THE STABILITY OF THE BOUNDARY LAYER FORMED BY SUPERSONIC FLOW PAST AXISYMMETRIC BODIES

A Thesis submitted to the University of Manchester
for the degree of Doctor of Philosophy
in the Faculty of Science

By

Stephen Shaw

Department of Mathematics

May 1992

Th17408
(D7RUY)
S James

√

Contents

List of Figures

List of Tables

Abstract

The supersonic flow past axisymmetric bodies is investigated; in particular the associated laminar boundary-layer flow (i.e. the velocity and temperature field) is computed and then analysed from the point of view of linear, temporal, inviscid stability theory. The basic, nonaxisymmetric disturbance equations are derived for general flows of this class and a so-called 'triply generalized' inflexion condition is determined for the existence of certain classes of neutral modes of instability. This condition is analogous to the well-known generalized inflexion condition found in planar compressible flows, although in the present case the condition depends on both axial and azimuthal wavenumbers.

Extensive numerical results are presented for the stability problem at freestream Mach numbers of $M_\infty = 2.8$ and $M_\infty = 3.8$, for the particular cases of a long thin, straight circular cylinder, subject to heated, cooled or adiabatic wall conditions, and a sharp cone for adiabatic wall conditions, at a range of streamwise locations and different azimuthal wavenumbers. The stability analysis reveals that curvature and choice of wall temperature conditions both have a significant effect on the stability of the flow. These results also reveal that a new mode of instability may occur, peculiar to flows of this type involving lateral curvature. This mode occurs at small wavenumbers, but under certain circumstances may in fact be the most unstable (and hence important) mode.

Both the asymptotic, large azimuthal wavenumber solution and asymptotic analyses valid close to the tip of the axisymmetric body and far downstream are presented, and compared with numerical results.

The effects of a viscous linear critical layer and a nonlinear-non-equilibrium critical layer on the temporal evolution of certain classes of axisymmetric instability modes on the compressible axisymmetric boundary layer formed on a thin cylinder, are then considered. In the case of the nonlinear critical layer, matching the inner solution with the flow outside the critical layer it is shown that the instability wave amplitude is governed by an integro-differential equation with a cubic-type nonlinearity. Numerical solutions are presented and the wave amplitude is found in each case to terminate in a singularity after a finite time evolution for all the calculations conducted.

Additionally, the effects of a viscous nonlinear-non-equilibrium critical layer are considered and a corresponding amplitude evolution equation is derived.

Declaration

No portion of the work referred to in this Thesis has been submitted in support of an application for another degree or qualification of this or any other University or institution of learning.

S. Shaw
Stephen Shaw

Acknowledgements

I would like to acknowledge the able help, assistance and guidance of Dr. P.W. Duck throughout the duration of this work. I would also like to thank the Department of Education, Northern Ireland for their financial support, and Professor J. Blake, who made available the time to complete this work.

I would also like to acknowledge my parents and brothers for putting up with me all these years and all the DP's and footy players I have known in my time in Manchester. Lastly I would like to thank the 400+ landlords who have made my fluids research time in Manchester a rather pleasant one.

The work presented in Chapters 2 and 3 of this thesis has been published in Duck and Shaw (1990) and Shaw and Duck (1992) (to appear).

Chapter 1

Introduction and Historical Background

1.1 Incompressible Stability theory

It is well known from experiment that laminar flows do not persist at very large Reynolds numbers, indeed, it is found that most naturally occuring flows are either fully or partially turbulent. The process of transition from a laminar to a turbulent state has interested research workers for well over a century now. Since turbulent flows result in considerably greater skin frictions and heat transfer coefficients than corresponding laminar flows, any method by which a boundary layer may be stabilized is worthy of investigation (although in many technological applications, such as in turbines, engines, instability is of course desirable as it aids mixing of different fluids and enhances heat transfer). It has been found that stability analysis is a useful tool to study this process, especially the early stages of transition. The early research carried out using stability analysis was conducted on a purely linear basis, but later more involved nonlinear stability theories were developed to help try and explain more accurately experimential observations.

In the case of linear stability theory, as applied to the laminar boundary layer, the problem that is actually solved turns out to be somewhat idealized, physically speaking. The laminar flow is assumed to be parallel implying that the basic flow variables do not change in the flow direction, but only depend on their distance from the fixed boundary. The stability analysis is usually treated in terms of the small amplitude perturbation method; the disturbance terms, which are linearized, are in the form of travelling waves whose amplitude varies either with respect to time or distance travelled, depending on the approach being used. The mathematical problem is to determine the eigenvalues of the stability problem for a fixed Reynolds number (or in the limit of infinite Reynolds numbers for the inviscid stability problem), the phase velocity, rate of amplification, and wave number for a range of disturbances. Generally, the amplification rates will either be growing, neutral or decaying disturbances, although generally it has been found that only damped disturbances exist below a critical Reynolds number.

The phenomenon of transition from laminar to turbulent flow was first investigated by Helmholtz (1868), Kelvin (1871), Rayleigh (1880) and Reynolds (1883), at the end of the last century. Reynolds (1883) proposed that, based on experimental evidence he had determined, transition was only possible if instability was developed in the laminar flow.

In a series of papers, Rayleigh (1880, 1887, 1892, 1895, 1913, 1916) produced a number of notable results concerning the stability of inviscid flows. Using physical reasoning, he determined that if the effect of viscosity is ignored, then the motion of rotating fluids is either stable or unstable, depending on whether the square of the circulation increases monotonically outwards. He also demonstrated, that if parallel flows are to be unstable, then an inflexion point must occur in the velocity profile within the flow.

Until this point in time it was commonly thought that viscosity only acted to stabilize flows - although Reynolds (1883) did conclude from his experiments that viscosity could be a cause of instability. The first workers to include the effects of viscosity in stability theory, were Orr (1907) and Sommerfeld (1908), who independently derived a single fourth-order differential equation which governs the disturbance amplitude in parallel, viscous flow of constant density, and now bears their name. Although Taylor (1915) had already indicated that viscosity could be a cause of instability, when Prandtl (1921) independently made the same discovery, conjecturing that viscous forces are capable of inducing a Reynolds stress, which in turn could convert energy from the basic flow into the disturbance, thus inducing instability, it set into motion investigations that finally led to a viscous theory of boundary-layer stability, several years later (Tollmien (1929)).

Heisenberg (1924) investigating the stability of plane Poiseuille flow to two-dimensional disturbance terms, deduced that instability did exist for viscous flow at sufficiently high Reynolds numbers, but was unable to determine the critical Reynolds number, above which instability began. Heisenberg's solution of the Orr-Sommerfeld equation was based on the method of successive approximations, employing two different approaches, the first using convergent series and the second using asymptotic series. In the first method solutions were obtained in terms of the small parameter $\epsilon = (\alpha Re)^{-1/3}$ (where α represents the spatial wavenumber and Re the Reynolds number), and a fundamental system of four solutions involving Hankel functions of order 1/3 are obtained. In the second approach, two asymptotic solutions in the small parameter $(\alpha Re)^{-1}$ are obtained. The initial approximation in this method satisfies the inviscid equation which Heisenberg solved by expanding in powers of α^2. The resultant integrations in Heisenberg's solution are required to pass under the singularity.

Using the analysis of Heisenberg (1924) and the work of Tietjens (1925), Tollmien (1929) obtained the first solution for the stability of the boundary layer formed on a flat plate, by an asymptotic approach. He accounted for the importance of viscosity in the neighbourhood of the wall and in a distinct critical layer (the region which exists in the neighbourhood of the point where the mean flow velocity equals the wavespeed of the disturbance). Tollmien applied the method of Frobenius to obtain two solutions to the inviscid equation about the critical point. The first solution is found to be regular in the neighbourhood of the critical point, but due to the presence of a logarithmic term, the second solution will generally be multi-valued. In the case of inviscid, neutral disturbances which are inflexional, this logarithmic term disappears, resulting in the second solution becoming regular. However, when inviscid theory is applied to obtain two of the four solutions in the asymptotic viscous theory, inflexional theories are invalid, resulting in an ambiguity regarding which branch of the logarithm show be taken for points below the critical point. Tollmien resolved this problem by introducing a viscous correction term in the neighbourhood of the critical point, which was required to match the solution away from the critical layer. This procedure results in the correct jump across the critical layer being '$+i\pi$', meaning the path of integration for the inviscid solution must pass under the critical point.

A series of papers by Schlichting (1933a, 1933b, 1935, 1940) and a second paper by Tollmien (1935) resulted in a fairly well developed viscous theory with a small number of numerical results. In his paper, Tollmien (1935) showed that not only was the existence of an inflexion point in the velocity profile a necessary condition for the existence of inviscid instability, but for certain flows, eg. symmetric profiles in a channel and for monotone profiles of the boundary layer, it also provides a sufficient condition. However, the sufficiency condition is not valid for all flow types as can been seen from the simple counter-example provided by Tollmien of a basic flow which has the form $\sin y^*$, where y^* denotes the transverse coordinate. He also determined that for neutral disturbances, the mean flow velocity is equal to the wavespeed of the disturbance term. However, any hopes that instability and transition to turbulence are synonymous for the boundary layer, were dashed as a result of the low values of the critical Reynolds number obtained. Tollmien's value for the critical Reynolds number in the case of the Blasius boundary layer was 60,000, and even in the high turbulence wind tunnels of that time, transition was observed to occur between values of 3.5×10^5 and 1×10^6.

Schlichting's first paper (1933a) contains one of the earliest applications of linear stability theory to transition predications, in which for the case of the Blasius boundary layer, he calculated the amplitude ratio of the most amplified frequency as a function of the Reynolds number.

In the above work only two-dimensional disturbances are considered. In 1933 Squire proved that the problem of three-dimensional disturbances of a plane flow is equivalent to a problem with two-dimensional disturbances at a lower Reynolds number, so the minimum critical Reynolds number is always given by two dimensional analysis.

Outside of Germany, this early development of stability theory was not met with much enthusiasm, and in some quarters with much scepticism, due mainly to the inability to experimentally observe the predicted disturbance waves and mathematical obscurities in the theory, particularly in the asymptotic developments. In the background of this hostile atmosphere, one of the most celebrated of all fluid dynamics experiments was carried out by Schubauer and Skramstad (1947). Through their experiments they demonstrated that instability waves did indeed exist in boundary layers, demonstrated their connection with turbulence and indicated that the theories of Tollmien and Schlichting were correct.

The early unconvincing mathematics of stability theory (namely the asymptotic theories), was clarified by Lin (1945) and he presented detailed calculations of the neutral stability curve. Tollmien (1929) had argued that in the case of inviscid disturbances, since the critical point will be located off the real axis for both growing and decaying instabilities, then there will be nothing to hinder integration along the real axis. Lin showed that if inviscid solutions are regarded as the infinite number limit of viscous solutions, that contrary to the statement made by Tollmien, the proper path of integration must be below the critical point, regardless of whether this point is above (amplified), on or below (damped) the real axis. This resolved ambiguities that existed in considering viscous disturbances in the limit of infinite Reynolds numbers, and demonstrated that a consistent inviscid theory could be constructed in which damped solutions exist that are not complex conjugates of the amplified solutions. The arguments used by Lin were physical and heuristic but a more rigorous justification of the results he obtained was given by Wasow (1948). Tollmien (1947) presented improved solutions for the neutral stability case, for real values of y^* (the coordinate normal to the surface). Solutions of a similar nature were obtained by Wasow (1953) for complex values of y^* and c^* (the complex wavespeed) and he also gave a complete proof of the construction he used.

Rayleigh's necessary inflexion condition was strengthened by Fjørtoft (1950) to give a condition which is equivalent to requiring that the *modulus* of the gradient of the streamwise mean velocity term must possess a maximum somewhere in the boundary layer for instability to occur.

The above work of Heisenberg (1924), Tollmien (1929, 1947) and Lin (1945, 1955) gave first approximations to the Orr-Sommerfeld equation for large values of Reynolds number by somewhat heuristic methods. Although these approximations have been successful for many computational purposes, it has long been recognised that heuristic approximations are not uniformly valid. Subsequent attempts to improve on these results have generally been based on either the comparison equation method or the method of matched asymptotic expansions.

Comparison equation methods of approximation to the solutions of the Orr-Sommerfeld equation have been extensively studied by Wasow (1953), Langer (1957, 1959), Lin (1957a,b, 1958) and Lin and Rabenstein (1960, 1969). In all these works the major aim was to obtain asymptotic approximations which are uniformly valid in a bounded domain containing one single turning point and to develop an algorithm by which higher approximations could be obtained systematically. The actual method employed seeks to express the solutions of the Orr-Sommerfeld equation asymptotically in terms of the solutions of an appropriately chosen comparison equation. Lakin and Reid (1970) also used the technique to obtain first order approximations to the Stokes multiplers for the Orr-Sommerfeld equation and hence to obtain outer expansions which were complete in the sense of Watson.

The method of matched asymptotic expansions was first applied to the Orr-Sommerfeld equation by Graebel (1966). A more systematic application of the method, based on the general theory developed by Fraenkel (1969) was later given by Eagles (1969). Eagles introduced a new independent variable involving the 'Langer' variable, which has the important consequence of bringing the Stokes and anti-Stokes lines associated with the inner and outer expansions into coincidence. Although a preliminary transform of this form is not an important proponent of the theory of matched asymptotics, and as Eagles' work shows, is it strictly necessary, the subsequent solution of the central matching problem and formation of composite approximations is substantially simplified. The introduction of the 'Langer' variable also leads to a larger domain in which matching between the inner and outer expansions occurs. A rigorous justification of the results obtained by Eagles has been given by De Villers (1975).

The method of matched asymptotics was also used by Reid (1972) to obtain composite approximations to the solutions of the Orr-Sommerfeld equation. It was customary, in the older work carried out on the Orr-Sommerfeld equation to express the inner expansions in terms of modified Hankel functions of order one-third. Reid (1972) pointed out the limitations in this approach and instead obtained the inner expansions, to all orders, in terms of a certain class of generalized Airy functions, the resultant expansions being used to derive approximations to the connection formulae. After matching the inner and outer expansions in certain sectors of the complex plane, Reid (1972) considered the consequences of combining them to form composite expansions, subject to the usual rules (Van Dyke (1964)) for additive or multiplicative composition. He determined that the 'modified' viscous solutions of Tollmien (1947) emerge as first-order composite approximations obtained by multiplicative composition and the 'viscous correction' to the singular inviscid solutions conjectured by Reid (1965) emerge as first order additive composite approximations. However, because of the completeness requirement, the composite expansions are only valid in certain restricted domains containing just one turning point: connection formulae must be used to obtain approximations valid in the complementary sectors.

Asymptotic work in connection with the Orr-Sommerfeld equation, has been successfully continued by Reid and his collaborators (Lakin, Ng and Reid (1978)).

Smith (1979a), considering the stability of growing boundary layers found that for sufficiently high Reynolds numbers, the linear disturbance can be described by a triple-deck structure. Triple-deck theory was initially developed to better understand and explain the separation of boundary layers; the initial success of triple-deck theory was in dealing with the Goldstein (1930) singularity at the trailing edge of a flat plate (see Stewartson (1969), Messiter (1970)), but it has since been applied to a wide variety of problems. It should be noted that the raw material for triple-deck theory can actually be found in Lin's (1955) book. Through matching procedures, Smith was able to obtain an asymptotic relationship between the Reynolds number and the neutral frequency for both parallel and non-parallel flow types. However his analysis was limited to the lower branch of the neutral stability curve. In a second paper, Smith (1979b) using asymptotic theory, considered the nonlinear stability of small disturbances to the Blasius boundary layer within a rational, high Reynolds number framework, for a wide range of disturbance sizes. He found that the nonlinear properties of the small disturbances are profoundly affected by non-parallel flow effects.

The book on hydrodynamic stability by Drazin and Reid (1981) explains in further detail the methods required for the application of the two asymptotic approximation approaches described above.

The first application of modern numerical methods was carried out by Pretsch (1942), during the war. He provided the only really large body of numerical results for exact boundary-layer solutions, before the advent of computers, by calculating the stability characteristics of the Falkner-Skan family of velocity profiles. The first attempt to make use of the digital computer in solving a laminar stability problem was made by Thomas (1953). Using a finite difference method, Thomas investigated the stability of plane Poiseuille flow which Lin (1945) predicated was unstable, although other authors thought was stable. Thomas was able to obtain 16 eigenvalues which confirmed Lin's predictions.

Around 1960 the advances in the digital computer field had reached the stage where the first direct solution of primary differential equations could be obtained. The development of ever sophisticated numerical techniques, coupled with the rapid progress of the computer, have made it possible to obtain numerical results for many different types of boundary-layer flows.

Brown (1959) was probably the first to apply numerical methods, using the digital computer to obtain solutions for three-dimensional boundary layers. Using finite difference techniques, Kurtz and Crandall (1962) obtained numerical solutions of the Orr-Sommerfeld equation in their study of the stability of the Blasius boundary layer and of free convection boundary layers on a vertical heated wall.

Neutral stability curves for the two-dimensional laminar boundary layer on a flat plate under zero pressure gradient, have been numerically determined by Kurtz

(1961), Kaplan (1964), Osborne (1967), Wazzan *et al.* (1968) and Jordinson (1970). The results of these calculations, obtained by means of slightly different numerical methods, are all sufficiently consistent to justify the statement that the neutral curve eigenvalues of the Orr-Sommerfeld equation are now well established. It should be noted, however, that since all these workers solved the Orr-Sommerfeld equation, then the parallel flow approximation is a key element of their work, but since realistically if any amount of viscosity is present there is no such thing as a parallel flow, then their results are inaccurate especially for low Reynolds numbers.

Jordinson (1970) applied the numerical techniques of Osbourne (1967) in his study of the spatial stability of the boundary layer for a wide range of values of frequency and Reynolds number.

Jordinson (1971), Mack (1976) and Corner, Houston and Ross (1976), using different numerical methods, determined the higher eigenvalues of the Orr-Sommerfeld equation for Blasius flow in their studies of discrete stable eigenmodes. Jordinson calculated eigenvalues for both spatially and temporally growing or decaying waves for a single Reynolds number and a single wavenumber (temporal approach) or a single frequency (spatial approach). Mack determined eigenvalues for a number of different values of wavenumber and Reynolds number, for the temporal problem only. Corner *et al.* recalculated the spatial modes. All these authors agreed on one conclusion - for any Reynolds number there exists only a finite and small spectrum of discrete eigenvalues. Grosch and Salwen (1978) proved the existence of a continuous spectrum of eigenvalues of the Orr-Sommerfeld equation in the case of Blasius boundary layers, for both temporal and spatial developments.

More recent work on incompressible flows has focused on three-dimensional boundary layers, in response to the renewed interest in laminar-flow control for swept wings. Srokowski and Orszag (1977) were the first to apply computational numerical techniques to calculate the suction required to avoid transition to turbulence. Mack (1979b), using a three-dimensional stability formulation, which he first presented in 1977, studied the three-dimensional Falkner-Skan-Cooke incompressible boundary layer.

Using linear stability theory, Lekoudis (1980) examined the effect of wall cooling in the leading edge region of a transonic swept wing. For both the temporal and spatial cases he determined that wall cooling has a stabilizing effect on cross flow disturbances, but that this stabilization is mild in comparison to the stabilizing effect wall cooling has on Tollmien-Schlichting waves.

Except for the asymptotic suction boundary layer, it is observed that most boundary layers grow in thickness in the downstream direction. Therefore, discrepancies still existed between the existing theoretical predictions of the neutral stability curve for the Orr-Sommerfeld equation and the experimental observations of Schubauer and Skramstad (1947) and the later work of Ross *et al.* (1970), especially for low Reynolds numbers.

The first real attempt to include boundary-layer growth into stability theory was made by Barry and Ross (1970) using a somewhat heuristic approach. They obtained an estimate of the the effect making the parallel flow assumption had, by performing computations on a modified form of the Orr-Sommerfeld equation in which the more important terms representing the growth of boundary layer thickness were included. The results obtained, however, still over-predicted the experimentally determined value for the critical Reynolds number, in the case of Blasius flow, by 25%.

Wazzan *et al.* (1974) using the Barry-Ross model, calculated the stability of Falkner-Skan flows and performed an analysis of the effects boundary-layer growth has on the Reynolds number, frequency and pressure gradient.

Bouthier (1972, 1973), Gaster (1974) and Saric and Nayfeh (1975) independently determined expansions that partially accounted for all non-parallel flow effects. Gaster (1974) considered the effect boundary-layer growth has on stability theory using an iterative method to generate an asymptotic series solution in terms of the inverse Reynolds number to the power one half. He obtained neutral-stability boundaries given by the first two terms of this series and compared the results with existing experimental data. Saric and Nayfeh (1975) used the method of multiple scales to analyse the spatial stability of two-dimensional incompressible boundary-layer flows, for both Blasius and Falkner-Skan profiles. It was found that for the Blasius flow, the non-parallel analytical results were in good agreement with the experimental data.

Gaster's neutral-stability curve calculations for the Blasius boundary layer were verified to be correct by Van Stijn and Van de Vooren (1983), and have the added advantage of being based on quantities that can be readily measured experimentally.

As previously mentioned, Smith's (1979a) triple-deck, asymptotic stability analysis is valid for both parallel and non-parallel flows. Bodonyi and Smith (1981) carried out an asymptotic analysis on the upper branch of the neutral stability curve using a quintic-deck analysis, which led to an asymptotic result for the neutral frequency, taking into consideration the effect of boundary-layer growth. In 1985, in an appendix to their paper, Smith and Burggraf considered examples of practical importance arising from non-parallel flow effects, for example, breakaway separation and flow over surface-mounted obstacles, in an asymptotic study based on a two-dimensional triple-deck.

1.2 Compressible Stability Theory

With the development of high speed flight vehicles interest soon turned to supersonic and hypersonic flows. Researchers tried to develop a stability theory for compressible flows, similar to that which had been developed for incompressible flows, and determine whether there was any relation between compressible stability theory and the important problem of transition to turbulence.

The major difference between incompressible and compressible boundary layers is that in the compressible case there will be an appreciable interchange of mechanical and thermal energies. Generally it is found in the case of supersonic boundary-layer flows that inviscid disturbances are more important (i.e. more unstable) than viscous disturbances. Here, we characterise inviscid disturbances as being those with wavelengths comparable to the boundary-layer thickness, whilst viscous disturbances possess much longer wavelengths; this is probably the broadest definition, although the alternative definition in which inviscid disturbances are characterised as having finite growth rates in the limit of large Reynolds number, is equivalent for most purposes. This is in contrast to the situation encountered in many incompressible boundary-layer flows where viscous instabilities are generally dominant.

Experiments performed by Laufer and Vrebalovich (1960) and Kendall (1967, 1975) demonstrated the existence of instability waves in supersonic and hypersonic boundary layers, but they were unable to show any real connection between linear instability and transition. A series of stability experiments with 'naturally' occuring transition in wind tunnels was carried out by Demetriades (1977) and Stetson *et al.* (1983, 1984), but many of their observations have yet to be explained theoretically. Dougherty and Fisher (1980) in a flight experiment obtained probably the best evidence yet that transition at supersonic speeds, in a low disturbance environment, is caused by linear instability.

One of the earliest attempts to develop a compressible stability theory was carried out by Küchemann (1938). In his work, Küchemann, neglected the effects of viscosity, curvature of the velocity profile, and the mean temperature gradient. As one would expect, the latter two assumptions were shown to be too restrictive to allow for any plausible argument to be developed. The most important early theoretical work on the stability of compressible flows, was carried out by Lees and Lin (1946). In their rigorous mathematical investigation of the stability of two-dimensional boundary layers to two-dimensional linear disturbances, they developed an asymptotic theory in close analogy to the incompressible asymptotic work of Lin (1945), and, in addition, gave detailed consideration to a purely inviscid theory. They concluded that for subsonic and slighly supersonic flow, stability characteristics are relatively unaffected by boundary conditions on temperature fluctuations, and are determined by satisfying velocity disturbance boundary conditions.

Lees and Lin (1946) solved the viscous problem using the two methods of solution of Heisenberg (1924), namely they determined solutions in terms of convergent series and asymptotic series. The initial approximation in the asymptotic series method gives the inviscid equation. It is found that the asymptotic solutions appear to be multi-valued, but the solutions are only valid for certain regions of the complex plane, determined by comparing them with the asymptotic expansions of the convergent solutions. Lees and Lin solved the inviscid equation in terms of a power series in α^2, the resultant integrations possessing integrands which are singular at the critical layer. Consequently the path of integration must be indented into the complex plane in the neighbourhood of the critical point. To determine the correct path of integration Lees and Lin obtained two linearly independent solutions in the vicinity of the singular point, $y^* = y_i^*$, one of which is regular in $(y^* - y_i^*)$, the other possessing a logarithmic singularity (where y^* denotes the coordinate normal to the surface). Since these solutions are restricted to the same regions of complex space as the asymptotic solutions, Lees and Lin determined that in passing from $Rl(y^* - y_i^*) < 0$ to $Rl(y^* - y_i^*) > 0$, the correct path lies below the critical point, $y^* = y_i^*$, where Rl denotes the real part. Because the second solution obtained in the neighbourhood of the critical point possesses a logarithmic singularity, this solution will undergo a phase change of '$+i\pi$' going from below the critical point to above. This singularity gives rise to strong velocity gradients and has the consequence that viscous (and conductivity) effects cannot be neglected in the critical layer.

Extending the Rayleigh/Tollmien theorems to compressible flow, Lees and Lin found that the quantity $\frac{\partial}{\partial y^*}[\rho^* \frac{\partial u^*}{\partial y^*}]$ (where u^* denotes velocity tangential to the surface, and ρ^* the fluid density) plays a role very similar to that of $\frac{\partial^2 u^*}{\partial y^{*2}}$ in inviscid incompressible theory. In particular, at the point where the above expression is zero ($y^* = y_i^*$), termed the generalized inflexion point, then there may exist a neutral mode with wavespeed $u^*(y_i^*)$; neutral modes are classed as being 'subsonic', 'sonic', or 'supersonic' depending on how the freestream Mach number is related to the wavespeed (Mack (1984)). If the neutral disturbance is subsonic then the mode decays in the far-field; supersonic, neutral disturbance modes exhibit an oscillatory behaviour in the far-field; a sonic mode occurs at the crossover point between subsonic and supersonic cases. Mathematically, these classifications are directly related to the non-dimensional wavespeed c (defined in Chapter 3 below), and the free-stream Mach number, M_∞. For subsonic disturbances we have

$$1 - \frac{1}{M_\infty} < c < 1 + \frac{1}{M_\infty},$$

for sonic disturbances we have

$$c = 1 - \frac{1}{M_\infty} \qquad \text{or} \qquad c = 1 + \frac{1}{M_\infty},$$

and for supersonic disturbances we have

$$c < 1 - \frac{1}{M_\infty} \qquad \text{or} \qquad c > 1 + \frac{1}{M_\infty}.$$

Arguments relating to generalized inflexion points have no implications for supersonic neutral modes.

If the critical point is found to coincide with a generalized inflexion point, the second velocity solution obtained by Lees and Lin (1946), valid in the critical layer, no longer possesses a logaritmic singularity and thus this solution is now regular.

Lees (1947) considered the effect that wall cooling has on the stability of compressible boundary layers on the basis of asymptotic theory. He predicted that with sufficient wall cooling the boundary layer could be completely stabilized and presented a criterion whereby the ratio of wall temperature to the recovery temperature at which the critical Reynolds number becomes infinite, can be computed. Even though Lee's original work contained numerical errors, subsequent authors including Van Driest (1952) and Dunn and Lin (1955) showed that Lee's predictions appeared to be correct.

Van Driest (1952) calculated the cooling required to completely stabilize the boundary layer on the flat plate at supersonic speeds with zero pressure gradient. Whereas Lee's investigations were limited to slightly supersonic flows, Van Driest predicated that complete stabilization was achieved by wall cooling over a wide range of Mach numbers up to hypersonic flow. He found, however, that for Mach numbers greater than 9, it is impossible to stabilize the boundary layer with any amount of cooling when a Prandtl number of 0.75 and the Sutherland viscosity-temperature law are assumed.

The above predications of Lees (1947) and Van Driest (1952) were based on the asymptotic theory of two-dimensional disturbances. Dunn and Lin (1955) extended this work to include three-dimensional disturbances. They determined that the conclusion of Lees and Lin regarding boundary conditions on temperature fluctuation terms was invalid for moderately high supersonic flows and in this Mach number range they indicated that cooling was indeed an effective method by which the boundary layer could be stabilized. Based on their asymptotic analysis, Dunn and Lin, however (wrongly) concluded that at supersonic free-stream Mach numbers the boundary layer can never be completely stabilized by cooling with respect to all three-dimensional disturbances.

Lees and Lin (1946) used an ordering procedure valid in the neighbourhood of the critical layer to obtain viscous solutions and then used these solutions to satisfy wall conditions. Such a procedure can only be sensibly valid if the critical layer is close to the wall. In the Dunn and Lin (1955) ordering procedure, the wall layer is assumed to be distinct from the critical layer, which leads to a set of reduced equations valid near the wall, but not necessarily valid at the critical layer.

In all of the above asymptotic compressible stability analysis, the authors assumed that the boundary layer was 'a nearly parallel flow'. In fact, Dunn (1953) and Cheng (1953) showed that the mean vertical velocity does not enter until the second asymptotic approximation to the viscous solution. Thus, if only the leading terms need to be considered the parallel flow approximation is a valid one.

In 1962 Lees and Reshotko presented a more accurate theoretical analysis; the ideas developed in this work are presented in more detail in the thesis of the junior author, Reshotko (1960). In their work, they considered two-dimensional disturbance terms in the two-dimensional boundary layer only, but they do include the effect of temperature fluctuations on viscosity and thermal conductivity and also introduced the viscous dissipation term that had been previously omitted.. Following the asymptotic expansion method of Heisenberg (1924) to solve the viscous disturbance equations, they obtained the inviscid equation in terms of the pressure fluctuation amplitude. They showed that instead of solving the inviscid equation in terms of a convergent series in powers of α^2, the correct expansion parameter for the compressible inviscid solutions is $(\alpha T_{\text{ref}})^2$ (where $T_{\text{ref}} \sim M_\infty^2 T_\infty^*$, T_∞^* representing the mean flow temperature in the far-field) or $(\alpha M_\infty^2)^2$. The most important result they obtained was to show that temperature fluctuations have a marked influence on the stability characteristics for compressible flow at Mach numbers greater than two, for both the viscous and the more slowly varying inviscid disturbances. Considering the behaviour of the inviscid disturbance terms in the neighbourhood of the critical point, by obtaining series solutions using the method of Frobenius, they determined that the temperature fluctuation terms possess an algebraic singularity at the critical point, which is independent of whether or not the profiles contained a generalized inflexion point. They also found the first indication of higher modes of inviscid instability, and determined that instead of being constant, as had previously been assumed, the inviscid pressure disturbance amplitude decreases abruptly with movement away from the wall for Mach numbers greater than three. The numerical examples given compared favourably with the experimental results of Laufer and Vrebalovich (1958, 1960) and Demetriades (1958, 1960).

Reshotko (1962) generalized the above analysis, in his study of the stability of three-dimensional boundary layer to three-dimensional disturbance terms. Introducing a suitable transform, he reduced the problem to a two-dimensional system.

Because of the close adherence of Lees and Lin to the incompressible theory and inadequacies of the asymptotic methods of Lees and Lin (1946), Dunn and Lin (1955) and Lees and Reshotko (1962), which turned out only to be valid up to low supersonic Mach numbers, major differences between incompressible and compressible stability analyses were not uncovered until extensive calculations had been carried out by numerical solution of the differential equations.

The first numerical calculation of normal mode eigenvalues directly from the viscous stability equations was carried out by Brown (1962). In a series of papers Mack (1963, 1964, 1965a, 1965b) obtained a large number of numerical results for different types of boundary layer. Until Mack's study it was widely thought that there only existed a single unstable inviscid mode. Through his extensive numerical work, Mack determined that for an insulated surface, if there exists a region of supersonic flow in the boundary layer relative to the phase velocity, then there exists multiple unstable modes.

The numerical schemes employed by Mack to solve both the viscous and inviscid boundary-layer problems, are explained clearly in his work published under reference 1965a. In this work he considered two-dimensional disturbances in a parallel compressible flow to obtain a system of linearized stability equations. The resulting sixth-order system of ordinary differential equations was then written as six first order equations to aid numerical integration. These equations were solved numerically in the free stream at a specified Reynolds number and the three independent solutions which decayed as $y^* \to \infty$ (normal component to the free surface) were used as the initial conditions for the numerical integration, which was taken from the edge of the boundary layer to the wall. At the wall, all but one of the homogeneous boundary conditions can be satisfied for an arbitrary choice of the complex wave parameters, α^*, β^*, ω^*, where α^* and β^* are the streamwise and spanwise wavenumber components and ω^* is the frequency. The remaining boundary condition was satisfied by a Newton-Raphson eigenvalue search procedure for one of the complex wave parameters. Considering the limit of infinite Reynolds number, Mack derived a system of equations analogous to those obtained by Lees and Reshotko (1962) which he then integrated numerically, deviating the contour of integration into the complex y-plane in the neighbourhood of the critical point, to deal with problems encountered due to the presence of this singularity. The indentation scheme used was based on a method developed by Zaat (1958) which Mack (1965a) generalized to include compressibility effects.

It soon became evident, however, that two-dimensional stability theory was inadequate to explain the observations of Laufer and Vrebalovich (1960), so the numerical method was extended by Mack (1969) to include three-dimensional normal modes. The viscous stability equations were written in the tilde-coordinates of Dunn and Lin (1955), forming an eighth-order system, but Mack managed to obtain good results from the eigenvalues of a sixth order system, formed when the single coupling term responsible for the increase in order was dropped. In the same work, Mack (1969) considered the effects of wall cooling on the corresponding inviscid boundary-layer problem. In the light of this study the predictions of Lees (1947) that cooling the fixed boundary acts to stabilize the boundary layer, was found to be slightly misleading. Mack found that in the case of the first mode of instability, even when oblique waves were considered, complete stabilization could be achieved with sufficent cooling for the Mach numbers he presented, which vindicated Lees (1947) predications. However, in the case of the second mode of instability, Mack observed that wall cooling had the reverse effect, causing this mode to undergo destabilization. Mack determined that the complete stabilization of mode I instabilities was a result of sufficient cooling causing the complete eradication of the subsonic generalized inflexion points. The higher modes of instability, being dependent only on a relative supersonic region, will remain.

Gill (1965) applied numerical techniques to consider the stability of jets or wakes in a compressible fluid for 'top-hat' velocity profiles, where the jet or wake is a region of uniform velocity separated on either side from the external flow by a vortex sheet. Like Mack's study of the compressible boundary layer, Gill (1965) determined that there existed multiple neutral solutions.

In further papers, Mack (1979a, 1982) applied compressible stability theory to sweptback wings. In 1984, Mack conducted a review of previous work carried out on the influence of Mach number on viscous and inviscid instabilities of flat plate boundary layers, and presented new spatial calculations. He concluded that viscosity only stabilizes two and three dimensional, first mode waves above a Mach number of 3.0, but stabilizes all mode II waves for all Mach numbers. In 1985, Mack presented a review of inviscid compressible stability theory paying particular attention to additional solutions that arise when there is a region of supersonic flow relative to the phase velocity. Mack gave an example of viscous multiple solutions, along with calculations of higher viscous modes and the compressible counterparts of the Squire mode.

In spite of their great practical importance, basic flows of the supersonic type, but involving lateral curvature, have received little attention. Duck and Hall (1989) showed how, in supersonic flows, curvature interacting with viscosity could provoke additional instabilities (axisymmetric in form), provided the body radius was below some critical value. Duck and Hall (1990) then went on to show how a similar effect occured with non-axisymmetric modes (which, in fact, turn out to be generally more unstable than corresponding axisymmetric modes.

Recently, Duck (1990) considered the effects that curvature has on the inviscid, axisymmetric linear stability of the boundary layer associated with supersonic flow past a thin circular cylinder. Duck determined that curvature has a stabilizing effect, causing the first mode of instability to ultimately disappear, and greatly reducing the amplification rates of the second mode. Extending the theorems of Lees and Lin (1946), Duck determined an inflexion condition that includes curvature terms and termed it the 'doubly generalized' inflexion condition.

In Chapters 2 and 3 of this thesis the work of Duck (1990) is extended to include non-axisymmetric disturbances which, indeed, turn out to be more important than axisymmetric disturbances considered previously. Further, rather than studying/applying our techniques to a thin straight circular cylinder, a somewhat more practical configuration, namely that of a sharp cone, is considered. In Chapter 2 the boundary-layer flow is determined, whilst in Chapter 3 a full linear stability anaylsis is conducted. Extensive numerical results are presented for both axisymmetric and non-axisymmetric disturbances in the compressible boundary layers formed on adiabatic, heated and cooled cylindrical surfaces and on adiabatic cones. At the end of Chapter 3 asymptotic studies valid for large azimuthal wavenumbers (for the cylinder only) and analyses valid close to the tip of the cone and far downstream from the cone tip are presented.

Mack (1987b) has performed some computations for the stability of the flow over a cone in supersonic flow, at finite Reynolds numbers, but found little difference with corresponding planar results. Here, we deliberately allow curvature to occur throughout the study, both in the equations governing the basic flow, and in the disturbance equations.

1.3 The Effect of Critical Layers

It is well known that for large Reynolds numbers, the Orr-Sommerfeld equation reduces in a singular manner to the Rayleigh equation. In addition to the necessity of wall boundary layers if no slip boundary conditions are assumed, the Rayleigh equation has a singular point, termed the critical point, anywhere in the boundary layer where the mean flow velocity is equivalent to the wavespeed of the disturbance. Consequently the dynamics in the region surrounding the critical point will be expected to differ considerably from those of other regions of the fluid. By the method of Frobenius, two solutions of the Rayleigh equation can be obtained in the neighbourhood of the critical point, but it is found that in the case of non-inflexional profiles the second solution contains a logarithmic singularity of the form $\log(y^* - y_c^*)$ (where y^* denotes the coordinate normal to the surface and y_c^* the critical point). To obtain any form of neutral stability curve, connection formulae relating the solutions either side of the critical layer must be determined. Another consequence of the logarithmic singularity is that the eigenvalue problem associated with the inviscid equation cannot be resolved until it is decided how to express $\log(y^* - y_c^*)$ for $y^* < y_c^*$.

Essentially two theories exist to tackle the difficulties of the singularity arising in the linearized inviscid problem. By re-introducing viscous terms in a small region around the critical point of thickness $Re^{-1/3}$ (where Re represents the Reynolds number) - termed the viscous critical layer - Heisenberg (1924), Tollmien (1929), and Lin (1944, 1945) for incompressible flows and Lees and Lin (1946) and Lees and Reskohto (1962) for compressible flows, determined that the logarithmic term undergoes a jump of '$+i\pi$' crossing the critical layer (where we are going from below to above the critical layer). The correct branch of the logarithmic term, in viscous theory, is determined by the range of validity of the Hankel functions found in the viscous solutions.

The possibility of an alternative resolution was first mentioned by Lin and Benney (1962). In seperate work, Benney and Bergeron (1969) and Davis (1969) observed that the Rayleigh equation was in fact the result of two limiting processes, as opposed to just one. They noted that although the Reynolds number is large, the stability analysis equations have actually been linearized, insuring the disturbance amplitudes are small. Consequently, they suggested that in the neighbourhood of the critical layer nonlinear terms could be retained as opposed to viscous terms, to resolve the singularity problem. The major result of this analysis is that in nonlinear theory the logarithmic phase shift has vanished crossing the critical layer.

Considering two-dimensional disturbance terms, Benney and Bergeron (1969) re-introduced nonlinear terms in a critical layer of thickness $O(\epsilon^{1/2})$, where ϵ is a measure of the perturbation amplitude, and determined that nonlinear theory yields an important class of wave-like solutions not found in viscous theory. Assuming that the critical layer and the wall layer were asymptotically distinct, they noted that though viscosity is small, it could not be completely ignored. It was found that the relative importance of nonlinear to viscous effects is measured by the parameter $\lambda = Re^{-1}\epsilon^{-3/2}$. Making use of a viscous secularity condition to ensure spatial periodicity, together with matching conditions, Benney and Bergeron determined the dominant structure in the nonlinear critical layer is the Kelvin cat's eyes solution, namely a shear plus an oscillation. In the case of non-inflexional profiles, discontinuities in vorticity occur at the cat's eye boundaries, which Benney and Bergeron treated by restoring viscosity in thin layers around the cell boundaries. Even though they found vorticity was discontinuous, Benney and Bergeron derived conditions to ensure the velocity is continuous and from these conditions and matching with the far-field they determined that the phase change across the critical layer is zero. They also briefly considered the possibility of more than one critical point existing and the case of oblique modes. Benney and Bergeron's work only determined neutral waves - it was unclear from their work whether 'near neutral' solutions would be stable or unstable - and they computed $c(\alpha)$ for a variety of velocity profiles.

As noted above, the nonlinear and viscous theories result in different neutral modes. Haberman (1972) considering critical layer effects in parallel flows for two specific problems - symmetric flows between rigid walls and boundary-layer flows - linked the two theories. He showed the leading order flow is indeed the cat's eyes pattern, and at the order which governs the jump conditions across the critical layer he included both viscous and nonlinear terms in his analysis, considering the full range of the parameter λ. Haberman determined that the asymptotic expansions of Benney and Bergeron (1969) should be modified due to an $O(\epsilon^{1/2})$ distortion of the mean and fundamental harmonic. This distortion of the mean flow, in the limit when nonlinear effects are dominant, results in both the velocity and vorticity being continuous across the cat's eyes, yielding Haberman to conclude that thin viscous layers, as introduced by Benney and Bergeron (1969), are not necessary. Haberman also determined that provided the critical layer and the wall boundary layer are distinct, then the phase shift of the logarithmic term depends on the local vertical Reynolds number in the critical layer, varying monotonically from the value of 0 in nonlinear theory to $+i\pi$ as determined by viscous theory. Haberman provided an argument by which his analysis linking the two theories is valid for long wave neutral modes and shorter modes if the critical and wall layers are distinct. In the case when the two layers are indistinguishable, the fully nonlinear boundary-layer equations must be considered.

During the late 70's and early 80's the idea of including nonlinearity in critical layer analysis was used by a number of authors to determine evolution equations for small disturbance terms in a range of problems. Benney and Maslowe (1975) and Huerre and Scott (1980) applied the technique to the problem of homogeneous shear flows, while Redekopp (1977), Maslowe and Redekopp (1979, 1980) and Stewartson (1981) considered disturbances with large horizontal scale to flows where critical layers are present. Stewartson (1978, 1981), Brown and Stewartson (1978a, 1980, 1982a, b) and Warn and Warn (1978) considered forced disturbances in flows which contained critical layers. The problem of stratified shear flows was studied by Brown and Stewartson (1978b), and Hickernell (1984) studied the effects of nonlinearity in the critical layers in shear flows on the beta-plane of a Rossby wave.

The work of Benney and Bergeron (1969) and Haberman (1972) deals exclusively with steady waves, for which at most the time dependence involves simple translations of the wave in the direction of propagation. Benney and Maslowe (1975) considered extending the nonlinear critical-layer analysis to include time dependence.

Prior to their study a time-dependent finite amplitude analysis existed which was valid for $\lambda >> 1$, i.e. a viscous critical layer, namely the weakly nonlinear Stuart-Watson theory (1960), but no such amplitude equation exists for nonlinear critical layers ($\lambda << 1$). Treating a wave that evolves both spatially and temporally, Benney and Maslowe applied the technique of multiple scales to obtain an amplitude equation valid for $\lambda << 1$, and determined that to the order considered no phase change occurs across the critical layer. If nonlinear effects are to produce any phase change, then the slow time dependence must have some relation to the viscous scale. In their analysis, since their inhomogeneous operator equation is singular at the critical level, they employed a modified solvability condition (which involves the correct matching conditions at the boundaries) to determine the Landau constant. Based on their analysis, they concluded that in order to obtain instability either the effects of viscosity at a lower order or alternately a stronger time dependence must be employed in the theory.

Using the temporal nonlinear stability approach of Schade (1964), Huerre (1980) considered the temporal and spatial evolution of weakly amplified waves in shear flow, considering a critical layer where viscosity is incorporated to smooth the singularity. To successively apply the method of matched asymptotic expansions, he determined that the effect of viscosity could not be neglected in the outer layer, resulting in mean flow distortion. To counteract viscous diffusion of the basic flow, he found it necessary to apply an artificial body force. Huerre determined that for large Reynolds numbers and $\lambda >> 1$, the weakly amplified waves do not approach an equilibrium amplitude as time evolves or with movement downstream. He concluded that this instability was not a result of introducing the artificial body force, since in all previous studies this was implicitly present, and consequently for sufficiently small amplitudes the waves will not be stabilized by weakly nonlinear interactions, but as the wave amplitude grows, λ becomes smaller and nonlinear effects become important in the critical layer.

In light of Huerre's work (1980), Huerre and Scott (1980) considered the case where both viscous and nonlinear effects are important in the critical layer for the same problem. They derived an amplitude equation representative of these combined effects which they also determined to be dependent upon the phase shift in the logarithmic singularity, as determined in the work of Haberman (1972). Huerre and Scott observed that growth of the instability waves results in the critical layer gradually spreading, the thickness varying as the square of the amplitude. They also determined that wave amplification increases the importance of nonlinear effects (as noted by Huerre (1980)) which in turn, for the particular case of spatial growth, causes the wave growth rates to gradually decrease resulting in the high amplitude fluctuations only having algebraic growth. They note, however, that these temporal and spatial changes will occur slowly when compared to the time scale of Stewartson (1978) and Warn and Warn (1978), resulting in their quasi-steady approach being self-consistent.

Warn and Warn (1978) considered the evolution of inviscid Rossby waves on a parallel flow in the presence of a critical layer, whose source corresponds to a switch-on forcing at a lateral boundary at time, $t = 0$, and they determined that for earlier times the waves will be governed by linear inviscid stability theory. For all finite times they assumed that a layer of transient fluid exists in the neighbourhood of the critical point which diminishes in thickness as time increases; note that the outer solution will be steady. By $t = O(\epsilon^{-1/2})$ the transient layer thickness is reduced to $O(\epsilon^{1/2})$ and nonlinearity is found to be significant at leading order in the critical layer. Warn and Warn determined that at this time, the regular expansion in ϵ from which the linear inviscid equations are obtained, is now non-uniform in t, yielding a situation similar to the steady nonlinear solutions of Benney and Bergeron (1969) and Haberman (1972). Thus, the domain of validity of the expansion must be extended to the nonlinear regime. This is achieved using a combination of the methods of multiple scales and matched asymptotic expansions, resulting in a nonlinear critical layer which when solved numerically yields the connection formulae as functions of time.

In his study of the finite amplitude free disturbances of an inviscid shear flow on the beta-plane of a Rossby wave, Hickernell (1984) applied perturbation theory and the method of matched asymptotics to obtain an evolution amplitude equation of a singular neutral mode of the Kuo equation. Hickernell's critical-layer analysis included the effects of time-dependence, nonlinearity and viscosity and he determined that as time evolves the effects of small viscosity and nonlinearity become important much earlier inside the critical layer, than outside. Three distinguishing features of Hickernell's analysis are that his vorticity equation depends explicitly on time, the flow inside the critical layer is determined to be weakly nonlinear in that nonlinear terms enter as non-homogeneities, and nonlinear effects are stronger inside the critical layer than outside. Hickernell determined a governing equation for the evolution of the amplitude in which the nonlinear term is a type of convolution integral as opposed to a simple polynomial. He postulated that the nonlinearity is of this form because the equation for the critical-layer flow is first order in time and nonhomogeneous. Hickernell also stated that any problem where the critical layer is described by a first or higher order differential equation in time and where nonlinear interactions are stronger within the critical layer than outside, will possess a similar form of singularity.

Over recent years research with regard to nonlinear critical layers and their effects on the stability of flows has been carried out by three main groups, who we shall term the 'Goldstein' group, the 'Gajjar' group and the 'Russian' group.

Goldstein and his group have studied the effects of a nonlinear critical layer for a number of different problems - spatial growth of Tollmien-Schlichting waves (Goldstein and Durbin (1986)); the roll-up of vorticity in adverse-pressure-gradient boundary layers (Goldstein, Durbin and Leib (1987)); spatial evolution of waves on shear layers (Goldstein and Hultgren (1988), Goldstein and Leib (1988, 1989), Goldstein and Choi (1989) and Leib (1991)); and spatial evolution of waves on hypersonic boundary layers (Goldstein and Wundrow (1990)).

The first work of this type carried out by Goldstein and his group was conducted by Goldstein and Durbin (1986). Considering the effects of a nonlinear viscous critical layer on the spatial growth of a time harmonic Tollmien- Schlichting wave, they determined that nonlinearity acted to alter the linear disturbance terms through its effect on the instabilities phase jump across the critical layer. This phase jump could be determined from the Haberman result, provided the Haberman parameter was interpreted correctly. Thus they determined that nonlinearity drives the phase jump to zero. Nonlinearity was also found to eliminate the upper branch of the neutral stability curve in the Blasius boundary layer.

Goldstein *et al.* (1987) considered the mutual effects of critical-layer nonlinearity and adverse-pressure-gradients on the spatial growth of time periodic inviscid instability waves in boundary-layer flows. Adjusting the appropriate scalings on the pressure gradient and the instability wave amplitude to ensure that the growth rate and nonlinear terms occuring within the critical-layer vorticity equation are of the same order of magnitude, the critical layer is found to be both nonlinear and unsteady. Matching the outer and critical layers, they determined that the instability wave amplitude now appears as a variable coefficient rather than as a parameter, as occurs in the Haberman problem. Consequently the critical-layer vorticity equation must be solved simultaneously with the external instability wave amplitude equation. They achieved this by employing a spectral method to solve the system numerically. Goldstein *et al.* find that even though the critical-layer dynamics are quite different from that determined by Goldstein and Durbin (1986), nonlinearity still causes the scaled velocity jump to be driven toward zero. They also determined that nonlinearity causes the instability wave to be ultimately stabilized, even though the adverse pressure gradient is strong enough to cause the linear wave to grow indefinitely. This is in constrast to the model studied by Goldstein and Durbin, who determined that nonlinearity caused the indefinite growth of the disturbance term for the Blasius boundary layer. The reason for the difference they concluded lay in the sign of the vorticity gradient at the critical layer; a positive value causes the phase jump to result in unstable growth whilst a negative value causes stabilization. Since nonlinearity drives the phase jump to zero in both cases, anyway, ultimately both effects will be eliminated if nonlinear effects are sufficiently large.

The first work conducted on shear layers by the Goldstein group was carried out by Goldstein and Leib (1987). In an effort to account for the 'roll-up' of shear layers in the neighbourhood of the linear stability point, Goldstein and Leib developed a nonlinear solution which is valid in the neighbourhood of the linear stability point, and which they also required to match onto the upstream linear (but weakly non-parallel) instability wave solution. For this to be achieved, they noted that there must exist an overlap domain where the two solutions could match correctly in an asymptotic sense. Because of the necessity of introducing an artificial body force term to counteract viscous spreading in the Stuart (1960) - Watson (1960) - Landau theory approach (where nonlinear terms are introduced by means of a multiple scales method), Goldstein and Leib instead opted for the nonlinear critical-layer approach. Considering the position where the local Strouhal number (or normalized frequency) differs from its neutral value by $O(\epsilon^{1/2})$, they derived an inviscid critical-layer vorticity equation which contained both nonlinear and non-equilibrium terms, and this choice of scaling also allowed for the correct matching with the upstream linear instability wave. Comparing their work with Robinson's (1974), they noted that since Robinson considered the case where the normalized frequency differed from the neutral value by $O(\epsilon)$, then his solution will not match with the strictly linear solution. Solving the critical layer vorticity equation, which represents a balance of spatial evolution and (linear and nonlinear) convection terms, by means of a spectral decomposition, they determined that nonlinearity in this case acts to reduce the growth rates to zero over a very short streamwise distance, well upstream of the linear neutral stability point, but with the final instability wave amplitude oscillating about a finite non-zero value. They could not determine whether or not a final equilibrium solution was reached, although the numerical observations suggested that the critical-layer vorticity equation continues to develop smaller and smaller lengthscales. Goldstein and Leib noted the close similarity between their results and the related Rossby wave critical-layer solution obtained by Stewartson (1978) and Warn and Warn (1978).

Based on the observations of Stewartson (1978, 1981) that even a very small amount of viscosity causes the nonlinear critical-layer Rossby wave solutions to evolve into an equilibrium critical layer of the Benney and Bergeron (1969) type, Goldstein and Hultgren (1988) introduced a small amount of viscosity into the work of Goldstein and Leib (1987). They assumed that viscous effects are of the same order as the spatial evolution and nonlinear convection terms, noting that outside the critical layer viscous effects are still unimportant. Consequently, the lowest order growth rate that they determined ($O(\epsilon^{1/2})$) is much larger than the corresponding equilibrium critical-layer ($O(\epsilon^{3/2})$) solution as determined by Huerre and Scott (1980). As had been determined previously by the Goldstein group, Goldstein and Hultgren noted that since the instability growth rates are proportional to the phase jump across the critical layer and nonlinearity drives the phase jump to zero, then clearly nonlinearity can be seen to force the growth rates to zero, i.e. cause stabilization. However, since viscosity prevents the phase jump from being eradicated entirely, then the growth rates of the disturbance increase asymptotically downstream. They determined that even though this growth is weak - algebraic now as opposed to exponential - nonlinear terms within the critical layer can become unbounded, resulting in a new dominant critical layer balance between linear and nonlinear convection terms. On the face of it, it appears that Goldstein and Hultgren have arrived at the Benney and Bergeron (1969) state. However, through careful analysis, Goldstein and Hultgren demonstrated that because of rapid spatial development, viscosity is not given a chance to fully act on the flow. Consequently, the vorticity in the closed streamline region within the cat's eye boundary is variable. Making use of a generalized Prandtl-Batchelor theorem, obtained from their non-equilibrium critical-layer vorticity equation, they showed how singular eigensolutions can be precluded, while there is variable vorticity at the cat's eye. Since the instability wave is now growing slowly, the mean flow will diverge noticeably, resulting in the critical-layer structure being altered and the $O(1)$ amplitude instability never being reached. Goldstein and Hultgren determined that mean-flow spreading ultimately dominates nonlinearity, forcing the growth rates towards zero, and then the wave begins to decay. Nonlinearity is found not to effect the location of the neutral stability point.

In his linear work on the effects of compressiblity in shear layers, Gropengeisser (1969) determined that oblique modes grow faster than two-dimensional modes. He also determined that due to calculated reductions in the linear growth rates as the Mach number is increased, then for supersonic flows, nonlinear critical layers have an increased importance. Consequently, Goldstein and Leib (1989) extended their incompressible work into the compressible regime and treated the more general case of three-dimensional disturbances, although to simplify their analysis they employed Squire transforms. They determined that the inclusion of compressibility causes critical-layer nonlinearity to behave quite differently. The major reason for this difference is that the temperature disturbance terms possess an algebraic singularity in the critical layer (as first noted by Reshotko (1960)) resulting in these terms being very large relative to velocity components. Consequently, critical-layer nonlinearity is found to occur at much smaller amplitudes than in analogous subsonic flows. Goldstein and Leib determined that nonlinearity effects will first become important when the instability wave growth rate is $O(\epsilon^{2/5})$. As a result, the critical-layer flow is governed by linear dynamics to lowest order, nonlinearity effects only entering through the higher-order (inhomogeneous) terms. They showed that the instability wave amplitude is governed by an integro-differential equation, similar to that derived by Hickernell (1984), the coefficients of which are determined numerically from the linear solution. Solving the inviscid amplitude equation numerically, Goldstein and Leib determined that it terminates in a singularity at a finite downstream distance. The reason why a blow up occurs in the compressible case is because the Hickernell type amplitude equation is a form of convolution integral, implying that history effects are important. It is these cumulative history effects which eventually cause the amplitude to grow and terminate in a singularity. Restoring viscosity in the critical layer only (although taking into account possible mean-flow spreading over this lengthscale), Goldstein and Leib determined a generalized amplitude evolution equation where the additional viscous effects are contained solely within an exponential factor whose argument is always negative. They determined that the viscous amplitude equation admits an equilibrium solution if the amplitude equation coefficients lie in certain regions of determined parameter space. In these parameter space regions it is found that the history effects of the convolution integral are damped by the exponential term, resulting in the solution being a more local one.

In the above work, the unsteady flow was assumed to evolve from a single oblique mode growing in its propagation direction, thus allowing Squire transforms to be employed. Goldstein and Choi (1989), considering the same overall problem, instead assumed an initial instability wave growing in the streamwise direction. In order to represent a fixed spanwise structure, they suppose that there are two oblique modes with the same frequency and steamwise wavenumber but with equal and opposite (real) spanwise wavenumbers. It is found that cross-flow velocity fluctuations, which possess the same form of algebraic singularity in the critical layer as the temperature disturbance terms, couple with the velocity fluctuations in the plane of the wave, causing critical-layer nonlinearity to be more important at smaller amplitudes than in the corresponding incompressible two-dimensional case. Goldstein and Choi also observed that nonlinear oblique-mode interaction causes nonlinear critical layer effects to occur at even smaller amplitudes than in the single mode compressible case - nonlinearity becomes important when the instability wave growth rate is $O(\epsilon^{1/3})$ as opposed to $O(\epsilon^{2/5})$, as explained in the preceeding paragraph. Because the nonlinearity in this case is due to oblique mode interaction, the nonlinear critical-layer dynamics will be unaffected by compressibility effects, and consequently Goldstein and Choi conducted their analysis for the incompressible case only. They obtained an integro-differential equation with a cubic nonlinearity governing the instability amplitude, similar to that obtained by Goldstein and Leib (1989) (although the structure of the nonlinear kernel function is somewhat different), which is valid also for supersonic shear layers. As in the single oblique mode case, nonlinearity causes the amplitude to grow rapidly, terminating in a finite downstream distance.

