of order ($\frac{1}{R}$ as $R \to \infty$.

As $R \to \infty$, $\frac{\partial \underline{u}}{\partial R} \to \underline{0}$, $p \to 0$ and $\underline{u} \to 0$.

Therefore as $R \to \infty$, then the integral I over the circular contour circumference tends to zero.

Therefore as $R \to \infty$, then the integral I over the circular contour circumference tends to zero.

Chapter 6

The flow in the far field.

6.1 Solutions for far field Laminar wake flow.

6.1.1 The Lagerstrom solution.

Lagerstrom considers solutions for the flow due to a singular drag element and a singular lifting element.

The singular drag element is due to a force D located at the origin in the negative x-axis direction. This is equivalent to a flow considering only the terms

$$\phi = \frac{D}{2\pi\rho U}\log r\ , \quad \text{and} \quad \Psi = \Psi_1$$

This solution satisfies the condition that the streamfunction of the flow ψ is single valued, and is such that there is a force D on the body located at the origin.

The singular lifting element is due to a lift L located at the origin in the negative y-direction. This is equivalent to the flow due to the potential term

$$\phi = -\frac{L(\theta - \pi)}{2\pi\rho U}$$

However, Lagerstrom considers the force located at a point rather than a body surface and so an extra condition that the velocity does not become singular at the origin must be considered. Therefore we consider the flow due to the terms

$$\phi = -\frac{L(\theta - \pi)}{2\pi\rho U}\ , \quad \text{and} \quad \Psi = me^{-kx}K_0(kr)$$

such that

$$\lim_{r\to 0}\left\{\frac{\partial\phi}{\partial x} + \frac{\partial\Psi}{\partial y}\right\} = 0$$

and

$$\lim_{r\to 0}\left\{\frac{\partial\phi}{\partial y} - \frac{\partial\Psi}{\partial x}\right\} = 0$$

As $r \to 0$, $me^{-kx}K_0(kr) \to -m\log r$ and so $m = -\frac{D}{2\pi\rho U}$.

6.1.2 The Landau and Lifshitz solution.

We follow exactly the method given for obtaining the far field wake flow in Landau and Lifshitz, applying it to two dimensional rather than three dimensional flow.

However, we shall see that this method applied to two dimensional flow presumes the wrong order for the velocity component u_2 in the far field wake. Hence this method gives invalid results.

We first consider the function of the form

$$\frac{a}{\sqrt{x}}e^{-\frac{ky^2}{2x}}$$

This function satisfies the diffusion equation

$$\left(\frac{\partial^2}{\partial y^2} - 2k\frac{\partial}{\partial x}\right) = 0$$

which the modified Helmholtz equation

$$\left(\nabla^2 - 2k\frac{\partial}{\partial x}\right) = 0$$

reduces to in the far field wake.

We presume as in the method given in Landau and Lifshitz that the lift and drag come from functions of the above form. Therefore to leading order we assume the solution

$$\underline{w} \sim \left(-\frac{d}{\rho U}\sqrt{\frac{k}{2\pi x}}e^{-\frac{ky^2}{2x}}, -\frac{l}{\rho U}\sqrt{\frac{k}{2\pi x}}e^{-\frac{ky^2}{2x}}\right)$$

where d and l are the drag and the lift on the body for the Landau and Lifshitz solution. However, from section (3.6) equation (3.33) we see that the wake velocity component w_2 does not have this form in the far field and thus we obtain invalid results.

The condition $\underline{\nabla}.\underline{w}$ must be satisfied. Therefore we consider the function G of lower order than the function $-\frac{1}{\rho U}\sqrt{\frac{k}{2\pi x}}e^{-\frac{ky^2}{2x}}$ such that

$$w_2 \sim -\frac{l}{\rho U}\sqrt{\frac{k}{2\pi x}}e^{-\frac{ky^2}{2x}} + G$$

We want the function G to give no contribution to the lift and so we consider $G = \frac{\partial \Xi}{\partial y}$ where Ξ is a continuous function. (Thus $-\rho U \int_{-\infty}^{\infty} \frac{\partial \Xi}{\partial y} dy = 0$ and so G gives no contribution to the lift.)