In their analysis Goldstein and Leib (1989) considered oblique subsonic modes where the critical point coincides with a generalized inflexion point, thus allowing them to assume the Lees and Lin (1946) generalized inflexion condition. This results in the critical point being a regular singular point for the compressible Rayleigh equation. Leib (1991) generalized the above analysis to the case of supersonic modes, where it is found that a generalized inflexion condition can no longer be assumed, resulting in the outer Frobenius solutions containing logarithmic terms. Leib also removed the restrictions of unit Prandtl number and linear viscosity-temperature relation, as assumed by Goldstein and Leib (1989), and derived an integral condition for the coefficients appearing in the amplitude equation based on the modified solvability condition of Redekopp (1977). Leib conducted his analysis for a non-equilibrium nonlinear viscous critical layer and determined a Hickernell type amplitude evolution equation similar to that corresponding to Goldstein and Leib's (1989) generalized amplitude evolution equation, and containing terms that accounted for all the subsequent generalizations made by Leib. As before, Leib determined that all the inviscid solutions terminate in a singularity at a finite downstream location. In the case of the viscous solutions, he derived a necessary condition for the existence of an equilibrium solution, which is found to depend upon the Prandtl number, viscosity law, the viscous parameter and a real parameter derived from linear inviscid stability theory. From numerical observations, Leib determined that an equilibrium solution could not be achieved for subsonic modes unless the temperature ratio of the low-to high-speed streams exceeds a critical value, whilst in the case of the most rapidly growing supersonic modes, equilibrium solutions exist over most of the parameter range studied.

In their study of the spatial evolution of nonlinear acoustic mode instabilities on the hypersonic boundary layer, Goldstein and Wundrow (1990) determined that nonlinearity is important when the amplitude of the pressure disturbance terms is $O(1/M_\infty^4 \ln M_\infty^4)$, where M_∞ denotes the freestream Mach number. The linear inviscid disturbance terms outside the critical layer are found by extending the asymptotic analysis of Cowley and Hall (1990) into the nonlinear regime. Goldstein and Wundrow determined that this flow has a triple-layer structure, while the critical layer is contained in an adjacent outer layer, which they termed the edge layer. The resultant critical-layer nonlinearity is found to be strong in that it enters through a coefficient in the lowest order equations (similar to the incompressible work of Goldstein *et al.* (1987)) and by employing a variable change, Goldstein and Wundrow were able to express the critical-layer vorticity equation (along with the energy equation) in a similar form as that determined by the Goldstein group in the incompressible shear-layer case. The coupled set of nonlinear critical layer equations was then solved using a numerical method based on the method of characteristics. Unlike the incompressible work of Goldstein *et al.* (1987), it is found here that when nonlinear effects first become important the instability wave growth rates continue to increase, which is attributable to the effects of compressibility - as in the work of Goldstein and Leib (1989). As the amplitude continues to grow, however, transverse convection effects eventually become dominant, resulting in the growth rates decreasing and eventually oscillating about zero. This is similar to the incompressible work of Goldstein *et al.*, where again critical-layer vorticity roll-up generates smaller and smaller lengthscales resulting in viscous effects becoming important. Goldstein and Wundrow also noted that transverse convection effects must be strong enough to counter the growth enhancing features of compressibility, before the singularity of Goldstein and Leib (1989) (and Leib (1991)) is encountered. They concluded that the vorticity roll-up of this fully nonlinear solution must be strong enough to reverse the growth build up of the weakly nonlinear compressible theory before the singularity has a chance to form.

Making use of self-consistent asymptotic methods based on multi-deck ideas, Gajjar and co-workers have been successful in introducing nonlinear effects due to the presence of a critical layer, into a number of problems from the mid-eighties onwards - Bodonyi, Smith and Gajjar (1983), Gajjar and Smith (1983), Bassom and Gajjar (1988), Gajjar and Cole (1989) and Gajjar (1991a, 1991b).

In their work, Gajjar and Smith (1985) considered the problem of global nonlinear growth/decay, from both a spatial and temporal approach, of an unsteady non-neutral, small disturbance in the presence of a nonlinear viscous critical layer. The problem they treated may be regarded as global in the sense that previous studies assumed that outside the critical-layer travelling waves are neutral, with the nonlinear growth or decay being treated with respect to motion within the critical layer only - this is akin to a fixed critical layer. Gajjar and Smith removed this restriction, considering instead a disturbance whose amplitude varies over the whole flowfield. Considering three types of basic flow - steady quasi-parallel channel flow, boundary-layer flow and liquid-layer flow - at high Reynolds number, i.e. the flows considered have small but non-zero viscosities, they determined that assuming unsteadiness is important in the problem, then the effects due to the slowly moving critical layer (which moves to counter the divergence effect of viscosity) are more important, generally, with regard to the evolution of the instability wave, than corresponding effects developed within the critical layer due to unsteadiness factor. This is because the critical layer being considered in this regime is relatively thin and consequently the actual movement of this critical layer causes changes to the internal flow properties which are larger than those induced by the internal unsteadiness. Coupled with this slow movement of the critical layer, instability wave amplitudes are found to respond nonlinearly on faster space and time scales, both inside and outside the critical layer, i.e. physically speaking, the slow movement of the critical layer forces the disturbance to vary on much faster scales with respect to time or space. For the special case of fixed frequency disturbances, Gajjar and Smith determined that for initial disturbances whose amplitude is either above or below a certain subcritical threshold value, then these waves will be amplified/stabilized, respectively, by nonlinear effects at later times, further downstream. In the case of amplification, the instability waves become unbounded until they are governed by a new subsequent structure. For the case of the general moving time-dependent nonlinear critical layer, unlike the corresponding fixed equation, there is no significant jump in the mean vorticity across the critical layer possible, implying that no large mean flow disturbances can be induced outside.

In the incompressible shear layer problem treated by Goldstein and Hultgren (1988), the final region they study where both critical-layer nonlinearity and viscous spreading of the mean flow are important, they observed that this region corresponds to a regime similar to that studied by Gajjar and Smith (1985), where their critical layer moves across the shear layer to maintain the quasi-equilibrium state against changes in mean flow. Of course both effects of viscosity and nonlinearity see the ultimate downfall of the wave growth rates.

In dynamical situations where cross flow vortices arise, there exists the possibility of more than one critical point occuring in the basic profile. In their study of the stability of non-stationary cross flow vortices in three-dimensional boundary-layer flows, Bassom and Gajjar (1988) assumed that the basic flow was modelled by the Von-Kármán solution. The Von-Kármán solution only allows two critical points at most, but Bassom and Gajjar stated that their results are easily generalized to the Ekman boundary-layer flows solution, where an infinite number of critical points may exist (Lilly (1966)). Starting with the linear theory, Bassom and Gajjar demonstrated that if both critical points are present, then one will exist very close to the wall while the other occurs in the main part of the boundary layer, and balancing the critical layer jumps with the Stokes' layer shift yields a eigensolution for neutral modes. They also showed that the linear non-stationary modes they considered only exist for a limited range of wave numbers between 10.6^o and 39.6^o, but on including nonlinearity the wave angle range is significantly increased, since nonlinearity results in the lower limit now being amplitude dependent.

Considering the stability of compressible boundary layers for the specific examples of a pressure gradient boundary layer (subject to both heat transfer and insulated wall gradients) and the Blasius boundary layer (subject to insulated wall conditions only), Gajjar and Cole (1989), noting that in the compressible case inviscid disturbances are generally more important, conducted a multi-deck asymptotic study for upper-branch stability. They conducted the work for neutral states only, and hence equilibrium critical layers, but noted that their work provides the basis for a study with respect to growing modes, as conducted by Gajjar and Smith (1985) and, of course, the Goldstein group. After determining the linear structure, they introduced nonlinear terms through the action of a nonlinear viscous critical layer. Through their analysis, they derived a nonlinear viscous compressible critical equation for the neutral modes, noting that it differs from the incompressible Haberman (1972) result due to the addition of a forcing term resulting from large density fluctuations in the neighbourhood of the critical layer. It is found, however, if the fixed boundary is subject to insulated conditions then the problem has to be rescaled. The dominance of the density disturbance terms (and temperature instability terms) in the neighbourhood of the critical layer is a result of introducing compressibility into the critical layer, of course, and is akin to the observations of Reshotko (1960, 1962) and Goldstein and Leib (1989) regarding temperature disturbance terms. Through their analysis, Gajjar and Cole determined that another effect of compressibility is that the phase shift across the critical layer becomes positive for certain parameter values (where Gajjar and Cole considered crossing the critical layer from above to below), which they determined. Considering the limit of when the critical layer becomes strongly nonlinear they noted the necessity of viscous critical layers in the cat's eye boundaries due to discontinuities in the vorticity gradient, which are found to be even stronger in this case due to heat transfer.

Another problem to which Gajjar (1991b) has applied nonlinear critical layer theory to is the problem of the nonlinear evolution of Travelling Wave Flutter (TWF) modes in the boundary-layer flow over isotropic compliant surfaces. Using the linear work of Carpenter and Gajjar (1990), Gajjar (1991b) introduced nonlinear effects by means of an unsteady nonlinear critical layer. He obtained two equations governing the amplitude evolution of the TWF mode which are very similar to equations determined by Goldstein and Wundrow (1990) in their work on the hypersonic boundary layer. The main result of Gajjar's analysis is that nonlinearity drives the jump in the Reynolds stress across the critical layer to zero, and since this is directly related to the instability wave's growth rates, then the growth rates of the TWF mode are reduced as it evolves downstream resulting in the roll up of vorticity within the critical layer and the generation of harmonics. Unlike the shear flow results, however, the wave amplitude oscillates about a non-zero value implying that the wave amplitude is still growing. Gajjar noted that the path to vorticity roll-up is the same as that described by Goldstein and Leib (1987) and Goldstein and Hultgren (1988), although he points out that their work is in terms of transformed coordinates and it is found that the picture is somewhat distorted in terms of physical coordinates. However, regions of thin and intense shear layers will still be present.

Applying multi-deck theory to the problem of the nonlinear evolution of slowly growing modes in the compressible boundary layer, Gajjar (1991a) obtained a pair of coupled unsteady nonlinear equations that govern the amplitude evolution, which again closely resemble those obtained by Goldstein and Wundow (1990) for the hypersonic limit. Examining the linear growth rate he noted how wall heating can destabilize the boundary layer. Carrying out a preliminary numerical study, Gajjar determined that the nonlinear growth rates behaviour is dependent upon whether the wall conditions are heated, cooled or adiabatic.

Over the last few years a number of Russians have been considering the effect of nonlinear critical layers on fluid dynamical problems. Churilov and Shukhman (1987, 1988) and Troitskaya (1991) have considered the nonlinear stability of a weakly supercritical shear flow with vertical temperature stratification, whilst Shukhman (1989, 1991) has considered nonlinear effects on the stability of the shear layer in a rotating fluid for both the incompressible (1989) and compressible (1991) cases.

In the approach adopted by Churilov and Shukhman (1987), they specified the relevant scales for viscous, unsteady and nonlinear effects to be individually important in the critical layer and noted that the critical layer will be characterised by which of these scales is the largest. They presented a diagram of the various critical-layer regimes on the amplitude-supercriticality plane at a fixed Reynolds number, showing clearly the viscous, unsteady and nonlinear regions, with the relevant boundaries marked showing where more than one effect will be important. Considering the case of a viscous critical layer and assuming that the instability wave amplitude and supercriticality are small enough, Churilov and Shukhman (1987) determined an evolution equation which is governed by the Landau equation (Landau and Lifshitz (1959)), and showed that higher order amplitude terms will only be important in very narrow regions in the neighbourhood of where the Prandtl number, σ, is one. Through interactions of harmonics in the critical layer, they determined that the Landau constant is directly proportional to the Reynolds number, changing its sign when $\sigma = 1$. Nonlinearity is found to substantially affect the disturbance terms causing them to be stabilized for $\sigma < 1$, while for $\sigma > 1$ it causes destabilization.

In a second paper, Churilov and Shukhman (1988) considered the case where the critical layer is unsteady as opposed to viscous. They determined that, with respect to their critical layer region diagram, a viscous critical layer is adjacent only to a region of an unsteady critical layer, so therefore unlike the situation described in their first paper, a nonlinear critical layer would not exist just above a viscous critical layer, in their diagram. Through their analysis, Churilov and Shukhman derived an evolution equation which is a form of integro-differential equation possessing both cubic and quintic nonlinearity terms. They noted that Hickernell (1984) equations (although, as already stated above, Hickernell's equation only possesses a cubic nonlinearity term) of this type will arise in situations with unsteady critical layers where nonlinearity is competitive. Churilov and Shukhman showed that if the amplitude of the wave is small then the cubic term will dominate, but if it becomes of order the inverse of the square root of the Reynolds number, then the quintic term dominates resulting in explosive growth of the wave. This explosive growth is attributable to the convolution integral form of the nonlinear terms, where of course past histories will matter, and eventually accumulate.

Considering the problem of the nonlinear stability of a weakly supercritical mixing layer in a rotating fluid, Shukhman (1989) conducted a study of the waves' nonlinear evolution for the different critical-layer regimes of viscous, nonlinear and unsteady scalings. Extending these theories into the compressible regime, Shukhman (1991) conducted a study of the nonlinear evolution of spiral density waves generated by the instability of the shear layer in a rotating compressible fluid, with particular application to the problem of the structure of spiral galaxies. Considering disturbances which he regarded as acoustic waves in the far-field, it is found necessary to impose a far-field radiation boundary condition. Also, because of the form of the disturbances he treated, the critical point no longer coincides with a compressible inflexion point as in the work of Goldstein and Leib (1989). Consequently, the outer solutions contain logarithmic terms which will be singular at the critical point at the same ordering as those determined by Leib (1991). However, because of the particular model being treated, Shukhman assumed that the temperature terms will be homogeneous, i.e. $T_0'(y) = 0$, where T_0 represents the temperature field, resulting in a much simplified energy equation. A more major consequence of this assumption is Shukhman's temperature disturbance term will not be singular at the critical point. In Shukhman's case it is the logarithmic terms which determine the critical-layer dynamics, resulting in quite a different critical-layer structure from that determined by Goldstein and Leib (1989) (and Leib (1991)). Remarkably, however, the resultant evolution equation governing the instability wave amplitude possesses a nonlinear term which only differs from that determined by Goldstein and Leib (1989) in the form of the coefficient term. Considering seperately the cases of a viscous critical layer and an unsteady critical layer, Shukhman provided conditions on the amplitude for explosive growth for both cases.

Troitskaya (1991) considered the problem of a viscous-diffusion nonlinear critical layer in a stratified shear flow. Considering stationary finite amplitude wave disturbances, he introduced nonlinearity through the action of a critical layer whose structure depends upon nonlinearity, viscosity and a new factor that he introduced, thermal conductivity.

In Chapter 4 of this thesis the effects of the critical layer on the linear stability analysis conducted in Chapter 3 is studied. It should be noted that all the work in this Chapter is conducted for the case of a straight cylinder subject to axisymmetric disturbance terms only. We begin by determining the effects curvature has on the linear compressible viscous theory as determined by Lees and Lin (1946). Nonlinearity is then introduced into the inviscid problem by means of the instability wave's interaction with nonlinear effects developed within the critical layer. The actual method employed is based on the method developed by Goldstein and Leib (1989) and allows us to consider near-neutral amplified disturbance terms. This study will be conducted from a temporal basis, i.e. waves periodic in space but growing in time are treated, and the aim is to determine the effects curvature has on the Goldstein/Leib results. In the last Chapter of this thesis, the full non-equilibrium, nonlinear viscous critical layer amplitude equation is determined for the compressible boundary layer formed on a straight circular cylinder subject to axisymmetric disturbances. Again the aim of this study is to see the effects curvature has on the critical-layer dynamics and the resultant evolution equation governing the instability wave's amplitude.

Chapter 2

Basic Flow

2.1 Equations of Motion and State

The general layout of the problem is shown in Figure 2.1. The z^* axis lies along the cone axis, r^* denotes the radial coordinate, and θ the azimuthal coordinate. The velocity vector $\mathbf{v}^*$ has components v_1^*, v_2^*, v_3^* in the r^*, θ, and z^* directions, respectively. The temperature field is represented by T^*. Throughout it is assumed that the basic flow is independent of θ, and the mean azimuthal velocity component, v_2^*, is zero, although when we go on to consider the stability of the flow, we shall be concerned with non-axisymmetric disturbances.

In the cylindrical polar coordinate system as defined above, the full equations of continuity, momentum, and energy take on the following forms (Thompson (1972)):

$$\frac{\partial \rho^*}{\partial t^*} + \frac{\partial}{\partial r^*}(\rho^* v_1^*) + \frac{1}{r^*}\frac{\partial}{\partial \theta}(\rho^* v_2^*) + \frac{\partial}{\partial z^*}(\rho^* v_3^*) + \frac{\rho^* v_1^*}{r^*} = 0, \tag{2.1}$$

$$\rho^*\left[\frac{Dv_1^*}{Dt^*} - \frac{(v_2^*)^2}{r^*}\right] = -\frac{\partial p^*}{\partial r^*} + \frac{\partial \Sigma_{r^*r^*}}{\partial r^*} + \frac{1}{r^*}\frac{\partial \Sigma_{r^*\theta}}{\partial \theta} + \frac{\partial \Sigma_{r^*z^*}}{\partial z^*} + \frac{\Sigma_{r^*r^*} - \Sigma_{\theta\theta}}{r^*}, \tag{2.2}$$

$$\rho^*\left[\frac{Dv_2*}{Dt^*} + \frac{v_1*v_2*}{r^*}\right] = -\frac{1}{r^*}\frac{\partial p^*}{\partial \theta} + \frac{\partial \Sigma_{\theta r^*}}{\partial r^*} + \frac{\partial \Sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \Sigma_{\theta z^*}}{\partial z^*} + 2\frac{\Sigma_{r^*\theta}}{r^*}, \tag{2.3}$$

$$\rho^* \frac{Dv_3^*}{Dt^*} = -\frac{\partial p^*}{\partial z^*} + \frac{\partial \Sigma_{z^*r^*}}{\partial r^*} + \frac{1}{r^*}\frac{\partial \Sigma_{z^*\theta}}{\partial \theta} + \frac{\partial \Sigma_{z^*z^*}}{\partial z^*} + \frac{\Sigma_{z^*r^*}}{r^*}, \tag{2.4}$$

$$\rho^* \frac{D}{Dt^*}\big(c_p T^*\big) - \frac{Dp^*}{Dt^*} = \Gamma^* + \frac{1}{r^*}\frac{\partial}{\partial r^*}\big(K^* r^* \frac{\partial T^*}{\partial r^*}\big) + \frac{1}{(r^*)^2}\frac{\partial}{\partial \theta}\big(K^* \frac{\partial T^*}{\partial \theta}\big) + \frac{\partial}{\partial z^*}\big(K^* \frac{\partial T^*}{\partial z^*}\big). \tag{2.5}$$

Here ρ^* is the density of the fluid, p^* is the pressure, c_p is the specific heat (at constant pressure), and K^* is the coefficient of heat conduction. The Eulerian operator is defined as

$$\frac{D}{Dt^*} = \frac{\partial}{\partial t^*} + v_1^* \frac{\partial}{\partial r^*} + \frac{v_2^*}{r^*}\frac{\partial}{\partial \theta} + v_3^* \frac{\partial}{\partial z^*}, \tag{2.6}$$

and the viscous stress components, assuming Newtonian flow, are defined to be

$$\Sigma_{r^*r^*} = 2\mu^* \frac{\partial v_1^*}{\partial r^*} + \lambda^* \nabla \cdot \mathbf{v}^*, \tag{2.7}$$

$$\Sigma_{\theta\theta} = 2\mu^* \Big[\frac{1}{r^*}\frac{\partial v_2^*}{\partial \theta} + \frac{v_1^*}{r^*}\Big] + \lambda^* \nabla \cdot \mathbf{v}^*, \tag{2.8}$$

$$\Sigma_{z^*z^*} = 2\mu^* \frac{\partial v_3^*}{\partial z^*} + \lambda^* \nabla \cdot \mathbf{v}^*, \tag{2.9}$$

$$\Sigma_{r^*\theta} = \Sigma_{\theta r^*} = \mu^* \Big[\frac{1}{r^*}\frac{\partial v_1^*}{\partial \theta} + r^* \frac{\partial}{\partial r^*}\big(\frac{v_2^*}{r^*}\big)\Big], \tag{2.10}$$

$$\Sigma_{\theta z^*} = \Sigma_{z^*\theta} = \mu^* \Big[\frac{\partial v_2^*}{\partial z^*} + \frac{1}{r^*}\frac{\partial v_3^*}{\partial \theta}\Big], \tag{2.11}$$

$$\Sigma_{z^*r^*} = \Sigma_{r^*z^*} = \mu^* \Big[\frac{\partial v_3^*}{\partial r^*} + \frac{\partial v_1^*}{\partial z^*}\Big]. \tag{2.12}$$

The dissipation function Γ^* in (2.5) is given by

$$\Gamma^* = 2\mu^*\Big[D^2_{r^*r^*}+D^2_{\theta\theta}+D^2_{z^*z^*}+2D^2_{r^*\theta}+2D^2_{z^*\theta}+2D^2_{z^*r^*}\Big]+(\lambda^*-\frac{2}{3}\mu^*)(\nabla\cdot\mathbf{v}^*)^2. \quad (2.13)$$

Here the 'D' terms are the components of the rate-of-deformation tensor, and are given by,

$$D_{r^*r^*} = \frac{\partial v_1^*}{\partial r^*}, \qquad (2.14)$$

$$D_{\theta\theta} = \frac{1}{r^*}\frac{\partial v_2^*}{\partial \theta} + \frac{v_1^*}{r^*}, \qquad (2.15)$$

$$D_{z^*z^*} = \frac{\partial v_3^*}{\partial z^*}, \qquad (2.16)$$

$$D_{r^*z^*} = \frac{1}{2}\Big[\frac{\partial v_1^*}{\partial z^*} + \frac{\partial v_3^*}{\partial r^*}\Big], \qquad (2.17)$$

$$D_{r^*\theta} = \frac{1}{2}\Big[\frac{1}{r^*}\frac{\partial v_1^*}{\partial \theta} + r^*\frac{\partial}{\partial r^*}\Big(\frac{v_2^*}{r^*}\Big)\Big], \qquad (2.18)$$

$$D_{\theta z^*} = \frac{1}{2}\Big[\frac{\partial v_2^*}{\partial z^*} + \frac{1}{r^*}\frac{\partial v_3^*}{\partial \theta}\Big]. \qquad (2.19)$$

The coefficients μ^* and λ^* above denote the first coefficient of viscosity and bulk viscosity, respectively (which are assumed to be functions of temperature only) and $\lambda^* = \mu_2^* + \frac{2}{3}\mu^*$, where μ_2^* is the second coefficient of viscosity.

The equation of state is taken to be that which models a perfect gas, i.e.,

$$p^* = \rho^* R^* T^*, \qquad (2.20)$$

where R^* is the gas constant.

With reference to Figure 2.1, the surface of the cone is taken to lie along $r^* = a^* + \lambda_1 z^*$, $z^* > 0$ (later, important assumptions regarding the size of the slope parameter will be made), and so on this surface we require

$$v_1^* = v_2^* = v_3^* = 0. \tag{2.21}$$

If the surface of the cone is insulated, then the following boundary condition must also be imposed:

$$\frac{\partial T^*}{\partial n^*} = 0, \tag{2.22}$$

(where n^* denotes an outwards normal to the wall). In the case of heated/cooled walls, then the condition

$$T^* = T_w^*, \tag{2.23}$$

must be imposed at the surface.

We now specify conditions at $z^* = 0$. In this problem the overall intention is to investigate the effects of curvature, in particular how curvature changes planar results. Thus, at the cone tip it is assumed that the boundary layer has zero thickness, enabling planar conditions to be imposed at this position. This progressive introduction of curvature, starting from planar conditions, is a sensible way of explicity studying its effects. A similar assumption was made by Seban and Bond (1951) for the laminar boundary layer formed on a cylinder in axial incompressible flow and their comments regarding this assumption are found to be valid here. Following previous authors, such as Mack (1984), the effect of any shock that may occur is ignored. This is also expected to be important in the vicinity of the tip (i.e. for $z^*/a^* = O(1)$ where z^* is a much longer lengthscale than a^*), the significance of this region for our model being discussed above. Further, it is expected that for moderate Mach numbers (such as considered throughout this thesis) downstream of the tip, the chosen 'thinness' of the boundary layer is such that the shock wave will be located well outside of the boundary layer. Since our analysis focuses attention primarily on the boundary layer, the effects of the shock are assumed to be negligible (see also the work of Chang *et al* (1990) which confirms this in the case of planar flows).

Assuming the cone to be slender, then the far-field conditions are taken to be uniform, to leading order, with

$$v_1^* = v_2^* = 0, \tag{2.24}$$

$$v_3^* = U_\infty^*, \tag{2.25}$$

$$T^* = T_\infty^*. \tag{2.26}$$

Subscript ∞ denotes free-stream conditions.

We next go on to derive the basic (boundary-layer) flow on the surface of the cone, assuming curvature plays a key role in the physics of the problem.

2.2 The Boundary-Layer Flow

We define our Reynolds number on the tip radius of the cone, a^*, as follows:

$$Re = \frac{U_\infty^* a^* \rho_\infty^*}{\mu_\infty^*}, \tag{2.27}$$

and this will be assumed to be large throughout this thesis, thus allowing us to make the steady boundary-layer approximation. A key element in this work is the inclusion of curvature terms in the governing equations to leading order. To achieve this the tip radius of the cone is assumed to be generally of the same order as the boundary-layer thickness (a similar approach was adopted by Seban and Bond (1951), Stewartson (1955), Bush (1976), Duck and Bodonyi (1986) and Duck (1990) for cylindrical bodies), except at the tip of the cone where, as already mentioned, the boundary layer is assumed to have zero thickness. As already noted the z^* lengthscale is much longer than the body radius ($z^* = O(a^* Re)$, generally) and so the 'tip effects' of the cone on the mean flow will be expected to be confined to $z^*/a^* = O(1)$.

With the formation of a thin boundary layer, (comparable in thickness to the body radius) the following classical assumptions are expected to hold

$$\frac{\partial}{\partial r^*} >> \frac{\partial}{\partial z^*} \quad \text{and} \quad v_3^* >> v_1^*, \tag{2.28}$$

(these orders will be made more precise shortly). We also must have

$$\lambda_1 = Re^{-1}\overline{\lambda}, \tag{2.29}$$

where

$$\overline{\lambda} = O(1), \tag{2.30}$$

implying a slender cone.

As noted previously, the basic flow is taken to be independent of θ, and has no azimuthal velocity component (i.e., $v_2^* = 0$). Introducing non-dimensional parameters as follows:

$$(v_1, v_3, r, z, T, \rho, \mu) = \Big(\frac{Rev_1^*}{U_\infty^*}, \frac{v_3^*}{U_\infty^*}, \frac{r^*}{a^*}, \frac{z^*}{Rea^*}, \frac{T^*}{T_\infty^*}, \frac{\rho^*}{\rho_\infty^*}, \frac{\mu^*}{\mu_\infty^*}\Big), \tag{2.31}$$

then the leading-order governing equations may be written (assuming $Re \rightarrow \infty$)

$$\frac{\partial}{\partial r}\Big(\frac{v_1}{T}\Big) + \frac{v_1}{rT} + \frac{\partial}{\partial z}\Big(\frac{v_3}{T}\Big) = 0, \tag{2.32}$$

$$\frac{\partial p}{\partial r} = 0, \tag{2.33}$$

$$v_1 \frac{\partial v_3}{\partial r} + v_3 \frac{\partial v_3}{\partial z} = \frac{T}{r}\frac{\partial}{\partial r}\Big[r\mu \frac{\partial v_3}{\partial r}\Big], \tag{2.34}$$

$$v_1 \frac{\partial T}{\partial r} + v_3 \frac{\partial T}{\partial z} = \mu T(\gamma - 1)M_\infty^2 \Big[\frac{\partial v_3}{\partial r}\Big]^2 + \frac{T}{r}\frac{\partial}{\partial r}\Big[\frac{r\mu}{\sigma}\frac{\partial T}{\partial r}\Big], \tag{2.35}$$

where the result

$$\rho = \frac{1}{T}, \tag{2.36}$$

has been used and is a consequence of applying equation (2.33) to the equation of state, in non-dimensional form; γ denotes the ratio of specific heats, σ is the Prandtl number, namely

$$\sigma = \frac{\mu^* c_p}{K^*}, \tag{2.37}$$

(which is assumed to be a constant in this problem), and M_∞ is the freestream Mach number, namely

$$M_\infty = \frac{U_\infty^*}{(\gamma R^* T_\infty^*)^{1/2}}. \tag{2.38}$$

The boundary conditions are

$$v_1 = v_3 = 0 \qquad \text{on} \qquad r = 1 + \overline{\lambda} z, \tag{2.39}$$

$$v_3 \to 1, \qquad T \to 1, \qquad \text{as} \qquad r \to \infty, \tag{2.40}$$

together with a wall temperature condition; in the case of insulated walls (to leading order)

$$\frac{\partial T}{\partial r} = 0 \qquad \text{on} \quad r = 1 + \overline{\lambda} z, \tag{2.41}$$

whilst in the case of heated/cooled surfaces

$$T = T_w \qquad \text{on} \quad r = 1 + \overline{\lambda} z. \tag{2.42}$$

All that remains is for us to specify a viscosity/temperature law. For the purpose of this work we assume the linear Chapman law (Stewartson (1964)), namely

$$\mu = CT, \tag{2.43}$$

where C is taken to be constant (although, here, conceptually, there would be no difficulty in taking more complex variations of viscosity with temperature).

If we write

$$\begin{aligned} v_1 &= C\overline{v}_1, \\ z &= C^{-1}\overline{z}, \\ \overline{\lambda} &= C\lambda, \end{aligned} \tag{2.44}$$

whilst retaining other terms, then the system (2.32) - (2.35) becomes

$$\frac{\partial}{\partial r}\Big(\frac{\overline{v}_1}{T}\Big) + \frac{\overline{v}_1}{rT} + \frac{\partial}{\partial \overline{z}}\Big(\frac{v_3}{T}\Big) = 0, \tag{2.45}$$

$$\frac{\partial p}{\partial r} = 0, \tag{2.46}$$

$$\overline{v}_1 \frac{\partial v_3}{\partial r} + v_3 \frac{\partial v_3}{\partial \overline{z}} = \frac{T}{r}\frac{\partial}{\partial r}\Big[rT\frac{\partial v_3}{\partial r}\Big], \tag{2.47}$$

$$\overline{v}_1\frac{\partial T}{\partial r}+v_3\frac{\partial T}{\partial \overline{z}}=T^2(\gamma-1)M_\infty^2\Big(\frac{\partial v_3}{\partial r}\Big)^2+\frac{T}{r}\frac{\partial}{\partial r}\Big[\frac{rT}{\sigma}\frac{\partial T}{\partial r}\Big], \tag{2.48}$$

whilst the wall boundary conditions are to be applied on

$$r=1+\lambda\overline{z}.$$

As described previously, it is assumed that as $z\rightarrow 0$ the solution approaches planar conditions and at the cone tip the boundary layer is taken to have zero thickness. The problem is thus singular at $z(=\overline{z})=0$, and consequently scaled variables must be introduced in order to solve (numerically) the system (2.45) - (2.48) accurately. Specifically we write

$$\begin{aligned} \overline{v}_1 &= \zeta^{-1}\hat{v}_1(\eta,\zeta),\\ v_3 &= \hat{v}_3(\eta,\zeta),\\ T &= \hat{T}(\eta,\zeta), \end{aligned} \tag{2.49}$$

where

$$\zeta=\overline{z}^{1/2}, \tag{2.50}$$

and

$$\eta=\frac{r-1-\lambda\zeta^2}{\zeta}. \tag{2.51}$$

The 'hatted' quantities are expected to behave regularly as $\zeta\rightarrow 0$, approaching the planar solution. Equations (2.45) - (2.48) now take the following form:

$$\frac{\partial}{\partial\eta}\Big(\frac{\hat{v}_1}{\hat{T}}\Big)+\frac{\zeta\hat{v}_1}{r\hat{T}}+\frac{\zeta}{2}\frac{\partial}{\partial\zeta}\Big(\frac{\hat{v}_3}{\hat{T}}\Big)-\Big[\lambda\zeta+\frac{\eta}{2}\Big]\frac{\partial}{\partial\eta}\Big(\frac{\hat{v}_3}{\hat{T}}\Big)=0, \tag{2.52}$$

$$\frac{\partial p}{\partial \eta} = 0, \tag{2.53}$$

$$\hat{v}_1 \frac{\partial \hat{v}_3}{\partial \eta} + \frac{\zeta \hat{v}_3}{2}\frac{\partial \hat{v}_3}{\partial \zeta} - \hat{v}_3\Big[\lambda\zeta + \frac{\eta}{2}\Big]\frac{\partial \hat{v}_3}{\partial \eta} = \frac{\hat{T}}{r}\frac{\partial}{\partial \eta}\Big[r\hat{T}\frac{\partial \hat{v}_3}{\partial \eta}\Big], \tag{2.54}$$

$$\hat{v}_1 \frac{\partial \hat{T}}{\partial \eta} + \frac{\zeta \hat{v}_3}{2}\frac{\partial \hat{T}}{\partial \zeta} - \hat{v}_3\Big[\lambda\zeta + \frac{\eta}{2}\Big]\frac{\partial \hat{T}}{\partial \eta} = \hat{T}^2(\gamma - 1)M_\infty^2\Big(\frac{\partial \hat{v}_3}{\partial \eta}\Big)^2 + \frac{\hat{T}}{r}\frac{\partial}{\partial \eta}\Big[\frac{r\hat{T}}{\sigma}\frac{\partial \hat{T}}{\partial \eta}\Big], \tag{2.55}$$

where

$$r = 1 + \lambda\zeta^2 + \zeta\eta. \tag{2.56}$$

The boundary conditions in terms of the new variables are

$$\hat{v}_1 = \hat{v}_3 = 0 \qquad \text{on} \qquad \eta = 0, \tag{2.57}$$

$$\hat{v}_3 \to 1, \qquad \hat{T} \to 1 \qquad \text{as} \qquad \eta \to \infty. \tag{2.58}$$

In the case of insulated walls, the additional surface condition is

$$\frac{\partial \hat{T}}{\partial \eta} = 0 \qquad \text{on} \qquad \eta = 0, \tag{2.59}$$

whilst for heated/cooled walls

$$\hat{T} = T_w \qquad \text{on} \qquad \eta = 0. \tag{2.60}$$

On examination of the system of equations (2.52) - (2.55) it is found that unlike the planar case, the introduction a Howarth-Dorodnitsyn (Stewartson (1951), Moore (1951)) transformation does not simplify matters; in fact the equations become rather untidy in an algebraic sense. Therefore it is found necessary to seek a numerical solution to $\hat{v}_1$, $\hat{v}_3$, and $\hat{T}$ directly. Setting $\zeta = 0$ in equations (2.52), (2.54) and (2.55), the problem just reduces to the planar case, namely the ordinary differential system

$$\hat{v}_{1\eta}\hat{T} - \hat{v}_1\hat{T}_\eta - \frac{\eta}{2}\hat{v}_{3\eta}\hat{T} + \frac{\eta}{2}\hat{v}_3\hat{T}_\eta = 0, \tag{2.61}$$

$$\hat{v}_1\hat{v}_{3\eta} - \frac{\eta}{2}\hat{v}_3\hat{v}_{3\eta} = \hat{T}\hat{T}_\eta\hat{v}_{3\eta} + \hat{T}^2\hat{v}_{3\eta\eta}, \tag{2.62}$$

$$\hat{v}_1\hat{T}_\eta - \frac{\eta}{2}\hat{v}_3\hat{T}_\eta = \hat{T}^2(\gamma-1)M_\infty^2\hat{v}_{3\eta}^2 + \frac{\hat{T}\hat{T}_\eta^2}{\sigma} + \frac{\hat{T}^2\hat{T}_{\eta\eta}}{\sigma}, \tag{2.63}$$

(subject to conditions (2.57), (2.58) and (2.59) or (2.60)).

Defining the variables

$$\begin{aligned} \hat{v}_3^1 &= \hat{v}_{3\eta}, \\ \hat{T}^1 &= \hat{T}_\eta, \end{aligned} \tag{2.64}$$

the system (2.61) - (2.63) together with (2.64) can be written as a first-order system of ordinary differential equations, namely

$$\hat{T}\hat{v}_{1\eta} - \hat{v}_1\hat{T}^1 - \frac{\eta}{2}\hat{T}\hat{v}_3^1 + \frac{\eta}{2}\hat{T}^1\hat{v}_3 = 0, \tag{2.65}$$

$$\hat{v}_1\hat{v}_3^1 - \frac{\eta}{2}\hat{v}_3\hat{v}_3^1 = \hat{T}^2\hat{v}_{3\eta}^1 + \hat{T}\hat{T}^1\hat{v}_3^1, \tag{2.66}$$

$$\hat{v}_3^1 = \hat{v}_{3\eta}, \tag{2.67}$$

$$\hat{v}_1\hat{T}^1 - \frac{\eta}{2}\hat{v}_3\hat{T}^1 = \hat{T}^2(\gamma-1)M_\infty^2(\hat{v}_3^1)^2 + \frac{\hat{T}^2\hat{T}_\eta^1}{\sigma} + \frac{\hat{T}(\hat{T}^1)^2}{\sigma}, \tag{2.68}$$

$$\hat{T}^1 = \hat{T}_\eta, \tag{2.69}$$

subject to

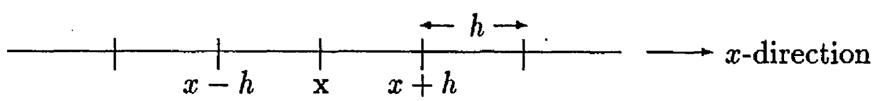
$$\begin{aligned}
\hat{T}^1 = 0 \quad &\text{on} \quad \eta = 0. \ \text{or} \ \hat{T} = T_w \ \text{on} \ \eta = 0, \\
\hat{v}_1 = \hat{v}_3 = 0 \quad &\text{on} \quad \eta = 0, \\
\hat{T} \to 1 \quad &\text{as} \quad \eta \to \infty, \\
\hat{v}_3 \to 1 \quad &\text{as} \quad \eta \to \infty.
\end{aligned} \tag{2.70}$$

This system of equations is now treated numerically by approximating each term by a second order finite difference scheme and the resultant truncated system is solved by means of a Newton iteration. At each iteration level, the algebraic system was of block-diagonal form, with each block comprising 10×5 elements.

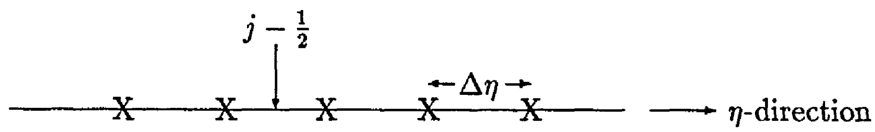
The finite difference scheme employed is the central difference analogue which for a function $u(x)$ has the form

$$\frac{du}{dx} = \frac{1}{2h}\Big[u(x+h) - u(x-h)\Big] + O(h^2), \tag{2.71}$$

where h is the step size and is determined by the fineness of the grid. The grid has the form



In this problem, to make the scheme more compact, instead of evaluating at successive grid points, all functions and finite differences are evaluated at a point half-way between the mesh points, although we still step up in amounts equivalent to the mesh size. In effect we are evaluating the functions and finite differences at two successive mesh points and then averaging the resultant. Note, the accuracy will still be the same as the error term is $O(h^2)$. Therefore the mesh employed to solve the ODE system under consideration has the form



where $j-\frac{1}{2}$ is the point where the central differencing is carried out. The central difference expansion for a function $v(\eta)$, in terms of the new grid, has the form

$$\frac{dv}{d\eta}=\frac{v(\eta+\frac{\Delta\eta}{2})-v(\eta-\frac{\Delta\eta}{2})}{\Delta\eta}+O(\Delta\eta^2), \tag{2.72}$$

where $\Delta\eta$ represents the step size.

Applying the central difference expansion (2.72) to the differential quantities in the ODE system (2.65) - (2.69) the following finite difference approximations are obtained. for the differential quantities

$$\hat{v}_{1\eta}=\frac{\hat{v}_1(j)-\hat{v}_1(j-1)}{\Delta\eta}+O(\Delta\eta^2),$$

$$\hat{v}_{3\eta}=\frac{\hat{v}_3(j)-\hat{v}_3(j-1)}{\Delta\eta}+O(\Delta\eta^2),$$

$$\hat{v}^1_{3\eta}=\frac{\hat{v}^1_3(j)-\hat{v}^1_3(j-1)}{\Delta\eta}+O(\Delta\eta^2),$$

$$\hat{T}_\eta=\frac{\hat{T}(j)-\hat{T}(j-1)}{\Delta\eta}+O(\Delta\eta^2),$$

$$\hat{T}^1_\eta=\frac{\hat{T}^1(j)-\hat{T}^1(j-1)}{\Delta\eta}+O(\Delta\eta^2). \tag{2.73}$$

To apply the Newton iteration to the truncated finite difference system, at the $(n+1)$th iteration level we write

$$f^{n+1}(j) = f^n(j) + \delta f(j), \tag{2.74}$$

with the understanding that terms of $O(\delta^2)$ and higher are neglected. Substituting (2.74) into the truncated finite difference system, and collecting δf terms on one side of the system of equations, the resultant system can be expressed in the matrix form

$$A.\mathbf{X} = B, \tag{2.75}$$

where A is the cofficient term matrix, B contains the matrix elements to be determined, and $\mathbf{X}$ contains the δf terms having the form

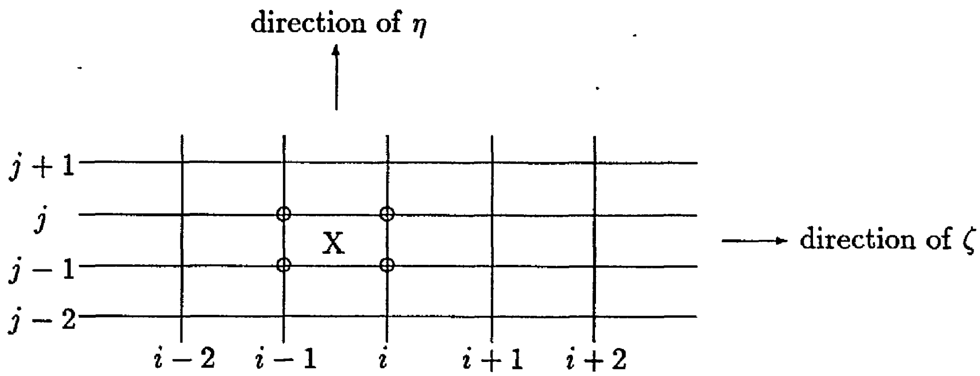
$$\mathbf{X} = \begin{pmatrix} \delta\hat{v}_1(1) \\ \vdots \\ \delta\hat{v}_1(j-1) \\ \delta\hat{v}_3(j-1) \\ \delta\hat{v}_3^1(j-1) \\ \delta\hat{T}(j-1) \\ \delta\hat{T}^1(j-1) \\ \delta\hat{v}_1(j) \\ \delta\hat{v}_3(j) \\ \delta\hat{v}_3^1(j) \\ \delta\hat{T}(j) \\ \delta\hat{T}^1(j) \\ \vdots \\ \delta\hat{T}^1(jmax) \end{pmatrix},$$

where $jmax$ represents the grid point where the far-field conditions are satisfied. The δf's are evaluated repeatedly until suitable convergence at a given j station is achieved.

The banded matrix system (2.75) is then solved using Gaussian elimination and banded matrices. Finally a relaxation condition is applied at each iteration level to speed up calculations.

Once obtained the solution to the ODE system then provides the initial conditions for a (straightforward) Crank-Nicolson scheme in ζ to solve equations (2.52) - (2.55). Values are determined for all η-stations along a row of ζ-stations and we march forward in ζ, determining each new row of η stations. Overall, this may be described as an implicit scheme.

The mesh used to approximate equations (2.52) - (2.55) has the following form



where X marks the point $(i-\frac{1}{2}, j-\frac{1}{2})$ about which the central differencing is carried out. The central difference expansion for a function $U_{i,j}$ in the η-direction has the form

$$\frac{\partial U_{i,j}}{\partial \eta} = \frac{U_{i,j} - U_{i,j-1} + U_{i-1,j} - U_{i-1,j-1}}{2\Delta\eta}, \tag{2.76}$$

whilst in the ζ-direction it takes the form

$$\frac{\partial U_{i,j}}{\partial \zeta} = \frac{U_{i,j} + U_{i,j-1} - U_{i-1,j} - U_{i-1,j-1}}{2\Delta\zeta}, \tag{2.77}$$

where $\Delta\eta$ and $\Delta\zeta$ represent the respective step sizes.

Applying (2.76) and (2.77) to the various differential quantities in the system of equations (2.52) - (2.55), yields the following Crank-Nicolson approximations

$$\frac{\partial \hat{v}_1}{\partial \eta} = \frac{\hat{v}_1(i,j) - \hat{v}_1(i,j-1) + \hat{v}_1(i-1,j) - \hat{v}_1(i-1,j-1)}{2\Delta\eta} + (\Delta\eta^2),$$

$$\frac{\partial \hat{v}_3}{\partial \eta} = \frac{\hat{v}_3(i,j) - \hat{v}_3(i,j-1) + \hat{v}_3(i-1,j) - \hat{v}_3(i-1,j-1)}{2\Delta\eta} + (\Delta\eta^2),$$

$$\frac{\partial \hat{v}_3^1}{\partial \eta} = \frac{\hat{v}_3^1(i,j) - \hat{v}_3^1(i,j-1) + \hat{v}_3^1(i-1,j) - \hat{v}_3^1(i-1,j-1)}{2\Delta\eta} + (\Delta\eta^2),$$

$$\frac{\partial \hat{T}}{\partial \eta} = \frac{\hat{T}(i,j) - \hat{T}(i,j-1) + \hat{T}(i-1,j) - \hat{T}(i-1,j-1)}{2\Delta\eta} + (\Delta\eta^2),$$

$$\frac{\partial \hat{T}^1}{\partial \eta} = \frac{\hat{T}^1(i,j) - \hat{T}^1(i,j-1) + \hat{T}^1(i-1,j) - \hat{T}^1(i-1,j-1)}{2\Delta\eta} + (\Delta\eta^2),$$

$$\frac{\partial \hat{v}_3}{\partial \zeta} = \frac{\hat{v}_3(i,j) - \hat{v}_3(i-1,j) + \hat{v}_3(i,j-1) - \hat{v}_3(i-1,j-1)}{2\Delta\zeta} + (\Delta\zeta^2),$$

$$\frac{\partial \hat{T}}{\partial \zeta} = \frac{\hat{T}(i,j) - \hat{T}(i-1,j) + \hat{T}(i,j-1) - \hat{T}(i-1,j-1)}{2\Delta\zeta} + (\Delta\zeta^2). \qquad (2.78)$$

The first set of mean flow results presented corresponds to a straight circular cylinder (i.e. $\lambda = 0$) subject to adiabatic wall conditions and are the same as those presented by Duck (1990). Distributions of wall temperature with axial coordinate ζ $(= \bar{z}^{1/2})$ for Mach numbers of $M_\infty = 2.8$ and $M_\infty = 3.8$,† denoted by curves (1) and (2) respectively, are shown in Figure 2.2; the corresponding distributions of wall shear $\hat{v}_{3\eta}|_{\eta=0}$ are shown in Figure 2.3. The fluid constants are assumed to have values of $\sigma = 0.72$ and $\gamma = 1.4$. Results are only presented up to an axial location of 10.0 as it is found that as we integrate downstream it becomes more difficult to determine accurate numerical solutions. The reason for this is that when the scaled radial mesh size, $\Delta\eta$, is expressed in terms of the unscaled variables, namely the radial coordinate r, we have $\Delta r = \Delta\eta\bar{z}^{1/2}$. Clearly the further downstream the integration proceeds in the axial direction, then the larger Δr becomes resulting in the radial integration becoming increasingly inaccurate. Studying Figure 2.2, the

† These Mach numbers have been chosen to compare the results directly with Mack's results for the planar case.

wall temperatures are observed to decline slightly from their planar values at $\zeta = 0$, for both Mach numbers presented, whilst the wall shear increases monotonically with movement downstream. Note, at $\zeta = 0$, the wall shears have small but non-zero positive values, for both Mach numbers.

Duck (1990) carried out a far downstream study of the basic flow following the incompressible work of Glauert and Lighthill (1955), Stewartson (1955) and Bush (1976). He determined that, in the limit of large ζ, there are two important radial lengthscales, namely $r = O(1)$ and $r = O(\overline{z}^{1/2})$. Matching between these respective layers and applying suitable boundary conditions, Duck determined that for adiabatic wall conditions,

$$T|_{r=1} = 1 + \frac{1}{2}\sigma(\gamma - 1)M_\infty^2 + O(\epsilon), \tag{2.79}$$

and

$$\frac{\partial v_3}{\partial r}\Big|_{r=1} = \frac{\epsilon}{1 + \frac{1}{2}\sigma(\gamma - 1)M_\infty^2} + O(\epsilon^2), \tag{2.80}$$

where $\epsilon = (\ln \zeta)^{-1}$ and the expansions for T and v_3 are valid in the inner layer, only.

For cooled/heated surfaces matching yields

$$\frac{\partial T}{\partial r}\Big|_{r=1} = \frac{\epsilon}{T_w}[1 + \frac{1}{2}\sigma(\gamma - 1)M_\infty^2 - T_w] + O(\epsilon^2), \tag{2.81}$$

and

$$\frac{\partial v_3}{\partial r}\Big|_{r=1} = \frac{\epsilon}{T_w} + O(\epsilon^2), \tag{2.82}$$

where T_w corresponds to the wall temperature.

The adiabatic asymptotic temperature results (i.e. (2.79) for respective Mach numbers) are displayed as broken lines in Figure 2.2. The wall shear as defined by expansion (2.80), in terms of the scaled coordinates η and ζ has the form $\zeta^{-1}\hat{v}_{3\eta}|_{\eta=0}$.

Plots of $\zeta^{-1}\hat{v}_{3\eta}|_{\eta=0}$ against ζ as determined numerically, are displayed in Figure 2.4, where the broken lines, again, represent asymptotically determined results. The agreement is satisfactory given the relative 'largeness' of the small parameter ϵ. If we precede further downstream it is found that the inner layer, i.e. the $r = O(1)$ layer, which can be regarded as an inner boundary layer, retains its thickness whilst the outer layer spreads due to the action of viscosity. As a result, comparison between asymptotically determined results and numerically determined solutions will become increasingly less accurate. This provides another reason why a mean flow study beyond $\zeta = 10.0$ is not conducted.

Examining Figure 2.4, it is noted that the $\zeta^{-1}\hat{v}_{3\eta}|_{\eta=0}$ distribution is singular in the planar limit as $\zeta \to 0$ and then appears to (slowly) fall continuously as ζ increases, for both Mach numbers considered.

Figure 2.5 displays variations of wall shear, $\hat{v}_{3\eta}|_{\eta=0}$, with ζ for both heated and cooled wall surfaces. It is apparent that cooling causes the wall shear to increase more sharply with movement downstream, whilst heating causes the converse effect. It also appears that wall heating causes the wall shear variation with ζ gradient to be asymptoting to some undetermined value.

Turning our attention to the boundary layer formed on a cone, we now present a few results for mean flow variations subject to adiabatic wall conditions only. It should be noted that in the case of the cone, the surface of the cone occurs at increasingly larger radii as we move downstream, as defined by (2.56), i.e. $O(\overline{z})$, but the far-field point used in the numerical integration is held fixed. It is found that the effect of body divergence conteracts the divergence of radial mesh size, Δr, as observed in the case of cylindrical bodies and the boundary layer growth, both of which are $O(\overline{z}^{1/2})$, thus making it possible to present results for relatively large downstream locations, with a high degree of accuracy.

Distributions of wall temperature with axial location ζ are shown in Figure 2.6

($M_\infty = 2.8$) and Figure 2.7 ($M_\infty = 3.8$), and the corresponding distributions of wall shear $\hat{v}_{3\eta}|_{\eta=0}$ are shown in Figure 2.8 ($M_\infty = 2.8$) and Figure 2.9 ($M_\infty = 3.8$), for the slope parameter values as shown. In all cases, these distributions are quite different to the corresponding $\lambda = 0$ distributions, as presented above, with the wall temperatures no longer undergoing monotonic decrease, and the wall shears no longer increasing monotonically. It is also quite clear that the results evolve from the planar case to the far downstream limit, as predicted by the Mangler transform (Mangler (1946), Stewartson (1955)), namely

$$\begin{aligned} \hat{v}_{3\eta}\Big|_{\eta=0,\zeta\to\infty} &\to \sqrt{3}\hat{v}_{3\eta}\Big|_{\eta=0,\zeta=0}, \\ \hat{T}\Big|_{\eta=0,\zeta\to\infty} &\to \hat{T}\Big|_{\eta=0,\zeta=0}. \end{aligned} \tag{2.83}$$

In the next chapter we investigate the stability of flows of this type, subject to small amplitude inviscid disturbances.

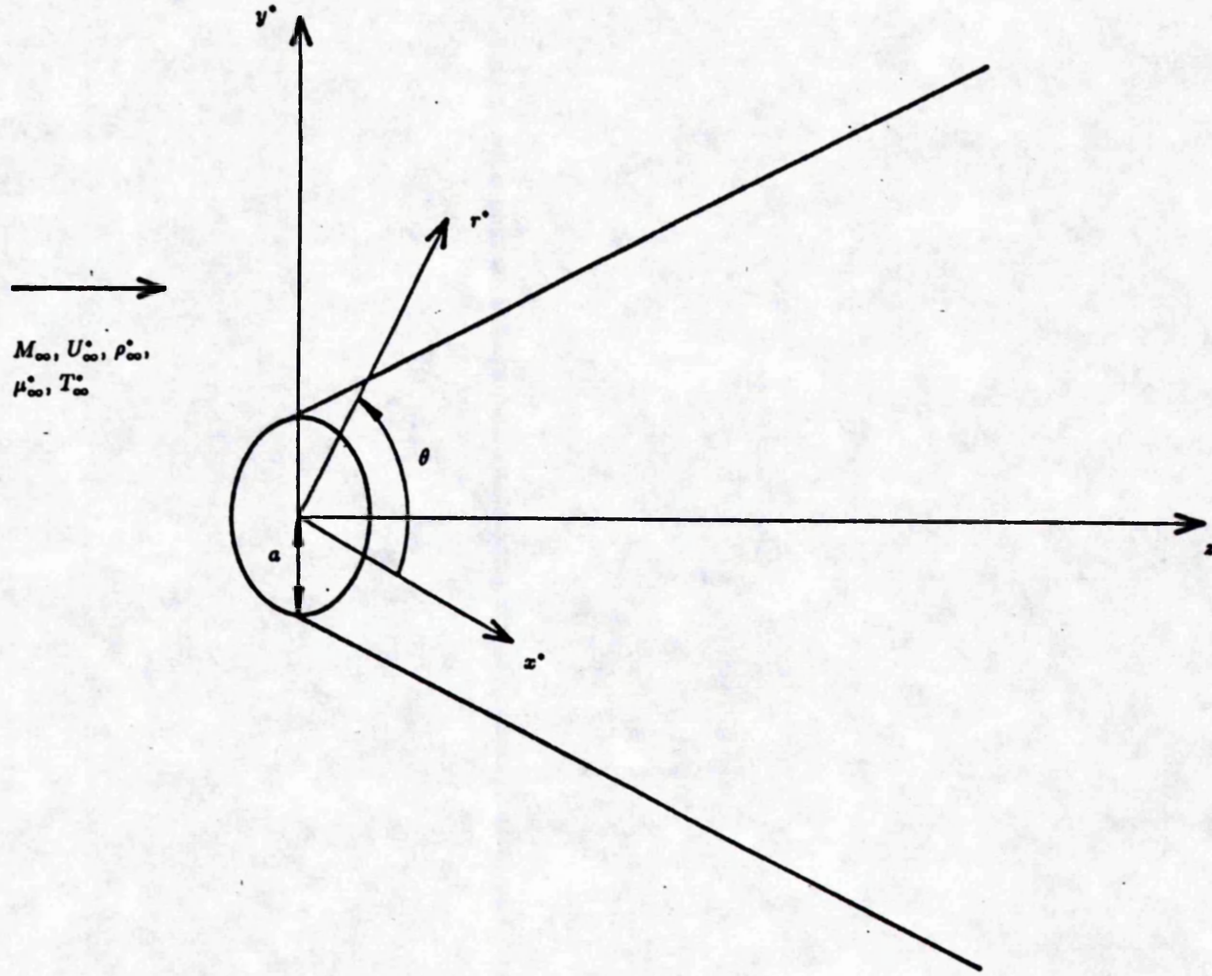


Figure 2.1: Layout

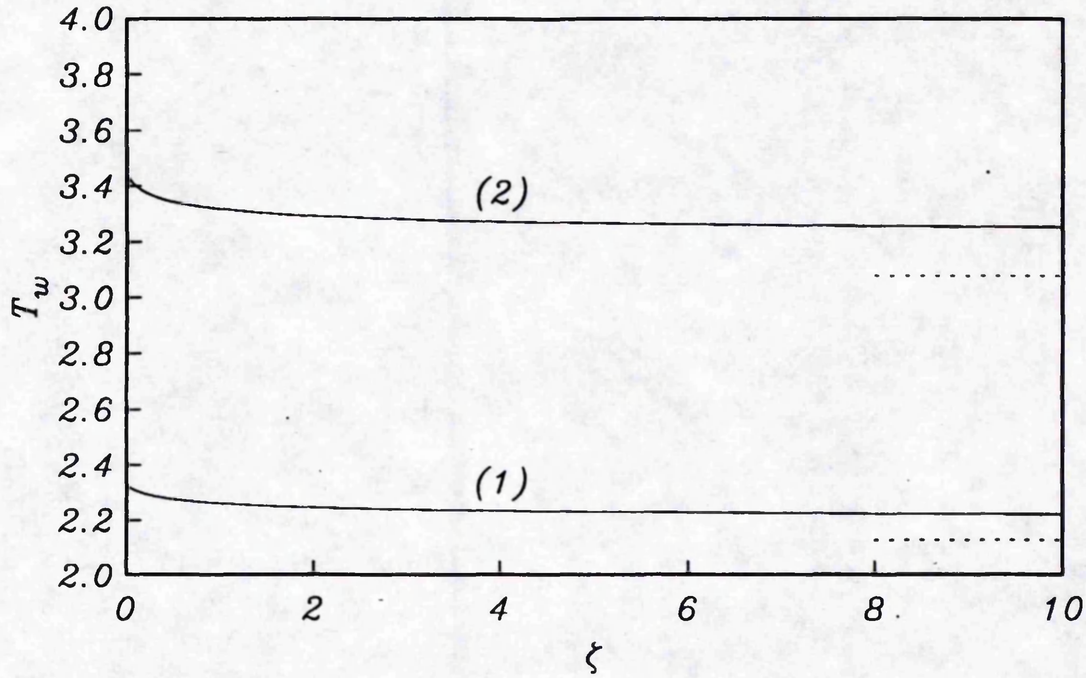


Figure 2.2: Axial wall temperature distributions for adiabatic cylinder.

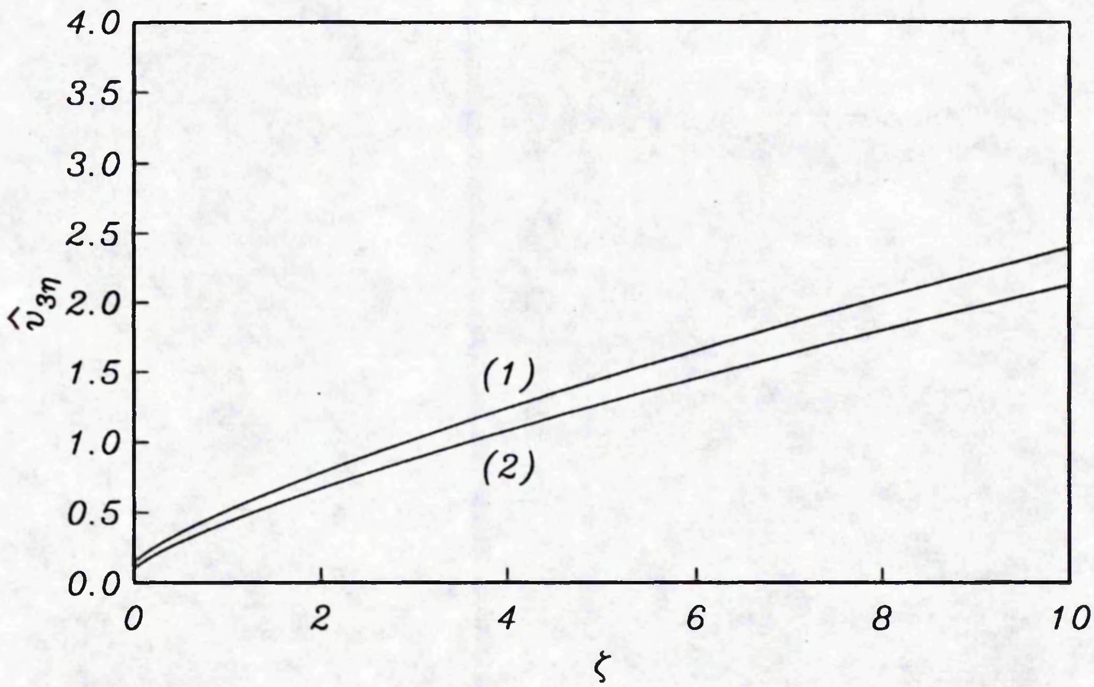


Figure 2.3: Axial distributions of $\hat{v}_{3\eta}|_{\eta=0}$ for adiabatic cylinder.

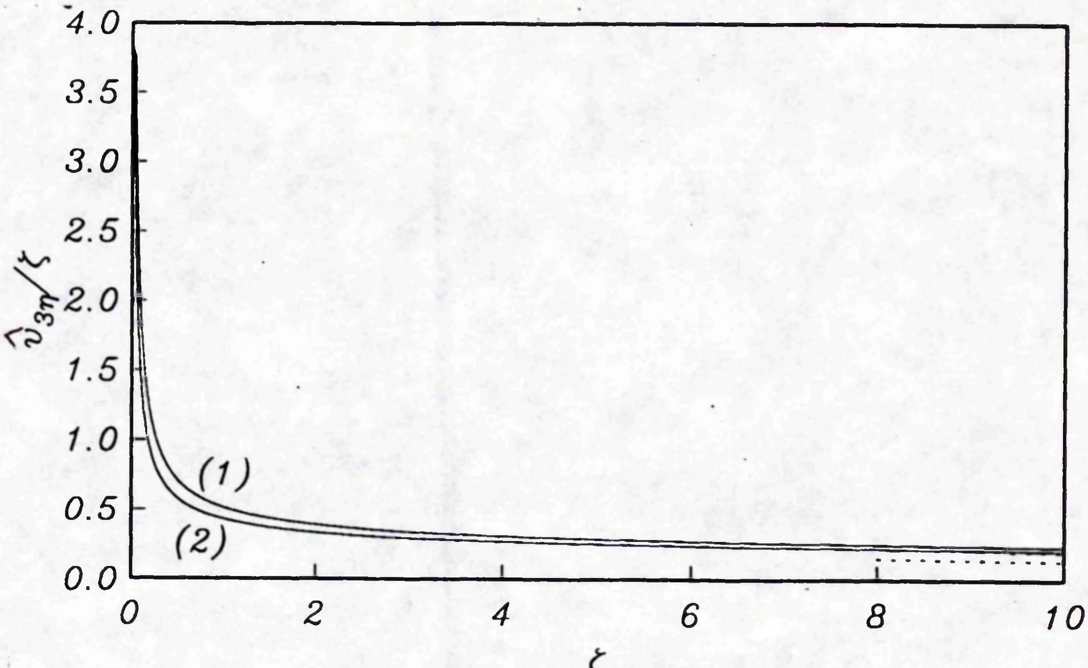


Figure 2.4: Axial distributions of $\zeta^{-1}\hat{v}_{3\eta}|_{\eta=0}$ for adiabatic cylinder.

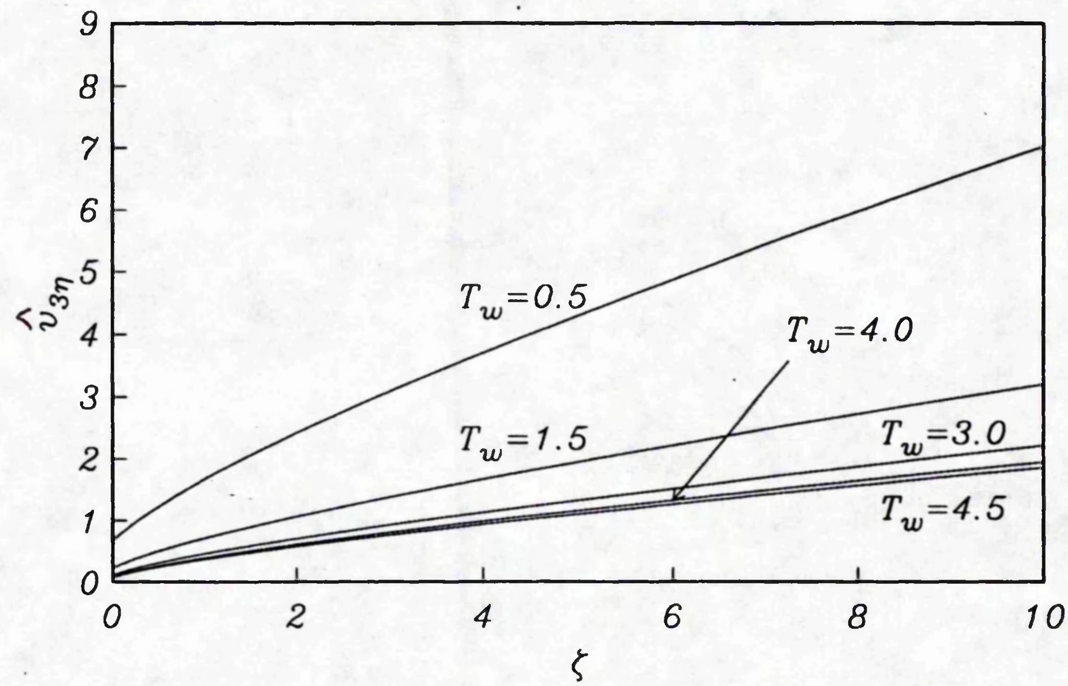


Figure 2.5: Axial distributions of $\hat{v}_{3\eta}|_{\eta=0}$ for heated/cooled cylinder.

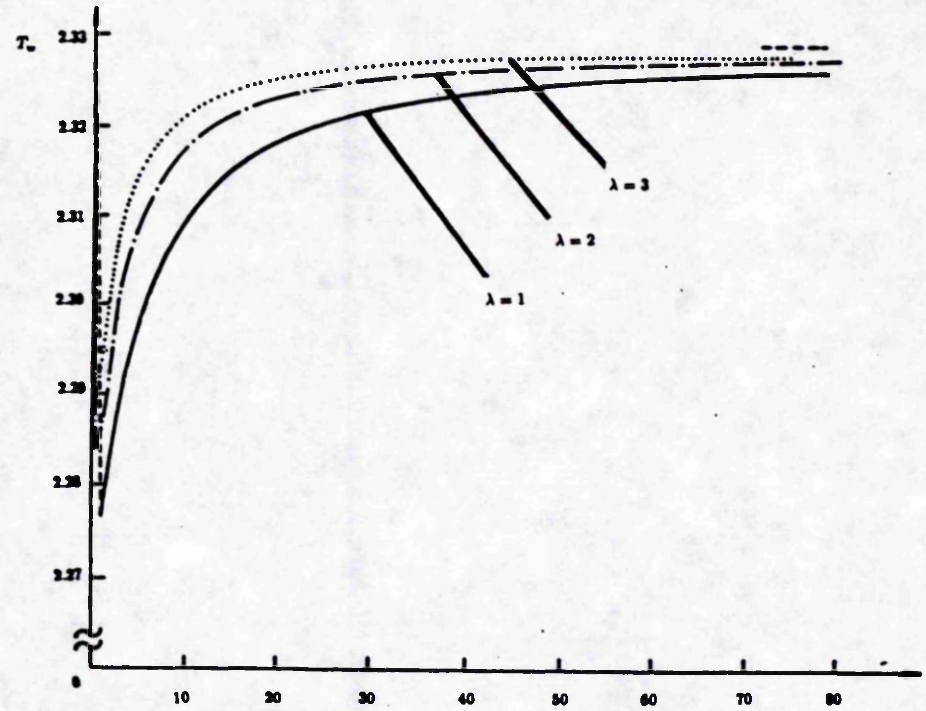


Figure 2.6: Axial wall temperature distributions for adiabatic cone, $M_\infty = 2.8$.

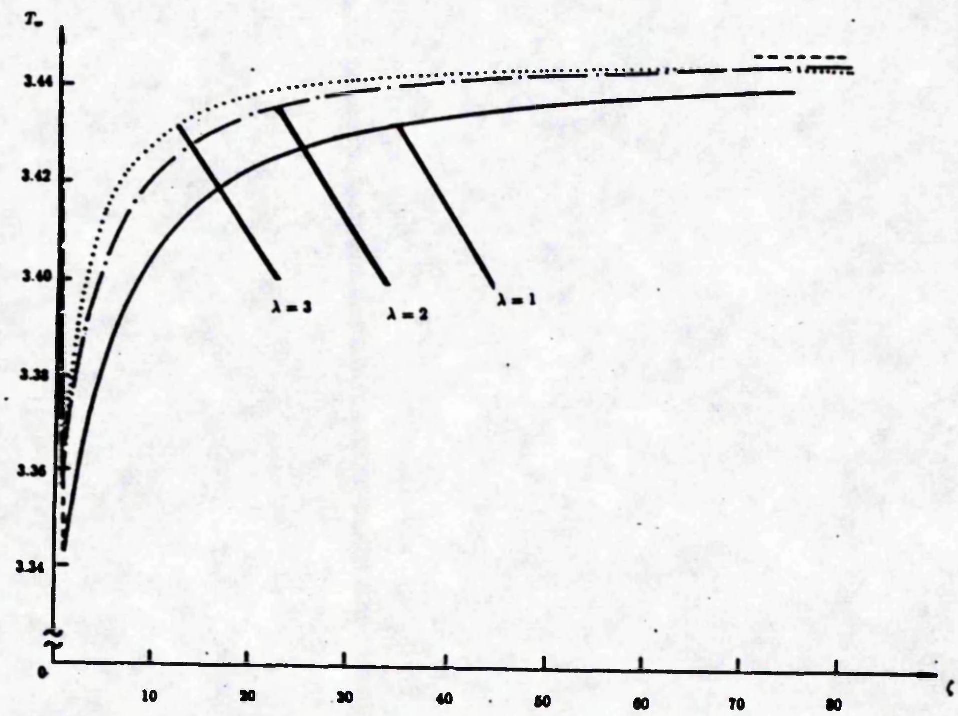


Figure 2.7: Axial wall temperature distributions for adiabatic cone, $M_\infty = 3.8$.

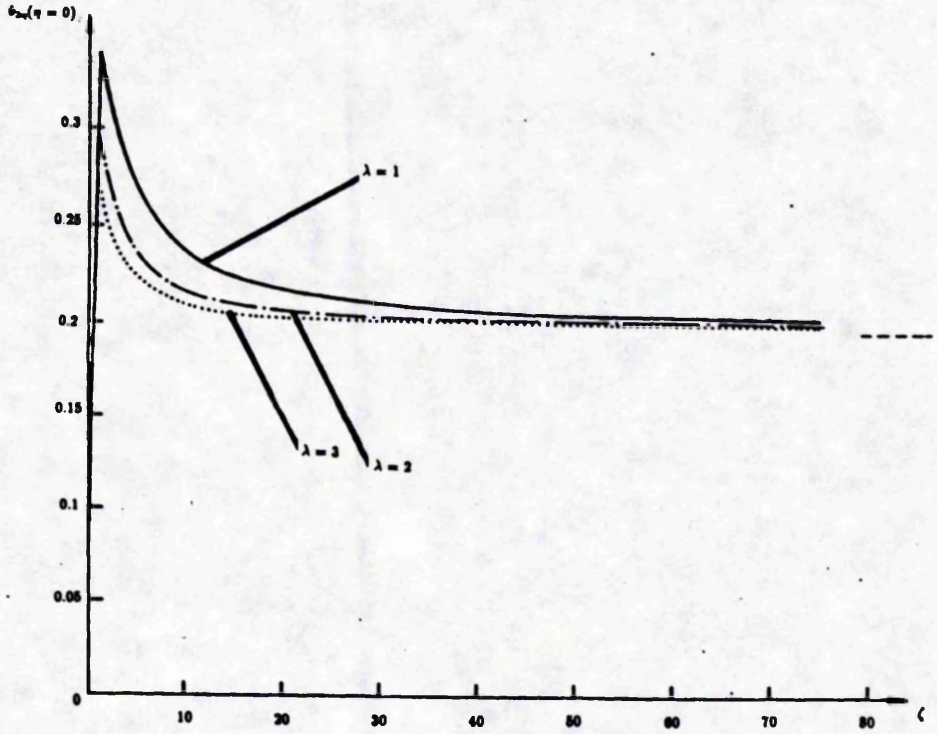


Figure 2.8: Axial distributions of $\hat{v}_{3\eta}|_{\eta=0}$ for adiabatic cone, $M_\infty = 2.8$.

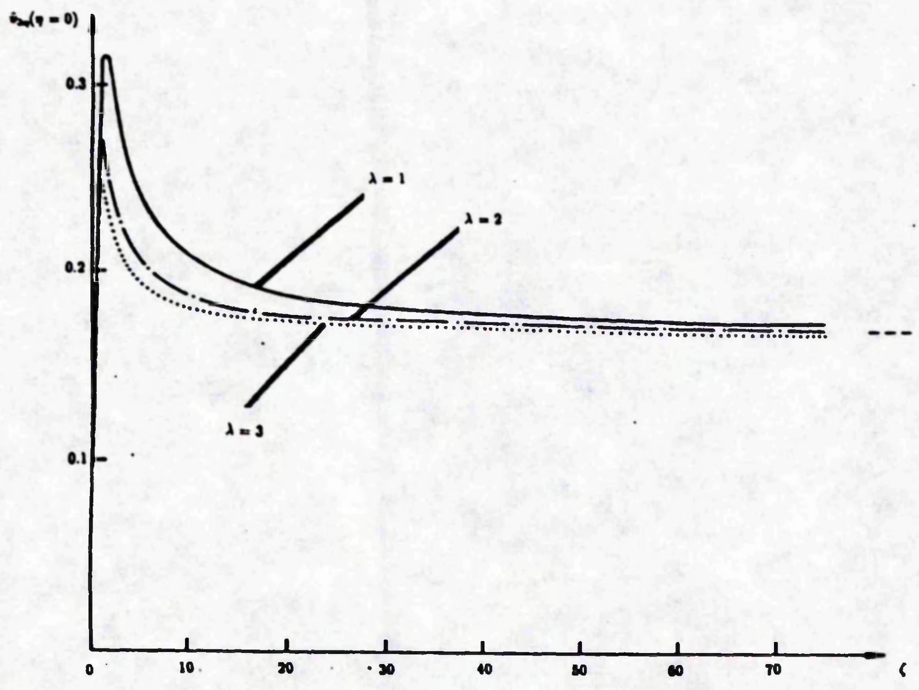


Figure 2.9: Axial distributions of $\hat{v}_{3\eta}|_{\eta=0}$ for adiabatic cone, $M_\infty = 3.8$.

Chapter 3

Linear Stability Theory

3.1 Inviscid Stability of the Flow

In order to study the stability of the basic flow determined in the previous chapter, we now investigate the effects of small amplitude disturbances. As mentioned in Chapter 1, for the supersonic boundary layer, inviscid disturbances are generally found to be more important (i.e. more unstable) than viscous disturbances. Therefore, the limit of infinite Reynolds numbers is assumed. It is also assumed that the disturbance wavelength is generally comparable to the boundary-layer thickness and therefore also of the (tip) radius of the cone ($O(a^*)$), in which case the parallel flow approximation is asymptotically correct. a^*, of course, is assumed small.

At a fixed $\overline{z}$ station the flow parameters of velocity, pressure, temperature and density are expressed as the sum of a mean flow term plus a small, first order disturbance term, i.e.

$$v_1^* = \delta\overline{\alpha}U_\infty^*\tilde{v}_1(r)E + O(\delta^2),$$

$$v_2^* = \delta U_\infty^*\tilde{v}_2(r)E + O(\delta^2),$$

$$v_3^* = U_\infty^*[w_0(r) + \delta\tilde{v}_3(r)E] + O(\delta^2),$$

$$T^* = T_\infty^*[T_0(r) + \delta\tilde{T}(r)E] + O(\delta^2),$$

$$\rho^* = \rho^*_\infty\Big[\frac{1}{T_0(r)} + \delta\tilde{\rho}(r)E\Big] + O(\delta^2),$$
$$p^* = \rho^*_\infty R^* T^*_\infty[1 + \delta\tilde{p}_1(r)E] + O(\delta^2), \tag{3.1}$$

where

$$E = \exp[i\overline{\alpha}(\hat{z} - ct) + in\theta], \tag{3.2}$$

and δ is the scale of the disturbance (taken to be diminishingly small), whilst

$$t = (\frac{U^*_\infty}{a^*})t^*,$$
$$\hat{z} = \frac{z^*}{a^*},$$
$$w_0(r) = \hat{v}_3(r, \overline{z}),$$
$$T_0(r) = \hat{T}(r, \overline{z}), \tag{3.3}$$

where $\hat{v}_3$ and $\hat{T}$ are determined from the computations in the previous section, $\overline{\alpha}$ is the non-dimensional spatial wavenumber, c the non-dimensional wavespeed and n the azimuthal wavenumber. We note the $\hat{z} = O(1)$ ($\Leftrightarrow z^* = O(a^*)$) scale is very much shorter than the basic flow scale $z = O(1)$ ($\Leftrightarrow \overline{z} = O(1) \Leftrightarrow z^* = O(a^* Re)$).

Substitution of the flow parameters into the full system of equations of continuity, momenta, energy and state (as defined in Chapter 2), discarding $O(\delta^2)$ terms and all but the largest terms in Re, yields the following linear system

$$-ic\tilde{\rho} + \frac{i\tilde{v}_3}{T_0} + \frac{1}{rT_0}\tilde{v}_1 + \frac{\tilde{v}_{1r}}{T_0} + in\frac{\tilde{v}_2}{rT_0\overline{\alpha}} + iw_0\tilde{\rho} - \tilde{v}_1\frac{T_{0r}}{T_0^2} = 0, \tag{3.4}$$

$$-\frac{ic}{T_0}\tilde{v}_3 + +i\frac{w_0\tilde{v}_3}{T_0} + \tilde{v}_1\frac{w_{0r}}{T_0} = -\frac{i\tilde{p}}{\gamma M^2_\infty}, \tag{3.5}$$

$$-\frac{ic}{T_0}\tilde{v}_2 + i\frac{w_0\tilde{v}_2}{T_0} = -\frac{in\tilde{p}}{\gamma M^2_\infty\overline{\alpha}r}, \tag{3.6}$$

$$-i\frac{\overline{\alpha}^2 c}{T_0}\tilde{v}_1 + i\frac{\overline{\alpha}^2 w_0 \tilde{v}_1}{T_0} = -\frac{\tilde{p}_r}{\gamma M_\infty^2}, \tag{3.7}$$

$$\frac{1}{T_0}[-ic\tilde{T} + iw_0\tilde{T} + \tilde{v}_1 T_{0r}] + \Big(\frac{\gamma-1}{\gamma}\Big)(ic\tilde{p} - iw_0\tilde{p}) = 0, \tag{3.8}$$

$$\tilde{p} = T_0\tilde{\rho} + \frac{\tilde{T}}{T_0}. \tag{3.9}$$

Writing

$$\tilde{v}_1 = \zeta\phi, \qquad \overline{\alpha} = \frac{\alpha}{\zeta}, \tag{3.10}$$

where ζ is defined by (2.50), after some algebra and using (2.56) we obtain

$$\phi_\eta + \frac{\zeta}{1+\lambda\zeta^2+\zeta\eta}\phi - \frac{w_{0\eta}\phi}{w_0 - c} = \frac{i\tilde{p}\Phi}{\gamma M_\infty^2(w_0-c)}, \tag{3.11}$$

together with

$$i\alpha^2(w_0-c)\frac{\phi}{T_0} = -\frac{\tilde{p}_\eta}{\gamma M_\infty^2}, \tag{3.12}$$

where

$$\Phi = T_0\Big[1 + \frac{n^2\zeta^2}{\alpha^2(1+\lambda\zeta^2+\zeta\eta)^2}\Big] - M_\infty^2(w_0-c)^2. \tag{3.13}$$

Equations (3.11) and (3.12) can be combined, eliminating $\tilde{p}$, to give

$$\frac{d}{d\eta}\Big\{\frac{(w_0-c)[\phi_\eta + (\zeta/(1+\lambda\zeta^2+\zeta\eta))\phi] - w_{0\eta}\phi}{\Phi}\Big\} = \frac{\alpha^2}{T_0}(w_0-c)\phi. \tag{3.14}$$

Alternatively ϕ may be eliminated to give

$$-\tilde{p}\Phi = \Big\{w_{0\eta} - \frac{\zeta(w_0-c)}{1+\lambda\zeta^2+\zeta\eta}\Big\}\frac{\tilde{p}_\eta T_0}{\alpha^2(w_0-c)} - (w_0-c)\frac{d}{d\eta}\Big\{\frac{T_0\tilde{p}_\eta}{\alpha^2(w_0-c)}\Big\}. \tag{3.15}$$

Equation (3.14) is very similar to the planar, inviscid, compressible disturbance equation, as obtained by Lees and Lin (1946), Lees and Reshotko (1962) and Mack (1984), the only difference being the inclusion of a non-axisymmetric term and a curvature term on the left-hand-side of the equation, in our case. Indeed, in the limit $\zeta \to 0$, the planar result can be recovered. We also observe that setting $n = 0$ (i.e. axisymmetric disturbances) and $\lambda = 0$ (zero cone angle), equations (3.14) and (3.15) reduce to those considered by Duck (1990). Similar results are to be found in the work on the stability of jets by Michalke (1971).

To close the problem appropriate boundary conditions need to be determined. On the surface of the cone we shall prescribe the impermeability condition, i.e.

$$\phi = \tilde{p}_\eta = 0 \qquad \text{on} \qquad \eta = 0. \tag{3.16}$$

The second condition is that ϕ be bounded as $\eta \to \infty$. This is achieved by considering equation (3.14) in this limit, i.e.

$$\phi_{\eta\eta} + \frac{\zeta\phi_\eta}{1+\lambda\zeta^2+\zeta\eta} - \frac{\phi\zeta^2}{(1+\lambda\zeta^2+\zeta\eta)^2} = \alpha^2[1 - M_\infty^2(1-c)^2]\phi, \tag{3.17}$$

which has the solution

$$\phi = \frac{1}{2}\phi_\infty\{K_{n+1}(\hat{\eta}) + K_{|n-1|}(\hat{\eta})\}, \tag{3.18}$$

where

$$\hat{\eta} = \pm\alpha[1 - M_\infty^2(1-c)^2]^{1/2}\Big(\frac{1}{\zeta} + \lambda\zeta + \eta\Big), \tag{3.19}$$

and $K_n(\hat{\eta})$ denotes the modified Bessel function of order n, the argument of which (i.e. the appropriate sign in (3.19)) is chosen to ensure boundedness in the far-field. ϕ_∞ is a constant. Substituting equation (3.18) into (3.12) gives the far-field boundary condition for the pressure disturbance term, namely

$$\tilde{p} \sim \mp \frac{\phi_\infty M_\infty^2 i\alpha\gamma(1-c)K_n(\hat{\eta})}{[1-M_\infty^2(1-c)^2]^{1/2}}. \tag{3.20}$$

In this problem, attention is focused on temporal stability for which the growth rate is αc_i, where c_i is the imaginary part of the wavespeed (most related work has also been temporal in nature, although there is no conceptual difficulty with the treatment of spatial stability). If $c_i > 0$ then the disturbance grows, if $c_i = 0$ the disturbance is neutral and if $c_i < 0$ the disturbance decays. The system of equations (3.11), (3.12) was solved using a fourth-order Runge-Kutta scheme for the eigenvalue c (generally complex), given n and α (real), subject to the boundary conditions (3.16), (3.18) and (3.20).

Consider a first order differential equation of the form

$$\frac{dy}{dx} = f(x, y). \tag{3.21}$$

The fourth-order numerical approximation to this equation using a Runge-Kutta scheme is

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4), \tag{3.22}$$

where

$$k_1 = f(x_n, y_n), \qquad k_2 = f(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_1),$$

$$k_3 = f(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_2), \qquad k_4 = f(x_n + h, y_n + hk_3), \tag{3.23}$$

and h represents the step size, which in our case is $\Delta\eta$.

We apply this method to the re-arranged equations

$$\phi_\eta = \frac{w_{0\eta}\phi}{w_0 - c} - \frac{\zeta\phi}{1 + \lambda\zeta^2 + \zeta\eta} + \frac{i\tilde{p}\Phi}{\gamma M_\infty^2(w_0 - c)}, \tag{3.24}$$

$$\tilde{p}_\eta = -\frac{i\alpha^2(w_0 - c)}{T_0}\gamma M_\infty^2 \phi, \tag{3.25}$$

where shooting begins at a suitably large value of η - where conditions (3.18) and (3.20) are applied - and the computation proceeds inwards towards $\eta = 0$. A suitable value of c is chosen in order to satisfy the impermeability condition on the surface of the cone. This is achieved by making a sensible initial guess for c and then by means of a Newton iteration scheme, the shooting process is repeated until the difference in c, between successive iterations, is sufficiently small to warrant convergence.

It should be noted that (3.24) possesses a simple pole at the point in the boundary layer where $w_0 = c$, i.e. at the the critical point. Since w_0 is always real this implies that the singularity will lie on the real axis in the complex η plane for neutral disturbance terms, i.e. $c_i = 0$. However, owing to the smallness of c_i in a number of the numerical calculations it is found necessary to divert the computation below the real η-axis (for neutral and damped disturbances only) in the neighbourhood of the critical layer. The technique used is based on the methods of Zaat (1958) and Mack (1965a).

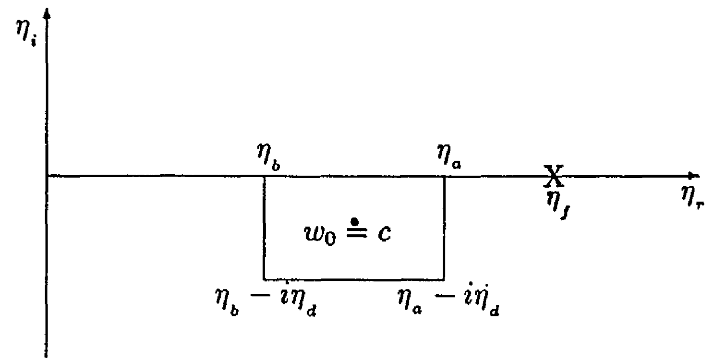
To continue the mean flow terms w_0, $w_{0\eta}$ and T_0 onto the indented contour, these are expressed as truncated power series, i.e. they are written in the form

$$w_0 = w_c + w_c'(\eta - \eta_c) + \frac{1}{2}w_c''(\eta - \eta_c)^2 + \frac{1}{6}w_c'''(\eta - \eta_c)^3 + O((\eta - \eta_c)^4),$$

$$w_{0\eta} = w_c' + w_c''(\eta - \eta_c) + \frac{1}{2}w_c'''(\eta - \eta_c)^2 + O((\eta - \eta_c)^3),$$

$$T_0 = T_c + T_c'(\eta - \eta_c) + \frac{1}{2}T_c''(\eta - \eta_c)^2 + \frac{1}{6}T_c'''(\eta - \eta_c)^3 + O((\eta - \eta_c)^4), \tag{3.26}$$

where the contour of integration has the form



The integration is started at some suitable point in the far-field η_f and continues inward on the real axis to η_a. We then follow the rectangular indentation below the real η axis to the point η_b, where the integration proceeds to $\eta = 0$ along the real axis once more. This method is found to be highly accurate.

Before carrying out a detailed numerical study of the eigenvalue problem posed above, we shall derive an important necessary condition for the possible existence of unstable modes, using the method of Lees and Lin (1946).

3.2 The 'Triply Generalized Inflexion Condition'

In stability theory, great emphasis is placed on the condition for the existence of neutral disturbance terms. Lord Rayleigh (1880), derived a condition for the existence of a neutral term in the case of incompressible flow, which he termed the 'inflexional condition', namely

$$w_{0\eta\eta}\Big|_{\eta=\eta_i} = 0, \tag{3.27}$$

(where η_i denotes the critical point) which Tollmien (1929) later demonstrated was also a sufficiency condition. It is also found that if the incompressible flow is to be unstable to disturbances, a point of inflexion must exist in the velocity profile at some point within the flow, i.e. (3.27) must be satisfied.

A series of authors, including Lees and Lin (1946)

generalized the inflexion condition to give a condition for the existence of neutral disturbance terms in compressible planar flows, namely

$$\frac{d}{d\eta}\Big[\frac{w_{0\eta}}{T_0}\Big]_{\eta=\eta_i} = 0. \tag{3.28}$$

Equation (3.28) also provides a necessary condition for the existence of subsonic amplified disturbance terms in the supersonic planar boundary layer.

In 1990, Duck generalized the inflexion condition further to include curvature terms, in the case of supersonic flow along a thin cylinder, determining an axisymmetric generalized condition (or 'doubly generalized' inflexion condition) of the form

$$\frac{d}{d\eta}\Big[\frac{w_{0\eta}}{T_0(1+\eta\zeta)}\Big]_{\eta=\eta_i} = 0. \tag{3.29}$$

We now wish to determine a corresponding condition for non-axisymmetric modes in the case of supersonic flow past a sharp cone. Multiplying equation (3.14) by $\phi^*/(w_0 - c)$ (where an asterisk denotes a complex conjugate) yields

$$\frac{\alpha^2}{T_0}\phi\phi^* = \frac{\phi^*}{w_0 - c}\frac{d}{d\eta}\Big\{\frac{(w_0-c)(\phi_\eta + (1/R)\phi) - w_{0\eta}\phi}{\chi}\Big\}, \tag{3.30}$$

where we have written

$$\chi = T_0\Big\{1 + \frac{n^2}{\alpha^2 R^2}\Big\} - M_\infty^2(w_0-c)^2, \tag{3.31}$$

and

$$R = \frac{1+\lambda\zeta^2+\zeta\eta}{\zeta}. \tag{3.32}$$

Subtracting the complex conjugate of equation (3.30) from itself gives

$$\frac{\phi^*}{w_0 - c}\frac{d}{d\eta}\left\{\frac{(w_0 - c)(\phi_\eta + (1/R)\phi) - w_{0\eta}\phi}{\chi}\right\} =$$
$$\frac{\phi}{w_0 - c^*}\frac{d}{d\eta}\left\{\frac{(w_0 - c^*)(\phi^*_\eta + (1/R)\phi^*) - w_{0\eta}\phi^*}{\chi^*}\right\}. \tag{3.33}$$

After some algebra, (3.33) can be written

$$\phi^*\frac{d}{d\eta}\left\{\frac{\phi_\eta + (1/R)\phi}{\chi}\right\} - \phi\frac{d}{d\eta}\left\{\frac{\phi^*_\eta + (1/R)\phi^*}{\chi^*}\right\} =$$
$$R\phi\phi^*\left\{\frac{1}{w_0 - c}\frac{d}{d\eta}\left[\frac{w_{0\eta}}{\chi R}\right] - \frac{1}{w_0 - c^*}\frac{d}{d\eta}\left[\frac{w_{0\eta}}{\chi^* R}\right]\right\}. \tag{3.34}$$

Writing

$$c = c_r + ic_i, \tag{3.35}$$

then for neutral disturbances we consider the limit $c_i \rightarrow 0$. In this limit we can automatically deduce $\chi = \chi^*$, since n and α are both real by the temporal approach being considered. However, the limit must be applied with more care on the right-hand-side of the equation. We re-write equation (3.34) in the form

$$\frac{1}{R}\frac{d}{d\eta}\left\{\frac{R[\phi^*(\phi_\eta + (1/R)\phi) - \phi(\phi^* + (1/R)\phi^*)]}{\chi}\right\} = 2i\frac{R|\phi|^2 c_i}{|w_0 - c|^2}\frac{d}{d\eta}\left\{\frac{w_{0\eta}}{\chi R}\right\}. \tag{3.36}$$

In the limit $c_i \rightarrow 0$, the left-hand-side of equation (3.36) will tend to zero, except possibly at the point η_i, where $w_0 = c$, i.e. at the critical point. By the impermeability condition the term inside the parenthesis must be zero at the wall ($\eta = 0$) and asymptote to zero at infinity if the wave under consideration is to be subsonic, as this form of wave must be bounded in the far-field (Lees and Lin (1946)). Since the derivative term is always zero (except possibly at the critical point), then the term inside the parenthesis must be constant and by the form of the boundary conditions this constant must be zero.

Examining the right-hand-side of equation (3.36) again, it is noted that this acts as a delta function as $c_i \rightarrow 0$. This implies that the term in parentheses on the left-hand-side of (3.36) undergoes a finite jump. This clearly leads to contradication, unless the right-hand-side is zero at the critical point. This requires (re-casting the equation in terms of the original variables)

$$\frac{d}{d\eta}\left\{\frac{w_{0\eta}}{T_0[1+\lambda\zeta^2+\zeta\eta][1+n^2\zeta^2/(\alpha^2(1+\lambda\zeta^2+\zeta\eta)^2)]}\right\}_{\eta=\eta_i}=0. \tag{3.37}$$

This result represents a further generalization of the so-called 'doubly generalized inflexion condition' as determined by Duck(1990); setting $n=\lambda=0$ retrieves Duck's results. It should be noted that the general inflexion condition for any shape of cone surface described by

$$r=1+\lambda f(\zeta), \tag{3.38}$$

is easily determined by replacing '$\lambda\zeta^2$' in (3.37) by '$\lambda f(\zeta)$'.

We now move on to numerically solve the eigenvalue problem in the next section.

3.3 Numerical Results

Clearly there are many choices of parameter that can be made in this study. The strategy here is to carry out a detailed study for one choice of Prandtl number (0.72), ratio of specific heats (1.4) and in the case of the cone, one cone angle ($\lambda=1$). We begin the study by considering the stability of the compressible boundary layer formed on a cylinder, subject to adiabatic wall conditions, for a Mach number of 3.8 (Figs 3.1 - 3.22). Extensive results are given in this subsection for a range of values of n, at fixed values of ζ. At the end of the subsection we present some results for $M_\infty = 2.8$, for comparison (Figs 3.23 - 3.26). Most of subsection 3.3.2 will be devoted to the effects of wall-cooling on the stability of compressible boundary layers and its interaction with curvature as these conditions are likely to be of interest in important applications (for example, in the case of high-speed flight vehicles frictional forces result in metal heating which in turn causes buckling and corrosion). Note that here, we shall refer to the term 'wall cooling' as being relative to the adiabatic wall temperature. The main body of results presented are for a Mach number, $M_\infty = 3.8$ (Figs 3.27 - 3.41), although we also present a number of results for $M_\infty = 2.8$ (Figs 3.42 - 3.49) for comparison. At the end of this subsection we shall give some heated wall results (Figs 3.50 - 3.55) which exhibit additional interesting physical features. In the last subsection, the stability characteristics of supersonic flow along a somewhat more practical configuration, namely a sharp cone, are presented (Figs 3.56 - 3.71). For this stability problem, the results presented are for one Mach number ($M_\infty = 3.8$) and for insulated wall conditions.

All numerical computations were carried out on the Amdahl VP1100. All the results may be regarded as being independent of numerical grid. Generally, two grid sizes were used to check consistency, namely $\Delta\eta = 0.0046875$ extending out to $\eta = 30$, together with $\Delta\eta = 0.00234375$ extending out to $\eta = 15$. For the far downstream results (for example, $\zeta = 20.0$ and $\zeta = 75.0$), the grid in ζ is coarser than nearer the cone tip due to the maximum limit of 300 minutes CPU time allowed on the Amdahl VP1100. For $\zeta < 20.0$, $\Delta\zeta = 0.0005$, while for $\zeta = 20.0$, $\Delta\zeta = 0.0125$ and for $\zeta = 75.0$, $\Delta\zeta = 0.0395$.

3.3.1 Adiabatic Cylinder Results

We begin by attempting to numerically determine the existence of non-axisymmetric generalized inflexion points. In section 3.2 a condition was derived for the existence of the so called 'triply generalized' inflexion points. In the case of the cylinder the

$\zeta = 0.01$	α	η_1	w_1	η_2	w_2
$n = 1$	0.139165	0.8349	0.0854	8.2513	0.8440
	0.398645	0.8388	0.0858	8.2507	0.8440
$n = 3$	0.140835	0.8014	0.0820	8.2566	0.8445
	0.399315	0.8345	0.0853	8.2514	0.8440
$n = 5$	0.143525	0.7447	0.0762	8.2655	0.8453
	0.400665	0.8259	0.0845	8.2527	0.8442
$n = 8$	0.147715	0.6417	0.0657	8.2820	0.8467
	0.403995	0.8063	0.0825	8.2558	0.8444

Table 3.1: Triply Generalized Inflexion Points at $\zeta = 0.01$

non-axisymmetric inflexion condition has the reduced form ($\lambda = 0$)

$$\frac{d}{d\eta}\left\{\frac{w_{0\eta}}{T_0(1+\eta\zeta)[1+(n^2\zeta^2/\alpha^2(1+\eta\zeta)^2)]}\right\}_{\eta=\eta_i} = 0. \tag{3.39}$$

In previous works (Duck (1990), for example) numerical solutions of the mean flow yielded continuous plots of the radial position, η_i, at which inflexion points occur, against ζ and these plots clearly demonstrate the existence and behaviour of the inflexion points. However, for non-axisymmetric disturbances it can be clearly seen that due to the inter-dependence of the variables n, α and ζ, it is difficult to forecast, prior to any numerical investigation of the stability equations, the existence of neutral stability points of this kind. Equation (3.39) could be solved in a idealistic manner where for given mean flow characteristics, values of the ratio n/α which satisfy (3.39) could be determined. However, if solutions do exist, the value of α (for a given n) must still be determined by a full numerical solution of the disturbance equations. This has been conducted for the adiabatic cylinder case, although it is found that due to the discrete and almost unique form of the results, these are best presented in tabular form.

Table 3.1 shows the radial position of the non-axisymmetric inflexion points and the corresponding values of $w_0(\eta_i)$, at the location $\zeta = 0.01$, for the displayed

$\zeta = 0.05$	α	η_1	w_1	η_2	w_2
$n = 1$	0.127125	2.7123	0.3231	7.2544	0.8218
	0.517785	3.0499	0.3620	7.1936	0.8160
$n = 3$	0.152865	1.5936	0.1927	7.4410	0.8393
	0.529265	2.8776	0.3422	7.2251	0.8190
$n = 5$	0.155735	0.8283	0.1015	7.5777	0.8518
	0.553865	2.6111	0.3114	7.2720	0.8235
$n = 8$	0.086265	0.1198	0.0149	7.7411	0.8662
	0.614795	2.2237	0.2665	7.3371	0.8297

Table 3.2: Triply Generalized Inflexion Points at $\zeta = 0.05$

$\zeta = 0.1$	α	η_1	w_1	η_2	w_2
$n = 1$	0.123635	3.0902	0.4126	6.5532	0.8091
	0.740695	4.4630	0.5781	6.1450	0.7672
$n = 3$	0.163505	1.0607	0.1507	6.9524	0.8477
	0.752825	4.0859	0.5334	6.2864	0.7820
$n = 5$	0.118255	0.2071	0.0305	7.1895	0.8692
	0.810095	3.5914	0.4739	6.4330	0.7970

Table 3.3: Triply Generalized Inflexion Points at $\zeta = 0.1$

azimuthal wavenumbers. For each value of n we observe that there are two values of α for which $c_i = 0$ (where $c = c_r + ic_i$). Note that a third neutral mode exists where the generalized inflexion condition is satisfied trivially, namely $c = 1$: this mode shall be considered more fully when the eigenvalue problem is treated. As in the axisymmetric work of Duck (1990), the non-axisymmetric points occur in pairs, the upper points (η_2) being an extension of the doubly generalized inflexion condition. In light of our earlier comments, we do have an additional condition for the existence of neutral subsonic disturbances, namely that

$$1 - \frac{1}{M_\infty} < c < 1 + \frac{1}{M_\infty}. \tag{3.40}$$

This has direct implications on the first mode of instability as this generally requires the presence of a generalized inflexion point in the profile satisfying (3.40). Therefore, we note, that for all the azimuthal numbers presented at this axial location, subsonic generalized inflexional modes of instability will occur.

It is observed that an increase in n causes the lower inflexion point (η_1) to approach the surface of the cylinder for both values of α, while the upper inflexion points occur at increasingly large radii. From numerical evaluations carried out for higher azimuthal wavenumbers (not displayed here), it is observed that subsonic generalized inflexion points occur for azimuthal wavenumbers as high as $n = 29$, for this axial location.

Moving along the cylinder to the axial position, $\zeta = 0.05$, Table 3.2 displays values of η_i and $w(\eta_i)$ for the non-axisymmetric generalized inflexion points corresponding to same set of azimuthal wavenumbers. Again, it is found that for each value of n there exists two neutral modes, each corresponding to a pair of generalized inflexion points. Movement downstream has resulted in the lower generalized inflexion points occuring further from the wall, except for the $n = 8$ lower inflexion point corresponding to the smaller value of α, whilst the upper points occur at smaller radii; this effect which is due to curvature was also noted by Duck (1990) for the axisymmetric case. At this location, an increase in n, again, causes the inflexion points to diverge. This divergence is so marked for the lower inflexion points corresponding to the smaller values of α, that the effect dominants the convergence effect of curvature for the $n = 8$ generalized inflexional point, causing this point to occur closer to the cylinder surface than in the corresponding $\zeta = 0.01$ case, as previously observed. For azimuthal wavenumbers higher than $n = 8$, the triply generalized inflexional points are found to disappear. Consequently, it is expected mode I type instabilities will only be present for $n < 9$.

The final set of non-axisymmetric generalized inflexion point results presented are for the position $\zeta = 0.1$. The same trends as before are observed, with curvature causing the inflexion points to approach one another, whilst an increase in azimuthal wavenumber causes divergence, with the divergence effect dominating for the $n = 3$ and $n = 5$ lower inflexion points corresponding to the smaller values of α.

Duck (1990) determined that neutral subsonic axisymmetric inflexional modes will disappear approximately $0.013C^{-1}\mathrm{Re}$ body radii ($\zeta \approx 0.11$) downstream of the leading edge. Non-axisymmetric modes are found to be much more persistent, especially for small azimuthal wavenumbers. The first three non-axisymmetric modes are clearly present at $\zeta = 0.5$. The $n = 2$ mode disappears around $\zeta = 5.18$, while the $n = 1$ generalized inflexional mode is found to be still present for downstream locations as large $\zeta = 75.0$.

We now turn our attention to the eigenvalue problem. We shall focus our attention on unstable mode results. Initially, plots of c_r, c_i and the temporal growth rate, αc_i, against α will be presented to clarify the points being made, but the majority of the results presented in this subsection will be for variations of the growth rate with α only, as this is considered the important quantity as far as stability theory is concerned.

For planar flows, Mack (1984, 1987a) determined that there exists two important modes of instability which he termed 'Mode I' and 'Mode II' respectively. Figures 3.1, 3.2 and 3.3 display variations of c_r, c_i and αc_i with α respectively corresponding to the $\zeta = 0$ location (and hence corresponds to the planar case), for Mach numbers of $M_\infty = 2.8$ and $M_\infty = 3.8$, denoted by curves (1) and (2) respectively, where broken lines represent mode I results, and continuous lines represent mode II results. It is found that the planar results agree favourably with previously computed results (Mack (1987a), for example) and thus provides a useful check on the accuracy of the numerical scheme (which is found to be entirely satisfactory). Considering the $M_\infty = 2.8$ results first, we note the first mode of instability originates as a sonic neutral mode (with $c_i = 0$, $c_r = 1 - 1/M_\infty$), for $\alpha = 0$, rises to a maximum and terminates as a subsonic generalized inflexional neutral mode at $\alpha \approx 0.1$, where $c_r = w_0(\eta_i) \approx 0.73$. This mode is found to continue as a decaying disturbance. The mode II instability originates as a subsonic neutral mode at $\alpha \approx 0.4$, rises to a maximum and terminates at $\alpha \approx 1.13$ as a (second) subsonic generalized inflexional neutral instability. The neutral mode at which the mode II instability originates is special in that $c = (c_r = w_0(\eta_i)) = 1$; this corresponds to a critical layer in the freestream (where there is a trivial satisfaction of the generalized inflexion point condition). The second mode of instability is found, also, to continue as a decaying mode ($c_i < 0$) for larger values of α. Turning our attention to the $M_\infty = 3.8$ results we observe the same qualitative features as in the $M_\infty = 2.8$ case. We note that an increase in Mach number has significantly increased the importance of mode II, although the growth rates of mode I have also increased.

Duck (1990) determined that introducing curvature terms into the linear disturbance problem, has a stabilizing effect on both modes, causing the first mode of instability to ultimately disappear, and greatly reducing the amplification rates of the second mode. In this section Duck's (1990) work is extended to include non-axisymmetric disturbances.

The first set of results including curvature terms which we present, corresponds to a location very close to the cylinder tip, namely $\zeta = 0.01$. Figures 3.4, 3.5 and 3.6 correspond to variations of c_r, c_i and αc_i with α respectively, for the mode I instability, and for the azimuthal wavenumbers as shown. The axisymmetric (i.e. $n = 0$) mode, as expected, has the same qualitative features as the planar mode I, but even though these results are at a location very close to the cylinder tip, the growth rates have undergone reduction due to curvature. A further effect of curvature, is that the neutral point at which the instability originates, has been shifted slightly along the α-axis, occuring at a very small positive value of α. It is found that even though c_r at this point is very close to the sonic value, the numerical observations indicate that the mode has become very slighly supersonic in nature. Because α here is non-zero (albeit small), and this particular type of neutral mode is not associated with a generalized inflexion point, it *must* be supersonic in nature. The use of conditions (3.18) and (3.20) permits outgoing (or indeed incoming) waves at infinity, and consequently such modes present no difficulty to our numerical scheme. Indeed the slightly supersonic nature of these modes may be confirmed asymptotically by the work of section 3.5.

The first non-axisymmetric instability considered corresponds to an azimuthal wavenumber, $n = 1$. The instability resembles the axisymmetric mode in that it originates as a very slightly supersonic mode, occuring at small positive α, rises to a maximum and terminates as a subsonic generalized inflexional neutral mode. It is found that this mode is slightly more unstable than the axisymmetric mode.

The instability corresponding to an azimuthal wavenumber, $n = 3$, has a slightly different structure. Firstly the lower neutral mode occurs at $\alpha = 0$, and from Figure 3.4 c_r can be clearly seen to be quite supersonic in nature ($c_r \approx 0.4829$). Figure

3.5 shows that c is complex at this neutral mode ($c_i \approx 0.1063$). The instability in Figure 3.5 also differs in that it possesses two peaks; the significance of this will be discussed later. The mode terminates as before, as a subsonic neutral mode. It is observed that an increase in azimuthal wavenumber has resulted in the growth rates also being increased. This trend of an increase in azimuthal wavenumber causing a destabilization of mode I is repeated for the $n = 5$ and $n = 8$ results, which we present, although in the latter case, it is noted that initially in α, the value of c_i is less than those of the $n = 3$ and $n = 5$ instabilities. Both the $n = 5$ and $n = 8$ instabilities originate as supersonic neutral modes (for $c_i \neq 0$), although it is found that c_r becomes less supersonic in nature. The increase in azimuthal wavenumber also has the effect of smoothing out one of the peaks observed in the $n = 3$ case; in the case of the $n = 5$ mode there is only a slight hint of the second peak, whilst for the $n = 8$ results, it is found to have completely disappeared.

Figures 3.7, 3.8 and 3.9 display variations of c_r, c_i and αc_i with α, corresponding to the mode II instability for this axial location and azimuthal wavenumbers. As with the mode I instability, the axisymmetric mode II instability resembles the corresponding planar result, originating as a subsonic generalized inflexional mode ($c_r = c = w_0(\eta_i) = 1$) and terminating as a (second) subsonic inflexional neutral mode. The stabilizing effect of curvature in this case has resulted in the maximum axisymmetric growth rates being approximately halved. The introduction of non-axisymmetric terms is found initially to have no major effect and it is only when higher azimuthal wavenumbers are considered that any appreciable stabilization of the mode is observed. Indeed, there is found to be virtually no difference between the axisymmetric and $n = 1$ instability growth rates; generally, however, increase in azimuthal wavenumber causes a growth rate reduction (in contrast to the observed situation for the mode I results presented). It should be noted that the (second) generalized inflexional modes at which the non-axisymmetric disturbances terminate, can all be clearly seen to be subsonic from Figure 3.7, and in all cases the modes continue as decaying modes.

Other modes of instability are found to exist at this Mach number for larger values of α (as determined by Mack (1965b, 1987a) for the planar case), but have considerably smaller growth rates than the modes I and II shown here, and are consequently much less important from a practical point of view. For small α, it was noted that the axisymmetric and $n = 1$ modes I originate at very small positive α. In the case of the $n = 1$ results, it is found that when even smaller α is considered, a third mode of instability is seen to develop, not present in the corresponding axisymmetric (and indeed planar) results. We refer to this additional mode as mode $\mathrm{I_A}$, distributions of c_r, c_i and αc_i being shown in Figures 3.10, 3.11 and 3.12 respectively, for $n = 1$ and $n = 2$. One important distinction between the $n = 0$ results and those for $n \neq 0$ emerges in the limit as $\alpha \to 0$, for which $c_i \not\to 0$ if $n \neq 0$ (the limit as $\alpha \to 0$, $\zeta = O(1)$ is considered in Appendix B), although of course the temporal growth/decay rate αc_i is none the less zero at $\alpha = 0$. As α increases, mode $\mathrm{I_A}$ rises to a maximum and then quickly terminates as a supersonic neutral point (i.e., where $c_r < 1 - 1/M_\infty$). The $n = 2$ distribution is qualitatively similar to those of $n = 1$, although the the maximum growth rate is more unstable by a factor of about 8.7.

Comparing the growth rates of the modes I and $\mathrm{I_A}$, the mode I instability is found to be more important. For the $n = 3$ case, modes I and $\mathrm{I_A}$ are seen to amalgamate - hence the observed double peak structure in Figure 3.6. The two peaks would correspond to the maximum growth rates of the mode I and $\mathrm{I_A}$ instabilities if the modes were still distinct. The amalgamation also explains why the mode I originates at $\alpha = 0$, with complex c, for azimuthal wavenumber values of 3 and higher. Comparing Figures 3.4 and 3.10 it is observed that an increase in azimuthal wavenumber, increases the value of c_r for the neutral mode at $\alpha = 0$.

It is not surprising that, at $\zeta = 0.01$, the results obtained are very similar to corresponding planar results, as curvature will generally play a minor role in the physics at this location; indeed, a crude examination of (3.14) suggests that, as $\zeta \to 0$, the corresponding planar Rayleigh equation is attained. However, as $\alpha \to 0$ and $\zeta \to 0$, a nonuniformity is present; this aspect is taken up in some detail in section 3.5, where further light is shed on the additional mode $\mathrm{I_A}$.

Moving along the cylinder to the axial location, $\zeta = 0.05$, Figures 3.13 and 3.14 show variations of the temporal growth rate with wavenumber for the mode I and mode II instabilities respectively. Comparing Figures 3.13 and 3.6, the axisymmetric mode I is found to have been slightly stabilized due to the increasing effect of curvature, resulting from movement downstream (as noted by Duck (1990)). Introducing non-axisymmetric terms, we note that the $n = 1$ modes I and $\mathrm{I_A}$ have just combined, the two maximum growth rate peaks still being very prominent. Curvature is found to have a slight destabilizing effect on the $n = 1$ results. This destabilization effect is more marked for the $n = 3$ instability, the maximum rates having been enhanced by a factor of approximately three. For higher azimuthal wavenumbers, it is found that curvature has the reverse effect, re-stabilizing the mode I instability, although the $n = 5$ maximum growth rate is clearly more unstable than the corresponding $\zeta = 0.01$ mode.

The stabilization effect of curvature, at this axial location is more noticeable for the mode II instabilities, the maximum growth rates of the axisymmetric mode having been reduced by 1.3. Introduction of non-zero azimuthal terms can be clearly seen to have a further stabilizing effect, the maximum growth rate of the $n = 8$ mode being 11 times less than the axisymmetric mode.

The next set of results considered corresponds to the axial location, $\zeta = 0.1$. Comparing Figure 3.15 with 3.13, the axisymmetric mode I instability has undergone further stabilization due to curvature. Again, curvature is found to have a destabilizing effect for low azimuthal wavenumbers; the $n = 1$ result is clearly more unstable, although the double peak structure is less pronounced. The $n = 3$ modes I are very similar at both ζ-stations, but the mode I undergoes rapid stabilization for higher azimuthual wavenumbers, at the new axial location, finally becoming a stable mode for $n = 6$. This complete eradication of mode I instabilities is only to be expected since, as has already been noted, for azimuthal wavenumbers of $n = 6$ and higher, a subsonic triply generalized inflexion point does not exist at this axial location.

The mode II instabilities at $\zeta = 0.1$ (Figure 3.16), are qualitatively similar to those presented nearer the cylinder tip, although as before, they are found to have undergone further stabilization due to curvature. Due to the special nature of the generalized inflexional point at which the mode II instabilities originate and the fact that it is always present irrespective of axial location and value of azimuthal wavenumber, the $n = 8$ mode is still present in this case, unlike the situation encountered with the mode I instabilities. In fact, for large values of n the mode II instability is found to persist (this is found to be true for all axial locations considered), but with much diminished growth rates. In the next section we shall consider the asymptotic structure of the disturbance equations in this limit. It should be noted that the $n = 8$ instability terminates as a supersonic neutral mode, and does not continue as a decaying instability. This is found to be true for all modes where a triply generalized inflexion mode no longer exists.

At $\zeta = 0.5$, the axisymmetric mode has been completely stabilized as determined by Duck (1990). From Figure 3.17, the $n = 1$ instability is found to be the most unstable mode I; indeed this mode is more unstable than the corresponding $\zeta = 0.1$ result. Again, the mode II disturbances (Figure 3.18) show growth rate reductions due to the stabilizing effect of both curvature and increase in azimuthal wavenumber on these types of instability.

For larger distances downstream, the above trends are repeated. As noted at the beginning of this section, curvature ultimately results in the complete eradication of all non-axisymmetric generalized inflexional modes, for distances far enough downstream. This effect causes the eventual stabilization of all mode I instabilities. Figures 3.19 and 3.21 display results at $\zeta = 1.0$ and $\zeta = 5.0$, respectively.