Since

$$\frac{\partial}{\partial x}\left(e^{-\frac{ky^2}{2x}}\right) \ll \frac{\partial}{\partial y}\left(e^{-\frac{ky^2}{2x}}\right)$$

then the equation $\underline{\nabla}.\underline{w} = 0$ reduces to

$$\frac{\partial}{\partial y}\left\{\frac{l}{\rho U}\sqrt{\frac{k}{2\pi x}}e^{-\frac{ky^2}{2x}}\right\} = -\frac{\partial^2 \Xi}{\partial y^2}$$

where Ξ satisfies the modified Helmholtz equation $(\nabla^2 - 2k\frac{\partial}{\partial x})\Xi = 0$. In the far field wake, we expect changes in velocity terms in the x-direction to be much less than changes in the y-direction. Thus

$$\frac{\partial^2 \Xi}{\partial x^2} \ll \frac{\partial^2 \Xi}{\partial y^2}$$

and so the modified Helmholtz equation reduces to the diffusion equation

$$\left(\frac{\partial^2}{\partial y^2} - 2k\frac{\partial}{\partial x}\right)\Xi = 0$$

Therefore

$$-\frac{\partial^2 \Xi}{\partial y^2} = -2k\frac{\partial \Xi}{\partial x} = -\frac{ky}{x}\frac{l}{\rho U}\sqrt{\frac{k}{2\pi x}}e^{-\frac{ky^2}{2x}}$$

We consider the function $\Xi' = \frac{e^{-\frac{ky^2}{2x}}}{\sqrt{x}}$. Differentiating Ξ' with respect to x, we obtain

$$\begin{aligned}\frac{\partial \Xi'}{\partial x} &= \frac{e^{-\frac{ky^2}{2x}}}{\sqrt{x}}\left\{-\frac{1}{2x} + \frac{ky^2}{2x^2}\right\} \\ &\sim -\frac{e^{-\frac{ky^2}{2x}}}{2x\sqrt{x}}\end{aligned}$$

in the far field.

Thus we choose

$$\Xi = \frac{l}{\rho U}\sqrt{\frac{k}{2\pi x}}ye^{-\frac{ky^2}{2x}}$$

and the method given in Landau and Lifshitz applied to steady two dimensional far field wake flow gives

$$\underline{w} \sim \left(-\frac{d}{\rho U}\sqrt{\frac{k}{2\pi x}}e^{-\frac{ky^2}{2x}}, -\frac{l}{\rho U}\sqrt{\frac{k}{2\pi x}}e^{-\frac{ky^2}{2x}} + \frac{\partial}{\partial y}\left\{\frac{l}{\rho U}\sqrt{\frac{k}{2\pi x}}ye^{-\frac{ky^2}{2x}}\right\}\right)$$

The Landau and Lifshitz method assumes that the function w_2 is of the form $\frac{a}{\sqrt{x}}e^{-\frac{ky^2}{2x}}$ in the far field and that the potential ϕ gives no contribution to the lift. From equation (4.1.2), we see that there is a contribution to the lift from the potential ϕ, and from equation (3.33) we see that the order of w_2 in the far field is $\frac{1}{x\sqrt{x}}e^{-\frac{ky^2}{2x}}$ and so both these assumptions used in this method are invalid.

Thus the Landau and Lifshitz method cannot be applied to two dimensional flow. This raises questions as to the validity of the method in three dimensional flow, although the same results are given in Batchelor (Introduction to Fluid Mechanics, pg 377).

(The reason given by Landau and Lifshitz for introducing the function Ξ is that the term $-\frac{1}{\rho}\nabla p$ from Oseen's equation may be taken as the gradient $\underline{\nabla\Xi}$ of some scalar function Ξ. However, there is no relation between p and Ξ since p satisfies Laplace's equation and is of much lower order in the wake than Ξ which satisfies the diffusion equation .)