The mode II instabilities are found to behave slightly differently for large ζ. We have already noted the existence of mode II instabilities for large n (even when mode I instabilities have been eradicated) purely due to the special nature of the subsonic neutral mode at which they originate. The same is found to be true for large ζ. Consequently, even though curvature and increase in azimuthal wavenumber will continue to cause stabilization, as can be seen from Figures 3.20 and 3.22 (corresponding to $\zeta = 1.0$ and $\zeta = 5.0$, respectively), for very large ζ, the mode II instability will still be present, although with much diminished growth rates. This limit is considered in section 3.5. Note, in Figure 3.22 the presented axisymmetric instability could not be computed any closer to the neutral point than that displayed because the inaccuracies developed within the numerical scheme due to the increase in *zeta*, as mentioned in the previous chapter, make it almost impossible to compute instabilities around the critical point occuring in the freestream.

Before considering the effect heated/cooled walls have on the disturbance terms, we shall briefly consider a few results for $M_\infty = 2.8$, solely for comparison.

Figures 3.23 and 3.24 display growth rate variations with wavenumber for the mode I/IA and II instabilities, respectively, at the axial location, $\zeta = 0.05$. Note, the mode $\mathrm{I_A}$ distributions are represented by a dashed line in Figure 3.23. Duck (1990) determined that no axisymmetric generalized inflexional modes exist at this axial location, for $M_\infty = 2.8$, and consequently, as expected, no axisymmetric mode I instabilities are found. From Figure 3.23 it is apparent that the $n = 1$ mode I and $\mathrm{I_A}$ instabilities are still distinct. The same is found to be true for the corresponding

$n = 2$ results (not shown). The higher azimuthal wavenumber mode I instabilities are qualitatively similar to the corresponding $M_\infty = 3.8$ results, although the larger Mach number modes are more unstable. Complete stabilization of the mode I instabilities for $M_\infty = 2.8$ and this position is achieved for values of $n = 8$ and higher. The mode II instabilities closely resemble the higher Mach number instabilites, with an increase in azimuthal wavenumber causing stabilization, and similarly to their mode I counterparts the higher Mach number modes are more unstable. It should be noted that the $n = 8$ mode II terminates as a supersonic neutral mode; the other mode II instabilities presented end as subsonic generalized inflexional modes, continuing as stable modes for higher values of α.

The next set of results presented for $M_\infty = 2.8$, corresponds to $\zeta = 0.5$, where Figures 3.25 and 3.26 display the respective mode I and II distributions. Both types of stability have undergone stabilization due to the increased effect of curvature , this stabilization being quite marked in the case of the mode II's, and the mode I is found to be completely eradicated for values of $n = 3$ and greater. On comparison with the higher Mach number results, both instability types are qualitatively similar, although, again it is noted that the lower Mach number results are more stable.

3.3.2 Cooled Wall Cylinder Results

We begin by considering the effect that wall cooling has on the inflexion points for $M_\infty = 3.8$. As previously mentioned, because the 'triply generalized' inflexion condition involves the ratio of the azimuthal and streamwise wavenumbers n and α, it is difficult to forecast, prior to a numerical investigation, the existence of neutral stability points of this kind. However in the case of axisymmetric disturbances, this is no longer the case, since the condition reduces to

$$\frac{d}{dr}\left[\frac{w_{0r}(r)}{rT_0(r)}\right]_{r=r_i} = 0, \tag{3.41}$$

as determined by Duck (1990).

Figure 3.27 shows the axial variation of radial location of the generalized inflexion points for the (non-dimensional) temperatures shown. As in the insulated cylinder case (Duck (1990)), the graphs display two prominent features: (i) the inflexion points occur in pairs and (ii) there exists a critical value of ζ, downstream of which no such points exist. This was also found to be true in the previous subsection for non-axisymmetric generalized inflexion points. The point $\zeta = 0$ corresponds to the tip of the cylinder and as such corresponds to the planar case as studied by Mack (1984). It is found that as the surface of the cylinder is cooled, the lower inflexion point lifts up off the cylinder surface. For sufficient cooling the lower point coalesces with the upper inflexion point and further cooling results in the complete disappearance of the inflexion points. Therefore for a given ζ-station there exists a critical wall temperature below which no inflexion points exist.

Figure 3.28 shows the axial variation of $w_0(\eta_i)$ for the displayed wall temperatures. From stability theory, unstable subsonic modes exist only if a generalized inflexion point satisfying (3.41) occurs within the boundary layer. Examination of the curve for $T_w = 3.0$ reveals that subsonic generalized inflexion points only occur for $0 \leq \zeta \leq 0.0795$; consequently the mode has completely disappeared before the generalized inflexion points have merged. For a wall temperature of $T_w = 2.0$ the generalized inflexion points are always both supersonic in nature ($w_0(\eta_i) < 1 - \frac{1}{M_\infty}$), implying for this and all cooler wall temperatures the eradication of mode I instabilities.

We now turn our attention to the eigenvalue problem for both axisymmetric and non-axisymmetric disturbances. Again, only unstable modes are presented and all plots are for the growth rate αc_i variation with wavenumber α. The first set of results presented corresponds to the tip of the cylinder and as such are comparable to the planar results as obtained by Mack (1969, 1984, 1987a). The adiabatic wall temperature at the tip of the cylinder is $T_w \simeq 3.448$. Figure 3.29 displays distributions for the first mode of instabilty. As in the adiabatic case, all modes originate as sonic neutral modes (i.e. with $c = 1 - \frac{1}{M_\infty}$) and terminate as subsonic, generalized inflexional modes. We note that as the surface of the cylinder is cooled this mode undergoes stabilization, until, with sufficient cooling it becomes completely stable, thus verifying Mack's observations (and those of Lees (1947), Van Driest (1952) and Dunn and Lin (1955)). From our inflexion point results, this is to be expected, as for cool enough wall conditons, the larger value of $w_0(\eta_i)$ drops below $1 - \frac{1}{M_\infty}$ and there no longer exist the conditions necessary for a subsonic, generalized inflexional mode.

Figure 3.30 displays the distribution of the temporal growth rate with wavenumber α, for the second instabilty at the cylinder tip, $\zeta = 0$. Similarly to the adiabatic mode II distributions, all modes originate as the special wavenumber case $c\ (= c_r = w_0(\eta_i)) = 1$. Depending on whether or not a subsonic, generalized inflexional mode exists for the given wall conditions, this unstable mode terminates as a subsonic or supersonic neutral mode, and may continue as a decaying mode ($\alpha c_i < 0$), thereafter. Examination of our results reveals that as the cylinder surface is cooled, the maximum value of the growth rates increases to a peak for $T_w \simeq 1.095$ (not shown) and further cooling causes the maximum growth rates to decrease again. However it is observed that for larger values of α, cooling has a completely destabilizing effect on the mode II instabilty. Thus we deduce that wall cooling generally destabilizes the second mode of instability, in line with Mack's observations (1969, 1984, 1987a), but there does appear to be a critical amount of cooling beyond which the maximum growth rates undergo stabilization again.

The next set of results presented corresponds to a relatively small distance from the tip of the cylinder in the axial direction, at the location $\zeta = 0.05$, and for an axisymmetric mode (i.e. n=0). The adiabatic wall temperature at this location is $T_w \simeq 3.42$. Figure 3.31 displays the mode I distributions and it is noted that with sufficient cooling, it is again possible to completely stabilize this mode. The neutral mode at which this instability originates is found to be consistent with the adiabatic results presented, being very slightly supersonic in nature, and occuring for α slightly greater than zero. When compared with the corresponding planar results curvature (as noted by Duck (1990) and for the adiabatic results) has a marked stabilizing effect. Even though this station is only a relatively short distance along the cylinder from the tip, curvature has reduced the value of maximum αc_i for the $T_w = 3.0$ curve by a factor of about 4.3 while for the $T_w = 2.8$ curve it is a much larger factor of about 12.3. Curvature results in the mode requiring less cooling to completely stabilize it. Figure 3.32 displays the axisymmetric mode II instability at this axial location. Again it is noted that curvature has had a stabilizing effect on the instability, but cooling causes the mode to become more unstable, in line with the planar results described previously, and it is observed that the critical value of T_w below which the maximum growth rates undergo stabilization again, has dropped, in this case, to a value of $T_w = 0.805$ (not shown).

The next set of results presented corresponds to the $\zeta = 0.05$ location, for azimuthal wavenumber $n = 1$. Figures 3.33, 3.34 display the mode I and II instabilities, respectively. We observe that in this case the mode I instability is substantially more unstable than the axisymmetric case. Again, with sufficient cooling this mode can be completely stabilized, although the mode does persist for cooler wall conditions. The mode II instability has the same qualitative features as the axisymmetric case, although it is less unstable than the axisymmetric instability.

In the previous subsection it was noted that near the cylinder tip, for non-axisymmetric disturbance terms, a third mode of instability is seen to develop, which was termed mode $\mathrm{I_A}$. At the $\zeta = 0.05$ location for $n = 1$ and adiabatic wall conditions, this new mode was found to have already amalgamated with the mode I instability. However, when cooled wall conditions are applied at this axial location and for $n = 1$ the mode $\mathrm{I_A}$ instability is still distinct (Figure 3.35) over the range of T_w shown; a partial explanation of this is provided in subsection 3.6. It is observed that wall cooling causes the mode $\mathrm{I_A}$ instability to become less unstable and with sufficient wall cooling it can be completely stabilized. Comparing with the mode I instability it is noted the maximum value of αc_i for the mode $\mathrm{I_A}$ instability is larger for corresponding wall temperatures and the mode persists for cooler wall temperatures. However since the mode I instability occurs over a much larger α-range it is felt that these growth rates are generally of more importance.

We now consider the situation for an azimuthal wavenumber of $n = 3$, at the same axial location. In this case the mode I and $\mathrm{I_A}$ instabilities have now amalgamated. Similarly to the combined adiabatic modes, the new combined mode originates as a neutral mode (but with $c_i \neq 0$ for $\alpha = 0$) and terminates as a subsonic generalized inflexional mode (Figure 3.36). Again it is noted that sufficient wall cooling can completely stabilize this mode, but the increase of n has resulted in the mode persisting for cooler wall temperatures. Comparing with the $n = 1$ results it is found that the increase in the value of n has also caused the mode to become less stable. The mode II instability (Figure 3.37), again has the same qualitative features, although the increase in n has resulted in further stabilization. However, it is found that cooling has the more dominant destabilizing effect here. The previous two comments regarding increase in azimuthal wavenumber mirror the corresponding effects observed for the adiabatic cylinder.

The next set of results presented corresponds to an azimuthal wavenumber of $n =$

5, at the same axial location. The same qualitative features as the the $n = 3$ results are observed for both modes (Figures 3.38, 3.39) although a marked stabilizing effect due to the increase in n is noted. The mode I instability is now completely stabilized for higher wall temperatures, while again, it is observed that cooling causes an even more marked destabilizing effect on the mode II instability.

As the azimuthal wavenumber n is further increased, both the mode I and II instabilities developed subject to cooled wall conditions, follow the same trend as observed for the insulated cylinder, undergoing additional stabilization, although cooling maintains a destabilizing effect on mode II. Consequently, for large n, the mode I is completely stabilized; indeed for cooled walls the mode I will ultimately disappear for smaller values of n owing to the additional stabilization effect. As expected, the mode II instability persists, although with much diminished growth rates. In the next section the possibility of cooled walls is also considered for large n.

The final set of results presented for $M_\infty = 3.8$ corresponds to the axial location $\zeta = 0.5$ and for an azimuthal wavenumber of $n = 1$. The adiabatic wall temperature for this axial location is $T_w = 3.343$. It is found that at this distance along the cylinder the mode I and $\mathrm{I_A}$ stabilities have now combined (Figure 3.40) for cooled walls as well . Comparison with the n=1 results at $\zeta = 0.05$, reveals that the mode has undergone destabilization (in line with the corresponding adiabatic observations). It is found that the combined mode prevails for cooler wall conditions, but again is completely stabilized with sufficient wall cooling. Figure 3.41 displays the mode II instability, and indicates that curvature has resulted in the growth rates being reduced, although cooling has a more marked destabilizing effect here.

Further along the cylinder, curvature continues to cause stabilization, as noted in the adiabatic case. For a given wall temperature, T_w, azimuthal wavenumber n, axial wavenumber α, there exists a critical value of ζ, beyond which no triply generalised inflexion points occur. Consequently, we expect the mode I instability to have disappeared for axial distances larger than this critical ζ, which is bourne out by our numerical results. The mode II instabilities persist for cooler wall conditions beyond this critical value of ζ, as expected, although with much reduced growth rates.

We now present a number of results for $M_\infty = 2.8$. The results for this Mach number are for direct comparison with the $M_\infty = 3.8$ results, and consequently the same wall temperatures as before are displayed, even though it is found that some of these conditions are higher than the adiabatic wall temperature (i.e. heated with respect to the adiabatic conditions). However, since the trend in wall temperatures is downward, we still feel that the results give a good indication of the effect wall cooling has on the instabilities. The first set of results presented is at the $\zeta = 0.05$ station and for a azimuthal wavenumber $n = 1$. The adiabatic wall temperature at this location and for the chosen Mach number is $T_w \simeq 2.317$. In this case it is observed that the mode I and $\mathrm{I_A}$ instabilities have just combined (Figure 3.42). It is found that the curves for the $T_w = 2.6$ and $T_w = 3.0$ conditions possess two peaks, which correspond to the maximum growth rates of the mode I and $\mathrm{I_A}$ instabilities (if the modes were still distinct). Again it is noted that with sufficient wall cooling this mode can be completely stabilized. Comparing with the corresponding $M_\infty = 3.8$ results for this axial location and azimuthal wavenumber it is observed that the modes are more unstable in this case. Figure 3.43 displays the mode II instability. In comparison with the $M_\infty = 3.8$ results it is found that mode II is less unstable, but cooling has had a more marked destabilizing effect. The wall temperature ($T_w \approx 0.47$) below which the maximum growth rates undergo stabilization again, is found to be lower.

The next set of results considered for this Mach number corresponds to an azimuthal wavenumber of $n = 3$ at this axial location. It is observed that the mode I instability (Figure 3.44) is more unstable than $n = 1$ results for this Mach number and that the mode persists for cooler wall temperatures. When compared with the corresponding $M_\infty = 3.8$ modes it is again observed that lower Mach number are the more unstable (except the $T_w = 1.6$ mode which is found to be very slightly less stable). The increase in azimuthal wavenumber has resulted in the mode II instability (Figure 3.45) becoming slightly more stable and on comparing with the corresponding $M_\infty = 3.8$ instablity it is again noted that for this type of instability, the $M_\infty = 3.8$ results are the more unstable.

We now consider the effect wall coooling has on the mode I and II instabilities (Figures 3.46 and 3.47, respectively) at this axial location and for an azimuthal wavenumber of $n = 5$. It is found that the increase in n has caused the mode I instability to undergo stabilization, although it is still more unstable than the corresponding $M_\infty = 3.8$ results (except for the $T_w = 1.8$ mode which is found to be more stable). The mode II instability has undergone further stabilization due to the increase in n and is still found to be more stable than the corresponding $M_\infty = 3.8$ modes.

The final set of results presented for $M_\infty = 2.8$ is for the axial location $\zeta = 0.5$ and an azimuthal wavenumber of $n = 1$. The adiabatic wall temperature for this axial location and Mach number is $T_w \simeq 2.277$. Figures 3.48 and 3.49 display the mode I and II instabilities, respectively. It is observed, in line with the $M_\infty = 3.8$ results, that the combined mode $\mathrm{I/I_A}$ instability is more unstable at this axial location than the $n = 1$ results at the $\zeta = 0.05$ location. We observe that this mode persists for cooler wall temperatures. In comparison with the $M_\infty = 3.8$ results, the mode I instabilities are more unstable for the wall temperatures shown, except for the $T_w = 1.4$ curve, which is slightly less unstable. The growth rates of the mode II instability have been reduced greater due to the stabilizing effect of curvature, although we do note that cooling has a very marked destabilizing effect in this case.

It is found that for the temperatures displayed the $T_w = 0.1$ curve has the largest maximum growth rate. As before, when compared with the corresponding $M_\infty = 3.8$ results, the higher Mach number results are observed to be the more unstable.

3.3.3 Heated Wall Cylinder Results

We begin by considering the effect that wall heating has on generalized inflexion points. We restrict our study to the case of axisymmetric disturbances, determining the effect wall heating has on condition (3.41) (as in the case of cooled walls). Only a Mach number of $M_\infty = 3.8$ is considered, although this is expected to be representative of moderate Mach numbers.

Figure 3.50 shows the axial variation of (radial) position of the generalized inflexion points for the temperatures shown. We observe again the same features seen in Figure 3.27. Close to the cylinder tip, however, it is found that for a small axial distance measured from the tip, there no longer exist any lower generalized inflexion points. As the surface of the cylinder is heated further, this axial distance is found to increase. It is also observed that wall heating causes the critical value of ζ, beyond which no generalized inflexion points exist, to increase. For $T_w = 4.5$, this critical value of ζ is about 0.216, while for $T_w = 6.0$, there is a substantial increase to a value of $\zeta \simeq 0.423$. This will have direct implications on the first mode of instability which is expected to persist for longer distances downstream. These effects are in many ways to be expected, being the converse of the cooling observations described earlier.

Figure 3.51 shows the axial variation of $w_0(\eta_i)$ for the displayed wall temperatures. The most marked feature of these curves is that as the cylinder surface is heated, the lower generalized inflexion point becomes subsonic beyond a critical value of ζ, which is temperature dependent, i.e. for axial distances greater than this critical value of ζ but upstream of the station beyond which no inflexion points occur, *both* generalized inflexion points are now subsonic in nature. The lowest wall temperature for which a lower, subsonic inflexion point is observed is for a wall temperature of about $T_w = 4.5$. It is found, however, that the critical value of ζ here, is very close to the stations where the generalized inflexion points coalesce. For $T_w = 5.0$ we observe that for the range $0.2635 \leq \zeta \leq 0.2720$ two subsonic inflexional modes exist, while for the hotter wall temperature of $T_w = 6.0$ we have the larger range $0.363 \leq \zeta \leq 0.423$ for which both generalized inflexion points are subsonic. In these ζ-ranges there exists the possibility of two subsonic generalized inflexional modes with the potential of a significant effect on the problem.

We now present growth rate results for axisymmetric disturbances at a Mach number of $M_\infty = 3.8$. As in the case of cooled wall conditions attention is focused on unstable modes. We begin by considering the effect wall heating has on the mode I and II instabilities for a ζ-station close to the cylinder tip ($\zeta = 0.05$) and consequently the lower generalized inflexion point is still supersonic in nature. Figure 3.52 shows the mode I instability for the temperatures shown. It is observed that all the modes originate as neutral modes at a value of α slightly greater than zero, which are very slightly supersonic in nature. As the wall is heated this neutral mode approaches the sonic value. All the modes terminate as subsonic generalized inflexional modes, continuing as stable modes ($\alpha c_i < 0$) for larger values of α. These observations are similar to the results obtained for both the axisymmetric cooled wall and adiabatic conditions cases at this ζ-station. It is found, as expected, heating the surface of the cylinder causes the mode I instability to become more unstable - converse to the effect of cooling on this mode.

Figure 3.53 displays the mode II instabilities at $\zeta = 0.05$, for the the temperatures shown. It is found that all the modes originate as subsonic generalized inflexional modes, rise to a maximum and terminate as subsonic generalized inflexional modes (which then continue in all the cases presented as stable modes). Heating the cylinder wall causes the mode II instability to become less unstable and the numerical evidence suggests that with sufficient heating this mode can be completely stabilized.

We now consider the effect that the lower generalized inflexion point becoming subsonic has on the mode I and II instablities. Figure 3.54 displays the growth rates of the mode I instability for a wall temperature of $T_w = 5.0$ and for the ζ-stations as indicated. At $\zeta = 0.26$ (where the lower generalized inflexion point is still supersonic) the mode originates as a very slightly supersonic mode and terminates as a subsonic generalized inflexional mode. For the $\zeta = 0.264$ station the lower generalized inflexion point has now become subsonic in nature, and it is found the mode I instability now originates as a lower subsonic generalized inflexional mode. Consequently the value of α for the neutral mode has increased correspondingly. As before, the mode I instability terminates as the upper generalized inflexional mode which is of course subsonic, as well. From the inflexion point curves we know that as we move upstream the inflexion points move closer together, eventually coalescing and this is reflected in the new form of the mode I instabilities. For the $\zeta = 0.27$ station the mode I instability occurs over a much smaller α-range and the growth rates are greatly diminished.

Figure 3.55 displays the mode I instability for a wall temperature of $T_w = 6.0$ and the indicated ζ-stations. Again it is noted that as the lower generalized inflexion point becomes subsonic the neutral point at which the instability originates transforms from being very slighly supersonic in nature, to this inflexional mode. Movement upstream causes the α-ranges and growth rates to be diminished, but the reduction is less marked (in comparison with the $T_w = 5.0$ results) due to the destabilizing effect brought on by wall heating.

The appearance of a second subsonic generalized inflexional mode is found to have little effect on the mode II instability as it always terminates as the upper generalized inflexional mode. It is found, however, that a third mode of instability exists, originating as the lower generalized inflexional mode and terminating as a slightly supersonic neutral mode. This new mode, which we shall term Mode II_A, occurs for values of α greater than the value of α for which the mode II instability terminates. It appears that the mode II instability continues as a stable mode and then becomes unstable again at the lower generalized inflexion point. The growth rates of the mode II_A are found to be very small. For a wall temperature of $T_w = 5.0$ and the station $\zeta = 0.27$ the growth rates are in the order 10^{-11}, while for $T_w = 6.0$ and $\zeta = 0.364$ the growth rates are of the order of 10^{-13}-10^{-14}.

3.3.4 Adiabatic Cone Results

We now consider the linear stability of the compressible boundary layer formed on a somewhat more practical configuration, namely a sharp cone. Comparing the non-axisymmetric generalized inflexion condition for a cone (3.37) with the corresponding cylinder condition, the former case is found to be even more complex, involving terms in the parameter λ as well as wavenumbers n and α. Consequently, we make no attempt to conduct a non-axisymmetric generalized inflexional mode study for the cone, and instead just present an eigenvalue study for the temporal growth rate variation with spatial wavenumber.

Since the cone surface is described by $r = 1+\lambda\zeta^2$, then for axial distances close to the cylinder tip, the growth rate distributions are expected to be very similar to those presented for the adiabatic cylinder study. Figures 3.56 and 3.57 display mode I and mode II instabilities, respectively, for $\zeta = 0.5$ and the azimuthal wavenumbers as shown. At this location the radius is 1.25 times the cylinder radius (and the cone tip radius). On comparing Figures 3.56 and 3.57 with the corresponding cylinder results, it is observed that the two sets of results are very similar. It is noted, however, that in the case of the mode I instability body radius divergence (with respect to the cylindrical body's results at this axial location, namely Figure 3.17) has caused the $n = 1$ instability (the most dangerous mode I) to be slightly stabilized whilst the higher azimuthal wavenumbers have undergone slight destabilization. In the case of the mode II instabilities body radius divergence has caused both the axisymmetric and all the non-axisymmetric modes presented to more noteably destabilized.

The next set of results presented corresponds to the $\zeta = 1.0$ location, Figures 3.58 and 3.59 display the respective mode I and II instabilities. Comparing these results with the corresponding adiabatic cylinder results (Figures 3.19 and 3.20), body divergence has caused a noticeable destabilization of the $n = 2$, mode I instability. It is found that the $n = 3$, mode I instability is still unstable (although the growth rates are so small that this mode is just visible in Figure 3.58), complete stabilization not being achieved until $n = 4$ (and higher). The mode II instabilities have all been significantly destabilized due to body divergence. Note, that on comparing these results with those obtained at $\zeta = 0.5$, curvature is still found to have a stabilizing effect.

At $\zeta = 2.0$, body radius divergence continues to cause destabilization to such an extent that this effect dominates the stabilizing effect of curvature. Comparing the mode I instabilities (Figure 3.60) with the corresponding results obtained at $\zeta = 1.0$, the $n = 1$ growth rates are observed to be similar, while the new $n = 2$ mode is found to be noticeably more unstable. On comparing the new locations's mode II instabilities (Figure 3.61) with corresponding $\zeta = 1.0$ results, all modes are markedly more unstable. It is also noted that the $\zeta = 2.0$ results are quite similar to $\zeta = 0.5$ instabilities.

Moving downstream to $\zeta = 5.0$, the 'recovery' in the maximum growth rates of the mode II instabilities (Figure 3.63) is found to continue, with all modes presented having undergone further destabilization. For the mode I instabilities (Figure 3.62), body divergence has caused the $n = 1$ mode to be stabilized, whilst the higher azimuthal wavenumbers have been destabilized. We also note the re-emergence of the $n = 4$, mode I instability, complete stabilization not being achieved until $n = 5$. Indeed, these results show some resemblence to the $\zeta = 0.2$ results (not shown); distributions of c_i versus α are found to be quite similar. The similarity, however, is found not to be as close for αc_i versus α distributions, because in the case of the mode II instabilities, for $\zeta = 5.0$, the modes occur at higher values of α and both instabilities are found to occur over larger α-ranges, for the further downstream location. This similarity is not surprising, given that, on account of the Mangler transformation (Mangler (1946), Stewartson (1964)), results as $\zeta \to \infty$ mirror those as $\zeta \to 0$ (except for a multiplicative factor of $\sqrt{3}$ in α; hence the higher α-results noted above).

This trend is confirmed in Figures 3.64 and 3.65 for $\zeta = 20.0$, which may be compared directly with the $\zeta = 0.05$ results displayed in Figures 3.13 and 3.14 for the adiabatic cylinder - this comparison can be made since the cone radius will only be 1.0025 times larger than the cylinder radius at this location. It is again noted, that the further downstream location results occur over larger α-ranges and the mode II instabilities occur at larger values of α, both these factors contributing to larger observed growth rates for mode I and II instabilities. Notice, also, the re-emergence of mode $\mathrm{I_A}$ for $n = 1$.

The last set of results presented is for the furthest downstream location studied, namely $\zeta = 75.0$. At this axial location, as well as growth rate variations (Figures 3.67 and 3.69 corresponding to mode I and II instabilities, respectively), distributions of c_i with spatial wavenumber, α, (Figures 3.66 and 3.68 corresponding to respective modes) are presented to clarify comparisons being made with results near the cone tip. A close resemblence is noted between Figures 3.66 - 3.69 and results presented for the cylinder at $\zeta = 0.01$ (with the factor $\sqrt{3}$ multiplying α, in the former case). The axisymmetric modes now correspond closely with the planar results of Mack (1984, 1987a, for example) (apart from the α-multiplicative factor), whilst mode $\mathrm{I_A}$ is clearly visible for $n = 1$ and $n = 2$ - Figures 3.70 and 3.71 display the respective mode $\mathrm{I_A}$ instabilities for these azimuthal wavenumbers on an enhanced scale to clarify the modal structure. Inspecting Figures 3.66 and 3.67 the union of modes I and $\mathrm{I_A}$ is quite noticeable, for $n = 3$, from the double peaked structure of the newly combined growth rate curves.

We now move on to consider the form of the disturbance equations in the limit of large azimuthal wavenumbers in the next section.

3.4 Disturbance Equations for Large n

In this section we consider the form of the disturbance equations in the asymptotic limit of large azimuthal wavenumber, n, guided partly by our numerical observations. It should be noted that the theory developed in this section is valid for the cylindrical case only.

3.4.1 Formulation of the Problem

The pressure disturbance equation for supersonic flow past a cylinder, as derived in section 3.2 (and setting $\lambda = 0$), has the form

$$\frac{w_0 - c}{\alpha^2}\frac{d}{d\eta}\Big[\frac{T_0 p_\eta}{w_0 - c}\Big] + \Big(\frac{(w_0 - c)\zeta}{1 + \eta\zeta} - w_{0\eta}\Big)\frac{T_0 p_\eta}{\alpha^2(w_0 - c)} = \Phi p, \tag{3.42}$$

where

$$\Phi = T_0\Big[1 + \frac{n^2\zeta^2}{\alpha^2(1 + \eta\zeta)^2}\Big] - M_\infty^2(w_0 - c)^2. \tag{3.43}$$

In the limit of large azimuthal wavenumber, n, our numerical observations suggest that the corresponding streamwise wavenumbers for the instability also increase, and that $\Phi(\eta = 0) \to 0$, asymptoting towards a constant value of for large n

$$\Phi \to 1 - M_\infty^2(1-c)^2. \tag{3.44}$$

Equation (3.43) suggests that if Φ is to be generally $O(1)$, we must have

$$\alpha = \overline{\alpha} n, \qquad \overline{\alpha} = O(1), \tag{3.45}$$

where $\overline{\alpha}$ is our (scaled) wavenumber, which we specify because of our temporal approach to the problem.

Assume asymptotic expansions of the form

$$c = c_0 + n^a c_1 + O(n^{2a}),$$
$$\Phi = \Phi_0 + n^a \Phi_1 + O(n^{2a}), \tag{3.46}$$

where $a < 0$ to ensure convergence and is to be determined.

In the limit of large n, to leading order, by the form of the asymptotic expansions (3.46) the pressure equation has the form

$$p_{\eta\eta} + \Big[\frac{T_{0\eta}}{T_0} + \frac{\zeta}{1+\eta\zeta} - \frac{2w_{0\eta}}{w_0 - c_0}\Big]p_\eta - \frac{\alpha^2}{T_0}\Phi_0 p = 0, \tag{3.47}$$

where

$$\Phi_0 = T_0[1 + \frac{\zeta^2}{\overline{\alpha}^2(1+\eta\zeta)^2}] - M_\infty^2(w_0 - c_0)^2. \tag{3.48}$$

Solutions of the WKBJ type satisfy (3.47), namely

$$p \sim \frac{f(\eta)}{(-\Phi_0)^{1/4}} \exp\Big[\pm i \frac{\alpha}{T_0^{1/2}} \int^\eta (-\Phi_0)^{1/2} d\eta\Big], \tag{3.49}$$

where $f(\eta)$ is determined by substitution. This is carried out in Appendix A, where $f(\eta)$ is determined to have the form

$$f(\eta) \sim \int^{\eta} \frac{\zeta(w_0 - c_0)^2}{T_0(1+\eta\zeta)} d\eta. \tag{3.50}$$

To ensure boundedness of disturbance terms in the far-field, we require a solution which decays as $\eta \to \infty$. The numerical observations suggest that $\text{Im}[(\Phi_0)^{1/2}] > 0$ for large η, therefore far from the cylinder wall the required solution is

$$p = \frac{D_0 f(\eta)}{(\Phi_0)^{1/4}} \exp\Big[+i\frac{\alpha}{T_0^{1/2}} \int_{\eta_0}^{\eta} (-\Phi_0)^{1/2} d\eta\Big], \tag{3.51}$$

where D_0 is a constant to be determined and η_0 is the transition point, i.e. the point where $\Phi_0 = 0$.

In the neighbourhood of the wall the numerical observations suggest that the pressure has an oscillatory nature, therefore in this region we expect a WKBJ solution of the form

$$p = \frac{A_0 f(\eta)}{\Phi_0^{1/4}} \exp\Big[\frac{\alpha}{T_0^{1/2}} \int_{\eta_0}^{\eta} \Phi_0^{1/2} d\eta\Big] + \frac{A_1 f(\eta)}{\Phi_0^{1/4}} \exp\Big[-\frac{\alpha}{T_0^{1/2}} \int_{\eta_0}^{\eta} \Phi_0^{1/2} d\eta\Big], \tag{3.52}$$

where A_0 and A_1 are constants and $0 < \eta < \eta_0$. (By the form of the wall pressure boundary condition (3.16) and (3.50), it is expected $f(0) = 1$.)

We require now only to determine the form of the solution in the neighbourhood of the transition point, $\eta = \eta_0$. Taylor expanding the Φ expansion about $\eta = \eta_0$, such that $\Phi \to 0$, yields

$$\begin{aligned} \Phi = \Phi_0(\eta_0) + (\eta - \eta_0)\Phi'(\eta_0) + \frac{1}{2}(\eta - \eta_0)^2 \Phi_0'' + \cdots \\ + n^a[\Phi_1(\eta_0) + (\eta - \eta_0)\Phi_1'(\eta_0) + \cdots] \\ + n^{2a}[\Phi_2(\eta_0) + \cdots]. \end{aligned} \tag{3.53}$$

We characterise the distance $\eta - \eta_0$, know as the transition layer, by the scale

$$\eta = \overline{\eta} n^a, \qquad \overline{\eta} = O(1). \tag{3.54}$$

Therefore expansion (3.53) simplifies to

$$\Phi = [\Phi_1(\eta_0) + \overline{\eta}\Phi_0'(\eta_0)]n^a + O(n^{2a}) + \cdots. \tag{3.55}$$

In the neighbourhood of the transition point the second order derivative and the right-hand-side of (3.42) are expected to be the important terms, resulting in the pressure disturbance equation having the following form, to leading order

$$\frac{T_0}{\overline{\alpha}^2 n^2}\frac{1}{n^{2a}}p_{\overline{\eta}\overline{\eta}} = n^a[\overline{\eta}\Phi_0'(\eta_0) + \Phi_1(\eta_0)]p. \tag{3.56}$$

Since a is chosen to ensure both sides of equation (3.56) balance it must take the value

$$a = -\frac{2}{3}, \tag{3.57}$$

which simplifies equation (3.56) to the form

$$p_{\overline{\eta}\overline{\eta}} = \frac{\overline{\alpha}^2}{T_0}[\overline{\eta}A - B]p, \tag{3.58}$$

where

$$A = \Phi_0'(\eta_0),$$

$$B = -\Phi_1(\eta_0). \tag{3.59}$$

Introducing the transformation variable

$$\tau = \Big(\frac{\overline{\alpha}^2}{T_0 A^2}\Big)^{1/3}\Big(A\overline{\eta} - B\Big), \tag{3.60}$$

equation (3.58) can be simplified to

$$p_{\tau\tau} = \tau p. \tag{3.61}$$

Clearly (3.61) is Airy's equation; therefore in the neighbourhood of the transition point the pressure disturbances must have the form

$$p = B_0 Ai(\tau) + B_1 Bi(\tau), \tag{3.62}$$

where B_0 and B_1 are constants to be determined.

We appear to have a three-layered structure for the form of the pressure as we move from the cylinder surface to the far field. All that remains is to match the solutions in the three regions, giving continuity. The inner WKBJ solution (equation (3.52)) region shall be refered to as I, the Airy solution (equation (3.62)) layer as II and the outer WKBJ solution (equation (3.51)) region as III.

Matching regions II and III immediately yields that $f(\eta_0) = 1$ (which will be confirmed *a posteriori*), $B_0 = D_0$ and $B_1 = 0$, since only decaying solutions can exist in region III. Consequently (3.62) simplifies to

$$p = D_0 Ai(\tau). \tag{3.63}$$

The asymptotic expansion of equation (3.63) in the limit $\tau \to -\infty$ has the form

$$p \sim \frac{D_0}{\pi^{1/2}\tau^{1/4}} \sin\Big[\frac{2}{3}\tau^{3/2} + \frac{\pi}{4}\Big], \tag{3.64}$$

which in $\overline{\eta}$-space can be written as

$$p \sim \frac{E_0}{\overline{\eta}^{1/4}} \sin\Big[\frac{2}{3}\Big(\frac{\overline{\alpha}A^{1/2}}{T_0^{1/2}}\Big)\overline{\eta}^{3/2} + \frac{\pi}{4}\Big], \tag{3.65}$$

where E_0 is a constant. Defining

$$z = \frac{2}{3}\Big(\frac{\overline{\alpha}A^{1/2}}{T_0^{1/2}}\Big)\overline{\eta}^{3/2}, \tag{3.66}$$

equation (3.65) can be re-written

$$p \sim \frac{D_0 e^{-i\pi/4}}{2i\overline{\eta}^{1/4}}[ie^{iz} - e^{-iz}]. \tag{3.67}$$

We shall now match equation (3.67) with the pressure solution in region I (equation (3.52)). The numerical observations suggest $\Phi_0 < 0$ in region I, therefore equation (3.52) can be re-written

$$p = \frac{e^{-i\pi/4}f(\eta)}{|\Phi_0|^{1/4}}\Big\{A_0 \exp\Big[\frac{i\alpha}{T_0^{1/2}}\int_{\eta_0}^{\eta}(-\Phi_0)^{1/2}d\eta\Big] + A_1 \exp\Big[-\frac{i\alpha}{T_0^{1/2}}\int_{\eta_0}^{\eta}(-\Phi_0)^{1/2}d\eta\Big]\Big\}. \tag{3.68}$$

In the limit $\eta \rightarrow \eta_0$ (for $\eta_0 \ll 1$) we can write

$$\Phi \sim \eta\Phi_0'(\eta_0) = n^{-2/3}\overline{\eta}\Phi_0'(\eta_0), \tag{3.69}$$

and remembering $f(\eta) \sim 1$ in this limit, (3.68) simplifies to

$$p \sim \frac{e^{-i\pi/4}}{\overline{\eta}^{1/4}}[A_2 e^{iz} + A_3 e^{-iz}], \tag{3.70}$$

where A_2 and A_3 are constants and z is defined by equation (3.66)

Matching equations (3.67) and (3.70) yields

$$A_2 e^{iz} + A_3 e^{-iz} = \frac{D_0}{2i}[ie^{iz} - e^{-iz}], \tag{3.71}$$

which implies

$$A_2 = iA_3 \qquad \Leftrightarrow \qquad A_0 = iA_1. \tag{3.72}$$

Therefore equation (3.68) can be written in the form

$$p = \frac{A_0 e^{-i\pi/4} f(\eta)}{|\Phi_0|^{1/4}} \Big\{ \exp\Big[\frac{i\alpha}{T_0^{1/2}} \int_{\eta_0}^{\eta} (-\Phi_0)^{1/2}\Big] + i \exp\Big[-\frac{i\alpha}{T_0^{1/2}} \int_{\eta_0}^{\eta} (-\Phi_0)^{1/2} d\eta\Big]\Big\}. \quad (3.73)$$

Imposing the boundary conditions at the wall, namely

$$p_\eta \mid_{\eta=0} = 0, \quad (3.74)$$

and remembering $f(0) = 1$, gives

$$\exp\Big[\frac{i\alpha}{T_0^{1/2}} \int_{\eta_0}^{0} (-\Phi_0)^{1/2} d\eta\Big] = i \exp\Big[-\frac{i\alpha}{T_0^{1/2}} \int_{\eta_0}^{0} (-\Phi_0)^{1/2} d\eta\Big], \quad (3.75)$$

which after manipulation yields

$$\frac{2i\alpha}{T_0^{1/2}} \int_{\eta_0}^{0} (-\Phi_0)^{1/2} d\eta = \frac{\pi}{2} + 2m\pi, \quad (3.76)$$

where m is a integer

From equation (3.76), if $m = O(1)$, then $(\frac{\pi}{2} + 2m\pi) = O(1)$. This implies the left-hand-side of the equation will also be $O(1)$. However as n becomes large, α also becomes large ($\alpha = \overline{\alpha} n$), therefore η_0 must be small and thus close to the surface of the cylinder. In this ordering Φ_0 is also very small, approaching zero for increasingly large n. Note, that if instead $m \to \infty$, Φ_0 would not be small.

The overall conclusion, therefore, is that for large n, η_0 collapses onto the cylinder surface, with $\Phi_0 \to 0$ at the wall, giving a two layered structure, as opposed to three. Therefore we have a two layered structure for the pressure disturbances consisting of an inner Airy solution and the outer WKBJ solution.

One further observation is that if m is large enough then η_0 would become large and move away from the cylinder and the WKBJ solution described above would become valid.

Returning attention to the asymptotic expansions for c and ϕ in the neighbourhood of the transition point, $\eta_0 = 0$, we have

$$
\begin{aligned}
c &= c_0 + n^{-2/3}c_1 + O(n^{-4/3}) + \cdots, \\
\Phi &= \Phi_0(0) + n^{-2/3}[\Phi_0'(0)\overline{\eta} + \Phi_1(0)] + \quad O(n^{-4/3}) + \cdots,
\end{aligned}
\tag{3.77}
$$

where

$$\Phi_0(0) = T_w[1 + \frac{\zeta^2}{\overline{\alpha}^2}] - M_\infty^2 c_0^2, \tag{3.78}$$

$$\Phi_1(0) = -2c_0c_1M_\infty^2, \tag{3.79}$$

and for insulated wall conditions

$$\Phi_0'(0) = -\frac{2\zeta^3 T_w}{\overline{\alpha}^2} + 2M_\infty^2 w_0'(0)c_0, \tag{3.80}$$

while for heated or cooled wall conditions

$$\Phi_0'(0) = T_{0\eta}(0)\Big[1 + \frac{\zeta}{\overline{\alpha}^2}\Big] - \frac{2\zeta^3 T_w}{\overline{\alpha}^2} + 2M_\infty^2 w_0'(0)c_0, \tag{3.81}$$

where T_w represents the wall temperature.

If $\Phi_0(0) = 0$, then we must have

$$c_0 = \frac{T_w^{1/2}[1 + \frac{\zeta^2}{\overline{\alpha}^2}]^{1/2}}{M_\infty}, \tag{3.82}$$

(which is clearly real). We now seek to determine the first order correction term to c, namely c_1. For this it is necessary to look at the wall layer pressure term. Transforming the boundary condition at the wall (equation (3.74)) to τ-space, yields

$$p_\tau[\tilde{\lambda}^{-1/3}\tau = -B] = 0, \tag{3.83}$$

where

$$\tilde{\lambda} = \frac{\overline{\alpha}^2}{T_w A^2}. \tag{3.84}$$

Since the solution to the pressure disturbance term in this region is given by Airy's function (3.63) then

$$Ai_\tau[\tau = -B\tilde{\lambda}^{1/3}] = 0. \tag{3.85}$$

Transforming back to η-space gives the result

$$\tau_i = \Big[\frac{\overline{\alpha}^2}{T_w(\Phi_0'(0))^2}\Big]^{1/3}(-\Phi_1(0)), \tag{3.86}$$

where τ_i (where $i = 1, 2, \cdots$) represents the solutions of the equation

$$Ai'(-\tau_i) = 0. \tag{3.87}$$

Substituting either result (3.80) or (3.81) and equation (3.79) into equation (3.86) yields the first order correction term for c namely

$$c_1 = \frac{T_w^{1/3}\{2[M_\infty^2 w_0'(0)c_0 - \frac{T_w\zeta^3}{\overline{\alpha}^2}]\}^{2/3}}{2\overline{\alpha}^{2/3}M_\infty^2 c_0}\tau_n, \tag{3.88}$$

for insulated walls, whilst for heated/cooled wall conditions we have

$$c_1 = \frac{T_w^{1/3}\{2[M_\infty^2 w_0'(0)c_0 - \frac{T_w\zeta^3}{\overline{\alpha}^2}] + T_0'(0)[1 + \frac{\zeta^2}{\overline{\alpha}^2}]\}^{2/3}}{2\overline{\alpha}^{2/3}M_\infty^2 c_0}\tau_n, \tag{3.89}$$

where c_1 is obviously real for both cases.

Since τ_i is a solution of equation (3.87), where $\tau_i > 0$, then there exists an infinite number of discrete, real possible values for τ_i, since the derivative of the Airy function has an infinite number of discrete roots confined to the negative real axis. This suggests that there are an infinite number of discrete modes.

We shall now compare these asymptotic results with numerically determined results for large values of n.

3.4.2 Numerical Results

All the results presented in this section are for a freestream Mach number of 3.8 and at the point $\zeta = 0.2$ along the cylinder. Only results for adiabatic wall conditions are presented.

Firstly consider the asymptotic expansion for c in the limit of large azimuthal wavenumber n. The leading order term in the c expansion, c_0, is given by equation (3.82) and using the numerically determined values

$$T_w \simeq 3.379,$$
$$\overline{\alpha} \simeq 0.1525, \tag{3.90}$$

we find

$$c_0 \simeq 0.7978. \tag{3.91}$$

The first order correction term, which for adiabatic wall conditions is given by equation (3.88), is computed to have the value

$$c_1 \simeq 0.3698\tau_{\ell}, \tag{3.92}$$

where we have used $w_{0\eta} \simeq 0.1904$ and τ_{ℓ} are the solutions of equation (3.87). The first six values of τ_{ℓ} are determined from tables (Abramowitz and Stegun (1965)) and the corresponding values of c_1 are shown in Table 3.4

Figure 3.72 shows a plot of $c(= c_0 + n^{-2/3}c_1)$, as determined asymptotically, against n for the different values of the first order correction term c_1, where the

c_1	
0.3698	(I)
1.2092	(II)
1.7750	(III)
2.5146	(IV)
2.9214	(V)
3.3282	(VI)

Table 3.4: Values of c_1

numbering refers to the numbering of the correction terms in Table 3.4. (It should be noted that only n integer has any physical significance, although Figs. 3.72 and 3.73 show c as a continuous function of n.) From here on we shall refer to these different values of c as order I to VI inclusively corresponding to the numbering convention of the correction terms in Table 3.4.

Now as observed above, the asymptotic analysis suggests the existence of an infinite, discrete number of possible values for c. When we searched for the eigenvalues numerically, for large n, we determined that there were indeed many modes. Figure 3.73 displays two plots of c_r against n for order I and order V correction terms. Graph (1) in each case represents the asymptotic curve and graph (2) is the numerically determined curve. It should be noted that in this range of n and α, $|c_i| << 1$ ($c_i \sim 10^{-10} - 10^{-12}$), comparable to the machine accuracy of our computations. From the two sets of plots it is noted that there is good agreement between the numerical solutions and asymptotic theory for large n.

Turning our attention now to the form of the pressure disturbance terms, as obtained numerically, it is found that they do indeed follow the pattern predicted by our asymptotic theory, being initially oscillatory in the Airy solution region but decaying to zero in the far field. It is also observed that increasing the order of the correction term has the effect of increasing the number of zeros of the eigensolution.

Figure 3.74 displays the distributions of Real$\{p\}$ for $n = 40$ corresponding to the orders as shown.

Examining the c expansion again, we have determined that both c_0 and c_1 are real and therefore the leading order imaginary term, c_i, is at most $O(n^{-4/3})$. This means that the leading order term in the growth rate (αc_i) is $O(n^{-1/3})$ at most. Therefore actual growth rates will decrease as $n \to \infty$ which is confirmed to be true by our numerical observations.

Note that even though c_0 may correspond to a subsonic (or supersonic) neutral mode, no generalized inflexion condition is necessary as this condition of neutrality is only reached asymptotically as $n \to \infty$ (and correspondingly $\alpha \to \infty$), and consequentially the generalized inflexion condition is not applicable/appropriate. Further to this point, generally the eigensolutions are to be expected to be exponentially small (compared with values close to the wall) in the neighbourhood of any critical layers, and these are expected to be generally of little consequence.

We now turn our attention to the form of the disturbance equations in the limit $\zeta \to \infty$.

3.5 Disturbance Equations for Large ζ - Cylindrical Bodies

In this section we consider the form of the disturbance equations in the far downstream region, guided by the numerical observations of section 3.3. It should be noted that this asymptotic analysis is valid only for cylindrical bodies. The case of cone-shaped bodies, in the limit of large ζ, is treated in the next section; it is found that because of the Mangler transformation (Mangler (1946); Stewartson (1964)), in this limit, the axisymmetric results closely resemble planar results, but for a multiplicative factor of $\sqrt{3}$ on α.

3.5.1 Formulation of the Problem

Consider the pressure disturbance equation as presented in the previous section, but written in terms of 'r' rather than η,

$$\frac{w_0 - c}{\alpha^2}\frac{d}{dr}\Big[\frac{T_0 p_r}{w_0 - c}\Big]\zeta^2 + \Big[\frac{w_0 - c}{r} - w_{0r}\Big]\frac{T_0 p_r \zeta^2}{\alpha^2(w_0 - c)} = \Phi p, \tag{3.93}$$

where

$$\Phi = T_0\Big[1 + \frac{n^2\zeta^2}{\alpha^2 r^2}\Big] - M_\infty^2(w_0 - c)^2, \tag{3.94}$$

and

$$r = 1 + \eta\zeta. \tag{3.95}$$

In the limit $\zeta \to \infty$, assume a scale on α of the form

$$\alpha = \overline{\alpha}^+\zeta, \tag{3.96}$$

where $\overline{\alpha}^+$ is to be determined.

Guided by Duck's (1990) work for the form of the basic flow in the far-field of the compressible boundary layer formed on a thin cylinder, we define a (small) parameter

$$\epsilon = (\frac{1}{2}\log \overline{z})^{-1} = (\log\zeta)^{-1}. \tag{3.97}$$

In the slow moving viscous region close to the wall (namely the $r = O(1)(\Leftrightarrow \eta = O(1/\zeta))$ lengthscale) we expect asymptotic expansions of the form

$$c = \hat{c}_0 + \epsilon\hat{c}_1 + O(\epsilon^2),$$

$$\Phi = \hat{\Phi}_0 + \epsilon\hat{\Phi}_1 + O(\epsilon^2),$$

$$T_0 = T_w + \epsilon\overline{T}_1 + o(\epsilon^2),$$
$$w_0 = \epsilon\overline{w}_0 + O(\epsilon^2). \tag{3.98}$$

Now, our numerical observations strongly suggest that $\hat{\Phi}_0 \rightarrow 0$ as ζ becomes large in the $r = O(1)$ region (which will be confirmed *a posteriori*) implying that $\Phi = O(\epsilon)$. Examination of the Φ expression (equation (3.94)) reveals that this is only possible in general if

$$\overline{\alpha}^+ = O(\epsilon^{-1/2}). \tag{3.99}$$

Therefore scale (3.96) can be redefined

$$\alpha = \overline{\alpha}\zeta(\log\zeta)^{1/2}, \tag{3.100}$$

where

$$\overline{\alpha} = O(1).$$

To leading order, equation (3.94) reduces to

$$\hat{\Phi}_0 = T_w - M_\infty^2\hat{c}_0^2, \tag{3.101}$$

but since it has already been assumed that $\hat{\Phi}_0 \rightarrow 0$, as $\zeta \rightarrow \infty$, for $r = O(1)$, then we must have

$$\hat{c}_0 = \frac{T_w^{1/2}}{M_\infty}, \tag{3.102}$$

which means $\hat{c}_0$ is real.

At first order in ϵ, equation (3.94) has the form

$$\hat{\Phi}_1 = \overline{T} + \frac{n^2T_w}{\overline{\alpha}^2r^2} - M_\infty^2(2\hat{c}_0\hat{c}_1 - 2\hat{c}_0\overline{w}_0), \tag{3.103}$$

while the $O(\epsilon)$ correction to the pressure equation (3.93) has the form

$$\frac{T_w}{\overline{\alpha}^2}p_{rr} + \frac{T_w}{\overline{\alpha}^2 r}p_r + \left[2\hat{c}_0 M_\infty^2(\hat{c}_1 - \overline{w}_0) - \overline{T}_1 - \frac{n^2 T_w}{\overline{\alpha}^2 r^2}\right]p = 0. \tag{3.104}$$

We now transform equation (3.104) using a similar transformation as used by Duck (1990) for the basic flow. Firstly employing the transform

$$\overline{r} = \ln r, \tag{3.105}$$

gives

$$\frac{T_w}{\overline{\alpha}^2}e^{-2\overline{r}}p_{\overline{r}\overline{r}} + \left[2\hat{c}_0 M_\infty^2(c_1 - \overline{w}_0) - \overline{T}_1 - \frac{n^2 T_w}{\overline{\alpha}^2}e^{-2\overline{r}}\right]p = 0. \tag{3.106}$$

The second transform used has the form

$$\overline{R} = \int \frac{d\overline{r}}{T_w}, \tag{3.107}$$

but since T_w is constant with respect to $\overline{r}$ the transform simplifies to

$$\overline{r} = \overline{R}T_w, \tag{3.108}$$

where the constant of integration is taken to be zero. Equation (3.106) can be re-written

$$p_{\overline{R}\overline{R}} + \{\overline{\alpha}^2 T_w[2\hat{c}_0 M_\infty^2(\hat{c}_1 - \overline{R}) - \overline{T}_1]e^{2\overline{R}T_w} - n^2 T_w^2\}p = 0, \tag{3.109}$$

where we have made use of the result obtained by Duck (1990) for the basic flow

$$\overline{w}_0 = \overline{R}. \tag{3.110}$$

Equation (3.109) is solved numerically to obtain a value for $\hat{c}_1$ subject to the condition at the wall

$$p_{\overline{R}} \mid_{\overline{R}=0} = 0, \tag{3.111}$$

and that p is bounded in the far-field. The second condition is obtained by taking $\overline{R} \to \infty$ limit of (3.109), i.e.

$$p_{\overline{R}\overline{R}} - \mu^2 \overline{R} e^{2\overline{R}T_w} p = 0, \tag{3.112}$$

where

$$\mu^2 = 2\hat{c}_0 M_\infty^2 \overline{\alpha}^2 T_w. \tag{3.113}$$

To leading order, this equation is found to have a decaying solution of the form

$$p \sim \mu^{-1/2} \overline{R}^{-1/4} \exp\Big[-\frac{\mu}{T_w} e^{\overline{R}T_w} . \overline{R}^{1/2} - \frac{1}{2} \overline{R} T_w \Big]. \tag{3.114}$$

We shall now compare these asymptotic results with numerically determined results for large values of ζ.

3.5.2 Numerical Results

All the results presented in this section are for a freestream Mach number of 3.8 and azimuthal wavenumber $n = 1$.

From the numerical observations the leading order term in the c expansion (3.98) is found to have the value

$$\hat{c}_0 = 0.4617922. \tag{3.115}$$

Using a fourth order Runge-Kutta scheme equation, (3.109) was solved subject to conditions (3.111) and (3.114) to determine the eigenvalues $\hat{c}_1$. We find that for a given value of $\overline{\alpha}$ there appears to be a large number of discrete, real values for $\hat{c}_1$.

Figure 3.75 displays a plot of $\hat{c}_1$ against $\overline{\alpha}$ corresponding to the first five modes, as shown.

As observed above, our asymptotic analysis implies the existence of a large number of discrete possible values for $\hat{c}_1$, which in turn implies the existence of a large number of discrete values for c_r. When we searched for the eigenvalues by solving the full system of equations numerically, for large ζ, we determined that there were indeed many modes and we managed to identify the first five modes. Figure 3.76 displays a comparison between the asymptotically determined value of c_r and the numerically determined value of c_r against ζ corresponding to the first mode. We have relatively good agreement, since the error term in the asymptotic theory is $O(\epsilon^2)$, which is quite large. Therefore the numerical results seem to confirm our asymptotic theory.

The asymptotic theory presented above tells us nothing about c_i and consequently reveals no information about the growth rate αc_i; such an investigation would require a prohibitive amount of algebra. However, our numerical observations strongly suggest that $\alpha c_i \to 0$ as $\zeta \to \infty$.

We now move on to consider the form of the disturbance equations in the limit of small (large) ζ for both adiabatic and heated/cooled wall conditions on a cone.

3.6 Disturbance Equations for Small ζ

In this section we consider the form of the disturbance equations in a number of limits for small (also large - for the cone only) ζ, to give us a better understanding of the details of the numerical results described in section 3.3. Note that this asymptotic theory is carried out for the general case of the cone, but is readily applicable to the cylinder problem (although this is only valid in the limit of small ζ - the large ζ limit has been considered in the previous section for this axisymmetric body) by setting $\lambda = 0$.

Perhaps the most intriging feature of the numerical results presented in section 3.3, is the emergence of an additional mode as $\zeta \to 0$ (or $\zeta \to \infty$) with $\alpha \to 0$. This feature is investigated first. Throughout this section we shall consider the velocity perturbation equation which has the form

$$\frac{d}{d\eta}\left\{\frac{(w_0 - c)[\phi_\eta + (\zeta/(1+\zeta\eta+\lambda\zeta^2))\phi] - w_{0\eta}\phi}{T_0[1 + n^2\zeta^2/(\alpha^2(1+\zeta\eta+\lambda\zeta^2)^2)] - M_\infty^2(w_0 - c)^2}\right\} = \frac{\alpha^2(w_0 - c)\phi}{T_0}. \quad (3.116)$$

3.6.1 $\zeta \to 0, \alpha = O(\zeta)$ or $\zeta \to \infty, \alpha = O(\zeta^{-1})$

Since the problem as posed is basically equivalent as $\zeta \to 0$ and $\zeta \to \infty$, we consider only the former limit, and later we show briefly how the results for the latter can be simply inferred.

As noted in Section 3.1, as $\zeta \to 0$, (3.116) is seen generally to reduce to the planar system as treated by Mack (1984, 1987a, for example). However, this will no longer be the case if $\alpha = O(\zeta)$, since then the denominator on the left-hand side of (3.116) no longer reduces to the planar case.

Specifically, let us write (consistent with (3.10))

$$\alpha = \zeta\overline{\alpha}, \quad (3.117)$$

where it is assumed $\overline{\alpha} = O(1)$ as $\zeta \to 0$. The results for mode $\mathrm{I_A}$ shown in the numerics section, together with other numerical results obtained, indicate that as $\zeta \to 0$, then $c \to 0$ also.

Partly guided by this, for $\eta = O(1)$ we choose expansions of the form

$$c = \zeta c_1 + \zeta^2 c_2 + \zeta^3 c_3 + \cdots,$$

$$\phi = \phi_0(\eta) + \zeta\phi_1(\eta) + \zeta^2\phi_2(\eta) + \zeta^3\phi_3(\eta) + \cdots,$$

$$w_0 = W_{00}(\eta) + \zeta W_{01}(\eta) + \zeta^2 W_{02}(\eta) + \zeta^3 W_{03}(\eta) + \cdots,$$

$$T_0 = T_{00}(\eta) + \zeta T_{01}(\eta) + \zeta^2 T_{02}(\eta) + \zeta^3 T_{03}(\eta) + \cdots, \tag{3.118}$$

(although see (3.228) below) where $W_{00}(\eta)$ and $T_{00}(\eta)$ represent the planar values of velocity and temperature profiles, respectively, and $W_{01}(\eta)$ and $T_{01}(\eta)$, etc., correspond to the perturbations to the basic flow due to curvature.

Substitution of expansions (3.118) into (3.116) to leading order gives

$$\frac{d}{d\eta}\Big\{\frac{W_{00}\phi_{0\eta} - W_{00\eta}\phi_0}{T_{00}[1 + n^2/\overline{\alpha}^2] - M_\infty^2 W_{00}^2}\Big\} = 0, \tag{3.119}$$

where it is assumed that $\lambda\zeta^2, \eta\zeta << 1$ for $\eta = O(1), \zeta \to 0$. Equation (3.119) can be re-written

$$W_{00}\phi_{0\eta} - W_{00\eta}\phi_0 = k_0\Big\{T_{00}\Big[1 + \frac{n^2}{\overline{\alpha}^2}\Big] - M_\infty^2 W_{00}^2\Big\}, \tag{3.120}$$

where k_0 is independent of η. By the form of the boundary conditions prescribed at the wall we have $\phi_0(\eta = 0) = W_{00}(\eta = 0) = 0$, which implies that on the cone surface the left-hand side of (3.120) is zero. In the far-field, namely $\eta \to \infty$, it is required that ϕ_0 does not grow exponentially. Both sets of conditions can only be satisfied if

$$k_0 = 0, \tag{3.121}$$

since generally $T_{00}[1 + (n^2/\overline{\alpha}^2)] - M_\infty^2 w_{00}^2 \neq 0$. This automatically implies

$$W_{00}\phi_{0\eta} = W_{00\eta}\phi_0 \quad \Leftrightarrow \quad \int \frac{d\phi_0}{\phi_0} = \int \frac{dW_{00}}{W_{00}}, \tag{3.122}$$

giving the result

$$\phi_0 = A_0 W_{00}(\eta), \tag{3.123}$$

where A_0 is independent of η and represents an arbitary constant (i.e., the unknown amplitude of the eigensolution).

However, (3.123) is not a uniformly valid approximation to (3.116) for all η; specifically, a breakdown occurs when $\eta = O(\zeta^{-1})$. Define

$$\hat{\eta} = 1 + \zeta\eta = O(1), \tag{3.124}$$

(i.e., $\hat{\eta}$ represents a scale comparable to the radius of the cone, and therefore corresponds to the region at the edge of the boundary layer), and on this scale ϕ expands as follows:

$$\phi = \hat{\Phi}_0(\hat{\eta}) + \zeta\hat{\Phi}_1(\hat{\eta}) + \cdots. \tag{3.125}$$

Equation (3.124) implies that we also have

$$\phi_\eta = \zeta\hat{\Phi}_{0\hat{\eta}} + \zeta^2\hat{\Phi}_{1\hat{\eta}} + \cdots. \tag{3.126}$$

Substituting expansions (3.125), (3.126) and the relevant parts of equation (3.118) into (3.116), to leading order yields

$$\frac{d}{d\hat{\eta}}\Big\{\frac{\hat{\Phi}_{0\hat{\eta}} + (1/\hat{\eta})\hat{\Phi}_0}{[1 + (n^2/\overline{\alpha}^2\hat{\eta}^2)] - M_\infty^2}\Big\} = \overline{\alpha}^2\hat{\Phi}_0, \tag{3.127}$$

where it is assumed in the limit $\hat{\eta} \to 1$, that $T_{00}, W_{00} \to 1$ (i.e. the far-field boundary conditions) and $\zeta^2\lambda << 1$. Also, since we are approaching the far-field, in this limit, the curvature perturbation terms for the temperature and velocity profiles (and their derivatives) will be expected to tend to zero (at least be exponentially bounded).

In the limit $\hat{\eta} \to 1$ ($\Leftrightarrow \eta \to \infty, \zeta \to 0$) (3.127) has a solution similar to (3.1[illegible]), namely

$$\hat{\Phi}_0 \sim \hat{\phi}_0 K_n'[i\overline{\alpha}(M_\infty^2 - 1)^{1/2}\hat{\eta}], \tag{3.128}$$

where $K_n(z_1)$ is the Bessel function of order n, argument z_1 (the $K_n(z_1)$ solution is chosen in perference to the $I_n(z_1)$ solution in order that disturbances are propagated along characteristics in the downstream direction (see Ward (1955); Kluwick et al (1984); Duck and Hall (1989, 1990)).

To determine $\hat{\phi}_0$ we match (3.128) with the leading order term of the inner solution, namely (3.123) which in the limit $\eta \to \infty$, has the form

$$\phi_0 \to A_0. \tag{3.129}$$

Therefore we have

$$\hat{\phi}_0 = \frac{A_0}{K_n'[i\overline{\alpha}(M_\infty^2 - 1)^{1/2}]}, \tag{3.130}$$

yielding the result

$$\hat{\Phi}_0 = \frac{A_0 K_n'[i\overline{\alpha}(M_\infty^2 - 1)^{1/2}\hat{\eta}]}{K_n'[i\overline{\alpha}(M_\infty^2 - 1)^{1/2}]}. \tag{3.131}$$

Returning to the $\eta = O(1)$ layer (which can be thought of as the main part of the boundary layer), we now wish to determine the $O(\zeta)$ correction to ϕ. Defining

$$\chi = T_0\Big[1 + \frac{n^2\zeta^2}{\alpha^2(1 + \eta\zeta + \lambda\zeta^2)^2}\Big] - M_\infty^2(w_0 - c)^2, \tag{3.132}$$

then substituting expansions (3.118) into (3.132), making use of the fact $\lambda\zeta^2, \zeta\eta <<$ 1 for $\zeta \to 0$, and applying the binomial theorem, yields

$$\frac{1}{\chi} = \frac{1}{T_{00}[1 + (n^2/\overline{\alpha}^2)] - M_\infty^2 W_{00}^2} - \zeta\frac{T_{01}[1 + (n^2/\overline{\alpha}^2)] - 2W_{00}M_\infty^2(W_{01} - c_1)}{(T_{00}[1 + (n^2/\overline{\alpha}^2)] - M_\infty^2 W_{00}^2)^2} + O(\zeta^2). \tag{3.133}$$

Therefore the $O(\zeta)$, ϕ equation has the form

$$-\frac{d}{d\eta}\Big\{\frac{(W_{00}\phi_{0\eta}-W_{00\eta}\phi_0)(T_{01}[1+(n^2/\overline{\alpha}^2)]-2W_{00}M_\infty^2(W_{01}-c_1))}{(T_{00}[1+(n^2/\overline{\alpha}^2)]-M_\infty^2W_{00}^2)^2}\Big\}$$
$$+\frac{d}{d\eta}\Big\{\frac{(W_{01}-c_1)\phi_{0\eta}+W_{00}[\phi_{1\eta}+\phi_0]-W_{01\eta}\phi_0-W_{00\eta}\phi_1}{T_{00}[1+(n^2/\overline{\alpha}^2)]-M_\infty^2W_{00}^2}\Big\}=0. \quad (3.134)$$

By relation (3.123) we have

$$W_{00}\phi_{0\eta}-W_{00\eta}\phi_0=W_{00}A_0W_{00\eta}-W_{00\eta}A_0W_{00}=0, \quad (3.135)$$

which reduces (3.134) to

$$\frac{d}{d\eta}\Big\{\frac{(W_{01}-c_1)\phi_{0\eta}+W_{00}[\phi_{1\eta}+\phi_0]-W_{01\eta}\phi_0-W_{00\eta}\phi_1}{T_{00}[1+(n^2/\overline{\alpha}^2)]-M_\infty^2W_{00}^2}\Big\}=0. \quad (3.136)$$

Integrating once with respect to η yields

$$(W_{01}-c_1)\phi_{0\eta}+W_{00}[\phi_{1\eta}+\phi_0]-W_{01\eta}\phi_0-W_{00\eta}\phi_1=k_1\Big\{T_{00}\Big[1+\frac{n^2}{\overline{\alpha}^2}\Big]-M_\infty^2W_{00}^2\Big\}, \quad (3.137)$$

where k_1 is an arbitrary constant. Matching (3.137) with the wall conditions gives

$$-c_1\phi_{0\eta}(\eta=0)=k_1T_{00}(\eta=0)\Big(1+\frac{n^2}{\overline{\alpha}^2}\Big)$$
$$\Leftrightarrow -c_1A_0W_{00\eta}(\eta=0)=k_1T_{00}(\eta=0)\Big(1+\frac{n^2}{\overline{\alpha}^2}\Big), \quad (3.138)$$

where the condition $W_{0i}=\phi_i=0$ for all integer $i\geq 0$, at the wall, has been used.

In the far-field, ϕ_1 must not be exponentially large and again, the basic flow terms behave as

$$W_{00},T_{00}\to 1,$$
$$W_{00\eta},W_{01},W_{01\eta}\to 0. \quad (3.139)$$

Consequently, we have

$$-c_1\phi_{0\eta}\Big|_{\eta\to\infty}+\phi_{1\eta}\Big|_{\eta\to\infty}+\phi_0\Big|_{\eta\to\infty}=k_1\Big\{1-M_\infty^2+\frac{n^2}{\overline{\alpha}}\Big\}, \tag{3.140}$$

which can be simplified using (3.123), to

$$\phi_{1\eta}\Big|_{\eta\to\infty}+A_0=k_1\Big\{1-M_\infty^2+\frac{n^2}{\overline{\alpha}^2}\Big\}. \tag{3.141}$$

Since both $\phi_{1\eta}$ and $\hat{\Phi}_{0\hat{\eta}}$ are of $O(\zeta)$, then to match correctly as $\eta\to\infty$, the following relation must hold

$$\phi_{1\eta}\Big|_{\eta\to\infty}=\hat{\Phi}_{0\hat{\eta}}\Big|_{\hat{\eta}\to 1}. \tag{3.142}$$

Therefore, matching yields

$$\phi_{1\eta}\Big|_{\eta\to\infty}=\frac{A_0 i\overline{\alpha}(M_\infty^2-1)^{1/2}K_n''[i\overline{\alpha}(M_\infty^2-1)^{1/2}]}{K_n'[i\overline{\alpha}(M_\infty^2-1)^{1/2}]}. \tag{3.143}$$

Substitution of (3.143) into (3.141) yields

$$k_1=A_0\Big\{1+\frac{i\overline{\alpha}(M_\infty^2-1)^{1/2}K_n''[i\overline{\alpha}(M_\infty^2-1)^{1/2}]}{K_n'[i\overline{\alpha}(M_\infty^2-1)^{1/2}}\Big\}/(1-M_\infty^2+\frac{n^2}{\overline{\alpha}^2}). \tag{3.144}$$

Eliminating k_1 from (3.138), we obtain the following result for c_1:

$$c_1=\Big\{1+i\overline{\alpha}(M_\infty^2-1)^{1/2}\frac{K_n''[i\overline{\alpha}(M_\infty^2-1)^{1/2}]}{K_n'[i\overline{\alpha}(M_\infty^2-1)^{1/2}]}\Big\}\Big\{\frac{T_{00}(\eta=0)(1+n^2/\overline{\alpha}^2)}{W_{00\eta}(\eta=0)[M_\infty^2-1-n^2/\overline{\alpha}^2]}\Big\}. \tag{3.145}$$

The asymptotic forms for this expression in the limit of large and small $\overline{\alpha}$ may be found readily. Firstly, as $\overline{\alpha}\to\infty$ we have $n^2/\overline{\alpha}^2\to 0$, which immediately simplifies the second bracketed term to

$$\Big\{\frac{T_{00}(\eta=0)}{W_{00\eta}(\eta=0)(M_\infty^2-1)}\Big\}. \tag{3.146}$$

Defining

$$z = i\overline{\alpha}(M_\infty^2 - 1)^{1/2}, \tag{3.147}$$

then in the limit $z \to \infty$, we have the Bessel function expansions

$$K_n'(z) \sim -(\frac{\pi}{2z})^{1/2}e^{-z}[1 + O(\frac{1}{z})], \tag{3.148}$$

and

$$\begin{aligned} K_n''(z) &\sim [\frac{1}{2}(\frac{\pi}{2z})^{1/2}e^{-z}\frac{1}{z} + (\frac{\pi}{2z})^{1/2}e^{-z}][1 + O(\frac{1}{z})] - (\frac{\pi}{2z})^{1/2}e^{-z}[O(\frac{1}{z^2})] \\ &= (\frac{\pi}{2z})^{1/2}e^{-z}[1 + O(\frac{1}{z}) + \cdots]. \end{aligned} \tag{3.149}$$

This means

$$\frac{K_n''[i\overline{\alpha}(M_\infty^2 - 1)^{1/2}]}{K_n'[i\overline{\alpha}(M_\infty^2 - 1)^{1/2}]} \sim -1 + O(\frac{1}{\overline{\alpha}}), \tag{3.150}$$

in the limit $\overline{\alpha} \to \infty$ and equation (3.145) has the asymptotic form

$$c_1 \to -i\overline{\alpha}\frac{T_{00}(\eta = 0)}{W_{00\eta}(\eta = 0)(M_\infty^2 - 1)^{1/2}} + O(\overline{\alpha}^{-1}). \tag{3.151}$$

Turning our attention to the $\overline{\alpha} \to 0$ limit, since $(n^2/\overline{\alpha}^2) >> M_\infty^2$ and 1, then in this case the second bracket has the reduced form

$$\{ - \frac{T_{00}(\eta = 0)}{W_{00\eta}(\eta = 0)} \}. \tag{3.152}$$

The Bessel function, $K_n(z)$, can be expressed as the ascending series

$$K_n(z) = \frac{1}{2}(\frac{1}{2}z)^{-n} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!}(-\frac{1}{4}z^2)^k + (-1)^{n+1} \ln(\frac{z}{2}) I_n(z)$$

$$+(-1)^n \frac{1}{2}(\frac{z}{2})^n \sum_{k=0}^{\infty} \{\psi(k+1) + \psi(n+k+1)\} \frac{(\frac{1}{4}z^2)^k}{k!(n+k)!}, \qquad (3.153)$$

where z is defined by (3.147),

$$I_n(z) = (\frac{z}{2})^n \sum_{k=0}^{\infty} \frac{(\frac{1}{4}z^2)^k}{k!\Gamma(n+k+1)}, \qquad (3.154)$$

and

$$\psi(n) = -\nu + \sum_{k=1}^{n-1} k^{-1} \qquad (n \geq 2), \qquad (3.155)$$

ν being Euler's constant. Therefore, as $z \to 0$, for $n = 1$, we have

$$K_1(z) \sim \frac{1}{z} - \frac{z}{2} \ln|z| + O(z). \qquad (3.156)$$

Differentiating with respect to z gives

$$K_1'(z) \sim -\frac{1}{z^2} - \frac{1}{2} \ln|z| + O(1),$$

$$K_1''(z) \sim \frac{2}{z^3} - \frac{1}{2z} + O(1), \qquad (3.157)$$

which in turn give

$$c_1 \to \frac{T_{00}(\eta = 0)}{W_{00\eta}(\eta = 0)} + O(\overline{\alpha}^2 \ln \overline{\alpha}). \qquad (3.158)$$

If $n > 1$, in the limit $z \to 0$, we have

$$K_n(z) \sim \frac{1}{2}(\frac{1}{2}z)^{-n}[\Gamma(n) + O(z^2)], \qquad (3.159)$$

which yields (remembering $\Gamma(n) = (n-1)!$)

$$\frac{K_n''(z)}{K_n'(z)} \sim -\frac{n+1}{z} + O(z). \tag{3.160}$$

Therefore

$$c_1 \rightarrow \frac{nT_{00}(\eta=0)}{W_{00\eta}(\eta=0)} + O(\overline{\alpha}^2), \qquad \text{for } n > 1. \tag{3.161}$$

To determine the leading order imaginary term for c, higher order terms in the Bessel function expansion have to be considered, namely

$$K_n(z) \sim \frac{1}{2}(\frac{z}{2})^{-n}[\Gamma(n) + O(z^2)] + \frac{(-1)^{n+1}}{\Gamma(n+1)} \ln\frac{z}{2}(\frac{z}{2})^n + \cdots, \tag{3.162}$$

which yields

$$\begin{aligned}\frac{K_n''(z)}{K_n'(z)} \sim \Big[-\frac{n+1}{z} + O(z) + \frac{(-1)^{n+2}}{2^{2n-1}n!(n-1)!}(n-1)z^{2n-1}\ln\frac{z}{2}\Big] \\ \times\Big[1 - \frac{(-1)^{n+2}}{n!(n-1)!2^{2n-1}}z^{2n}\ln\frac{z}{2}\Big]^{-1} + \cdots.\end{aligned} \tag{3.163}$$

Employing the binomial theorem and taking the principal value of the logarithmic term yields

$$\mathrm{Im}\{c_1\} \rightarrow -\frac{\pi}{2}\frac{\overline{\alpha}^{2n}(M_\infty^2-1)^n T_{00}(0)2^{2-2n}}{W_{00\eta}(0)[(n-1)!]^2}. \tag{3.164}$$

Equations (3.158), (3.161) are precisely the (real) values found by Duck and Hall (1990) for the downstream limit of a non-axisymmetric viscous mode (taking into account the different scalings used in Duck and Hall's paper). Consequently, as $\zeta \rightarrow 0/\infty$, on a scale smaller/larger than that of the cone tip radius, this mode is expected to become predominantly viscous in nature, and to be described by triple-deck theory.

Figure 3.77 displays $\mathrm{Im}\{c_1\}$ with α for $n = 1, 2, 3$ (with (3.151) shown also). Unfortunately (perhaps) it is seen that $\mathrm{Im}\{c_1\} < 0$ for all $\overline{\alpha}$ (confirmed by (3.151)). It is also observed that $\mathrm{Im}\{c_1\} = O(\zeta)$ as $\zeta \to 0$. Unfortunately, it is also found that $O(\zeta^2)$ and higher corrections to this mode would require an extensive amount of algebra. However, progress can be made, in particular, an estimate for $\mathrm{Im}\{c(\overline{\alpha} = 0)\}$ can be obtained by considering, instead, the limit $\zeta \to 0$ of (B.2), pertinent to the $\alpha \to 0$ case. (This aspect is considered in the next section)

Finally, for this subsection, note that the $\zeta \to \infty$ results may be inferred from these $\zeta \to 0$ results, simply by replacing the small parameter "ζ" in the various expansions by the small parameter "$1/\lambda\zeta$ ". More subtle differences between the $\zeta << 1$ and the $\zeta >> 1$ solutions only appear at higher orders. Note, this is only valid for the case of a cone.

3.6.2 $\overline{\alpha} = 0, \zeta \to 0$ (or $\zeta \to \infty$)

System (B.2) turns out to be easier to analyse as $\zeta \to 0$ than does the corresponding finite $\overline{\alpha}$ ($= \alpha\zeta^{-1}$) problem. We again utilize expansions (3.118) (although for the full ϕ expansion, see later).

To leading order, we have, for $\eta = O(1)$

$$\phi_0 = A_0 W_{00}(\eta), \tag{3.165}$$

where A_0 is some (arbitrary) amplitude parameter.

At the next order we have the following system:

$$W_{00}\phi_{1\eta} - A_0 W_{00\eta} c_1 + A_0 W_{00}^2 - W_{00\eta}\phi_1 + A_0 W_{01} W_{00\eta} - A_0 W_{01\eta} W_{00} = k_1 T_{00}, \tag{3.166}$$

where k_1 is a constant, and we have utilized (3.165). Setting $\eta = 0$ in (3.166), assuming $\phi_1(\eta = 0) = 0$, then

$$-A_0 c_1 W_{00\eta}(\eta = 0) = k_1 T_{00}(\eta = 0). \tag{3.167}$$

We now match with the far-field boundary conditions. Defining the outer scale

$$\bar{r} = 1 + \eta\zeta = O(1), \tag{3.168}$$

we assume outer expansions of the form

$$\phi = \phi_0^{\text{out}}(\bar{r}) + \zeta\phi_1^{\text{out}}(\bar{r}) + O(\zeta^2), \tag{3.169}$$

and so

$$\phi_\eta = \zeta\phi_{0\bar{r}}^{\text{out}} + \zeta^2\phi_{1\bar{r}}^{\text{out}} + \cdots. \tag{3.170}$$

The boundary conditions must be compatible with B.4, therefore we must have an outer solution of the form

$$\phi^{\text{out}} \sim \hat{A}.\bar{r}^{-(n+1)}, \tag{3.171}$$

where

$$\hat{A} = \hat{A}_0 + \zeta A_1 + \cdots. \tag{3.172}$$

Considering (3.165) in the limit $\eta \to \infty$, it is immediately evident that matching with the leading order term of equation (3.171) yields

$$\hat{A}_0 = A_0. \tag{3.173}$$

In the far-field, equation (3.166) reduces to

$$\phi_{1\eta}\Big|_{\eta\to\infty} = k_1 - A_0. \tag{3.174}$$

Since both $\phi_{1\eta}$ and $\phi_{0\bar{r}}^{\text{out}}$ are $O(\zeta)$, then for correct matching we must have

$$\phi_{1\eta}\Big|_{\eta\to\infty} = \phi_{0\bar{r}}^{\text{out}}\Big|_{\bar{r}=1} = -(n+1)A_0. \tag{3.175}$$

Therefore substituting (3.175) into (3.174) and eliminating k_1, using (3.167), gives

$$c_1 = \frac{nT_{00}(\eta=0)}{W_{00\eta}(\eta=0)}, \tag{3.176}$$

in accord with (3.161). In order to estimate complex values of c we must proceed to higher orders in ζ.

At the next order in ζ the following equation governing ϕ_2 is obtained:

$$\begin{aligned}
W_{00}\phi_{2\eta} + 2\lambda W_{00}\phi_{0\eta} + 2\eta W_{00}\phi_{1\eta} + W_{02}\phi_{0\eta} - W_{02\eta}\phi_0 + W_{01}\phi_{1\eta} + \eta^2 W_{00}\phi_{0\eta} \\
-c_1\phi_{1\eta} + 2W_{01}\eta\phi_{0\eta} + W_{01}\phi_0 - W_{01\eta}\phi_1 - 2\eta W_{01\eta}\phi_0 - 2\eta c_1\phi_{0\eta} - c_2\phi_{0\eta} \\
+W_{00}\phi_1 + \eta W_{00}\phi_0 - c_1\phi_0 - W_{00\eta}\phi_2 - 2\lambda W_{00\eta}\phi_0 - 2\eta\phi_1 W_{00\eta} - \eta^2 W_{00\eta}\phi_0 \\
= k_2 T_{00} + n^2 T_{00}\int_0^\eta \frac{\phi_0 W_{00}}{T_{00}}d\eta + k_1 T_{01},
\end{aligned} \tag{3.177}$$

where k_2 is a constant and k_1 is defined above. We defer any consideration of this equation, and move to the next order of ζ, which yields

$$
\begin{aligned}
W_{00}\phi_{3\eta} &+ 2\lambda W_{00}\phi_{1\eta} + 2\eta W_{00}\phi_{2\eta} + 2\lambda\eta W_{00}\phi_{0\eta} + \eta^2 W_{00}\phi_{1\eta} - c_1\phi_{2\eta} - 2\lambda c_1\phi_{0\eta} \\
&-2\eta c_1\phi_{1\eta} - \eta^2 c_1\phi_{0\eta} - c_2\phi_{1\eta} - 2\eta c_2\phi_{0\eta} - c_3\phi_{0\eta} + W_{00}\phi_2 + \lambda W_{00}\phi_0 + \eta W_{00}\phi_1 \\
&-c_1\phi_1 - c_1\eta\phi_0 - c_2\phi_0 - W_{00\eta}\phi_3 - 2\lambda W_{00\eta}\phi_1 - \eta^2 W_{01\eta}\phi_0 - 2\eta\phi_2 W_{00\eta} \\
&-\eta^2 W_{00\eta}\phi_1 + W_{01}\phi_{2\eta} + 2\lambda W_{01}\phi_{0\eta} + 2\eta W_{01}\phi_{1\eta} + \eta^2 W_{01}\phi_{0\eta} + W_{02}\phi_{1\eta} \\
&+2\eta W_{02}\phi_{0\eta} + W_{03}\phi_{0\eta} + W_{01}\phi_1 + W_{01}\eta\phi_0 + W_{02}\phi_0 - W_{03\eta}\phi_0 - 2\eta W_{02\eta}\phi_0 \\
&-2\lambda W_{01\eta}\phi_0 - 2\lambda\eta W_{00\eta}\phi_0 - W_{02\eta}\phi_1 - 2\eta W_{01\eta}\phi_1 - W_{01\eta}\phi_2 \\
&= k_2 T_{01} + k_1 T_{02} + k_3 T_{00} + n^2 T_{01}\int_0^\eta \frac{\phi_0 W_{00}}{T_{00}}d\eta \\
&+n^2 T_{00}\int_0^\eta \frac{[\phi_1 T_{00} - \phi_0 T_{01}]W_{00} - c_1 T_{00}\phi_0 + W_{01}\phi_0 T_{00}}{T_{00}^2}d\eta. \qquad (3.178)
\end{aligned}
$$

Our main goal here is to determine the leading-order *imaginary* component of the complex wavespeed c (we do, of course, already know the leading-order real term). Now since the above equations just contain real coefficients, any imaginaries must, of necessity, only arise at a critical point, where, $c = W_{00}$. Since $c = O(\zeta)$, this implies that in the neighbourhood of the critical point, $W_{00} = O(\zeta)$. However, $W_{00} = O(1)$, but $W_{00} = W_{00}(\eta)$; therefore this must occur when $\eta = O(\zeta)$. Consequently, consider a thin layer relative to the $\eta = O(1)$ scale, namely

$$\tilde{\eta} = \eta/\zeta = O(1). \qquad (3.179)$$

On this scale, the expansion for ϕ is expected to develop as

$$\phi = \zeta\Phi_0(\tilde{\eta}) + \zeta^2\Phi_1(\tilde{\eta}) + \zeta^3\Phi_2(\tilde{\eta}) + \cdots, \qquad (3.180)$$

where the Φ_i are expected to be normalized in such a way as to be generally $O(1)$ quantities.

From the governing equations for the basic flow (as defined in Chapter 2), the z-momentum equation has the form

$$\hat{v}_1 \frac{\partial w_0}{\partial \eta} + \frac{\zeta w_0}{2}\frac{\partial w_0}{\partial \zeta} - w_0\Big[\lambda\zeta + \frac{\eta}{2}\Big]\frac{\partial w_0}{\partial \eta} = \frac{T_0}{r}\frac{\partial}{\partial \eta}\Big[rT_0\frac{\partial w_0}{\partial \eta}\Big], \tag{3.181}$$

where

$$r = 1 + \zeta\eta + \lambda\zeta^2. \tag{3.182}$$

Substituting the basic flow expansions for T_0 and w_0, with respect to ζ, yields to leading order

$$\hat{v}_1 W_{00\eta} - \frac{\eta}{2}W_{00}W_{00\eta} = T_{00}\frac{\partial}{\partial \eta}[T_{00}W_{00\eta}]. \tag{3.183}$$

Evaluating on the surface of the cone and assuming adiabatic conditions gives

$$W_{00\eta\eta}(0) = 0. \tag{3.184}$$

The $O(\zeta)$ equation has the form

$$\begin{aligned}
&\hat{v}_1 W_{01\eta} + \frac{W_{00}W_{01}}{2} - \frac{\eta}{2}\Big(W_{00}W_{01\eta} + W_{01}W_{00\eta}\Big) + \lambda W_{00}W_{00\eta} = \\
&T_{00}T_{00\eta}W_{01\eta} + T_{00}^2 W_{01\eta\eta} + T_{00}T_{01\eta}W_{00\eta} + T_{00}T_{01}W_{00\eta\eta} + T_{00}^2 W_{00\eta} \\
&+\eta T_{00}T_{00\eta}W_{00\eta} + \eta T_{00}^2 W_{00\eta\eta} + (T_{01} - \eta T_{00})(T_{00\eta}W_{00\eta} + T_{00}W_{00\eta\eta}),
\end{aligned} \tag{3.185}$$

which when evaluating on the cone surface and adiabatic conditions are again assumed, yields

$$W_{01\eta\eta}(0) = -W_{00\eta}(0). \tag{3.186}$$

In the thin, critical layer, adjacent to the wall, basic flow expansions for velocity and temperature are Taylor expanded about $\eta (= \tilde{\eta}\zeta) = 0$, giving the critical layer expansions

$$w_0 = \zeta W_{00\eta}(0)\tilde{\eta} + \zeta^2[\frac{1}{2}W_{00\eta\eta}(0)\tilde{\eta}^2 + W_{01\eta}(0)\tilde{\eta}]$$

$$\zeta^3[\frac{1}{6}\tilde{\eta}^3 W_{00\eta\eta\eta}(0) + \frac{1}{2}W_{01\eta\eta}(0)\tilde{\eta}^2 + W_{02\eta}(0)\tilde{\eta}] + O(\zeta^4), \tag{3.187}$$

and

$$T_0 = T_{00}(0) + \zeta[T_{00\eta}(0)\tilde{\eta} + T_{01}(0)] + \zeta^2[T_{02}(0) + T_{01\eta}(0)\tilde{\eta} + \frac{1}{2}T_{00\eta\eta}(0)\tilde{\eta}^2]$$

$$\zeta^3[T_{03}(0) + \tilde{\eta}T_{02\eta}(0) + \frac{1}{2}\tilde{\eta}^2 T_{01\eta\eta}(0) + \frac{1}{6}\tilde{\eta}^3 T_{00\eta\eta\eta}(0)] + O(\zeta^4), \tag{3.188}$$

where the no-slip boundary condition has been applied. Also, since $W_{0\eta} = \frac{1}{\zeta}W_{0\tilde{\eta}}$, then

$$W_{0\eta} = W_{00\eta}(0) + \zeta[W_{00\eta\eta}(0)\tilde{\eta} + W_{01\eta}(0)]$$

$$+\zeta^2[\frac{1}{2}\tilde{\eta}^2 W_{00\eta\eta\eta}(0) + W_{01\eta\eta}(0)\tilde{\eta} + W_{02\eta}(0)] + \cdots. \tag{3.189}$$

Substituting expansions (3.180), (3.187) - (3.189), and the c expansion into (B.2), and integrating through with respect to $\tilde{\eta}$, to leading order in the critical layer, we have

$$\tilde{k}_0 T_{00}(0) = 0 \quad \Rightarrow \tilde{k}_0 = 0, \tag{3.190}$$

where $\tilde{k}_0$ is the leading order term in the constant of integration, the whole term having the form

$$\tilde{k} = \tilde{k}_0 + \zeta\tilde{k}_1 + \zeta^2\tilde{k}_2 + \cdots. \tag{3.191}$$

At the next order in ζ, we obtain

$$[W_{00\eta}(0)\tilde{\eta} - c_1]\Phi_{0\tilde{\eta}} - \Phi_0 W_{00\eta}(0) = \tilde{k}_1 T_{00}(0). \tag{3.192}$$

Integrating, gives

$$\Phi_0 = -\frac{\tilde{k}_1 T_{00}(0)}{W_{00\eta}(0)} + \tilde{A}_0[W_{00\eta}(0)\tilde{\eta} - c_1], \tag{3.193}$$

where $\tilde{A}_0$ is the constant of integration. Applying the wall condition $\Phi_0(\eta = 0) = 0$ yields

$$\frac{\tilde{k}_1 T_{00}(0)}{W_{00\eta}(0)} = -\tilde{A}_0 c_1. \tag{3.194}$$

Therefore

$$\Phi_0 = A_0 W_{00\eta}(0)\tilde{\eta}, \tag{3.195}$$

where we have matched with (3.165) taken in the limit $\eta \to 0$, obtaining $\tilde{A}_0 = A_0$.

At $O(\zeta^2)$ the following system governs Φ_1:

$$\begin{aligned}[\frac{1}{2}W_{00\eta\eta}(0)\tilde{\eta}^2 + W_{01\eta}(0)\tilde{\eta} - c_2]\Phi_{0\tilde{\eta}} + [W_{00\eta}(0)\tilde{\eta} - c_1]\Phi_{1\tilde{\eta}} - \Phi_1 W_{00\eta}(0)\\ -[W_{00\eta\eta}(0)\tilde{\eta} + W_{01\eta}(0)]\Phi_0 = \tilde{k}_2 T_{00}(0) + \tilde{k}_1[T_{00\eta}(0)\tilde{\eta} + T_{01}(0)].\end{aligned} \tag{3.196}$$

Utilizing (3.195), this can be simplified to

$$[W_{00\eta}(0)\tilde{\eta} - c_1]\Phi_{1\eta} - \Phi_1 W_{00\eta}(0) = \hat{k}_1 + \frac{1}{2}W_{00\eta\eta}(0)\tilde{\eta}^2 A_0 W_{00\eta}(0) + \tilde{k}_1 T_{00\eta}(0)\tilde{\eta}, \tag{3.197}$$

where the constant $\hat{k}_1$ is given by

$$\hat{k}_1 \doteq \tilde{k}_2 T_{00}(0) + \tilde{k}_1 T_{01}(0) + c_2 A_0 W_{00\eta}(0). \tag{3.198}$$

Assuming adiabatic conditions (3.197) reduces to

$$[W_{00\eta}(0)\tilde{\eta} - c_1]\Phi_{1\eta} - \Phi_1 W_{00\eta}(0) = \hat{k}_1. \tag{3.199}$$

Integrating with respect to η and applying the wall condition $\Phi_1(\eta = 0) = 0$ gives the solution

$$\Phi_1 = A_1 W_{00\eta}(\eta = 0)\tilde{\eta}, \tag{3.200}$$

where A_1 is a constant, linearly related to A_0.

At the next order in ζ the equation of Φ has the form

$$\begin{aligned}
&\Phi_{0\tilde{\eta}}[\frac{1}{6}W_{00\eta\eta\eta}(0)\tilde{\eta}^3 + \frac{1}{2}W_{01\eta\eta}(0)\tilde{\eta}^2 + W_{02\eta}(0)\tilde{\eta} - c_3] + \Phi_{1\tilde{\eta}}[\frac{1}{2}\tilde{\eta}^2 W_{00\eta\eta}(0) \\
&\quad +W_{01\eta}(0)\tilde{\eta} - c_2] + \Phi_{2\tilde{\eta}}[W_{00\eta}(0)\tilde{\eta} - c_1] + 2(\lambda + \tilde{\eta})\Phi_{0\tilde{\eta}}[W_{00\eta}(0)\tilde{\eta} - c_1] \\
&\Phi_0[W_{00\eta}(0)\tilde{\eta} - c_1] - \Phi_2 W_{00\eta}(0) - 2(\lambda + \tilde{\eta})\Phi_0 W_{00\eta}(0) - \Phi_1[W_{00\eta\eta}(0)\tilde{\eta} + \\
&\quad W_{01\eta}(0)] - \Phi_0[\frac{1}{2}W_{00\eta\eta\eta}(0)\tilde{\eta}^2 + W_{01\eta\eta}(0)\tilde{\eta}] - \Phi_0 W_{02\eta}(0) \\
&= \tilde{k}_2[T_{00\eta}(0)\tilde{\eta} + T_{01}(0)] + \tilde{k}_1[T_{02}(0) + T_{01\eta}(0)\tilde{\eta} + \frac{1}{2}T_{00\eta\eta}(0)\tilde{\eta}^2] + \tilde{k}_3 T_{00}(0).
\end{aligned} \tag{3.201}$$

Utilizing (3.195) and (3.200) and assuming adiabatic wall conditions this system reduces to

$$\begin{aligned}
&\Phi_{2\tilde{\eta}}[W_{00\eta}(0)\tilde{\eta} - c_1] - \Phi_2 W_{00\eta}(0) = \hat{k}_2 + 3\tilde{\eta}A_0 W_{00\eta}(0)c_1 \\
&+\frac{1}{3}\tilde{\eta}^3 W_{00\eta\eta\eta}(0)W_{00\eta}(0)A_0 - \frac{3}{2}A_0 W_{00\eta}^2(0)\tilde{\eta}^2 - \frac{T_{00\eta\eta}(0)}{2T_{00}(0)}\tilde{\eta}^2 A_0 c_1 W_{00\eta}(0),
\end{aligned} \tag{3.202}$$

where the constant $\hat{k}_2$ is given by

$$\hat{k}_2 = \tilde{k}_2 T_{01}(0) + \tilde{k}_1 T_{02}(0) + \tilde{k}_3 T_{00}(0) + c_3 A_0 W_{00\eta}(0) + A_1 W_{00\eta}(0) c_2 + 2\lambda A_0 c_1 W_{00\eta}(0), \qquad (3.203)$$

and $\tilde{k}_1$ has been eliminated from (3.202), using (3.194).

If we take (as we are quite at liberty to do) A_0 and $\hat{k}_2$ to be *real* constants (this is not essential for our arguments, but simplifies the following argument), then we now consider only Φ_2^i (where here and elsewhere a superscript i denotes an imaginary component). This quantity is triggered by the well-known $+i\pi$ jump in the logarithm (Mack, 1984, for example) across the critical layer. Specifically, here, this is caused by the η dependency on the right-hand side of (3.202) ($\hat{k}_2$ plays no role being η-independent). If (3.202) is written symbolically as

$$[W_{00\eta}(0)\tilde{\eta} - c_1]\Phi_{2\tilde{\eta}} - W_{00\eta}(0)\Phi_2 = \hat{R}, \qquad (3.204)$$

then

$$\Phi_2 = [W_{00\eta}(0)\tilde{\eta} - c_1]\int_0^{\tilde{\eta}} \frac{\hat{R}d\hat{\eta}_1}{[W_{00\eta}(0)\hat{\eta}_1 - c_1]^2}. \qquad (3.205)$$

Evaluating this integral, taking only the imaginaries together with the limit $\tilde{\eta} \to \infty$ yields

$$\Phi_2^i \sim B^i[W_{00\eta}(0)\tilde{\eta} - c_1], \qquad (3.206)$$

where

$$B^i = \pi A_0 c_1 \Big\{ \frac{W_{00\eta\eta\eta}(0)c_1}{W_{00\eta}^3(0)} - \frac{T_{00\eta\eta}(0)c_1}{T_{00}(0)W_{00\eta}^2(0)} \Big\}. \qquad (3.207)$$

Equation (3.206) then provides lower boundary conditions for the system (3.177) and (3.178).

Since (3.177) contains only real coefficients (taking c_2 to be real as well, which will be justified *a posteriori*) and assuming ϕ_0, ϕ_1 and all their derivatives are real, then we must have

$$W_{00}\phi^i_{2\eta} - W_{00\eta}\phi^i_2 = 0, \tag{3.208}$$

which on integration yields

$$\phi^i_2(\eta) = AW_{00}(\eta), \tag{3.209}$$

where A is a constant of integration. Matching the expansions for ϕ in the η and $\tilde{\eta}$ layers, at $O(\zeta^3)$ yields

$$\Phi^i_2 = AW_{00\eta}(0)\tilde{\eta} + \phi^i_3(0), \tag{3.210}$$

where the Taylor expansion of (3.209) about $\eta = 0$ has been utilized and the boundary condition $W_{00}(0) = 0$ has been applied. Matching (3.206) with (3.210) gives

$$A = B^i \qquad \text{and} \qquad \phi^i_3(0) = -B^i c_1. \tag{3.211}$$

Therefore (3.209) can be written

$$\phi^i_2(\eta) = B^i W_{00}(\eta). \tag{3.212}$$

The imaginary part of equation (3.178) has the form

$$\begin{aligned} &W_{00}\phi^i_{3\eta} + 2\eta W_{00}\phi^i_{2\eta} - c_1\phi^i_{2\eta} + W_{00}\phi^i_2 - W_{00\eta}\phi^i_3 \\ &+W_{01}\phi^i_{2\eta} - 2\eta\phi^i_2 W_{00\eta} - W_{01\eta}\phi^i_2 - c^i_3\phi_{0\eta} = k^i_3 T_{00}. \end{aligned} \tag{3.213}$$

In the far-field this equation reduces to

$$\phi^i_{3\eta}\Big|_{\eta\to\infty} + \phi^i_2\Big|_{\eta\to\infty} = k^i_3, \tag{3.214}$$

where it has been assumed $\phi^i_{2\eta} \to 0$, by the form of (3.212).

Equation (3.214) must match the far-field solution as given by (B.4). In the limit $\eta \to \infty$ all the perturbation terms are expected to behave like (B.4) (resulting in decay), therefore

$$\begin{aligned}\phi^i_{3\eta}\Big|_{\eta\to\infty} \sim -(n+1)\eta^{-(n+1)} &\sim -(n+1)\phi^i_2\Big|_{\eta\to\infty}\\ &\sim -(n+1)B^i,\end{aligned} \tag{3.215}$$

and so

$$k^i_3 = -nB^i. \tag{3.216}$$

Setting $\eta = 0$ in (3.213) yields

$$-c_1\phi^i_{2\eta}\Big|_{\eta=0} - W_{00\eta}(0)\phi^i_3\Big|_{\eta=0} - c^i_3\phi_{0\eta}\Big|_{\eta=0} = -nB^iT_{00}(0). \tag{3.217}$$

However, from (3.211) and (3.212) we have

$$\phi^i_{2\eta}\Big|_{\eta=0} = B^iW_{00\eta}(0), \tag{3.218}$$

$$\phi^i_3\Big|_{\eta=0} = -c_1B^i. \tag{3.219}$$

Consequently (after substituting for c_1 and $\phi_{0\eta}$),

$$c^i_3 = n^3\frac{T^2_{00}(0)\pi}{W^2_{00\eta}(0)}\Big\{\frac{T_{00}(0)W_{00\eta\eta\eta}(0)}{W^4_{00\eta}(0)} - \frac{T_{00\eta\eta}(0)}{W^3_{00\eta}(0)}\Big\}. \tag{3.220}$$

From the governing equations for the basic flow (as defined in Chapter 2), the energy equation has the form

$$\hat{v}_1\frac{\partial T_0}{\partial\eta}+\frac{\zeta w_0}{2}\frac{\partial T_0}{\partial\zeta}-w_0\big[\lambda\zeta+\frac{\eta}{2}\big]\frac{\partial T_0}{\partial\eta}=T_0^2(\gamma-1)M_\infty^2\Big(\frac{\partial w_0}{\partial\eta}\Big)^2+\frac{T_0}{r}\frac{\partial}{\partial\eta}\Big[\frac{rT_0}{\sigma}\frac{\partial T_0}{\partial\eta}\Big], \quad (3.221)$$

where r is defined by (3.182). Substituting the basic flow expansions for T_0 and w_0, with respect to ζ, yields to leading order

$$\hat{v}_1 T_{00\eta}-\frac{\eta}{2}W_{00}T_{00\eta}=T_{00}^2(\gamma-1)M_\infty^2W_{00\eta}^2+T_{00}\frac{\partial}{\partial\eta}\Big[\frac{T_{00}T_{00\eta}}{\sigma}\Big]. \qquad (3.222)$$

Evaluating on the surface of the cone and assuming adiabatic conditions gives

$$T_{00\eta\eta}(0)=-\sigma(\gamma-1)M_\infty^2W_{00\eta}^2(0). \qquad (3.223)$$

The governing equation for continuity has the form

$$\frac{\partial}{\partial\eta}\Big(\frac{\hat{v}_1}{T_0}\Big)+\frac{\zeta\hat{v}_1}{rT_0}+\frac{\zeta}{2}\frac{\partial}{\partial\zeta}\Big(\frac{w_0}{T_0}\Big)-\big[\lambda\zeta+\frac{\eta}{2}\big]\frac{\partial}{\partial\eta}\Big(\frac{w_0}{T_0}\Big)=0. \qquad (3.224)$$

Substituting the revelant terms of (3.118) and evaluating the resultant leading order equation on the body surface, yields

$$\hat{v}_{1\eta}\Big|_{\eta=0}=0, \qquad (3.225)$$

where, again, adiabatic wall conditions are assumed.

Differentiating the leading equation in ζ of the basic flow z-direction momentum equation, i.e. (3.183) with respect to η and evaluating on the boundary yields

$$W_{00\eta\eta\eta}(0)=-\frac{T_{00\eta\eta}(0)W_{00\eta}(0)}{T_{00}(0)}, \qquad (3.226)$$

where (3.225) has been utilized and again, the cone surface is assumed to act as an insulator. Substituting results (3.223) and (3.226) into (3.220) gives

$$c_3^i = 2\pi n^3 \sigma(\gamma - 1) M_\infty^2 \frac{T_{00}^2(0)}{W_{00\eta}^3(0)}. \tag{3.227}$$

In fact, the expansion for ϕ in (3.118) is not quite complete as it stands, since the analysis of the $\tilde{\eta} = O(1)$ layer above indicates the presence of logarithmic terms; specifically, we require

$$\phi = \phi_0(\eta) + \zeta\phi_1(\eta) + \zeta^2\phi_2(\eta) + \zeta^3\phi_3(\eta) + \cdots + \log\zeta[\zeta^2\phi_{21}(\eta) + \zeta^3\phi_{31}(\eta) + \cdots], \tag{3.228}$$

where

$$\phi_{21}(\eta) = A_{21} W_{00}(\eta), \tag{3.229}$$

with A_{21} a constant.

A comparison of the fully numerical computation of Real$\{c(\alpha = 0)\}$, with the asymptotic formula (3.176), as $\zeta \to 0$, is shown in Figure 3.78. The agreement is seen to be entirely satisfactory. Unfortunately, the correlation between the computed Im$\{c(\alpha = 0)\}$ and that obtained using (3.227) is found to be less agreeable. However, this poor correlation is not unexpected for two reasons. Firstly, accurate computations of Im$\{c\}$ in this limit become exceedingly difficult, as confirmed by the quite complex asymptotic structure detailed above, with both short ($\eta = O(\zeta)$) and long ($\eta = O(1/\zeta)$) lengthscales emerging. Secondly, the asymptotic form for Im$\{c_1\}$ is achieved very slowly as $\zeta \to 0$, at least in one particular configuration, where with $n = 1$, the imaginary wavespeed has a leading-order coefficient of approximately $3.898 \times 10^5\zeta^3$. A comparison between numerical and asymptotic results is not shown in this case.

In the case of $\zeta \to \infty$, we may only replace the small parameter 'ζ' in the above by the small parameter '$1/\lambda\zeta$' (valid for the cone only).

When we impose heated/cooled wall conditions, instead of adiabatic conditions, the asymptotic theory is rather different. In the case of heated/cooled walls results (3.184) and (3.186) are no longer valid. Equation (3.195) is found to be still valid, but equation (3.197) no longer simplifies. Using this equation as the starting point, in this case, we are now only interested in terms Φ_1^i, which following the theory set out for the adbiabatic case, will be triggered by the '$+i\pi$' jump in the logarithm term across the critical layer. Rewritting (3.197) in the form

$$[W_{00\eta}(0)\tilde{\eta} - c_1]\Phi_{1\tilde{\eta}} - \Phi_1 W_{00\eta}(0) = \tilde{R}, \tag{3.230}$$

where

$$\tilde{R} = \hat{k}_1 + \frac{1}{2}W_{00\eta\eta}(0)\tilde{\eta}^2 A_0 W_{00\eta}(0) + \tilde{k}_1 T_{00\eta}(0)\tilde{\eta}, \tag{3.231}$$

then

$$\Phi_1 = [W_{00\eta}\tilde{\eta} - c_1]\int_0^{\tilde{\eta}} \frac{\tilde{R}d\hat{\eta}_1}{[W_{00\eta}(0)\hat{\eta}_1 - c_1]^2}. \tag{3.232}$$

As before, this integral is evaluated, taking only the imaginaries together with the limit $\tilde{\eta} \to \infty$, yielding

$$\Phi_1^i \sim B_1^i[W_{00\eta}(0)\tilde{\eta} - c_1], \tag{3.233}$$

where

$$B_1^i = \pi A_0 c_1 \Big\{\frac{W_{00\eta\eta}(0)}{W_{00\eta}^2(0)} - \frac{T_{00\eta}(0)}{T_{00}(0)W_{00\eta}(0)}\Big\}. \tag{3.234}$$

Equation (3.233) provides lower boundary conditions for the system (3.166) and (3.177).

Assuming (3.166) contains real coefficients only (c_1 is assumed real, an assumption that may be justified *a posteriori*) and matching the expansion for ϕ in the η and $\tilde{\eta}$ layers, in a manner similar to the adiabatic case, yields

$$\phi_1^i = B_1^i W_{00}(\eta), \tag{3.235}$$

where B_1^i is defined by (3.234)

The imaginary part of system (3.177) taken in the limit $\eta \to \infty$, has the form

$$\phi_{2\eta}\Big|_{\eta\to\infty} + \phi_1^i\Big|_{\eta\to\infty} = k_2^i. \tag{3.236}$$

Following the adiabatic theory, by the form of ϕ in the far-field, it is required

$$\begin{aligned} \phi_{2\eta}^i\Big|_{\eta\to\infty} &\sim -(n+1)\phi_1^i\Big|_{\eta\to\infty} \\ &\sim -(n+1)B_1^i, \end{aligned} \tag{3.237}$$

yielding

$$k_2^i = -B_1^i. \tag{3.238}$$

The imaginary part of (3.177), evaluated on the fixed boundary is

$$-c_1\phi_{1\eta}^i\Big|_{\eta=0} - c_2^i\phi_{0\eta}\Big|_{\eta=0} - W_{00\eta}(0)\phi_2^i\Big|_{\eta=0} = k_2^i T_{00}(0). \tag{3.239}$$

Making use of the results

$$\begin{aligned} \phi_{1\eta}^i\Big|_{\eta=0} &= B_1^i W_{00\eta}(0), \\ \phi_2^i\Big|_{\eta=0} &= -B_1^i c_1, \end{aligned} \tag{3.240}$$

which are obtained by matching ϕ in the two layers, and equation (3.238), gives the result

$$c_2^i = n^2 \frac{\pi T_{00}^2(\eta = 0)}{W_{00\eta}^3(\eta = 0)} \left\{ \frac{W_{00\eta\eta}(\eta = 0)}{W_{00\eta}(\eta = 0)} - \frac{T_{00\eta}(\eta = 0)}{T_{00}(\eta = 0)} \right\}. \tag{3.241}$$

Evaluating equation (3.183) on the cone surface yields

$$W_{00\eta\eta} = -\frac{T_{00\eta} W_{00\eta}}{T_{00}}, \tag{3.242}$$

which simplifies (3.241) to

$$c_2^i = n^2 \frac{2\pi T_{00}^2(0) W_{00\eta\eta}(0)}{W_{00\eta}^4(0)}. \tag{3.243}$$

Comparing this result with the adiabatic result, as obtained above, it is noted that the first imaginary term in c is an order in ζ larger for heated/cooled wall conditions, implying larger growth rates in the present case. The ratio of the leading-order imaginary terms has the form

$$\frac{\text{Heated/Cooled } c_i}{\text{Adiabatic } c_i} = \frac{W_{00\eta\eta}(0)}{\zeta W_{00\eta}(0)\sigma M_\infty^2(\gamma - 1)n}. \tag{3.244}$$

A comparison of the fully numerical computations (solid lines) of $\mathrm{Re}\{c(\alpha = 0)\}$ and $\mathrm{Im}\{c(\alpha = 0)\}$, with the asymptotic formulae (3.176) and (3.243), (broken lines), as $\zeta \to 0$, are shown in Figures 3.79 and 3.80, respectively, for a wall temperature $T_w = 5.0$, and $M_\infty = 3.8$. The agreement is seen to be entirely satisfactory, although ζ is required to be quite small for these asymptotics to be valid. It is interesting to note that (3.243) predicts that if the cylinder surface is cooled, then $c_i^i < 0$, and hence this mode is stabilized, whilst heated cylinder surfaces exhibit c_i^i >0, and hence the mode remains unstable.

In the following subsection the behaviour of mode I as $\zeta \to 0$, is considered.

3.6.3 $\zeta \to 0$, $\alpha = O(\zeta^{1/2})$

The numerical results presented in section 3.3 strongly suggest that generally, as $\zeta \to 0$, mode I has a structure very similar to the planar case, for all values of n. However, there is one important exception found in the comparison, namely the behaviour of the lower neutral point in this limit. In the planar case, as $\alpha \to 0$, $c \to 1 - 1/M_\infty$, corresponding to the so-called 'sonic' mode. However, the numerical evidence (section 3.3) suggests that on introducing curvature terms there is a shift in the neutral point, along the positive real-α axis, and the neutral point becomes (slightly) supersonic, with $c < 1 - 1/M_\infty$, as $\zeta \to 0$.

A (sensible) balancing of terms suggests that we might look for a solution of the form

$$
\begin{aligned}
c &= \hat{c}_0 + \zeta\hat{c}_1 + \cdots, \\
\phi &= \phi_0 + \zeta\phi_1 + \cdots, \\
w_0 &= W_{00} + \zeta W_{01} + \cdots, \\
T_0 &= T_{00} + \zeta T_{01} + \cdots,
\end{aligned}
\tag{3.245}
$$

with

$$
\alpha = \zeta^{1/2}\hat{\alpha}, \qquad \hat{\alpha} = O(1). \tag{3.246}
$$

To leading order, (3.116) yields

$-W_{0z}\phi_0$

$$
\frac{d}{d\eta}\left\{\frac{(W_{00} - \hat{c}_0)}{\hat{\tau}_0}\right\} = 0, \tag{3.247}
$$

where

$$
\hat{\tau}_0 = T_{00} - M_\infty^2(W_{00} - \hat{c}_0)^2. \tag{3.248}
$$

Integrating (3.247) subject to the impermeability wall condition gives

$$\phi_0 = \hat{K}_0(W_{00} - \hat{c}_0) \int_0^{\eta} \frac{\hat{\tau}_0}{(W_{00} - \hat{c}_0)^2} d\eta, \tag{3.249}$$

where $\hat{K}_0$ is some arbitrary constant and the integral is to be taken underneath the critical point to avoid any singularities arising. In the limit $\eta \to \infty$, ϕ_0 is required to be at least expondentially bounded and by the form of the imposed boundary conditions must tend to a constant. If $\hat{\tau}_0 \sim$ constant, then the integral would be $O(\eta)$, which of course, does not converge as $\eta \to \infty$. Consequently the integral will only be convergent as $\eta \to \infty$, if $\hat{\tau}_0 = 0$, giving

$$\hat{c}_0 = 1 \pm \frac{1}{M_\infty}. \tag{3.250}$$

Further, the *negative* sign is taken to be appropriate with the numerical results and the comments made above: indeed, this is simply a repeat of the planar calculation (Lees and Lin (1946)).

Curvature plays an important role at the next order, namely $O(\zeta)$. The governing equation in this case is

$$\frac{d}{d\eta}\Big\{\frac{1}{\hat{\tau}_0}[(W_{00} - \hat{c}_0)(\phi_{1\eta} + \phi_0) - \hat{c}_1\phi_{0\eta} - W_{00\eta}\phi_1 + W_{01}\phi_{0\eta} - W_{01\eta}\phi_0]$$
$$-\frac{1}{\hat{\tau}_0^2}[(W_{00} - \hat{c}_0)\phi_{0\eta} - W_{00\eta}\phi_0]\Big[\frac{n^2 T_{00}}{\hat{\alpha}^2} + T_{01} + 2M_\infty^2(W_{00} - \hat{c}_0)\hat{c}_1$$
$$-2M_\infty^2(W_{00} - \hat{c}_0)W_{01}\Big]\Big\} = \frac{\hat{\alpha}^2(W_{00} - \hat{c}_0)\phi_0}{T_{00}}. \tag{3.251}$$

Integrating with respect to η and utilizing (3.249), yields

$$(W_{00}-\hat{c}_0)(\phi_{1\eta}-\phi_0)-\hat{c}_1\phi_{0\eta}-W_{00\eta}\phi_1-W_{01\eta}\phi_0+W_{01}\phi_{0\eta}$$
$$-\hat{K}_0\Big[\frac{n^2T_{00}}{\hat{\alpha}^2}+T_{01}+2M_\infty^2(W_{00}-\hat{c}_0)\hat{c}_1-2M_\infty^2(W_{00}-\hat{c}_0)W_{01}\Big]$$
$$=\hat{K}_1\hat{\tau}_0+\hat{\alpha}^2\hat{\tau}_0\int_0^\eta\frac{(W_{00}-\hat{c}_0)\phi_0}{T_{00}}d\eta, \qquad (3.252)$$

where $\hat{K}_1$ is a constant.

(3.252) is required to match correctly at the outer edge ($\eta\to\infty$) with (3.18). In the limit $\zeta\to 0$, (3.18) has the form

$$\phi\sim\frac{\phi_\infty}{2}K_n'[\hat{\eta}_0(1+\eta\zeta)+O(\zeta^2)], \qquad (3.253)$$

where

$$\hat{\eta}_0=M_\infty^{1/2}\hat{\alpha}(2\hat{c}_1)^{1/2}, \qquad (3.254)$$

the positive sign being taken for the argument of the modified Bessel function since the real part of $\hat{\eta}$ (as defined by (3.19)) is required to be positive in the limit $\eta\to\infty$ and by the form of (3.250).

Taylor expanding the Bessel function around $\hat{\eta}_0$, yields

$$\phi_0=\frac{\phi_\infty}{2}K_n'(\hat{\eta}_0), \qquad (3.255)$$

and

$$\phi_1=\eta\frac{\phi_\infty}{2}\hat{\eta}_0K_n''(\hat{\eta}_0), \qquad (3.256)$$

in the limit $\eta\to\infty$.