Chapter 7

Discussion.

We have found the complete expansions for the velocity and pressure in the far field where Oseen flow is valid for steady two dimensional flow.

Some of the coefficients in the expansion are related to the drag, lift and moment on the body.

The complete velocity and pressure expansions are found to be equivalent to the Oseen representation of velocity and pressure as a distribution of singularities (Oseenlets and multipoles) over a closed contour enclosing the body.

In the far field wake, the velocity components from the complete velocity expansion are the same as the velocity components given in Lagerstrom.

An important result for two dimensional flow is that although the drag is expressed in terms of a wake traverse, the lift is calculated from the circulation of the velocity potential $\underline{\nabla\phi}$.

7.0.3 Lamb's treatment of three dimensional steady Oseen flow.

Lamb considers the case of axisymmetric flow. He solves the Oseen flow past a sphere and introduces a function χ related to the vorticity $\underline{\omega}$ by $\underline{\omega} = (0, -\frac{\partial \chi}{\partial z}, \frac{\partial \chi}{\partial y})$. (From Lamb Hydrodynamics art. 342 equation 19.)

7.0.4 Goldstein's treatment of three dimensional steady Oseen flow.

[Goldstein, S. Proceedings of the Royal Society; volume 131 1931a pg 198-208]

Goldstein considers the singular needle and singular lifting element (defined in Lagerstrom, High speed aerodynamics and jet propulsion; volume 6 pg 92-98) which give rise to drag and lift forces respectively at the origin.

Hence he considers the equivalent of Oseen flow past an infinitely small body which has no shadow region. However, it is possible that the potential ϕ and the potential derivative $\frac{\partial \phi}{\partial y}$ are discontinuous on the infinite half line $x > 0$, $y = z = 0$.

Goldstein locates the terms in the Fourier expansion of the pressure $\frac{f(y,z)}{r^{2n+1}}$ (where $f(y, z)$ are rational integral harmonics of degree n in the Fourier expansion of the pressure and are funcions of y and z only) which give rise to the discontinuity in ϕ.

Goldstein argues that a general solution for the flow past a body is some distribution of singular needle and singular lifting elements. From section (4.3.1), we see that this is indeed the case.

7.0.5 Future work.

The aim of the work is to find the relations between ship manoeuvrability and hull design.

We have found the velocity and pressure in the far field. (Some of the coefficients in the expansion are related to the lift, drag and moment.) Thus these expansions may be asymptotically matched to near field expansions for the velocity and pressure.

We have also considered Oseenlets which are the singular functions valid in Oseen flow which cause the lift, drag and moment on the body. We may further consider the various body shapes generated by particular distributions of these.

Appendix A

Fourier's theorem for the expansion of a function.

We consider an expansion of a certain type of function $g(r, \theta)$ for $r > R$ where the radius R encloses any singularity in f and the function $g(r, \theta)$ is continuous in the region $r > R$. Thus in this region θ is defined from $0 \leq \theta \leq 2\pi$. We apply Fourier's theorem to give us an expansion of the function $g(r, \theta)$.

We state a form of Fourier's theorem below:

A certain type of function $f(t)$ defined in the region $-T < t < T$ can be expressed by a Fourier series in the form

$$f(t) = (1/2)a_0 + \sum_1^\infty a_n cos(\frac{n\pi t}{T}) + b_n sin(\frac{n\pi t}{T})$$

where the coefficients a_n and b_n are given by the formulas

$$a_n = \frac{1}{T}\int_{-T}^{T} f(\tau) cos(\frac{n\pi\tau}{T}) d\tau\,, \quad b_n = \frac{1}{T}\int_{-T}^{T} f(\tau) sin(\frac{n\pi\tau}{T}) d\tau$$

Letting $t = \theta$ and $T = \pi$, we have an expansion for θ in the range $-\pi \leq \theta < \pi$. Noting that $f(\theta) = f(\theta + 2\pi)$, we give the expansion for θ in the range $0 \leq \theta < 2\pi$.

Thus for fixed r, we expand $g(r, \theta)$ in terms of a fourier series. Different values of r give different coefficients a_n, and b_n. Thus a_n and b_n are functions of r and we write the expansion of g as:

$$g(r,\theta) = (1/2)a_0(r) + \sum_1^{\infty}(a_n(r)cosn\theta + b_n(r)sinn\theta)$$

where

$$a_n(r) = \frac{1}{\pi}\int_0^{2\pi} g(r,\theta)cosn\theta d\theta\,, \quad b_n(r) = \frac{1}{\pi}\int_0^{2\pi} g(r,\theta)sinn\theta d\theta$$

for θ in the range $0 \leq \theta < 2\pi$.

A.0.6 The Fourier expansion of Laplace's equation.

We consider the case for the function $g(r,\theta)$ satisfying Laplaces equation. In polar coordinates,

$$\nabla^2 g = \frac{1}{r}\left\{\frac{\partial}{\partial r}(r\frac{\partial p}{\partial r}) + \frac{\partial}{\partial \theta}(\frac{1}{r}\frac{\partial p}{\partial \theta})\right\} = 0$$

Applying the operator $\frac{d}{dr}(r\frac{d}{dr})$ to the function a_n we obtain

$$\begin{aligned}\frac{d}{dr}(r\frac{da_n}{dr}) &= \frac{1}{\pi}\int_0^{2\pi}\frac{\partial}{\partial r}(r\frac{\partial p}{\partial r})cosn\theta d\theta \\ &= -\frac{1}{\pi}\int_0^{2\pi}\frac{\partial}{\partial \theta}(\frac{1}{r}\frac{\partial p}{\partial \theta})cosm\theta d\theta\end{aligned}$$

by Laplace's equation.

Thus,

$$\begin{aligned}\frac{d}{dr}(r\frac{da_n}{dr}) &= -(1/\pi r)\left\{\left[cosn\theta\frac{\partial p}{\partial \theta}\right]_0^{2\pi} + n\int_0^{2\pi} sinn\theta dp\right\} \\ &= -(1/\pi r)\left\{[psinn\theta]_0^{2\pi} - n^2\int_0^{2\pi} cosn\theta p d\theta\right\} \\ &= \frac{n^2}{r}a_n\end{aligned}$$

(Since $g(\theta) = g(\theta + 2\pi)$, and $\frac{\partial g(\theta)}{\partial \theta} = \frac{\partial g(\theta+2\pi)}{\partial \theta}$, the square bracketed terms in the above equations are zero.)

Thus a_n satisfies the second order differential equation

$$r^2 a_n'' + r a_n' - n^2 a_n = 0 \tag{A.1}$$

This is satisfied by $a_n(r) = Ar^{\alpha_1} + Br^{\alpha_2}$ for $n \geq 1$, where α_1 and α_2 are different constants dependant on n.

Substituting into the above equation $a_n = r^\alpha$, we obtain

$$\begin{aligned} r^2\alpha(\alpha-1)r^{n-2} + r\alpha r^{n-1} - n^2 r^\alpha &= 0 \\ \alpha(\alpha-1) + \alpha - n^2 &= 0 \\ \alpha^2 &= n^2 \end{aligned}$$

Thus $\alpha_1 = n$, and $\alpha_2 = -n$.

For the case $n = 0$

$$\frac{d}{dr}(ra_0') = 0 \quad \Rightarrow \quad ra_0' = k$$

Thus $a_0 = 2k_1 lnr + 2k_2$ where k_1 and k_2 are constants.

Similarly, $b_n(r) = C_n r^n + D_n r^{-n}$.

Thus we obtain g as an expansion in the form

$$p(r,\theta) = k_2 + k_1 lnr + \sum_1^\infty \left\{(A_n r^n + B_n r^{-n})cosn\theta + (C_n r^n + D_n r^{-n})sinn\theta\right\} \quad \text{(A.2)}$$

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