Matching (3.249) with (3.255) taken in the limit $\eta\to\infty$, yields

$$\frac{\phi_\infty}{2} = \frac{\hat{K}_0 I}{M_\infty \hat{K}_n'(\hat{\eta}_0)}, \tag{3.257}$$

where

$$I = \int_0^\infty \frac{\hat{\tau}_0}{(W_{00} - \hat{c}_0)^2} d\eta, \tag{3.258}$$

and we have employed (3.250). This clearly yields

$$\phi_{1\eta}\Big|_{\eta\to\infty} = \frac{\hat{\eta}_0 K_n''(\hat{\eta}_0)\hat{K}_0 I}{K_n'(\hat{\eta}_0) M_\infty}. \tag{3.259}$$

(3.252) in far-field reduces to

$$\phi_{1\eta}\Big|_{\eta\to\infty} = -\hat{K}_0(1-\hat{c}_0)I + \frac{\hat{K}_0}{1-\hat{c}_0}\Big[\frac{n^2}{\hat{\alpha}^2} + 2M_\infty^2(1-\hat{c}_0)\hat{c}_1\Big]. \tag{3.260}$$

Matching (3.259) and (3.260) yields the following nonlinear dispersion relationship for $\hat{c}_1$:

$$\hat{\eta}_0 \frac{K_n''(\hat{\eta}_0)}{K_n'(\hat{\eta}_0)} + 1 = \frac{M_\infty^2}{I}\Big[\frac{n^2}{\hat{\alpha}^2} + 2M_\infty \hat{c}_1\Big]. \tag{3.261}$$

The integral (3.258) was evaluated numerically, and for the conditions prevailing in all the numerical results it was found that $I \approx -228.4 - 59.3i$. Equation (3.260) was solved using a Newton iteration, and results for Real$\{\hat{c}_1\}$ and Im$\{\hat{c}_1\}$ for various n are shown in Figures 3.81 and 3.82, respectively.

Making use of result $K_n''(\hat{\eta}_0)/K_n'(\hat{\eta}_0) \to -1 + O(\frac{1}{\hat{\eta}_0})$, in the limit $\hat{\alpha} \to \infty$, (3.260) predicts that one family of solutions has the form

$$\hat{c}_1 \to \frac{\hat{\alpha}^2 I^2}{2M_\infty^5}, \tag{3.262}$$

which is in agreement with the $\alpha << 1$ expansion for c in the planar case, namely

$$c = 1 - \frac{1}{M_\infty} + \frac{\alpha^2 I^2}{2M_\infty^5} + O(\alpha^4). \qquad (3.263)$$

Equation (3.262) is shown as an asymptote in Figures 3.81 and 3.82. Note, a (real) family of $\hat{c}_1$, which may exist is an exact solution of (3.261), namely

$$\hat{c}_1 = -\frac{n^2}{2M_\infty \hat{\alpha}^2}, \qquad (3.264)$$

where we have made use of the relation $K_n''(in)/K_n'(in) = -(1/(in))$. The importance of this mode is not thought to be great. The complex families of $\hat{c}_1$'s are seen to terminate at a finite value of $\hat{\alpha}$, corresponding to the (lower) neutral point of mode I. Notice that in all cases, because Real$\{\hat{c}_1\} < 0$ at the termination point, these modes correspond to supersonic modes.

From the result shown in Figure 3.82, we are therefore able to offer an estimate of the position of the lower neutral point of mode I as $\zeta \to 0$. In particular, for the freestream conditions considered throughout this paper, for $n = 0$ this position is given by $\alpha \approx 0.1\zeta^{1/2}$, $n = 1$ by $\alpha \approx 0.20\zeta^{1/2}$, and for $n = 2$ by $\alpha \approx 0.295\zeta^{1/2}$. Comparing these asymptotic results with the $\zeta = 0.01$ results displayed in Figures 3.5 and 3.6 (which even though are cylinder results, because of the smallest of ζ correspond very closely to the cone results at this axial location, as explained in subsection 3.4.4) reveals a fair degree of agreement.

In the case of $\zeta >> 1$, the above results may be easily transposed, by the replacement of 'ζ' by '$1/\lambda\zeta$'; the corresponding positions for lower neutral point are then $\alpha \approx 0.1(\lambda\zeta)^{-1/2}$ for $n = 0$, $\alpha \approx 0.20(\lambda\zeta)^{-1/2}$ for $n = 1$, and $\alpha \approx 0.295(\lambda\zeta)^{-1/2}$ for $n = 2$. These results are seen to agree quite well with the $\zeta = 75$ results shown in Figure 3.66 and 3.67.

Finally, it is found that altering the wall conditions from adiabatic conditions to heated/cooled wall conditions, has little significant effect on the asymptotic analysis for the mode I lower neutral point.

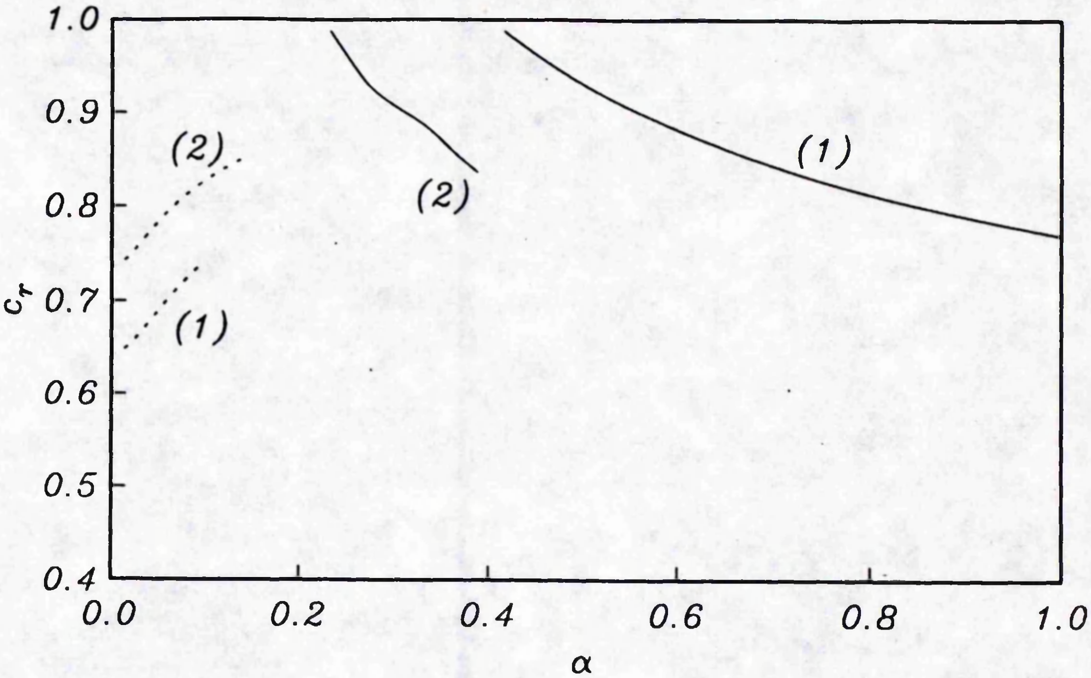


Figure 3.1: Variation of c_r with α for adiabatic cylinder, $\zeta = 0$ (Planar).

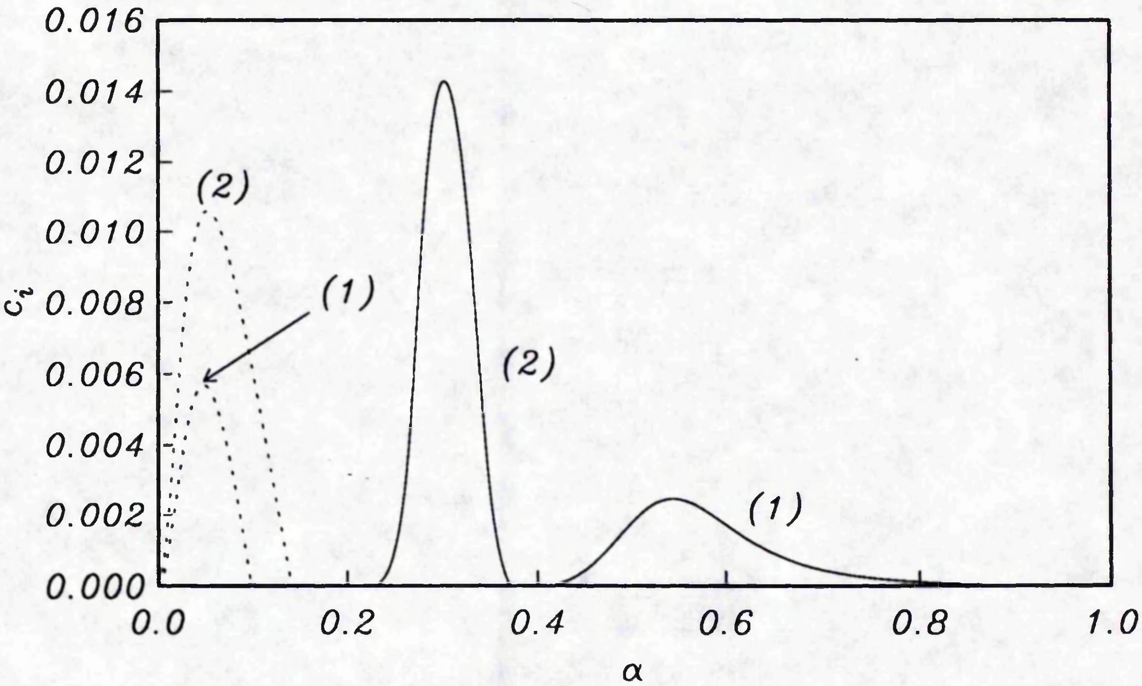


Figure 3.2: Variation of c_i with α for adiabatic cylinder, $\zeta = 0$ (Planar).

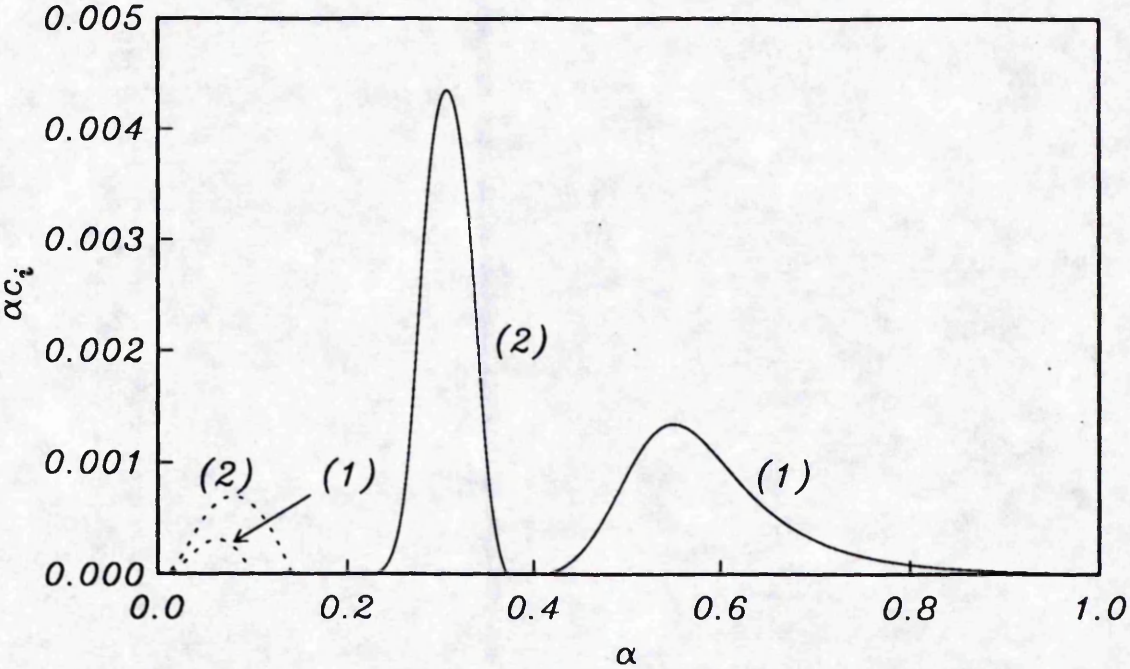


Figure 3.3: Variation of αc_i with α for adiabatic cylinder, $\zeta = 0$ (Planar).

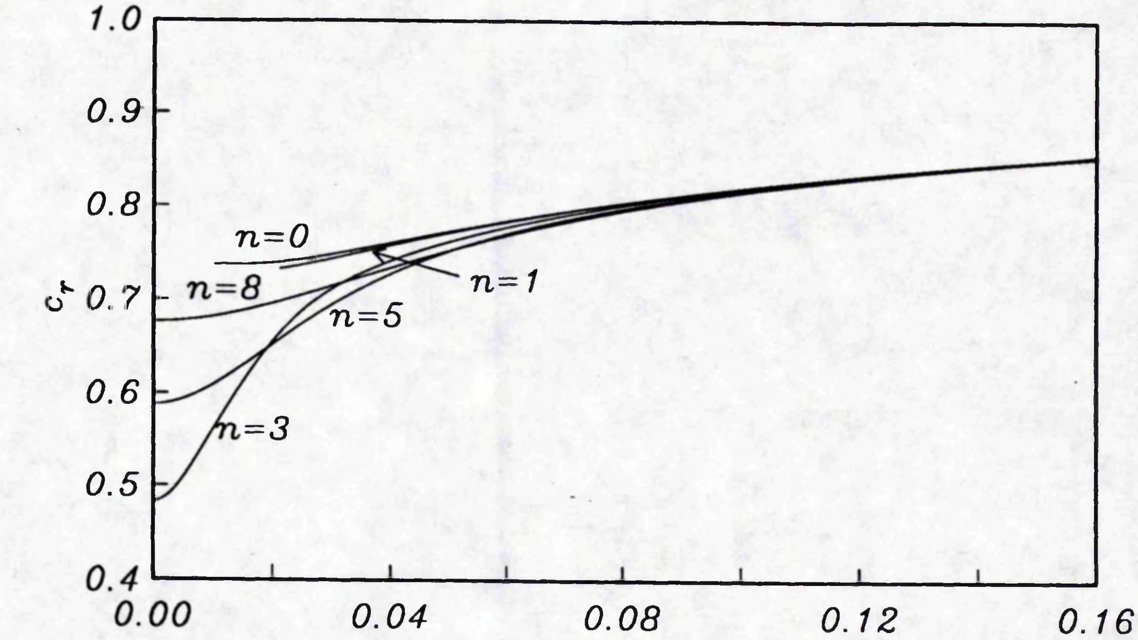


Figure 3.4: Variation of c_r with α for adiabatic cylinder, $M_\infty = 3.8$, $\zeta = 0.01$, Mode I.

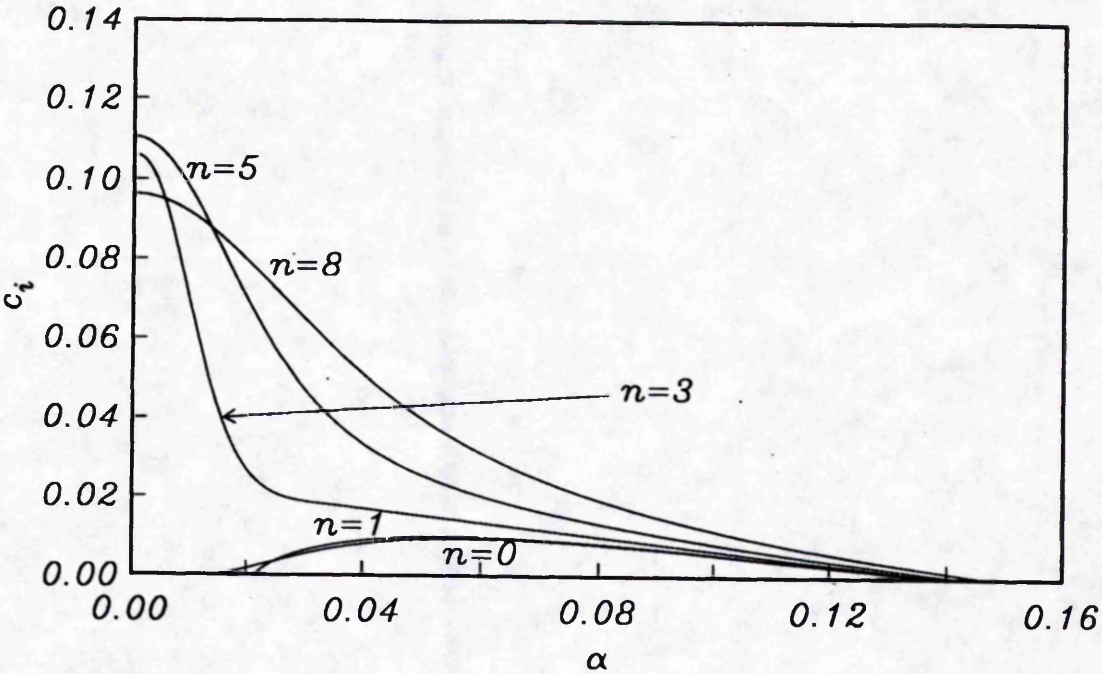


Figure 3.5: Variation of c_i with α for adiabatic cylinder, $M_\infty = 3.8$, $\zeta = 0.01$, Mode I.

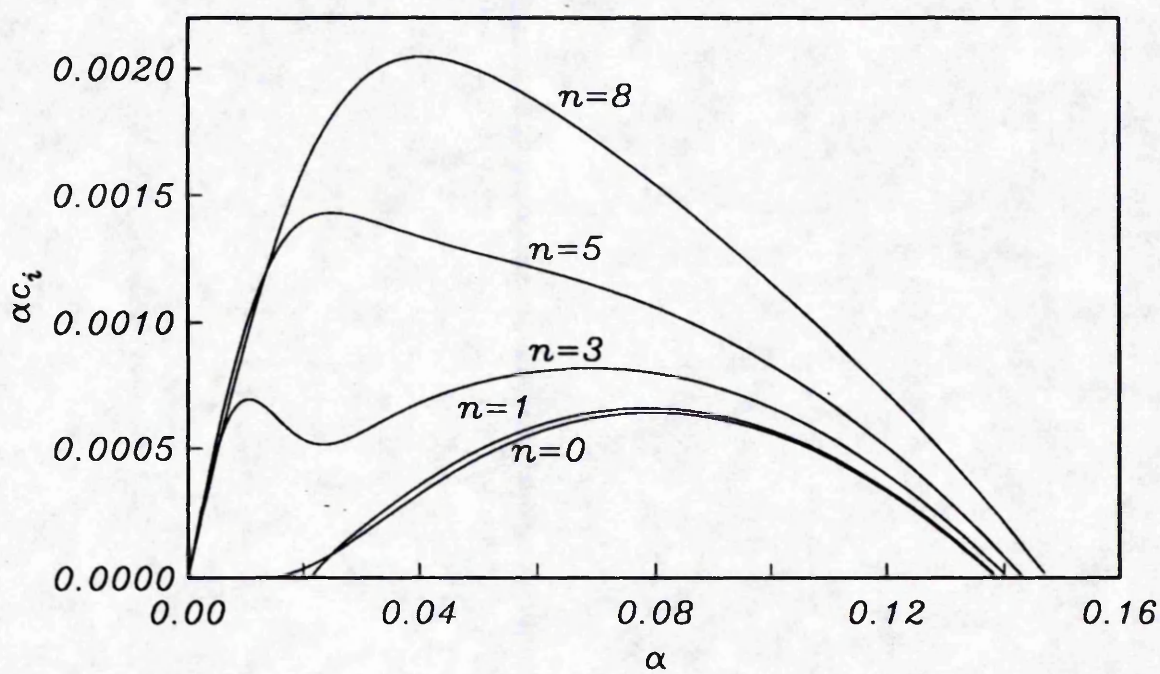


Figure 3.6: Variation of αc_i with α for adiabatic cylinder, $M_\infty = 3.8$, $\zeta = 0.01$, Mode I.

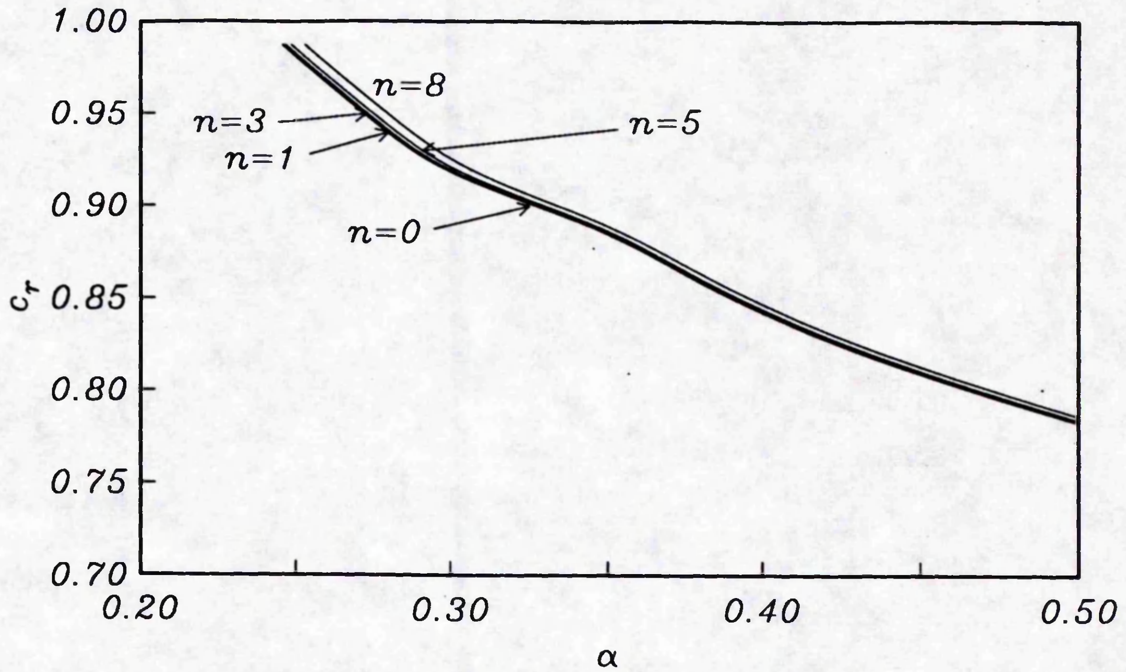


Figure 3.7: Variation of c_r with α for adiabatic cylinder, $M_\infty = 3.8$, $\zeta = 0.01$, Mode II.

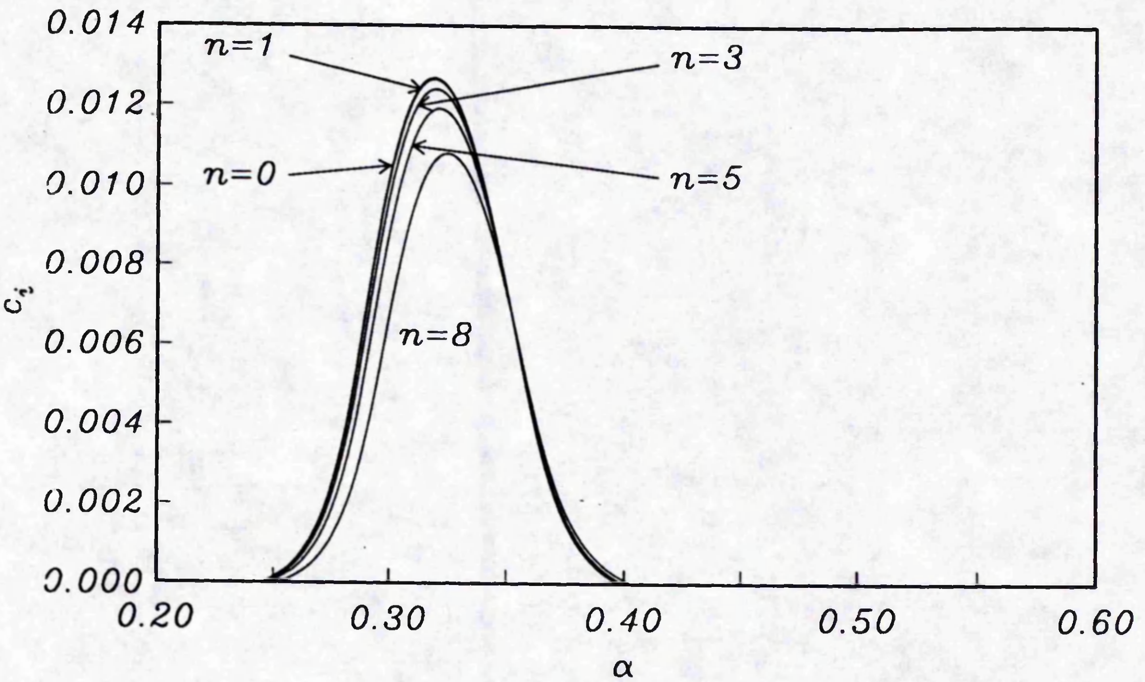


Figure 3.8: Variation of c_i with α for adiabatic cylinder, $M_\infty = 3.8$, $\zeta = 0.01$, Mode II.

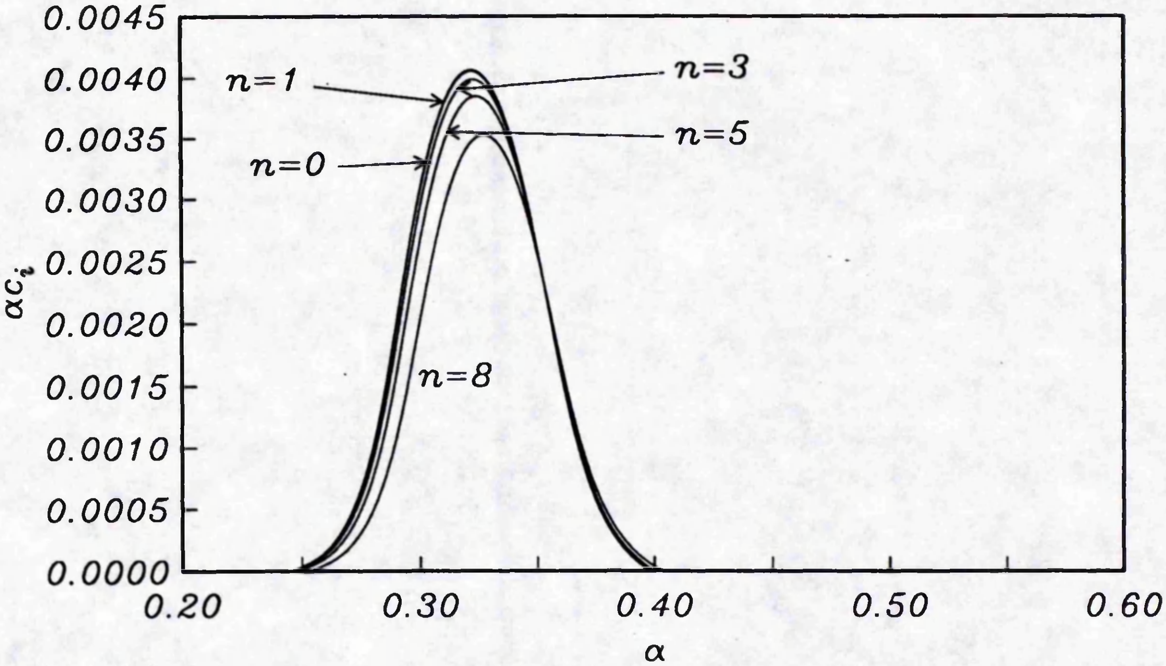


Figure 3.9: Variation of αc_i with α for adiabatic cylinder, $M_\infty = 3.8$, $\zeta = 0.01$, Mode II.

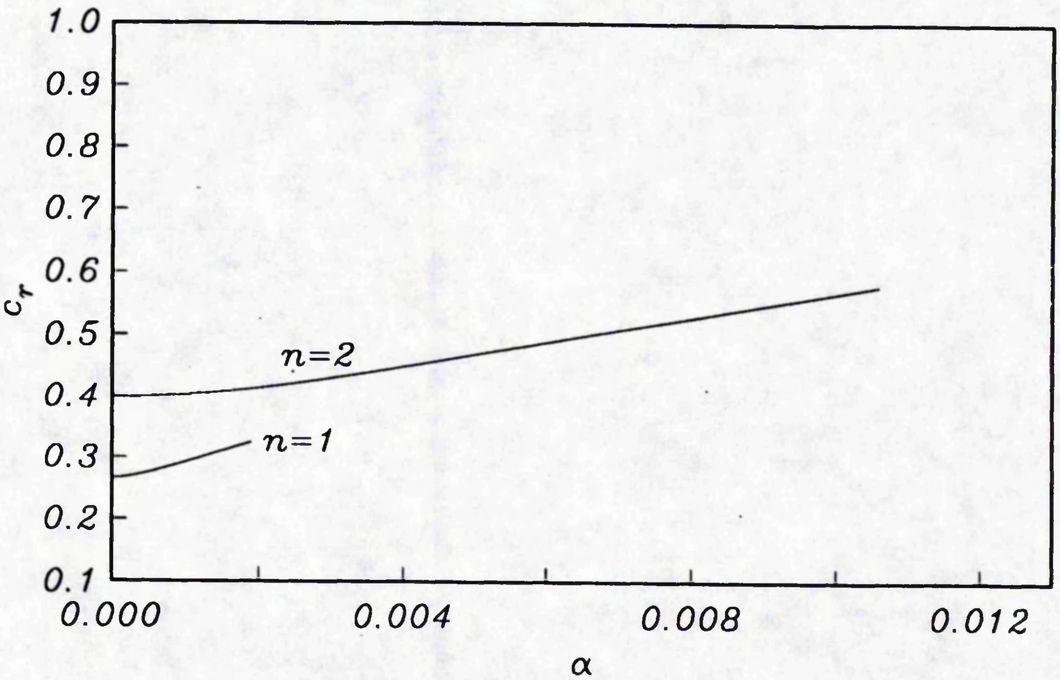


Figure 3.10: Variation of c_r with α for adiabatic cylinder, $M_\infty = 3.8$, $\zeta = 0.01$, Mode I_A.

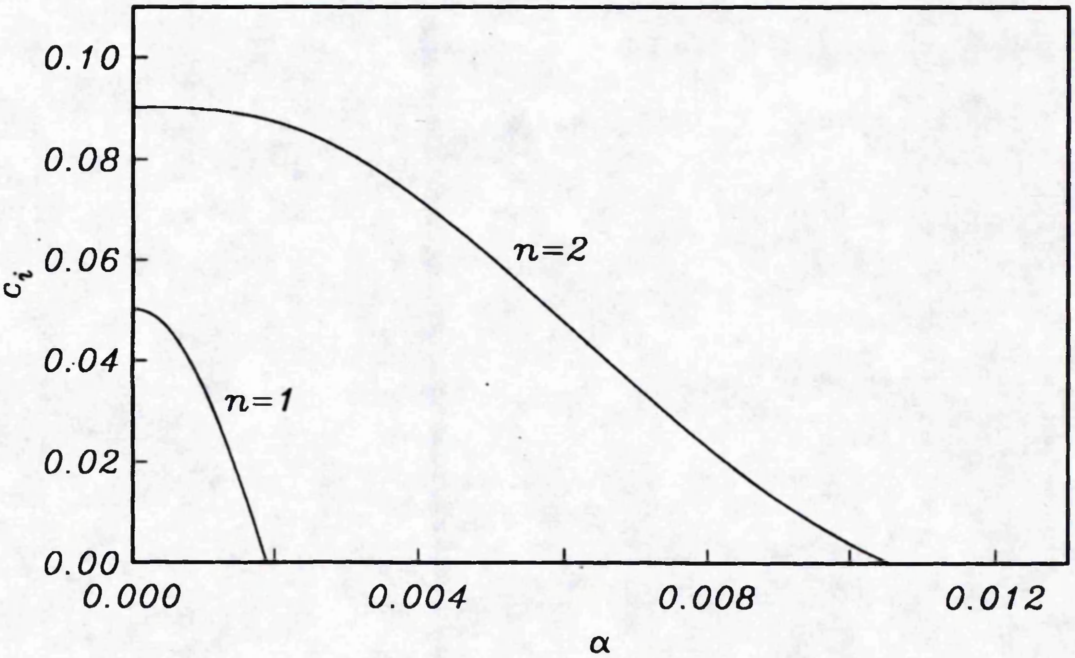


Figure 3.11: Variation of c_i with α for adiabatic cylinder, $M_\infty = 3.8$, $\zeta = 0.01$, Mode I_A.

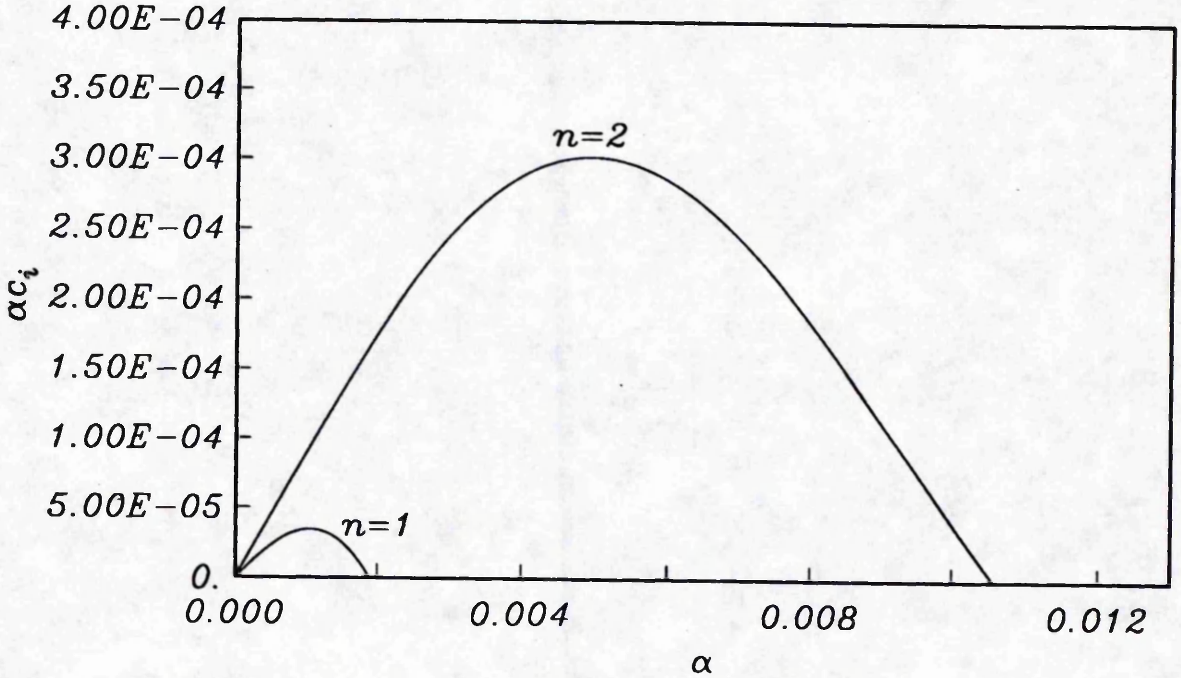


Figure 3.12: Variation of αc_i with α for adiabatic cylinder, $M_\infty = 3.8$, $\zeta = 0.01$, Mode I_A.

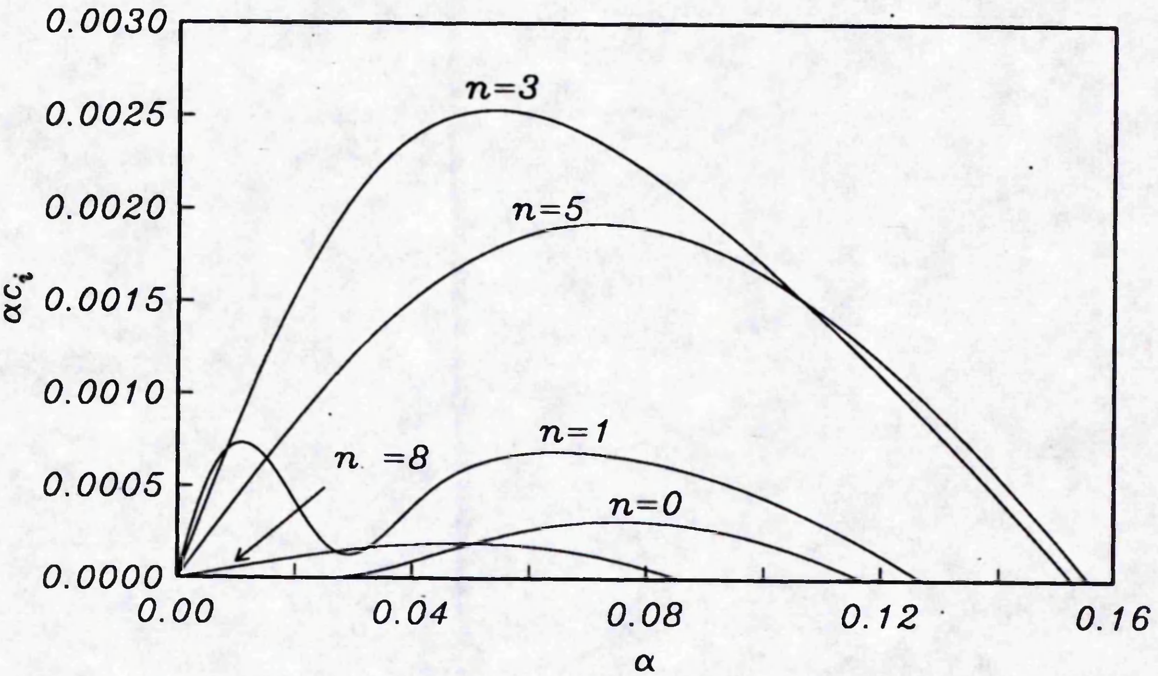


Figure 3.13: Variation of αc_i with α for adiabatic cylinder, $M_\infty = 3.8$, $\zeta = 0.05$, Mode I.

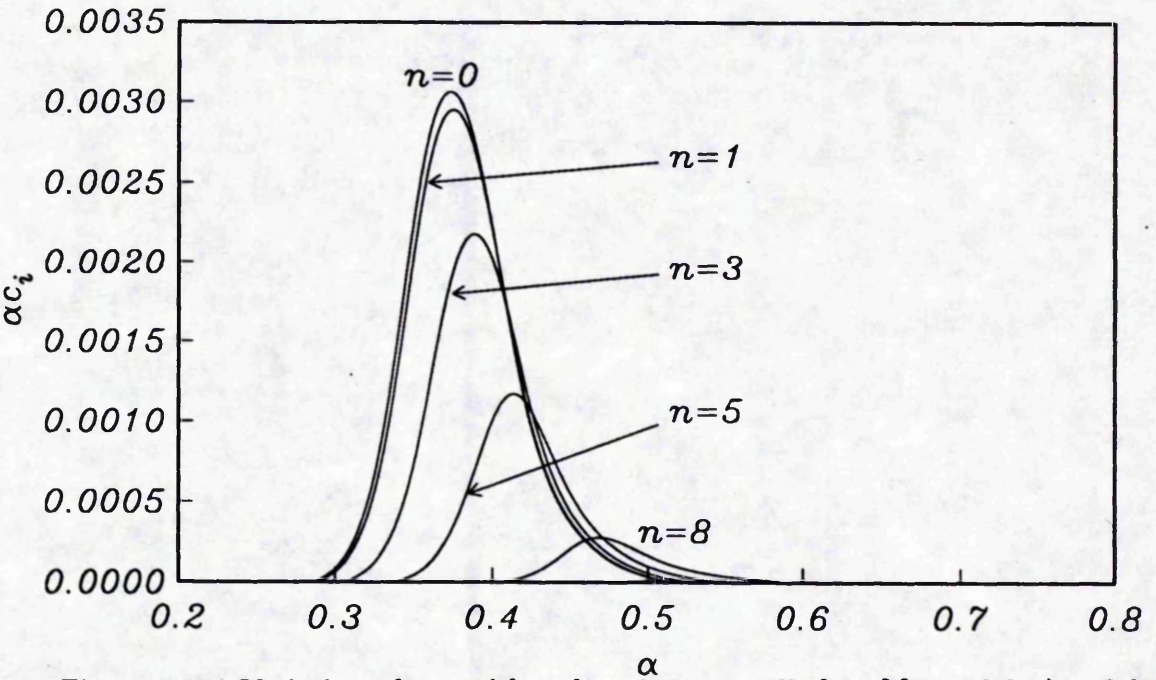


Figure 3.14: Variation of αc_i with α for adiabatic cylinder, $M_\infty = 3.8$, $\zeta = 0.05$, Mode II.

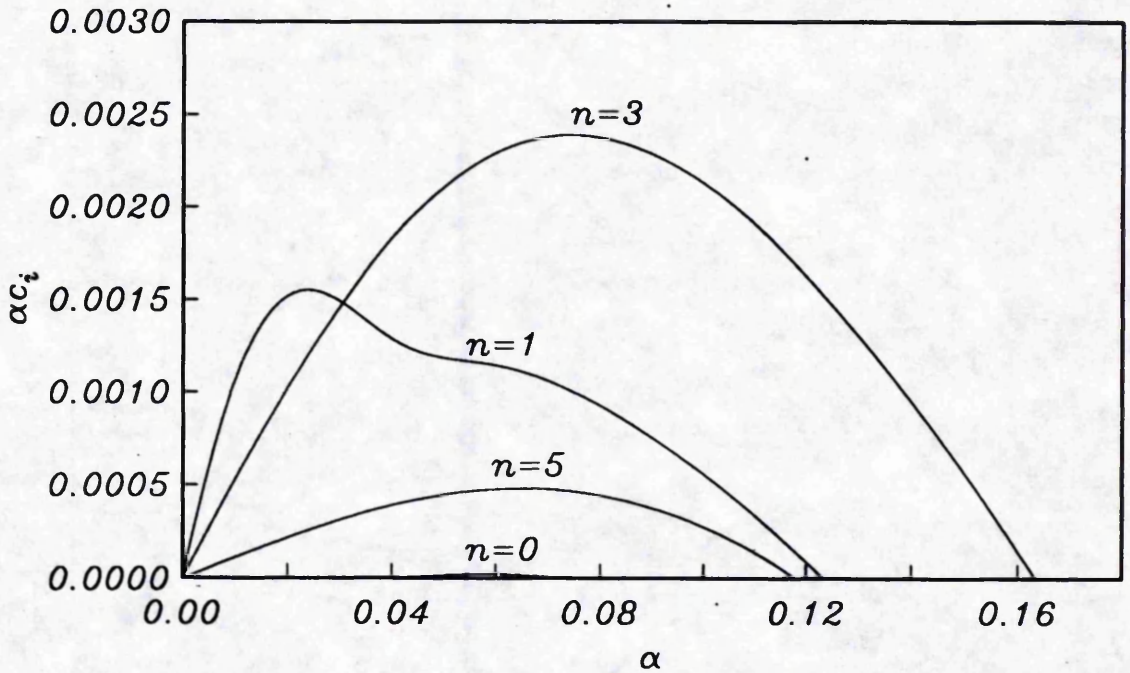


Figure 3.15: Variation of αc_i with α for adiabatic cylinder, $M_\infty = 3.8$, $\zeta = 0.1$, Mode I.

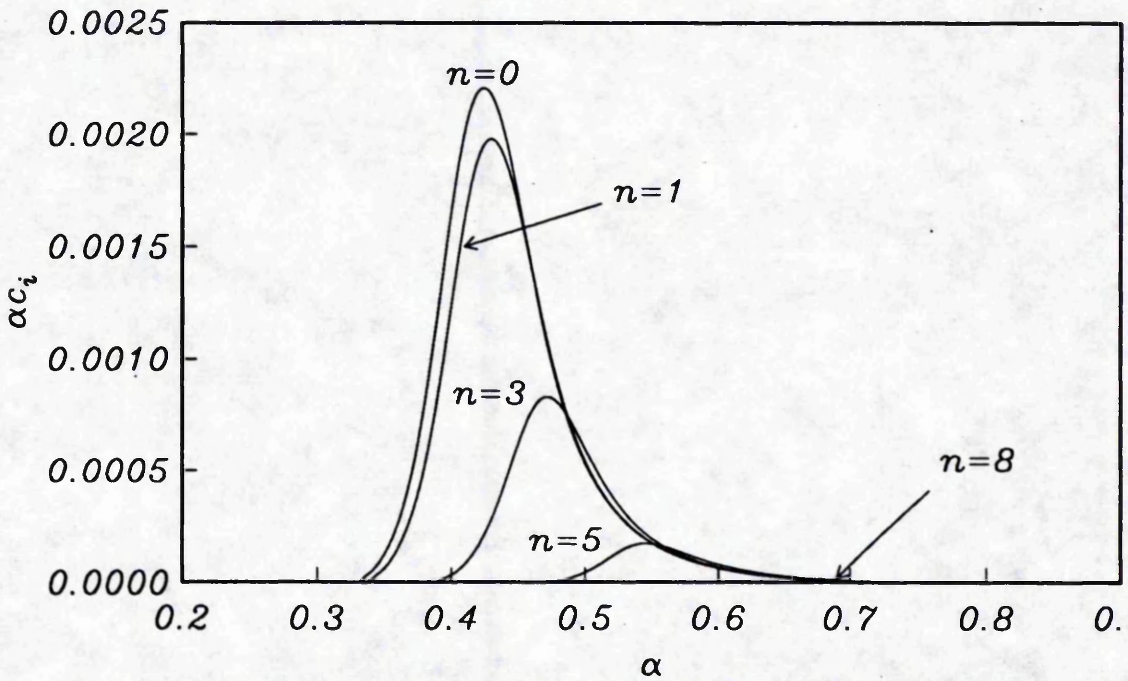


Figure 3.16: Variation of αc_i with α for adiabatic cylinder, $M_\infty = 3.8$, $\zeta = 0.1$, Mode II.

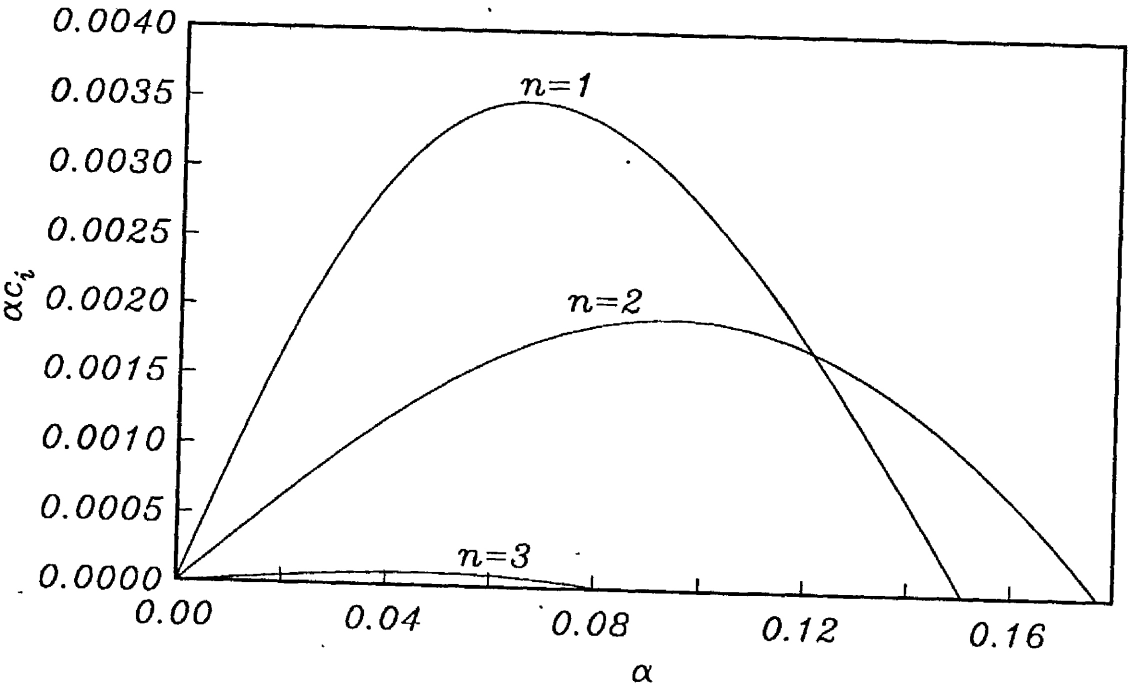


Figure 3.17: Variation of αc_i with α for adiabatic cylinder, $M_\infty = 3.8$, $\zeta = 0.5$, Mode I.

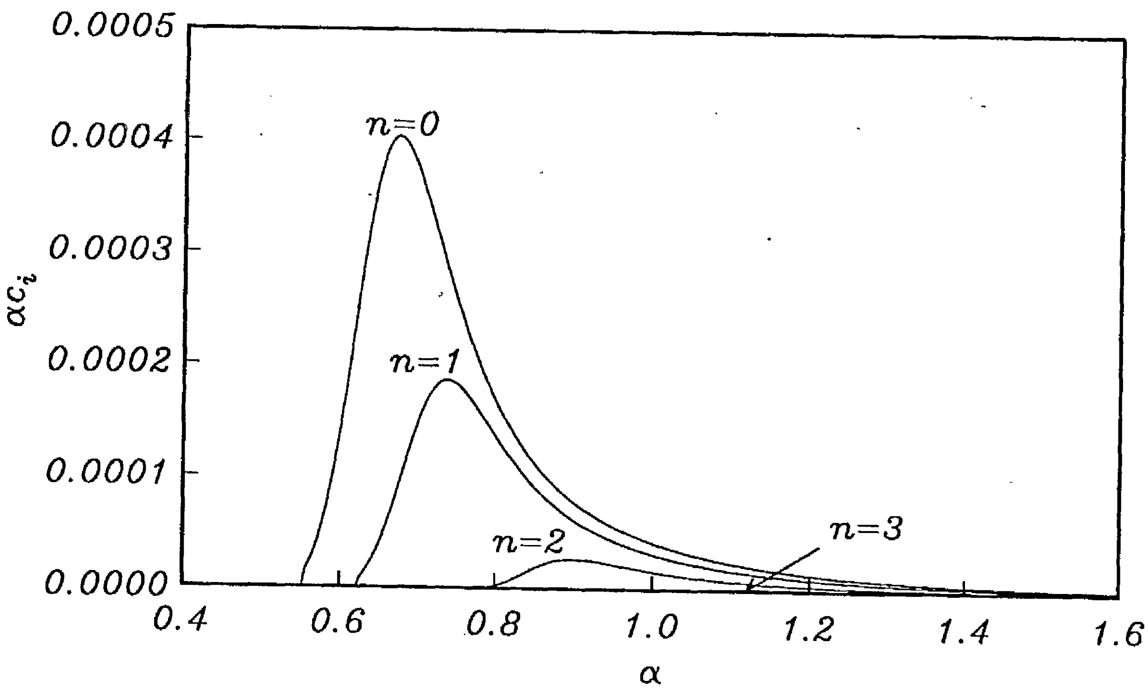


Figure 3.18: Variation of αc_i with α for adiabatic cylinder, $M_\infty = 3.8$, $\zeta = 0.5$, Mode II.

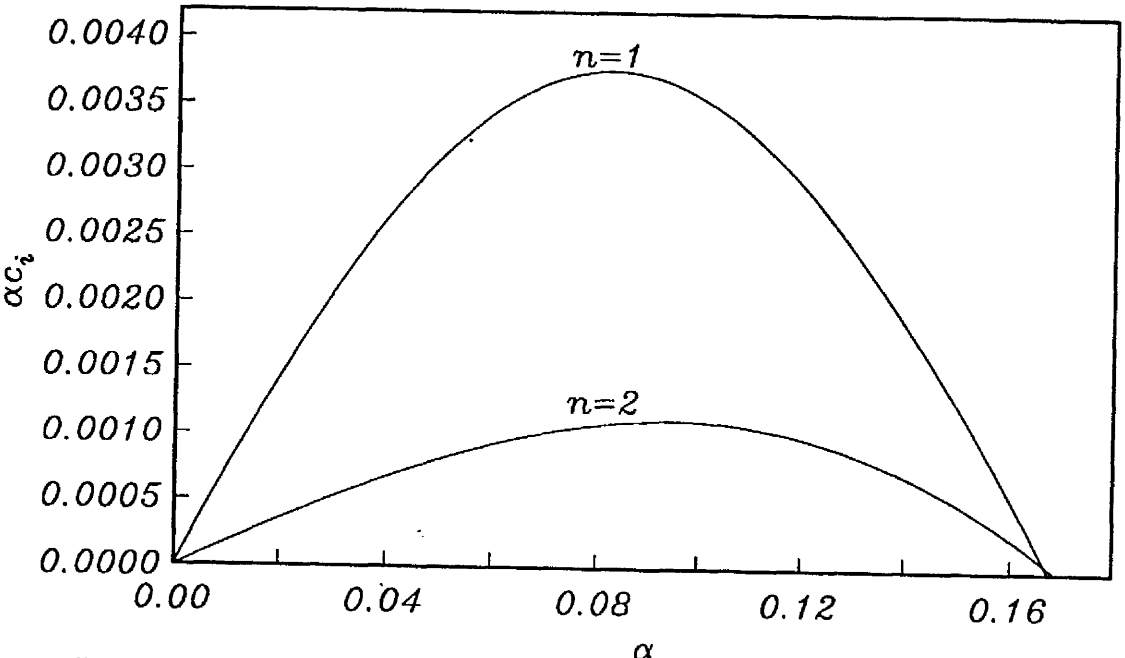


Figure 3.19: Variation of αc_i with α for adiabatic cylinder, $M_\infty = 3.8$, $\zeta = 1.0$, Mode I.

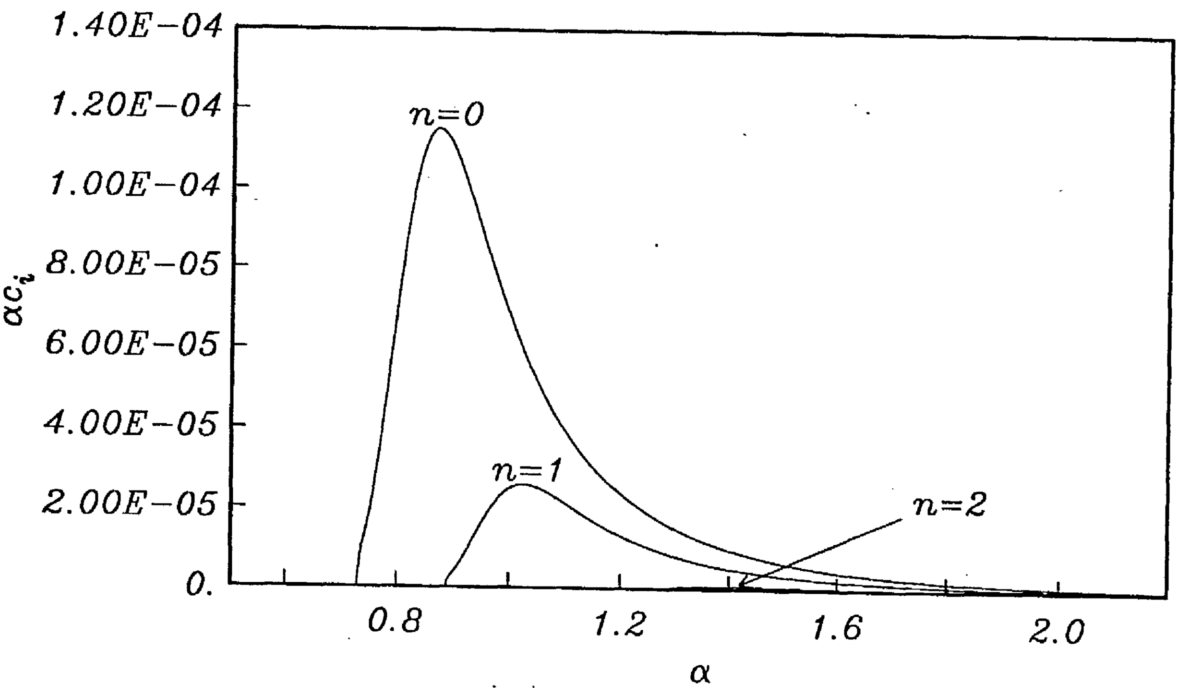


Figure 3.20: Variation of αc_i with α for adiabatic cylinder, $M_\infty = 3.8$, $\zeta = 1.0$, Mode II.

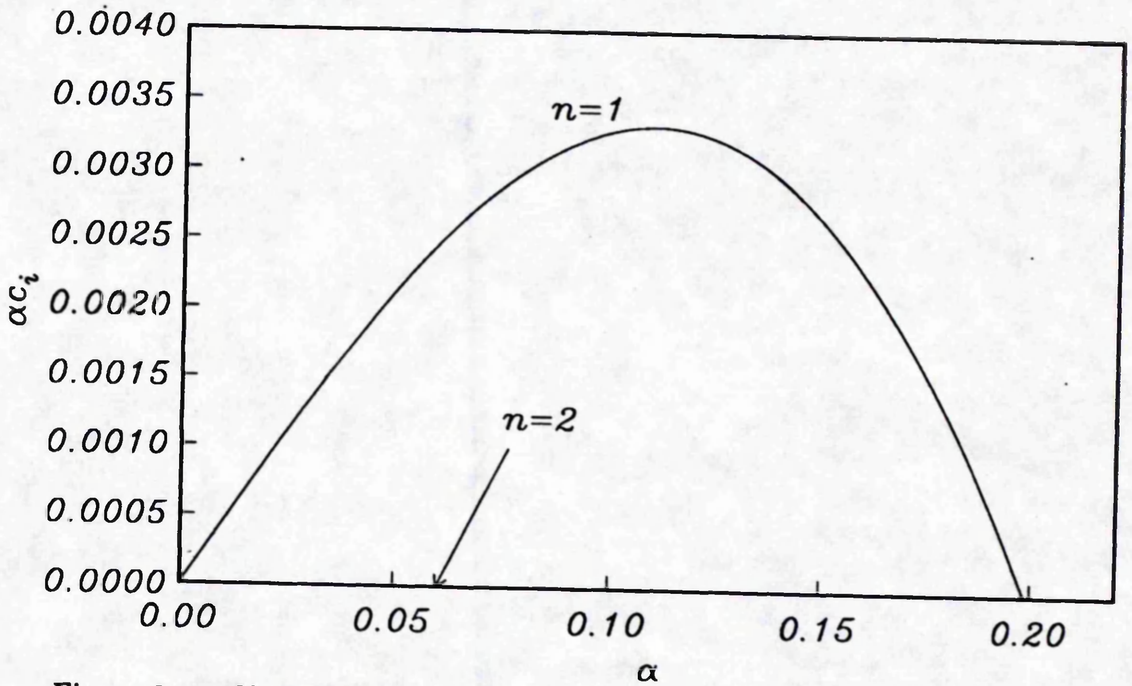


Figure 3.21: Variation of αc_i with α for adiabatic cylinder, $M_\infty = 3.8$, $\zeta = 5.0$, Mode I.

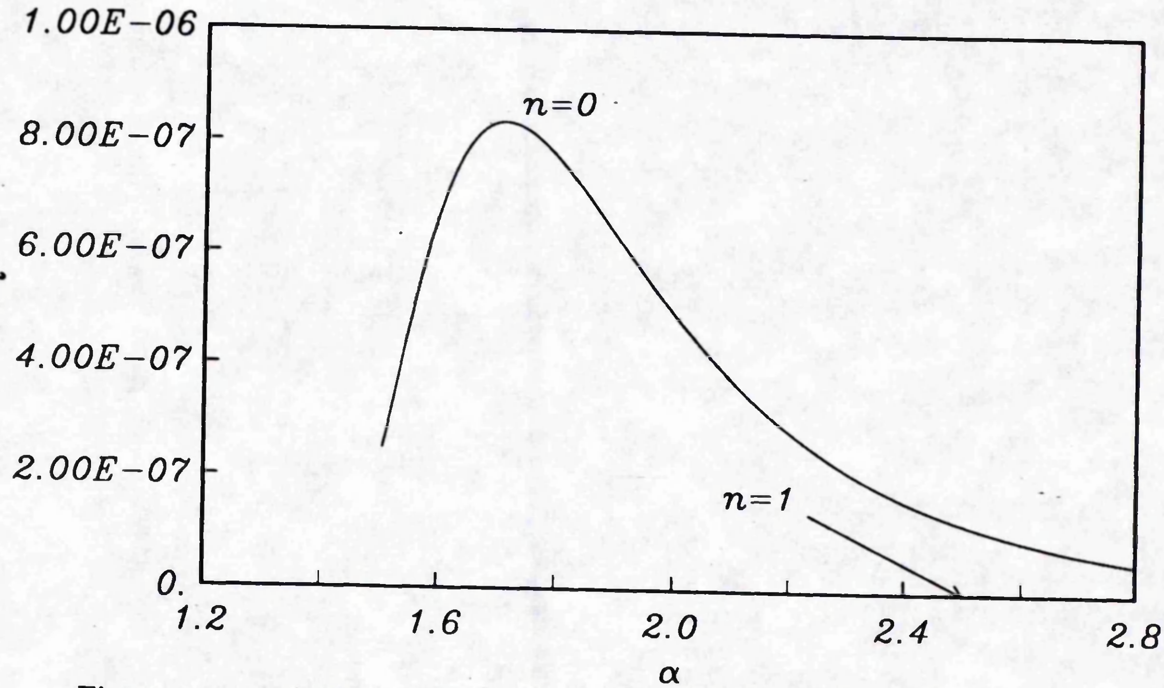


Figure 3.22: Variation of αc_i with α for adiabatic cylinder, $M_\infty = 3.8$, $\zeta = 5.0$, Mode II.

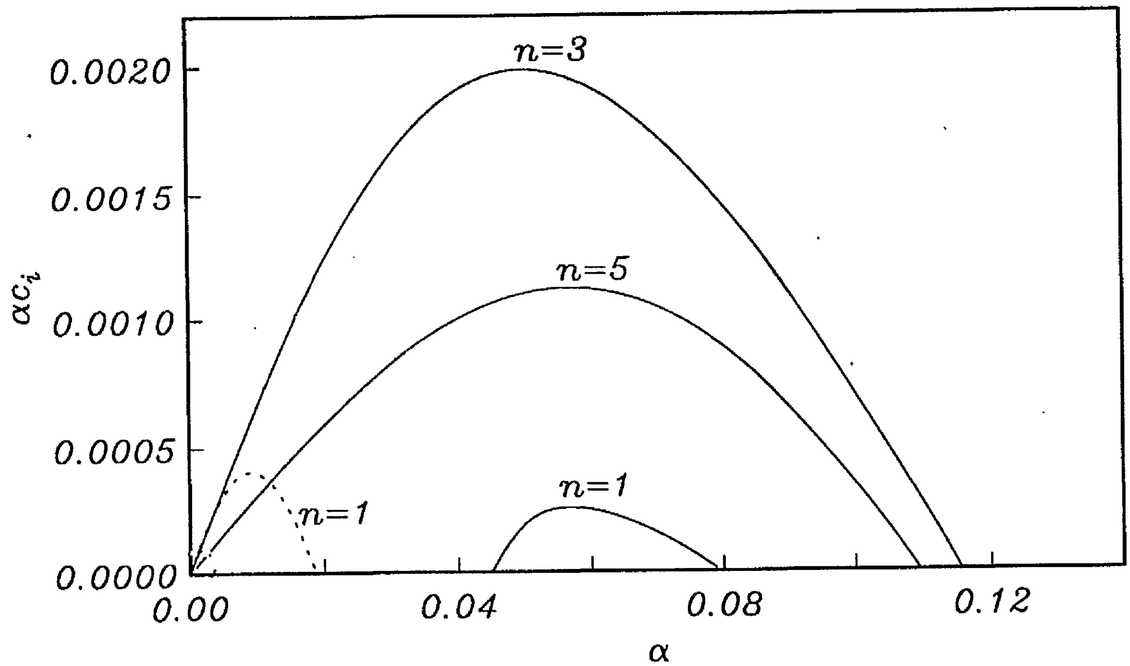


Figure 3.23: Variation of αc_i with α for adiabatic cylinder, $M_\infty = 2.8$, $\zeta = 0.05$, Mode I.

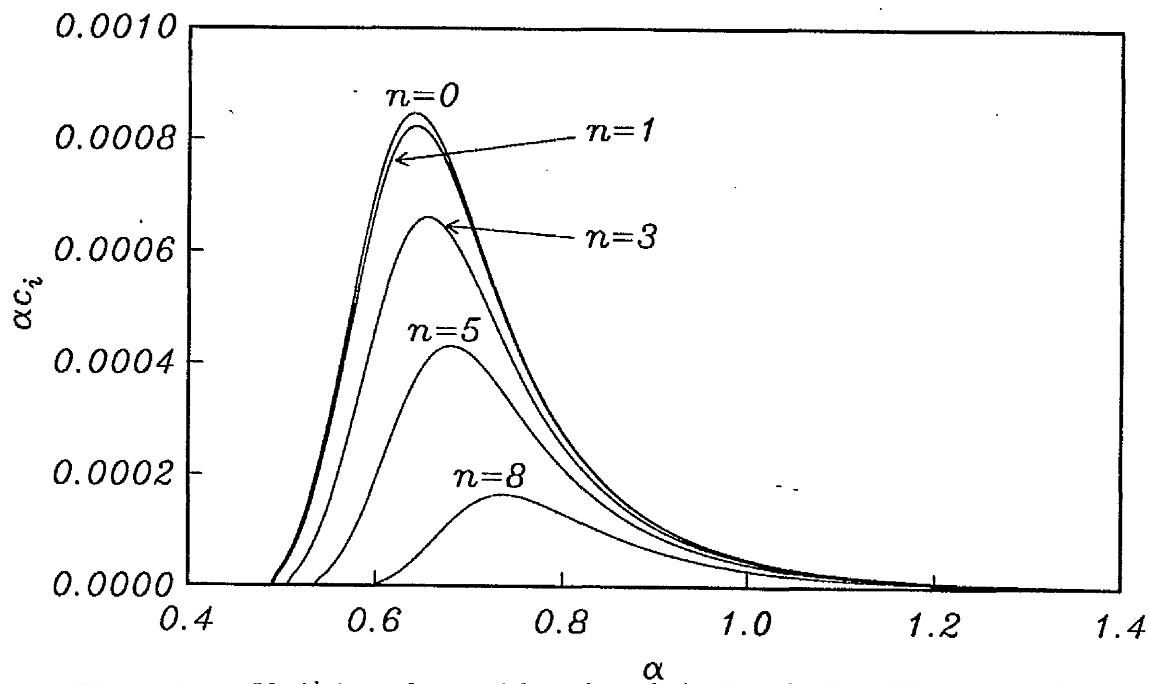


Figure 3.24: Variation of αc_i with α for adiabatic cylinder, $M_\infty = 2.8$, $\zeta = 0.05$, Mode II.

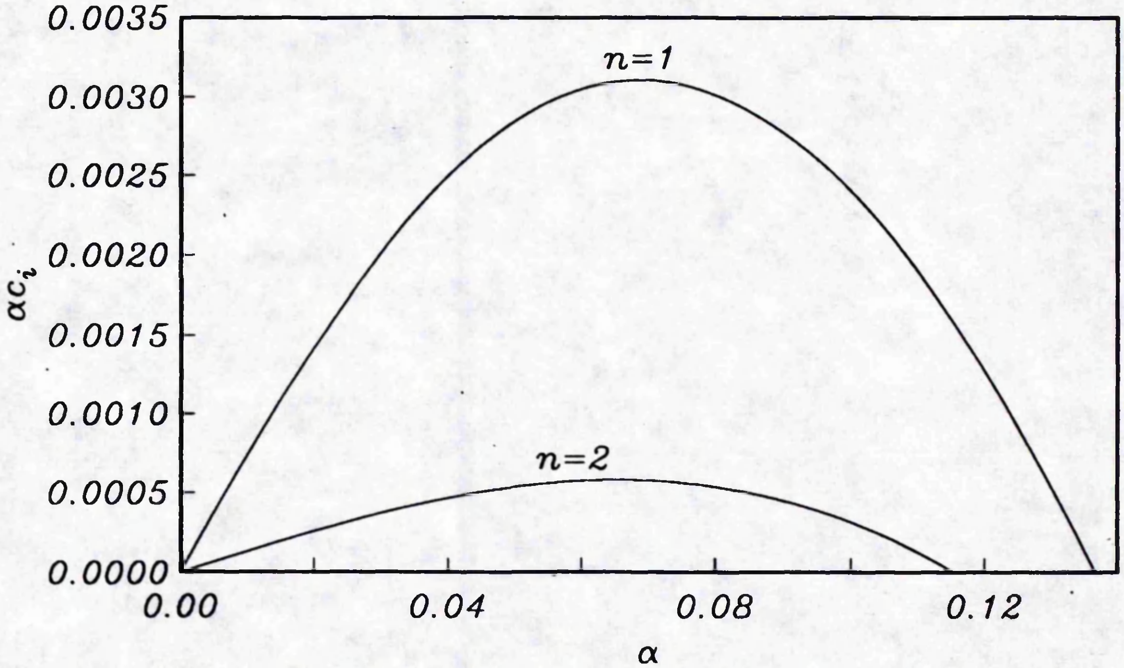


Figure 3.25: Variation of αc_i with α for adiabatic cylinder, $M_\infty = 2.8$, $\zeta = 0.5$, Mode I.

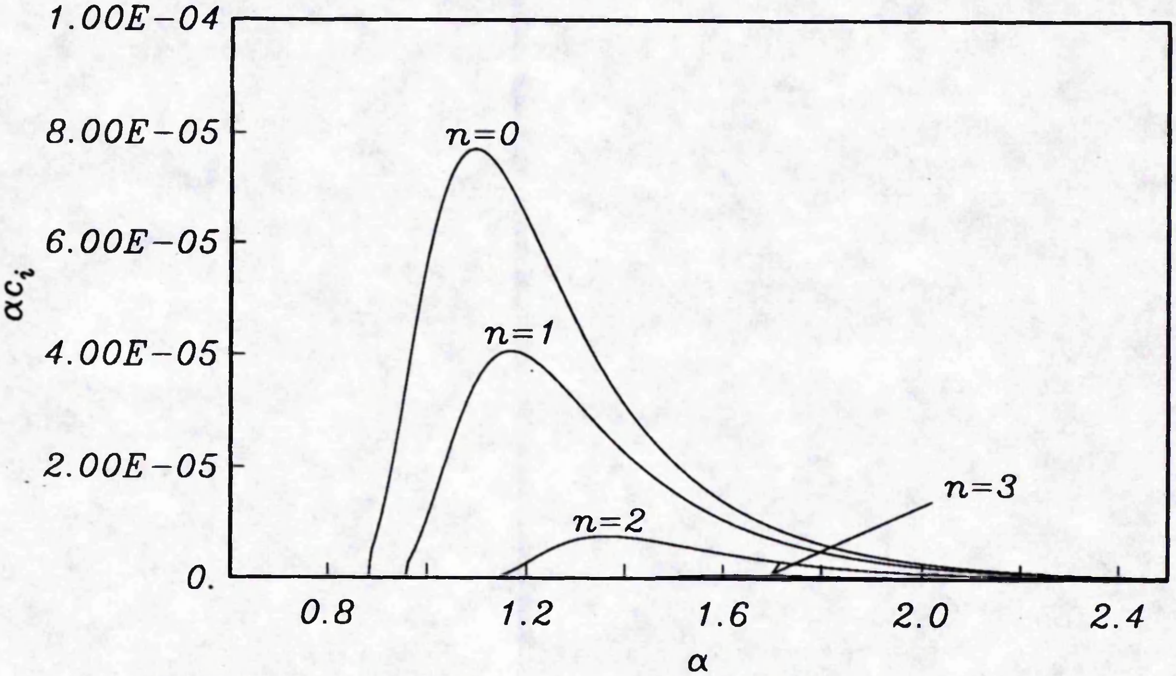


Figure 3.26: Variation of αc_i with α for adiabatic cylinder, $M_\infty = 2.8$, $\zeta = 0.5$, Mode II.

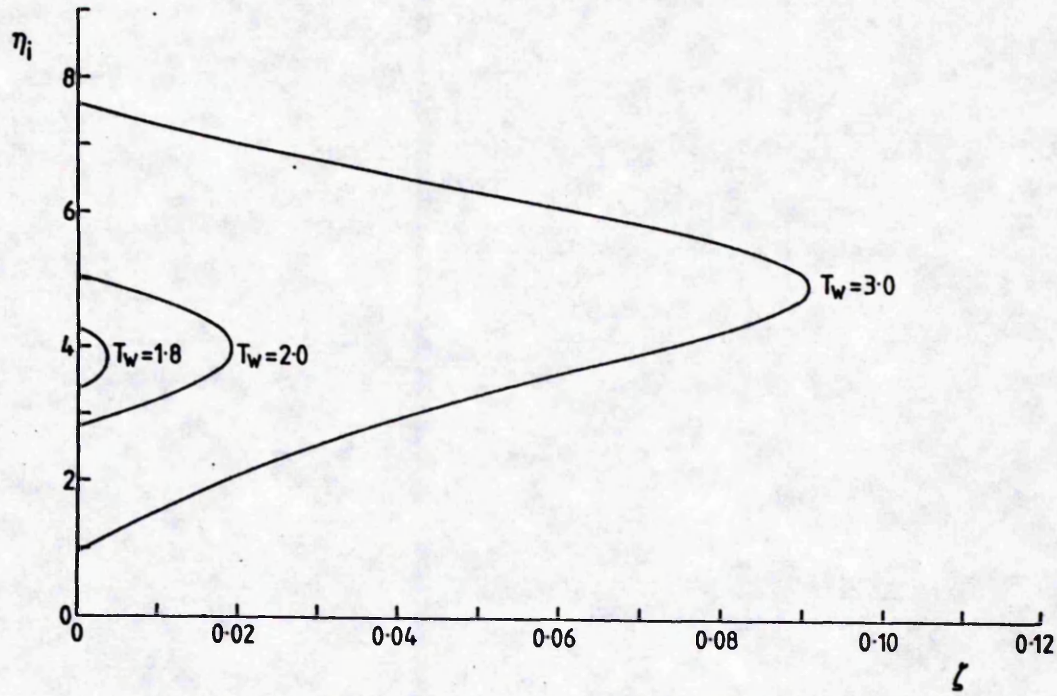


Figure 3.27: Variation of transverse positions of inflexion points (η_i) with axial locations (ζ) for cooled cylinder, $M_\infty = 3.8$.

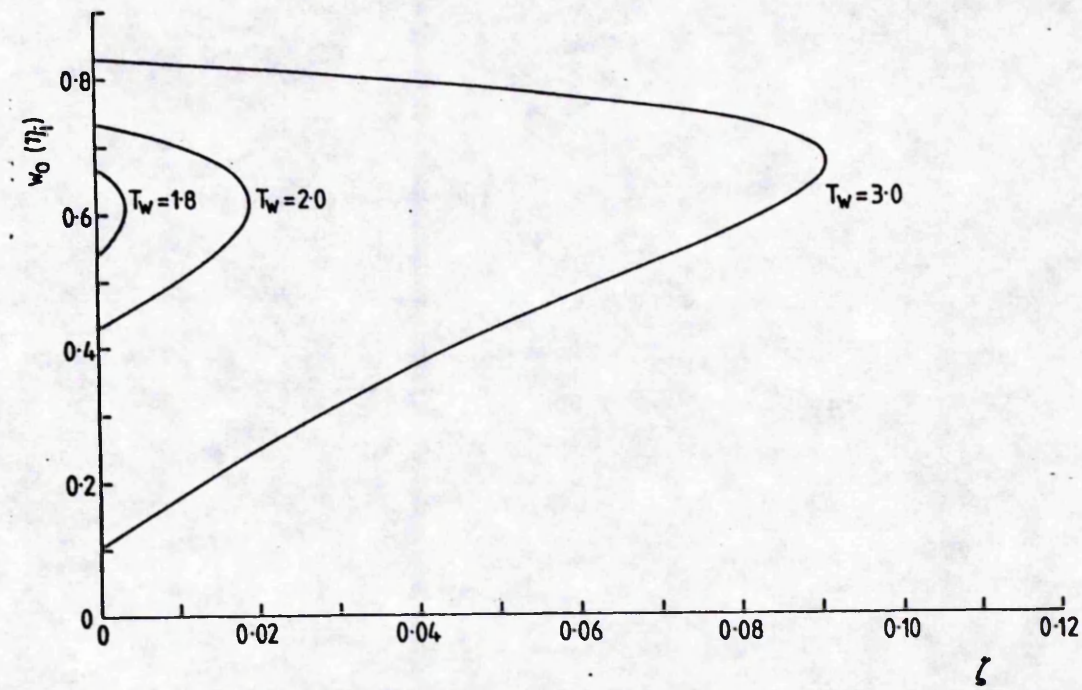


Figure 3.28: Variation of $w_0(\eta = \eta_i)$ with ζ for cooled cylinder, $M_\infty = 3.8$.

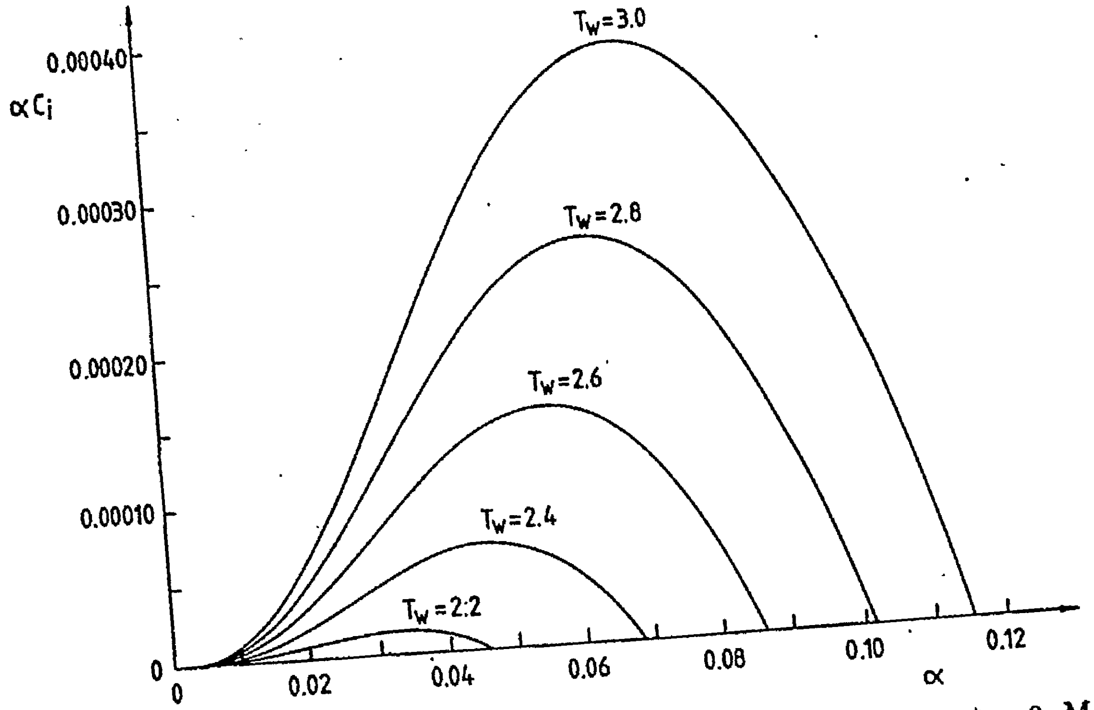


Figure 3.29: Variation of αc_i with α for cooled cylinder, $M_\infty = 3.8$, $\zeta = 0$, Mode I (Planar).

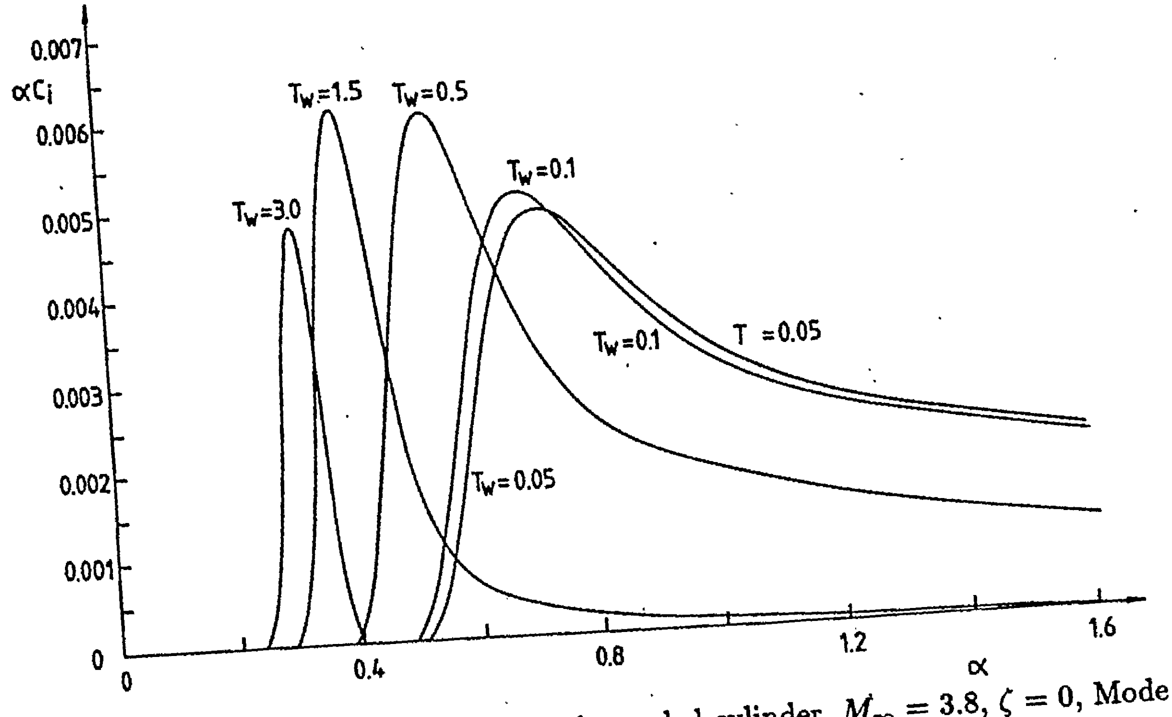


Figure 3.30: Variation of αc_i with α for cooled cylinder, $M_\infty = 3.8$, $\zeta = 0$, Mode II (Planar).

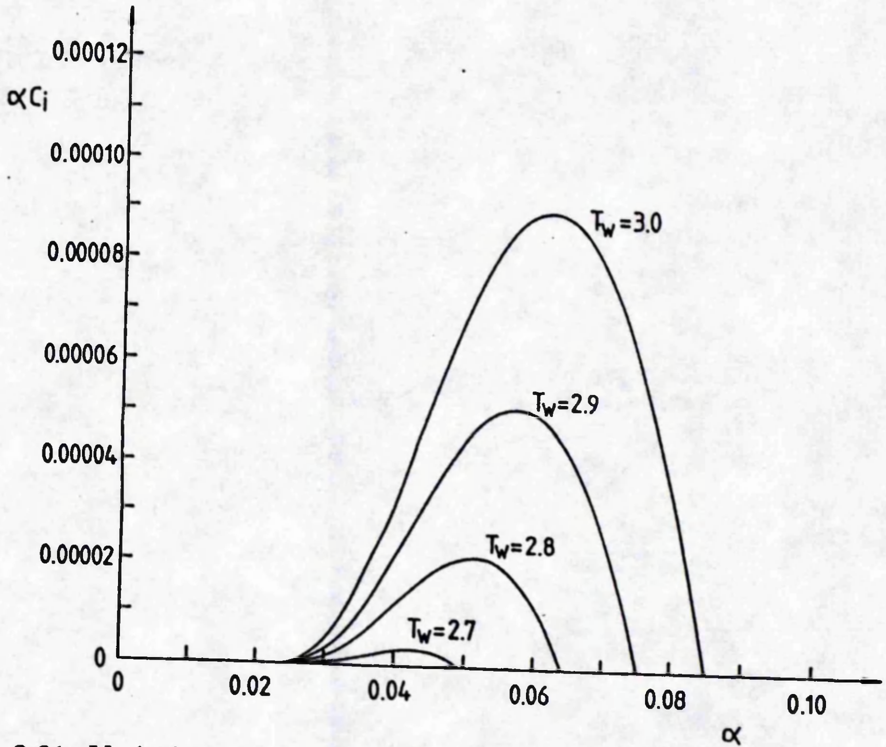


Figure 3.31: Variation of αc_i with α for cooled cylinder, $M_\infty = 3.8$, $\zeta = 0.05$, $n = 0$, Mode I.

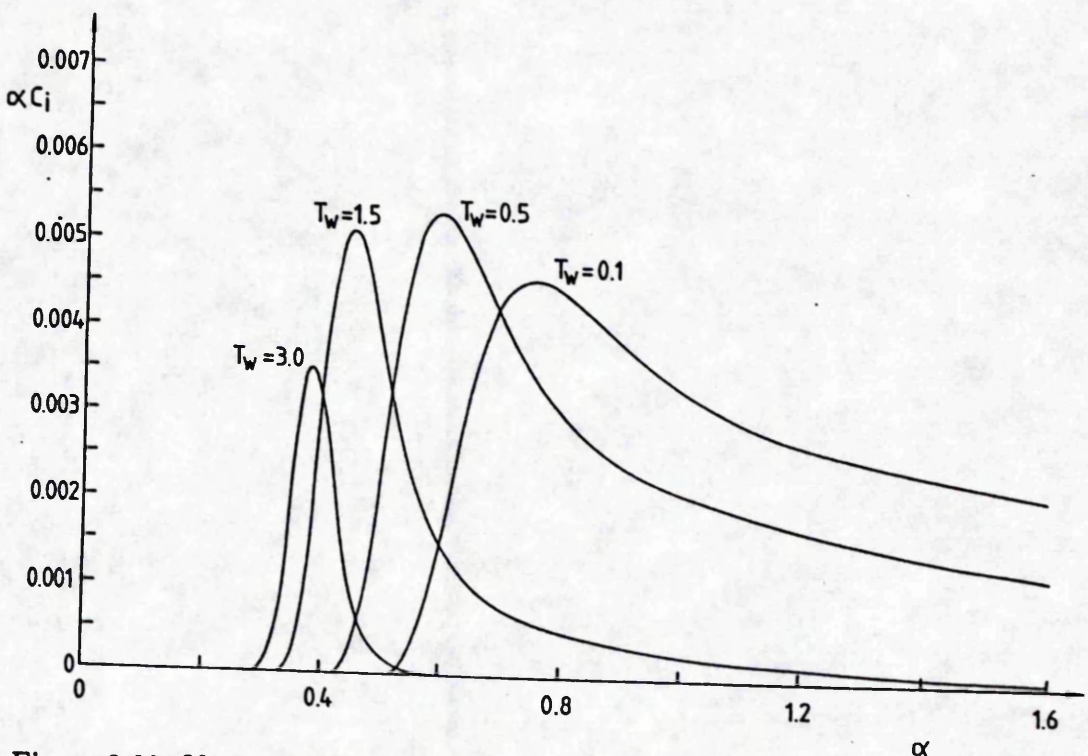


Figure 3.32: Variation of αc_i with α for cooled cylinder, $M_\infty = 3.8$, $\zeta = 0.05$, $n = 0$, Mode II.

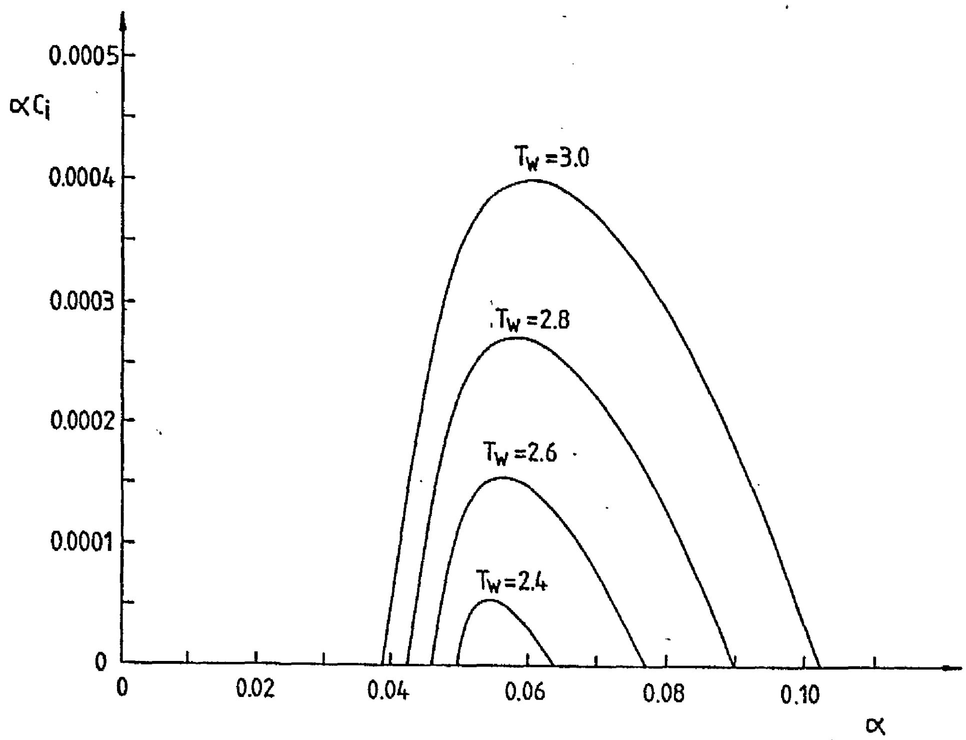


Figure 3.33: Variation of αc_i with α for cooled cylinder, $M_\infty = 3.8$, $\zeta = 0.05$, $n = 1$, Mode I.

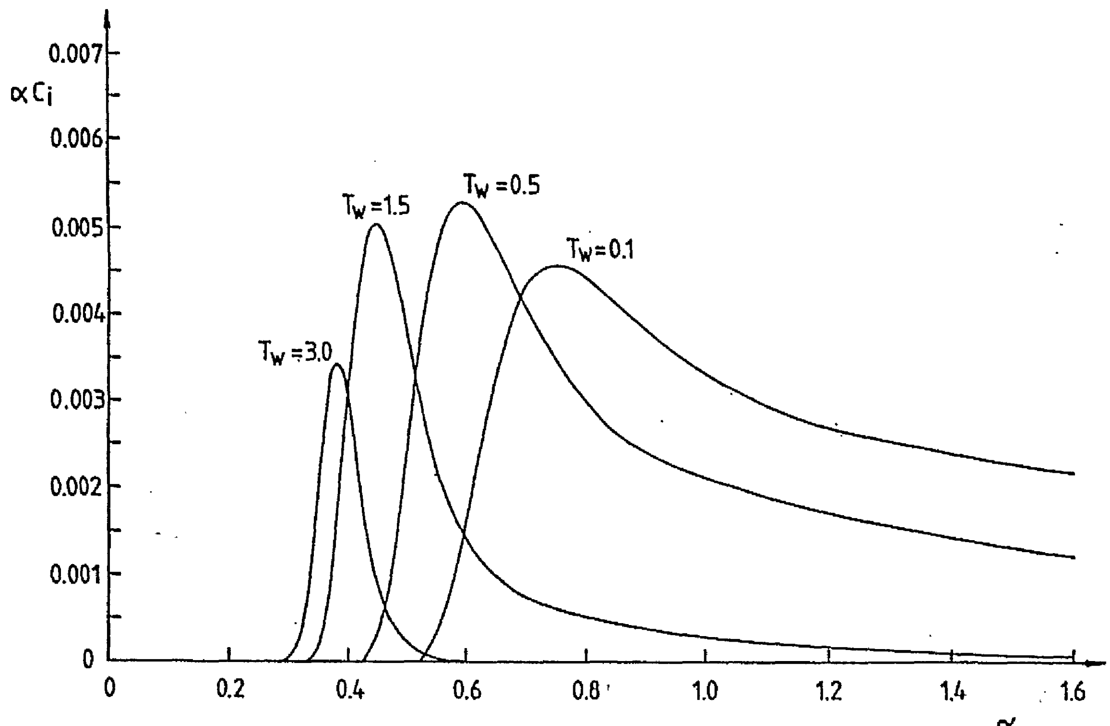


Figure 3.34: Variation of αc_i with α for cooled cylinder, $M_\infty = 3.8$, $\zeta = 0.05$, $n = 1$, Mode II.

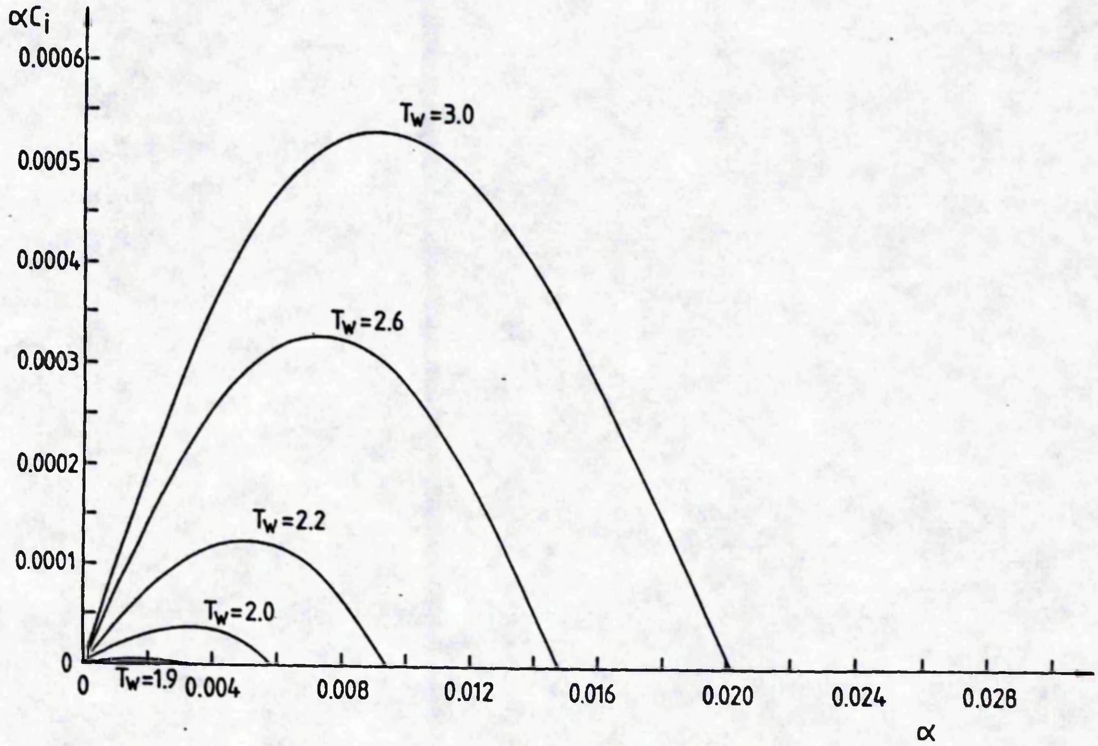


Figure 3.35: Variation of αc_i with α for cooled cylinder, $M_\infty = 3.8$, $\zeta = 0.05$, $n = 1$, Mode $\mathrm{I_A}$.

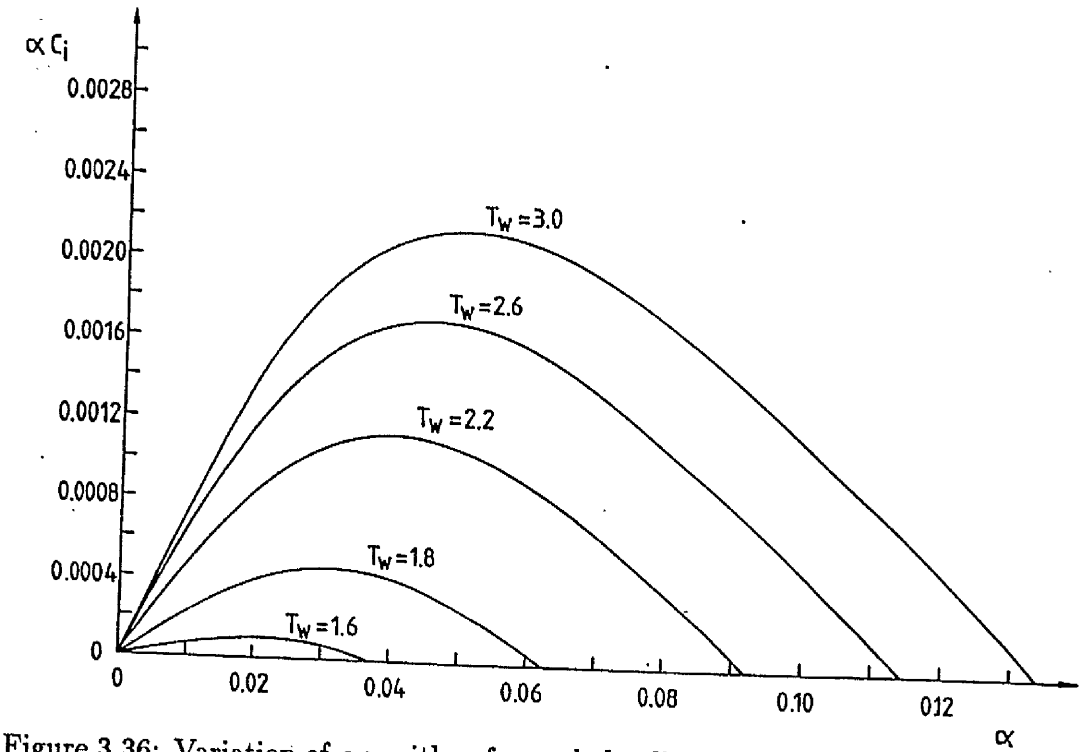


Figure 3.36: Variation of αc_i with α for cooled cylinder, $M_\infty = 3.8$, $\zeta = 0.05$, $n = 3$, Mode I.

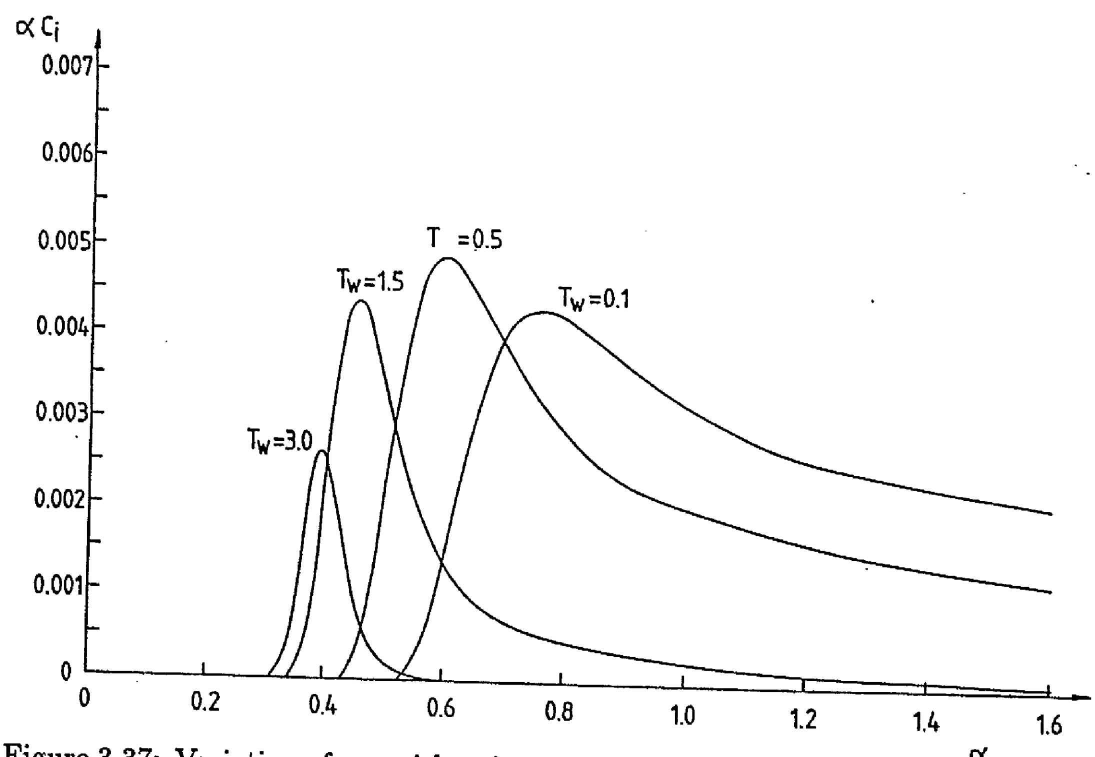


Figure 3.37: Variation of αc_i with α for cooled cylinder, $M_\infty = 3.8$, $\zeta = 0.05$, $n = 3$, Mode II.

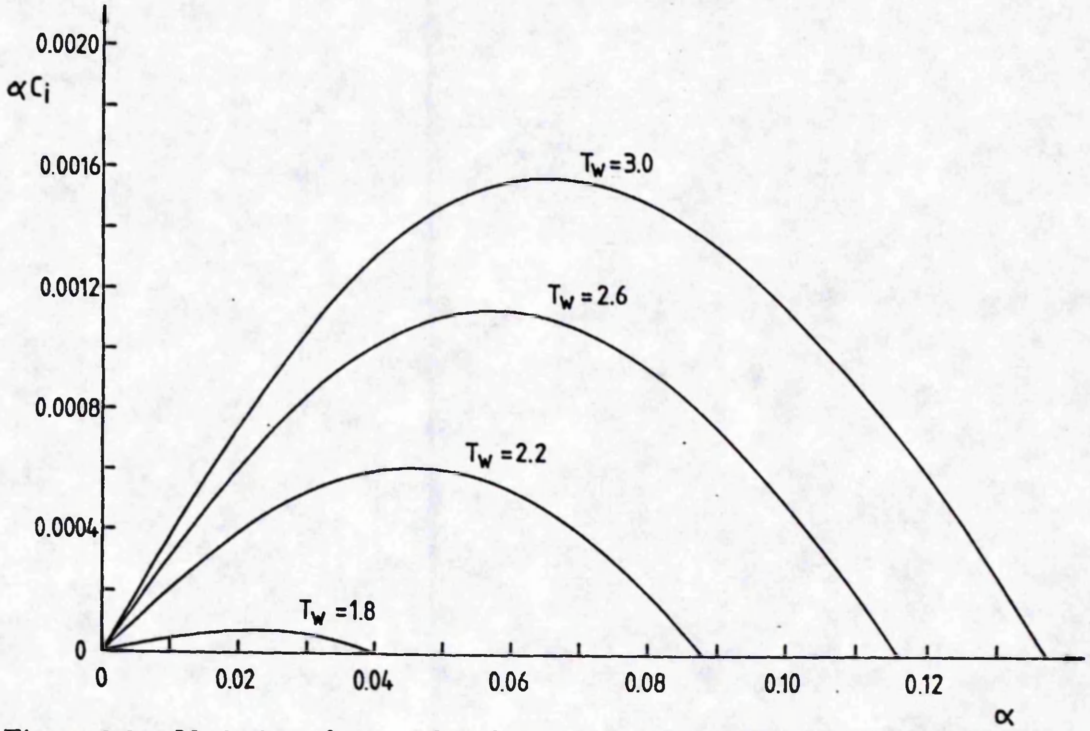


Figure 3.38: Variation of αc_i with α for cooled cylinder, $M_\infty = 3.8$, $\zeta = 0.05$, $n = 5$, Mode I.

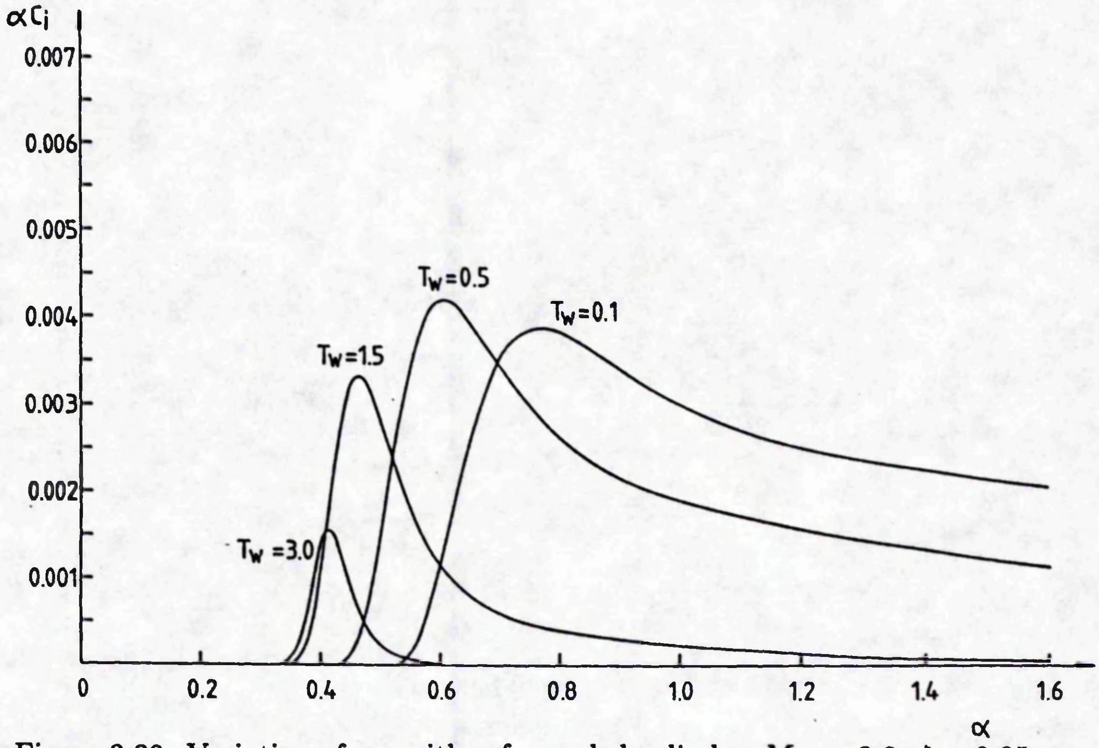


Figure 3.39: Variation of αc_i with α for cooled cylinder, $M_\infty = 3.8$, $\zeta = 0.05$, $n = 5$, Mode II.

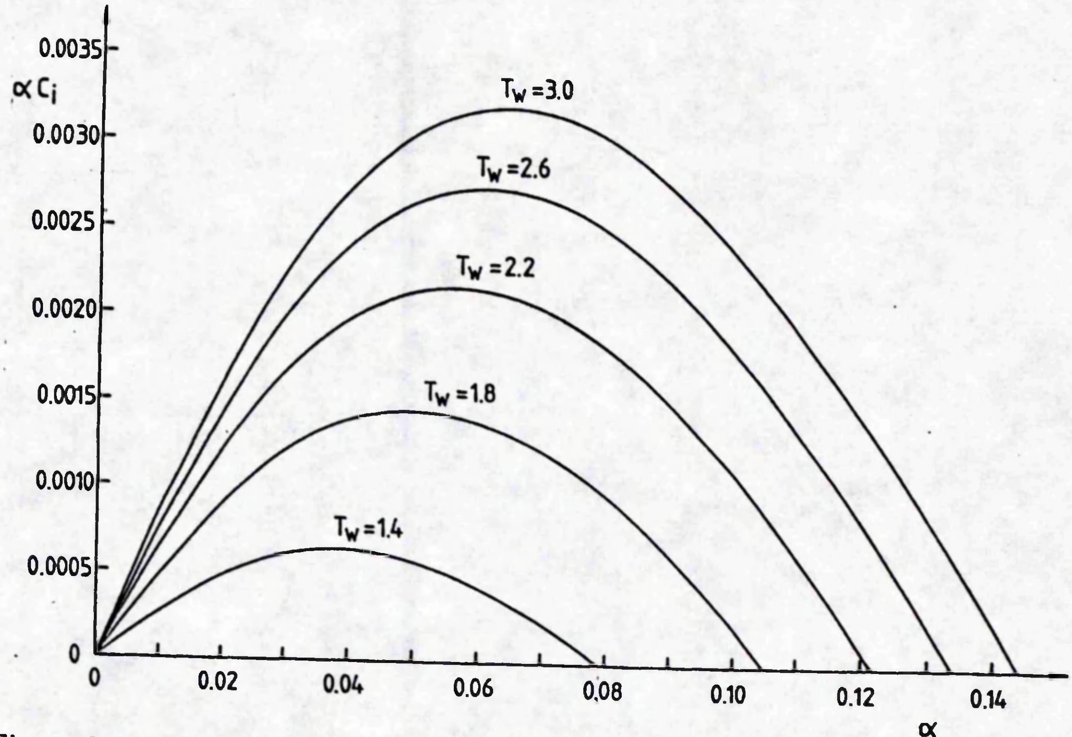


Figure 3.40: Variation of αc_i with α for cooled cylinder, $M_\infty = 3.8$, $\zeta = 0.5$, $n = 1$, Mode I.

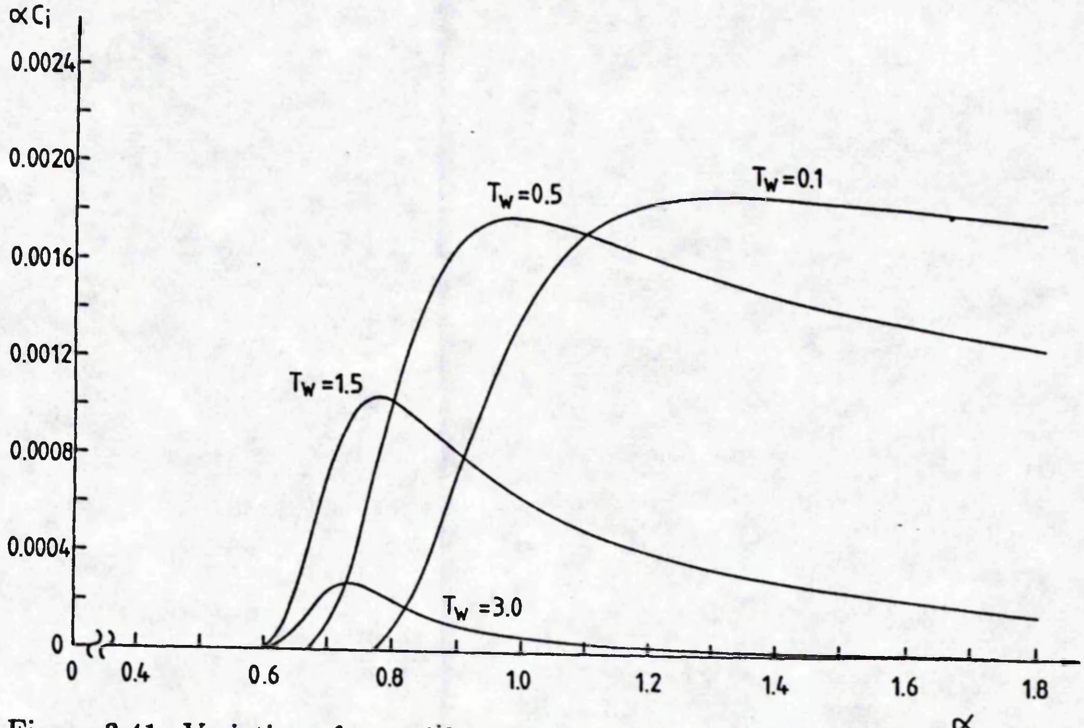


Figure 3.41: Variation of αc_i with α for cooled cylinder, $M_\infty = 3.8$, $\zeta = 0.5$, $n = 1$, Mode II.

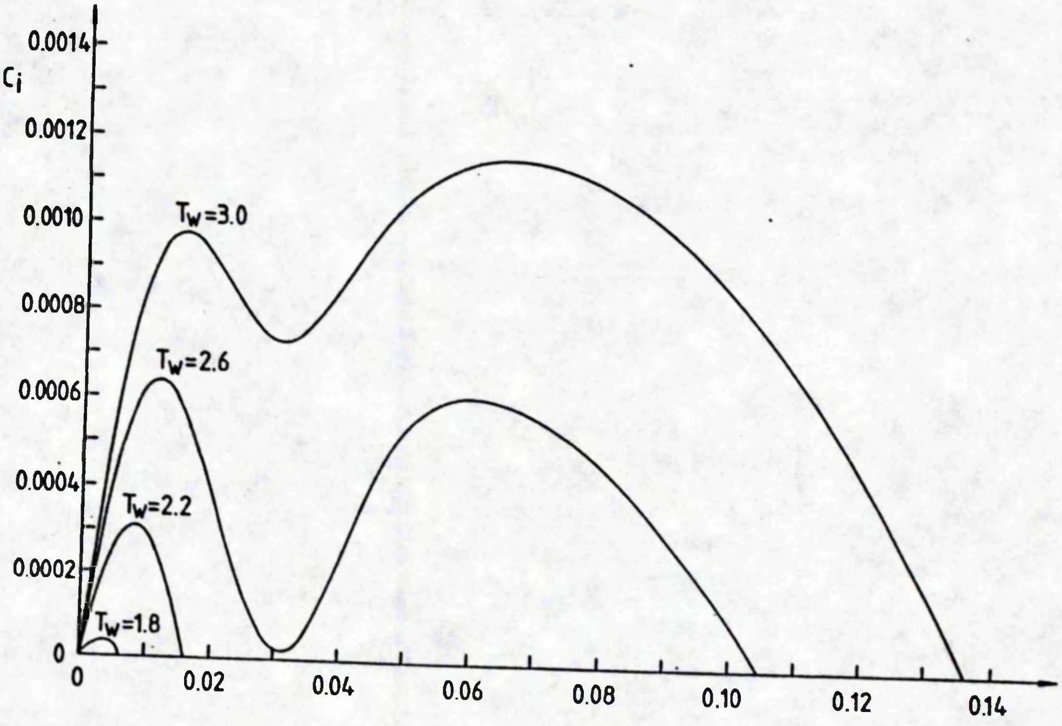


Figure 3.42: Variation of αc_i with α for cooled cylinder, $M_\infty = 2.8$, $\zeta = 0.05$, $n = 1$, Mode I.

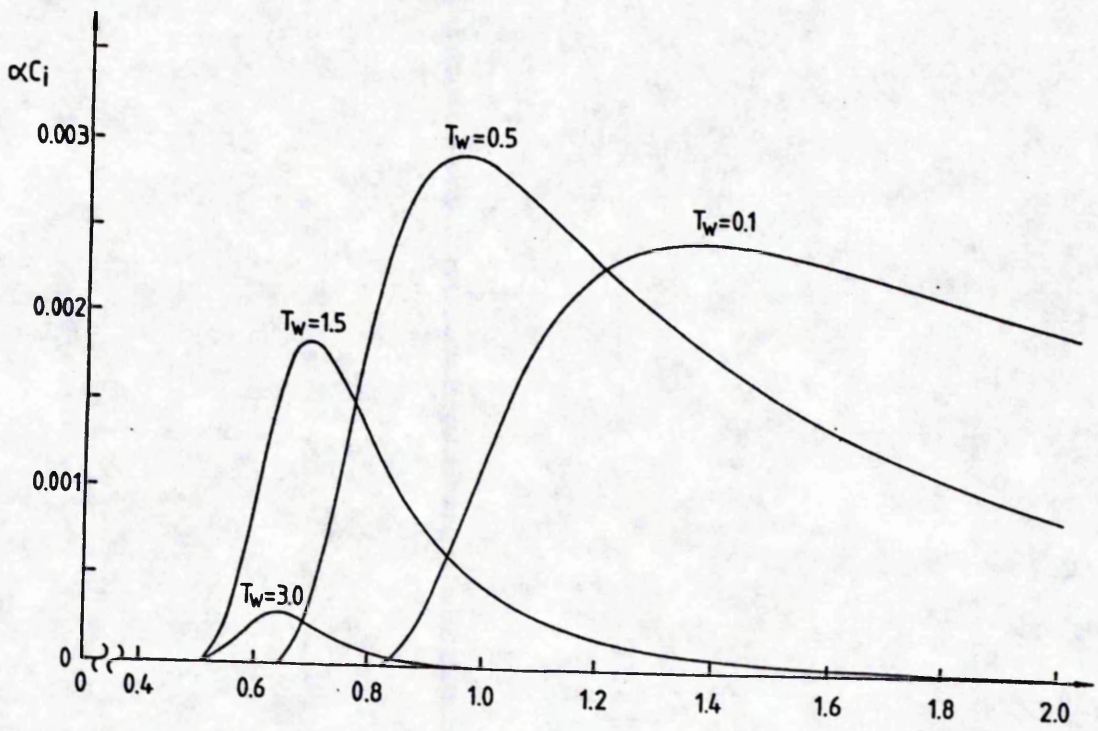


Figure 3.43: Variation of αc_i with α for cooled cylinder, $M_\infty = 2.8$, $\zeta = 0.05$, $n = 1$, Mode II.

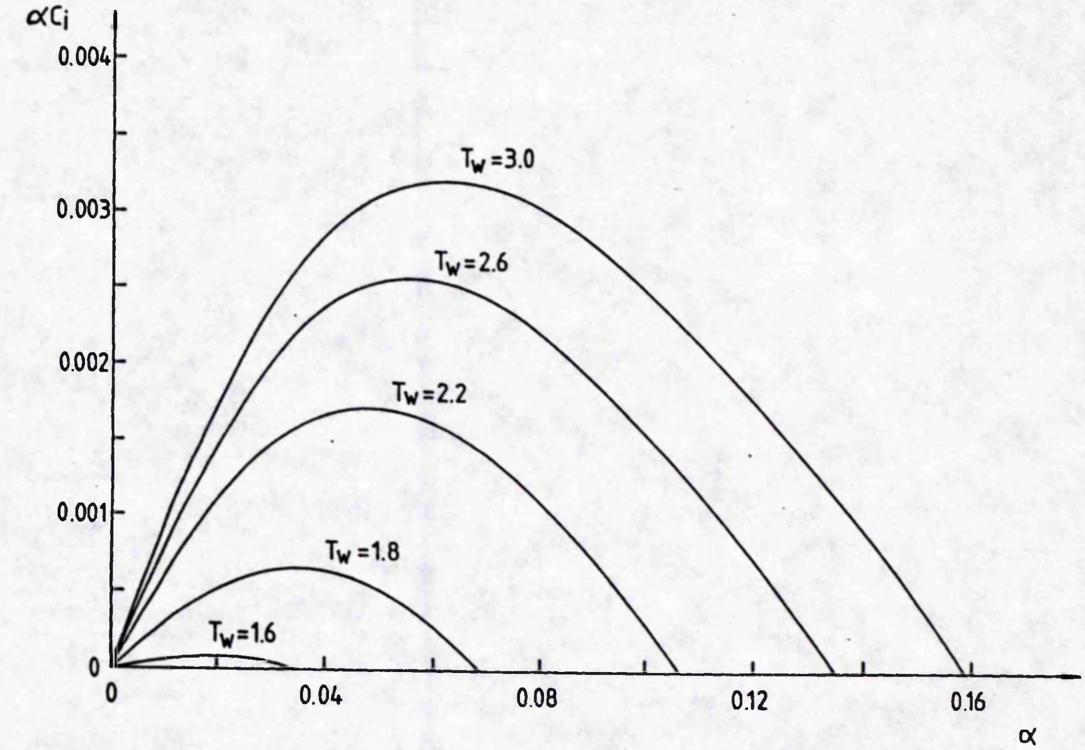


Figure 3.44: Variation of αc_i with α for cooled cylinder, $M_\infty = 2.8$, $\zeta = 0.05$, $n = 3$, Mode I.

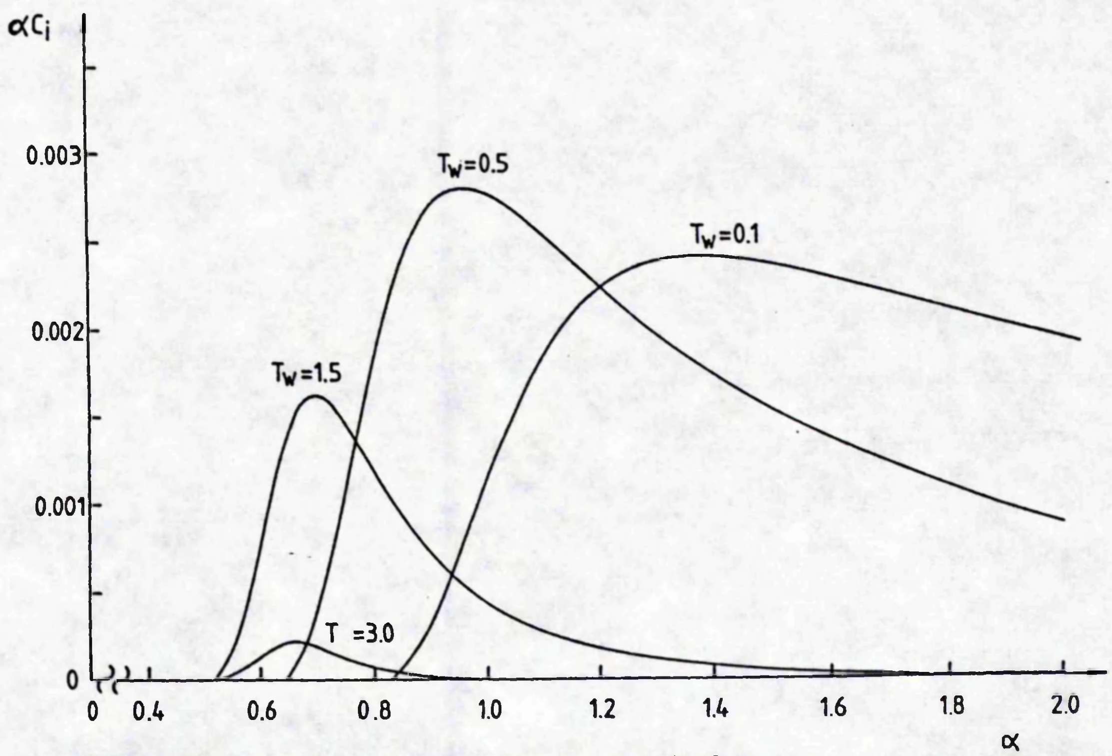


Figure 3.45: Variation of αc_i with α for cooled cylinder, $M_\infty = 2.8$, $\zeta = 0.05$, $n = 3$, Mode II.

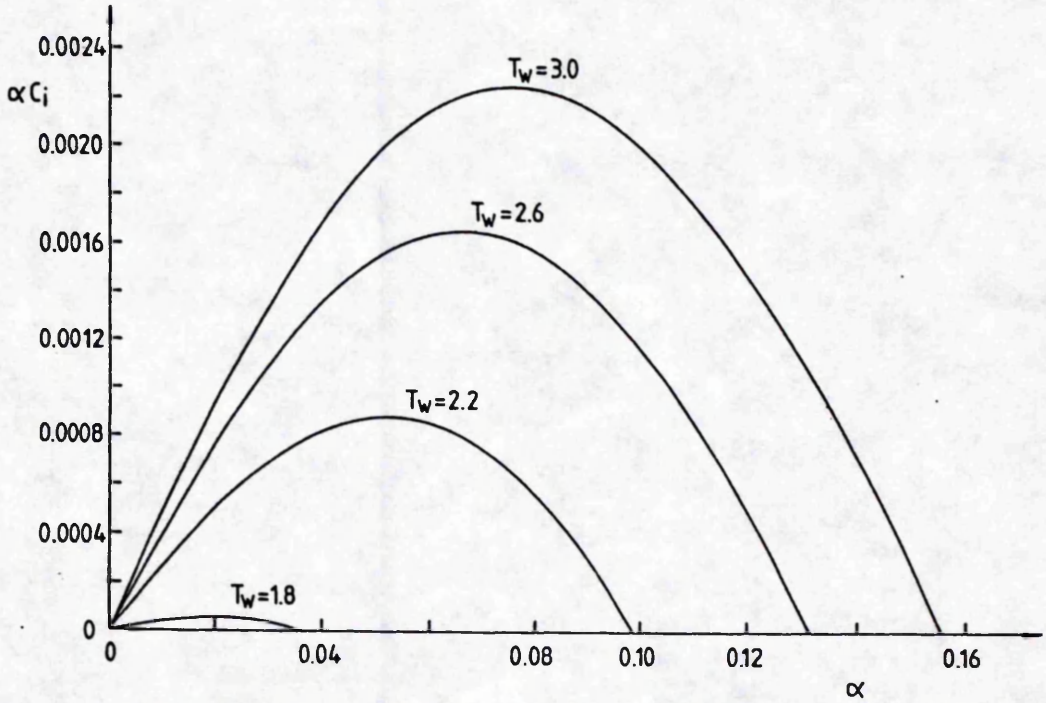


Figure 3.46: Variation of αc_i with α for cooled cylinder, $M_\infty = 2.8$, $\zeta = 0.05$, $n = 5$, Mode I.

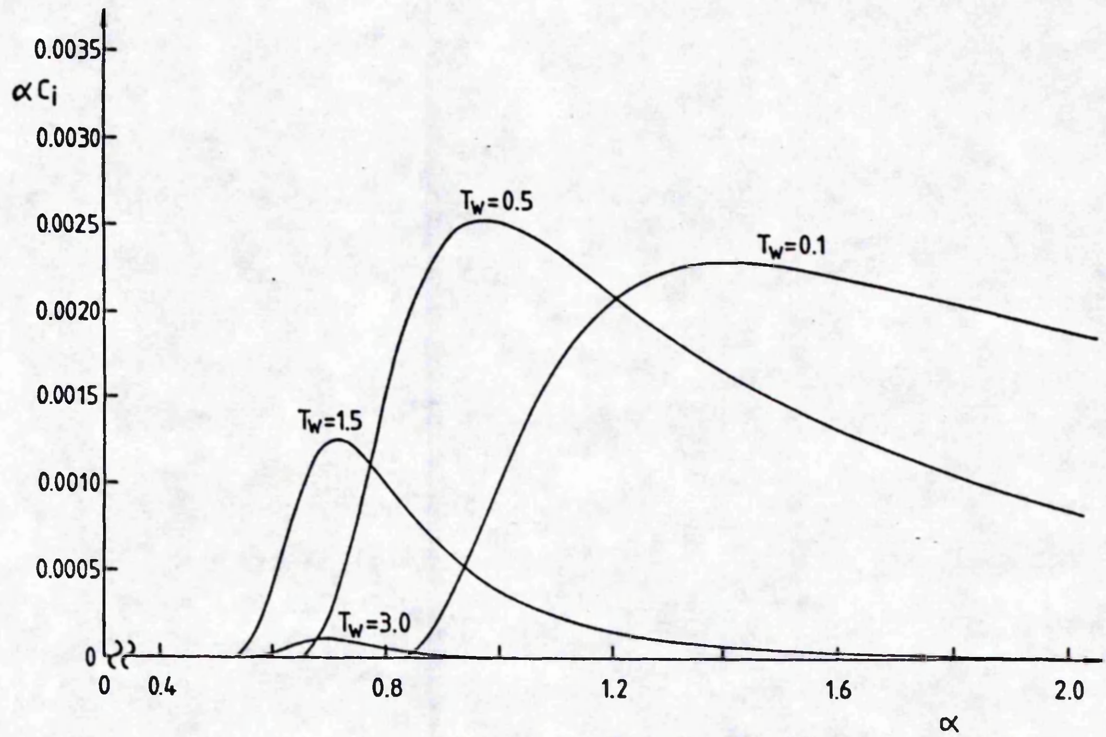


Figure 3.47: Variation of αc_i with α for cooled cylinder, $M_\infty = 2.8$, $\zeta = 0.05$, $n = 5$, Mode II.

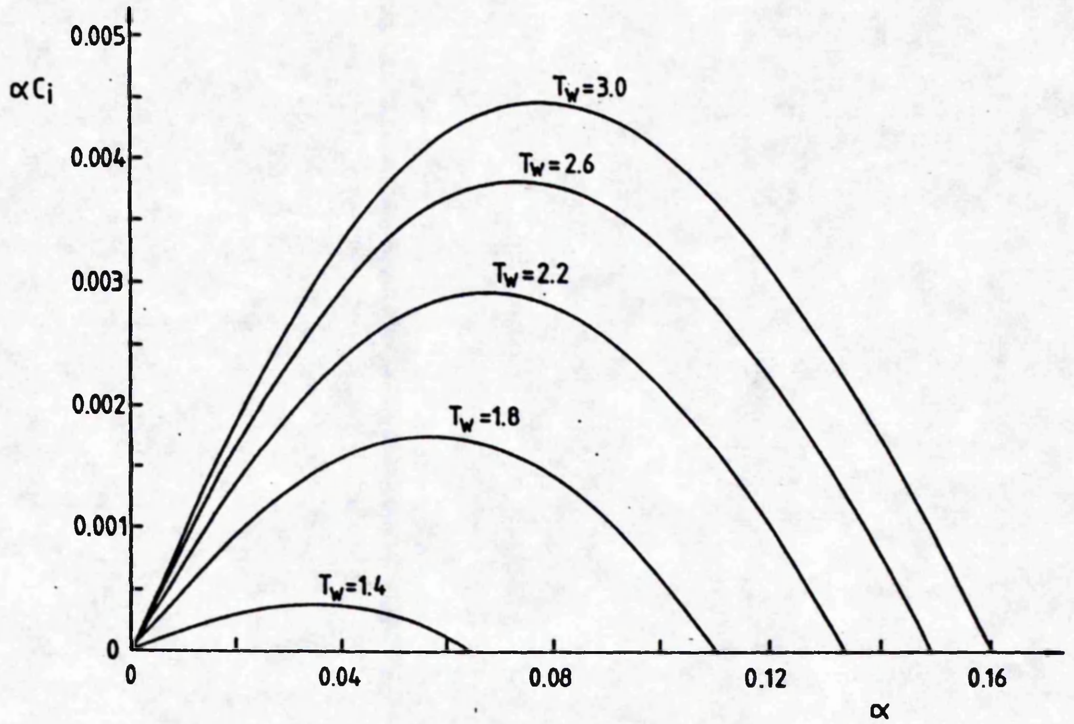


Figure 3.48: Variation of αc_i with α for cooled cylinder, $M_\infty = 2.8$, $\zeta = 0.5$, $n = 1$, Mode I.

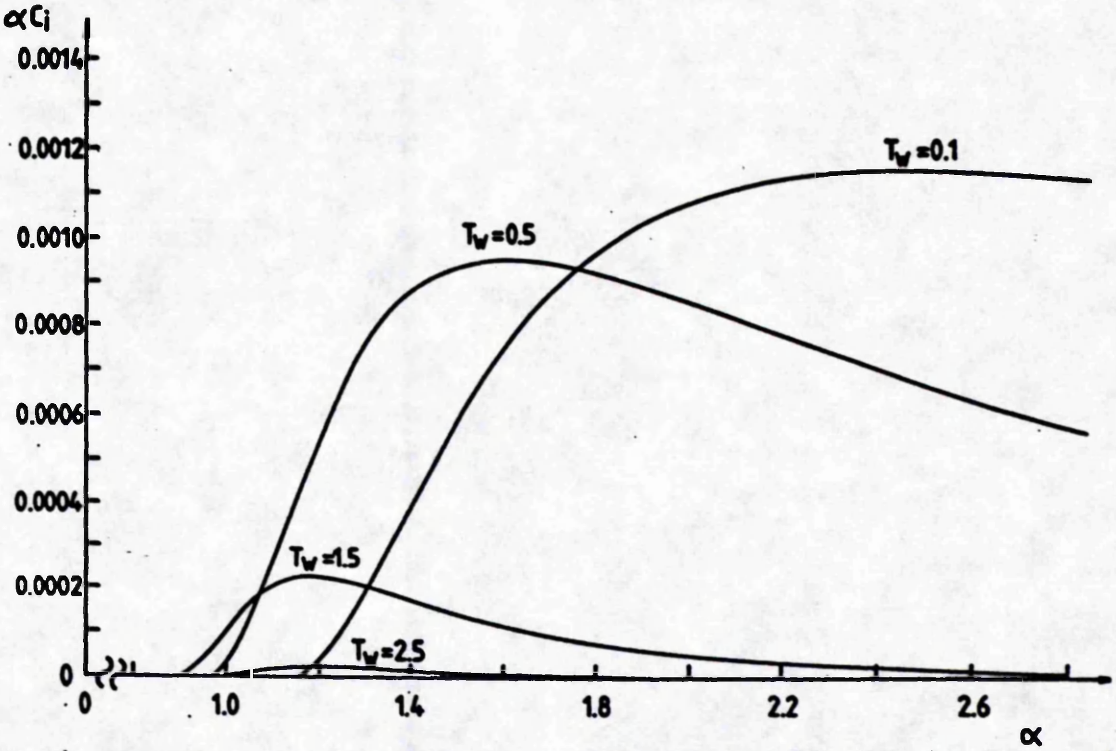


Figure 3.49: Variation of αc_i with α for cooled cylinder, $M_\infty = 2.8$, $\zeta = 0.5$, $n = 1$, Mode II.

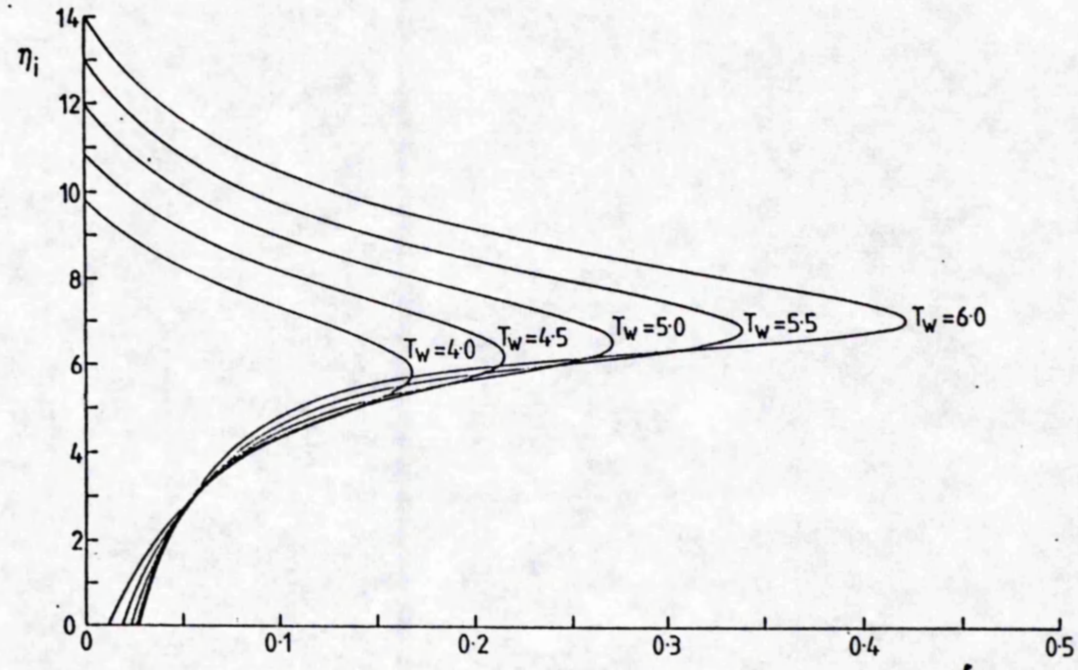


Figure 3.50: Variation of transverse positions of inflexion points (η_i) with axial locations (ζ) for heated cylinder, $M_\infty = 3.8$.

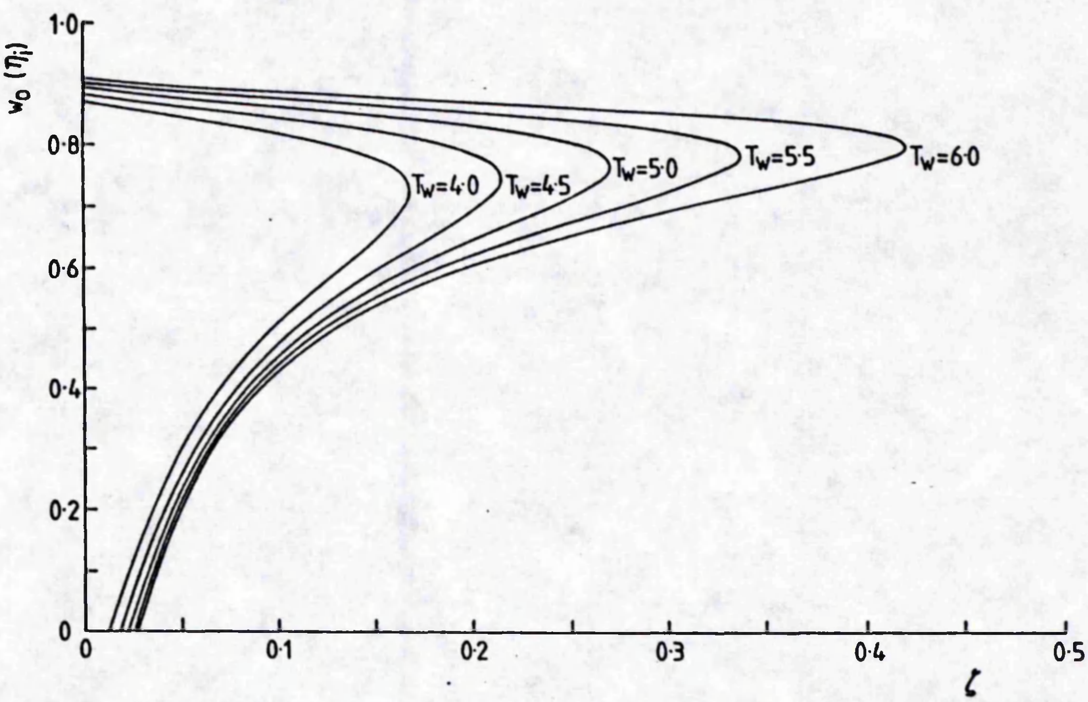


Figure 3.51: Variation of $w_0(\eta = \eta_i)$ with ζ for heated cylinder, $M_\infty = 3.8$.

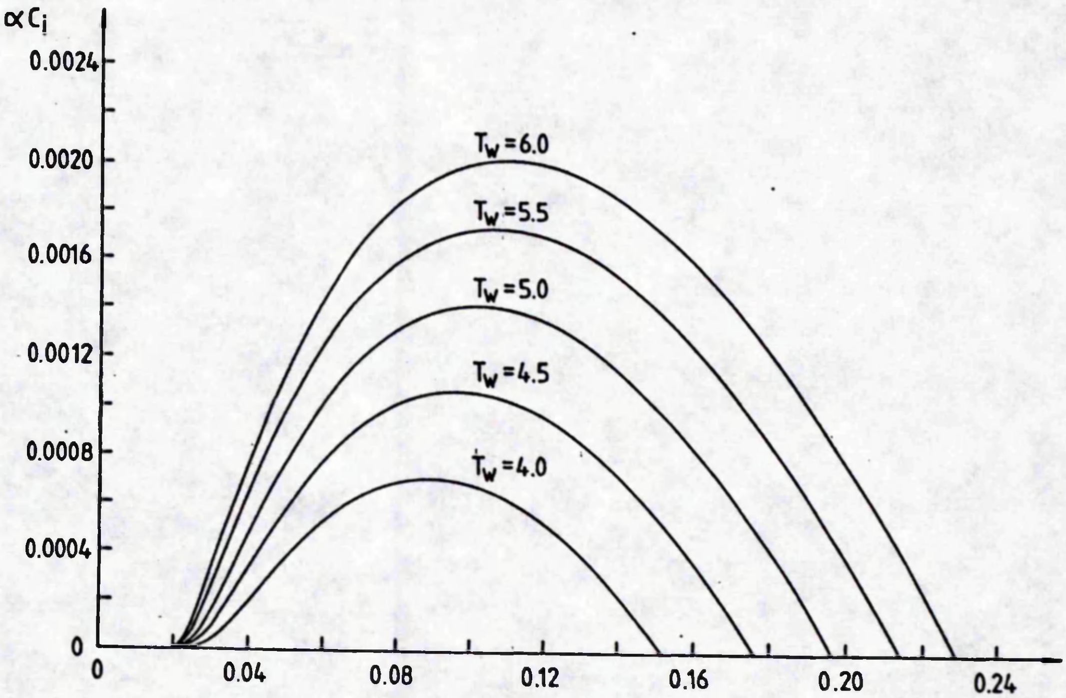


Figure 3.52: Variation of αc_i with α for heated cylinder, $M_\infty = 3.8$, $\zeta = 0.05$, $n = 0$, Mode I.

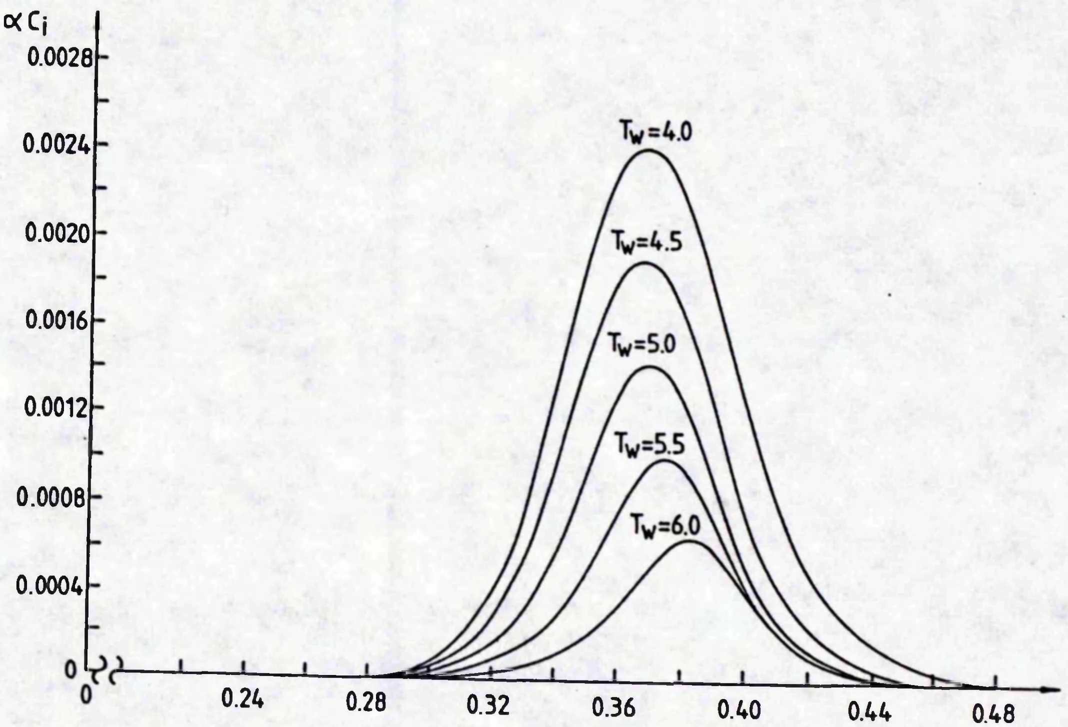


Figure 3.53: Variation of αc_i with α for heated cylinder, $M_\infty = 3.8$, $\zeta = 0.05$, $n = 0$, Mode II.

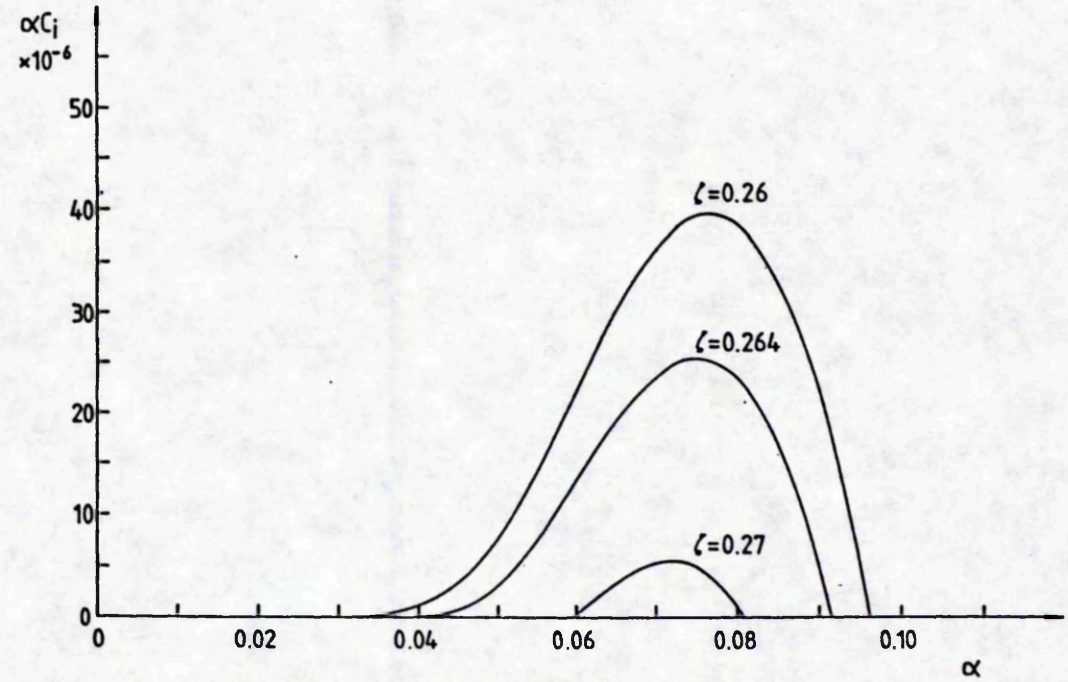


Figure 3.54: Variation of αc_i with α for heated cylinder, $M_\infty = 3.8$, $T_w = 5.0$, $n = 0$, Mode I.

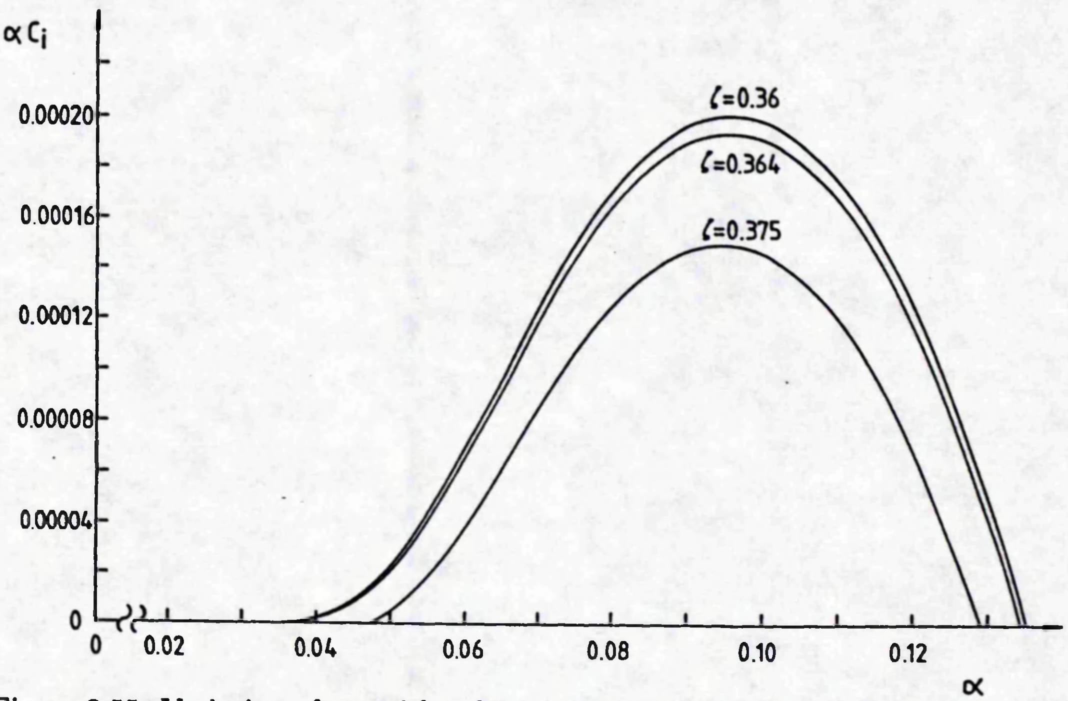


Figure 3.55: Variation of αc_i with α for heated cylinder, $M_\infty = 3.8$, $T_w = 6.0$, $n = 0$, Mode I.

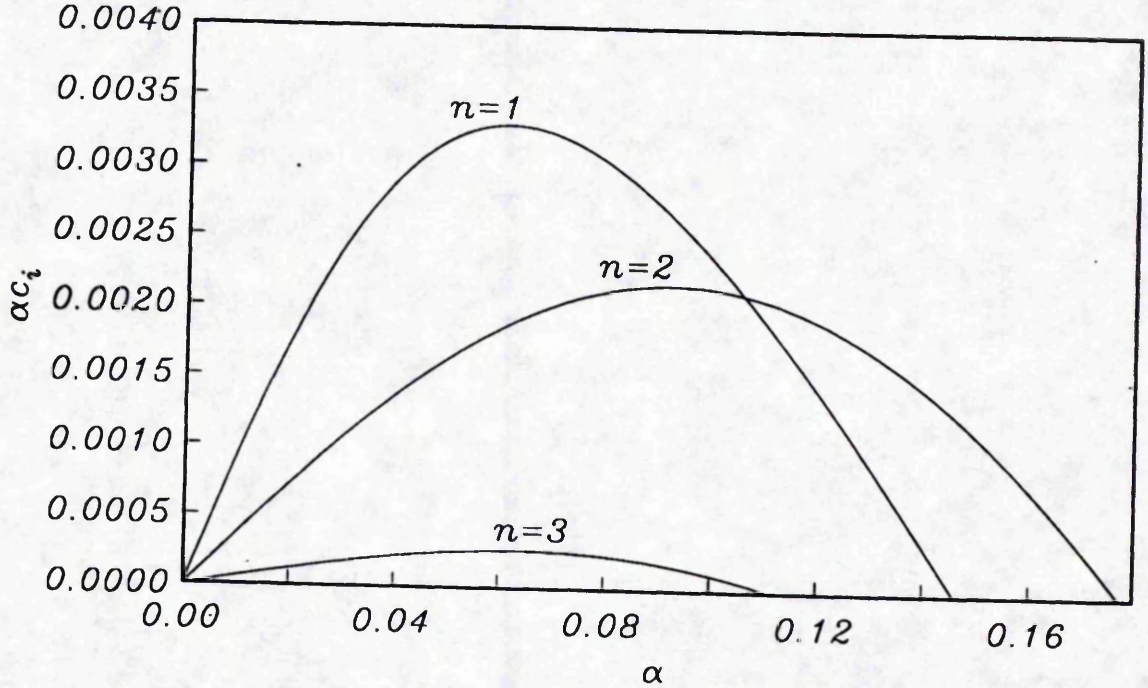


Figure 3.56: Variation of αc_i with α for adiabatic cone, $M_\infty = 3.8$, $\zeta = 0.5$, Mode I.

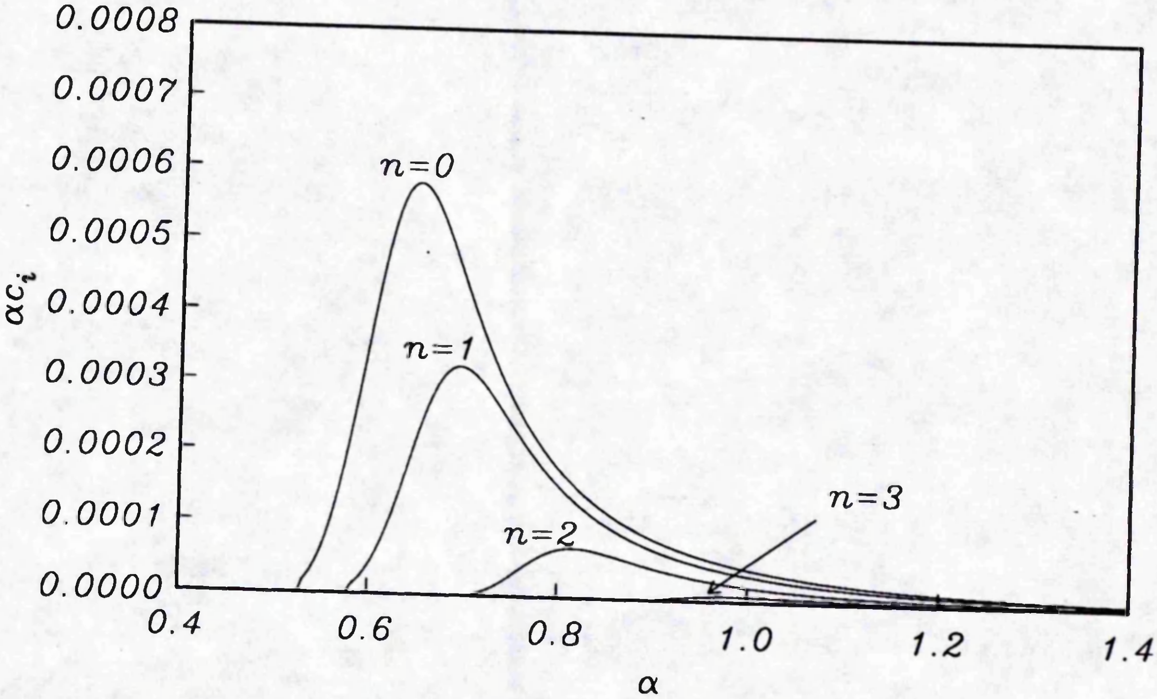


Figure 3.57: Variation of αc_i with α for adiabatic cone, $M_\infty = 3.8$, $\zeta = 0.5$, Mode II.

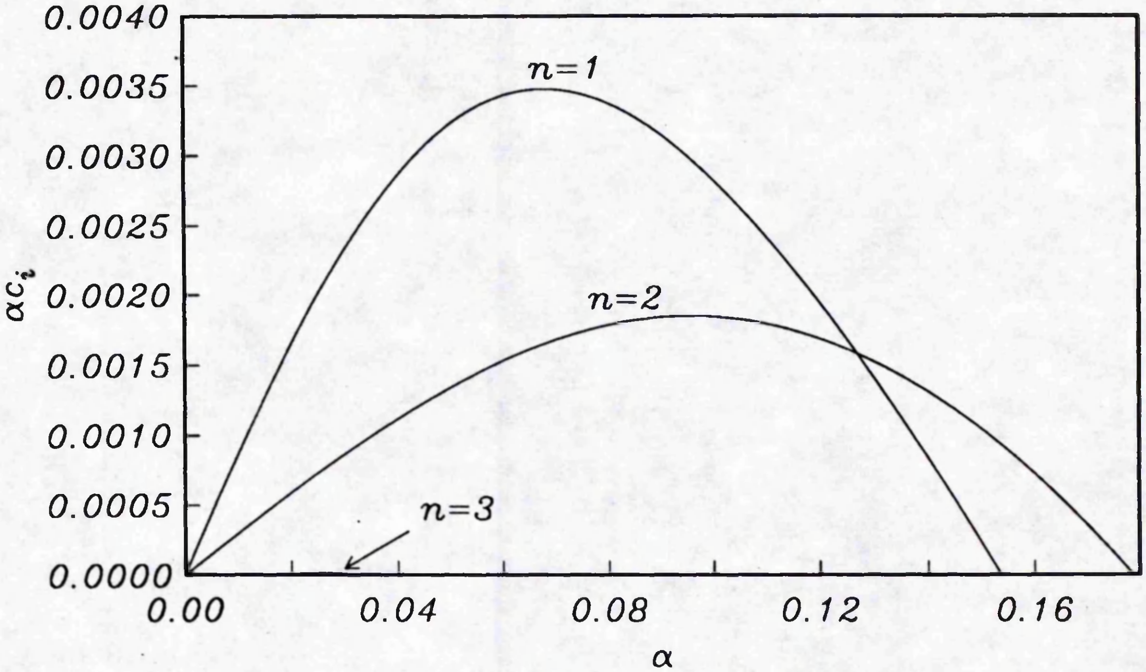


Figure 3.58: Variation of αc_i with α for adiabatic cone, $M_\infty = 3.8$, $\zeta = 1.0$, Mode I.

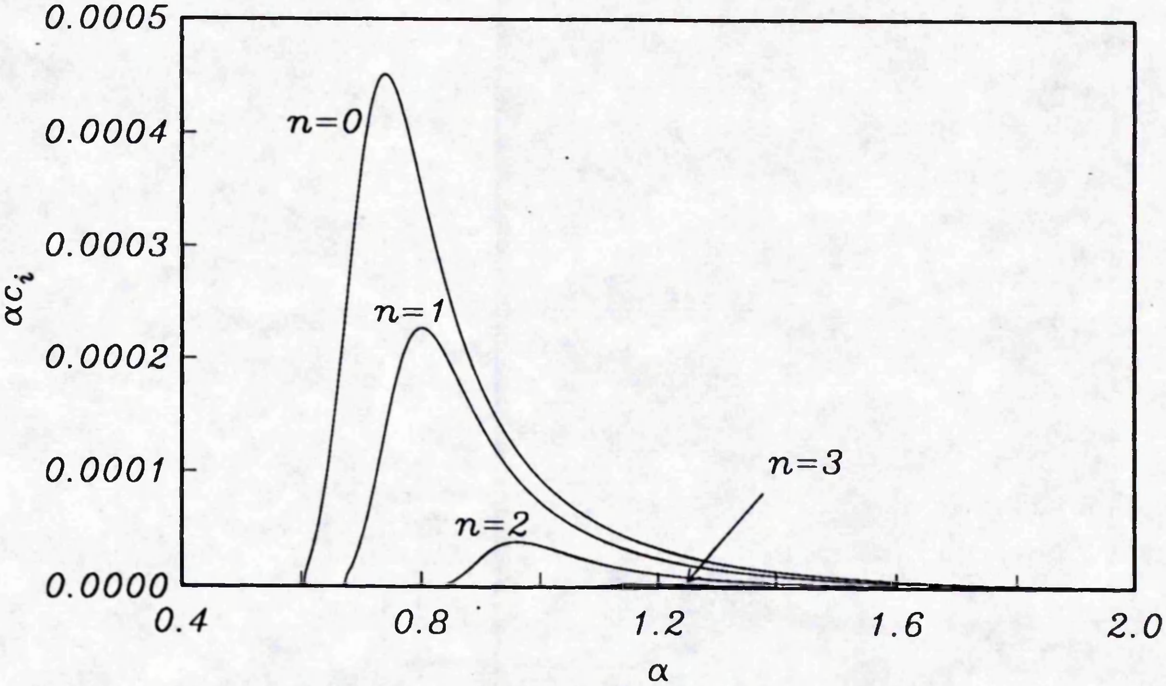


Figure 3.59: Variation of αc_i with α for adiabatic cone, $M_\infty = 3.8$, $\zeta = 1.0$, Mode II.

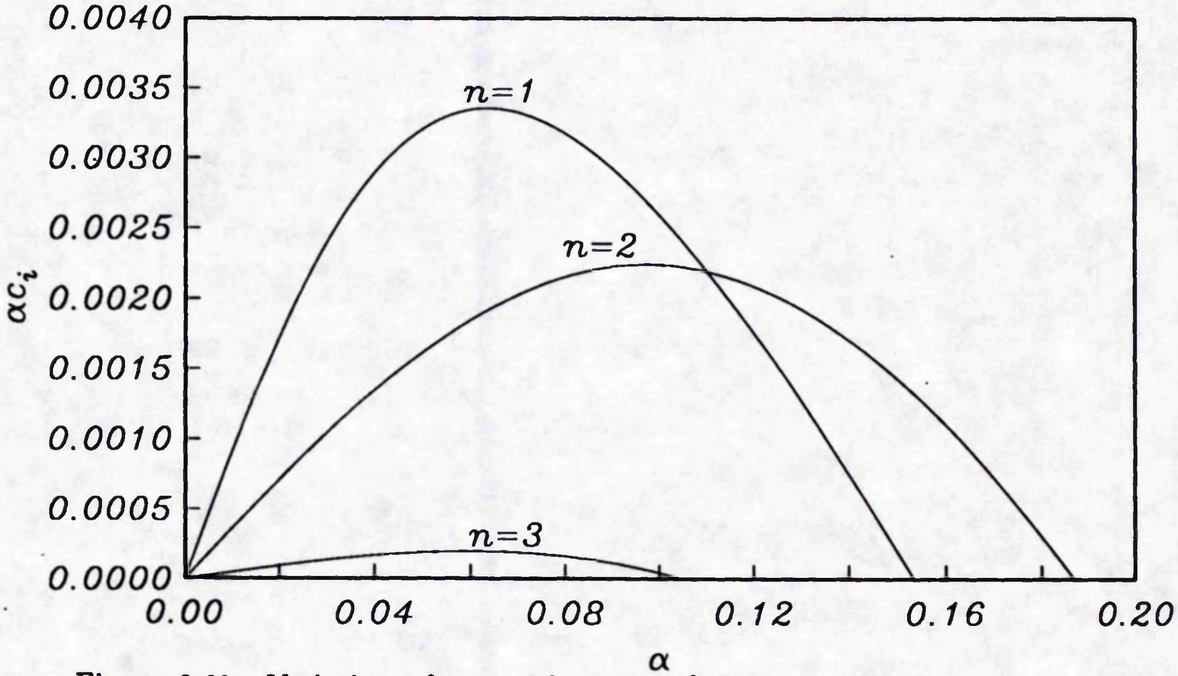


Figure 3.60: Variation of αc_i with α for adiabatic cone, $M_\infty = 3.8$, $\zeta = 2.0$, Mode I.

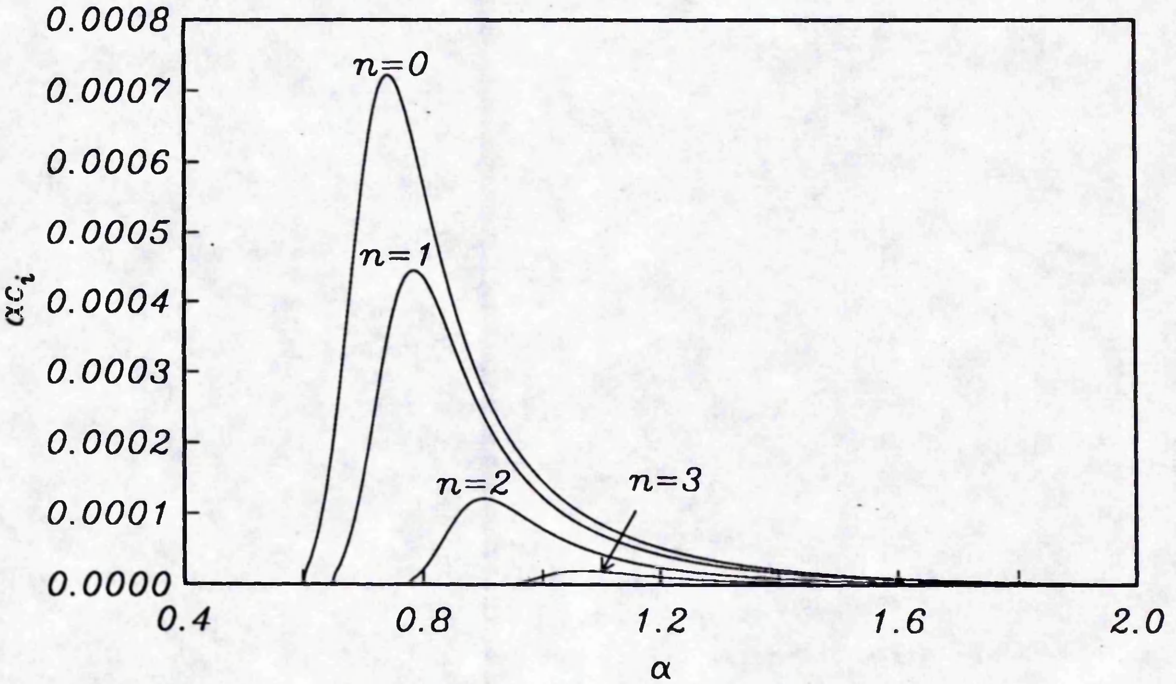


Figure 3.61: Variation of αc_i with α for adiabatic cone, $M_\infty = 3.8$, $\zeta = 2.0$, Mode II.

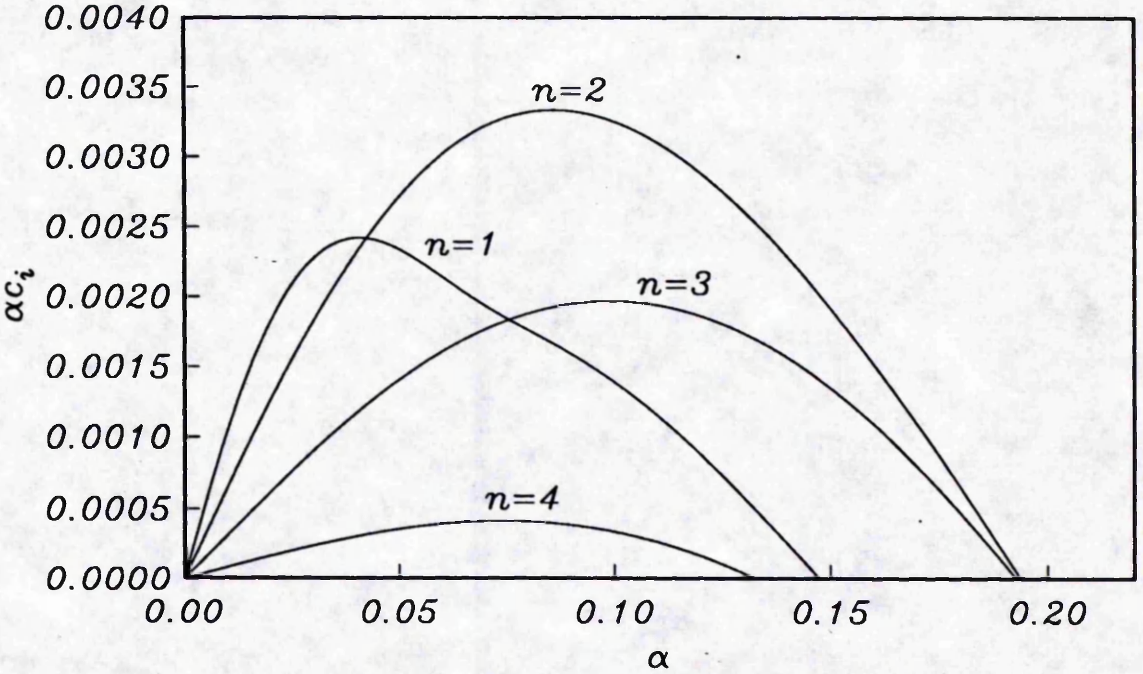


Figure 3.62: Variation of αc_i with α for adiabatic cone, $M_\infty = 3.8$, $\zeta = 5.0$, Mode I.

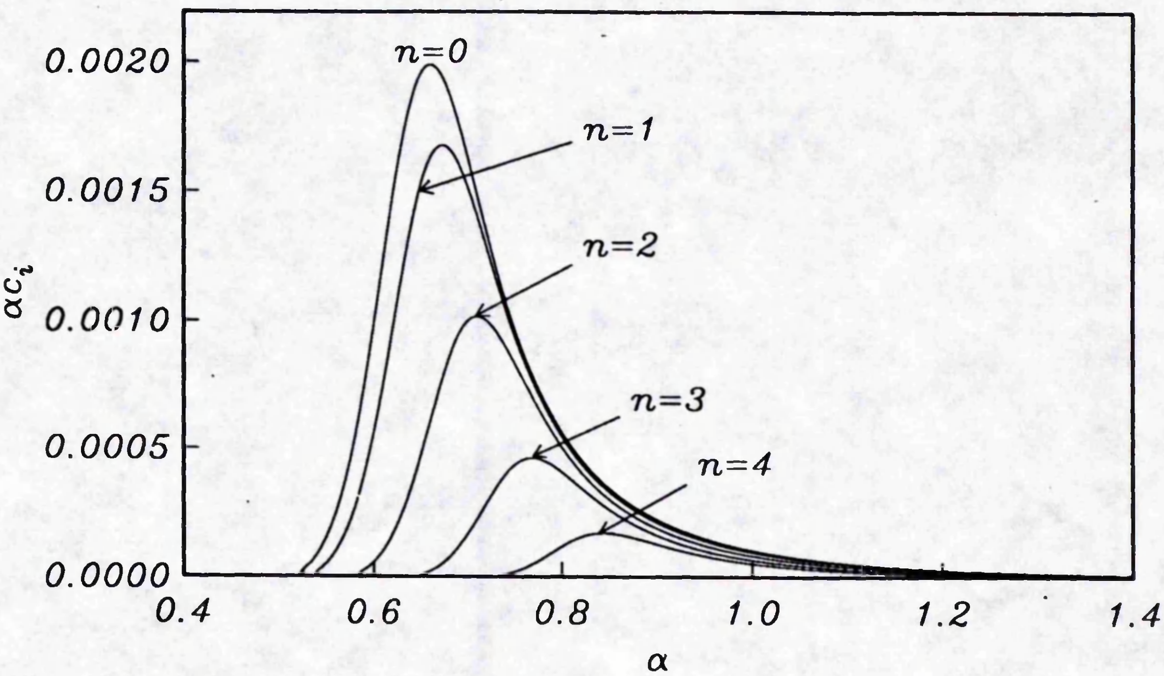


Figure 3.63: Variation of αc_i with α for adiabatic cone, $M_\infty = 3.8$, $\zeta = 5.0$, Mode II.

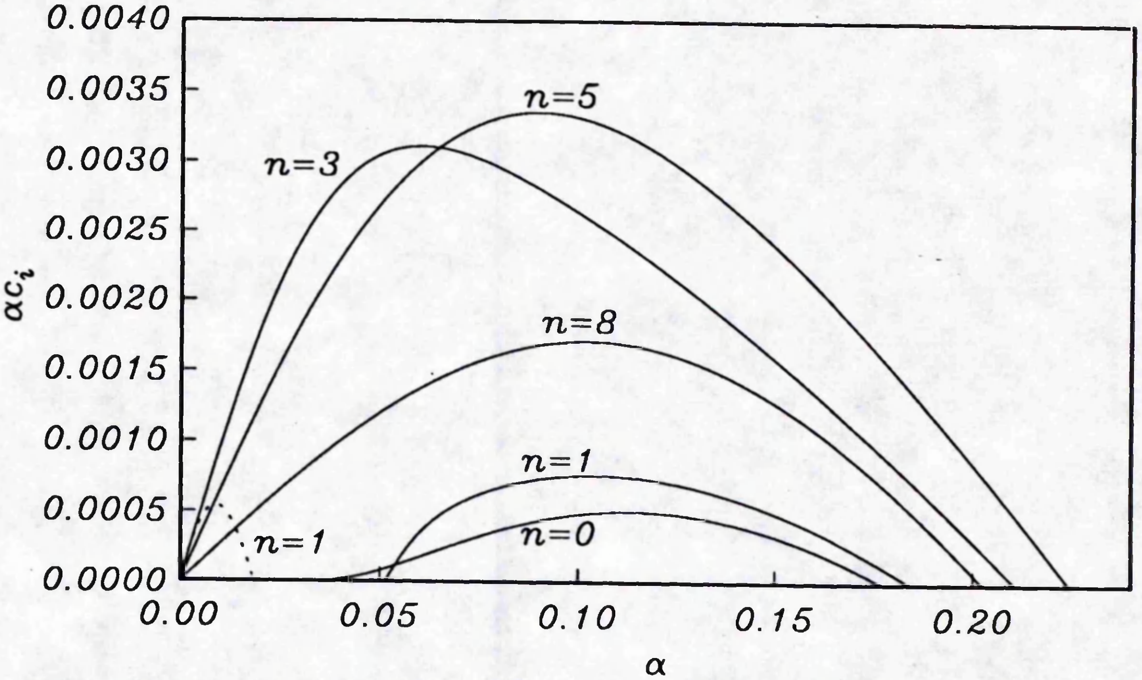


Figure 3.64: Variation of αc_i with α for adiabatic cone, $M_\infty = 3.8$, $\zeta = 20.0$, Mode I.

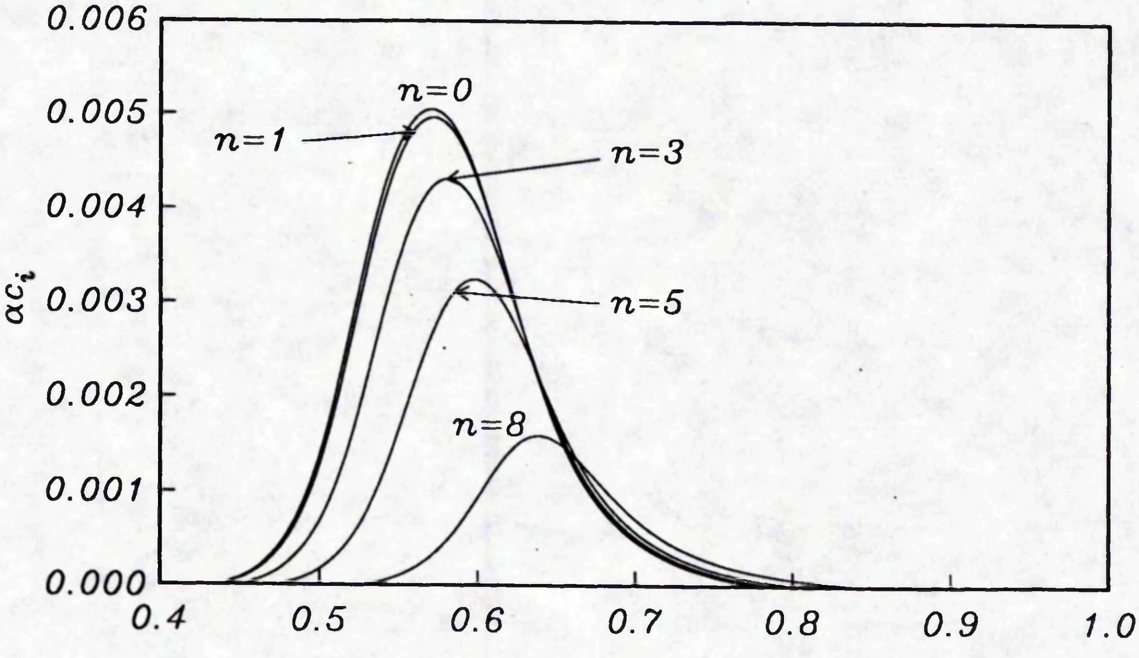


Figure 3.65: Variation of αc_i with α for adiabatic cone, $M_\infty = 3.8$, $\zeta = 20.0$, Mode II.

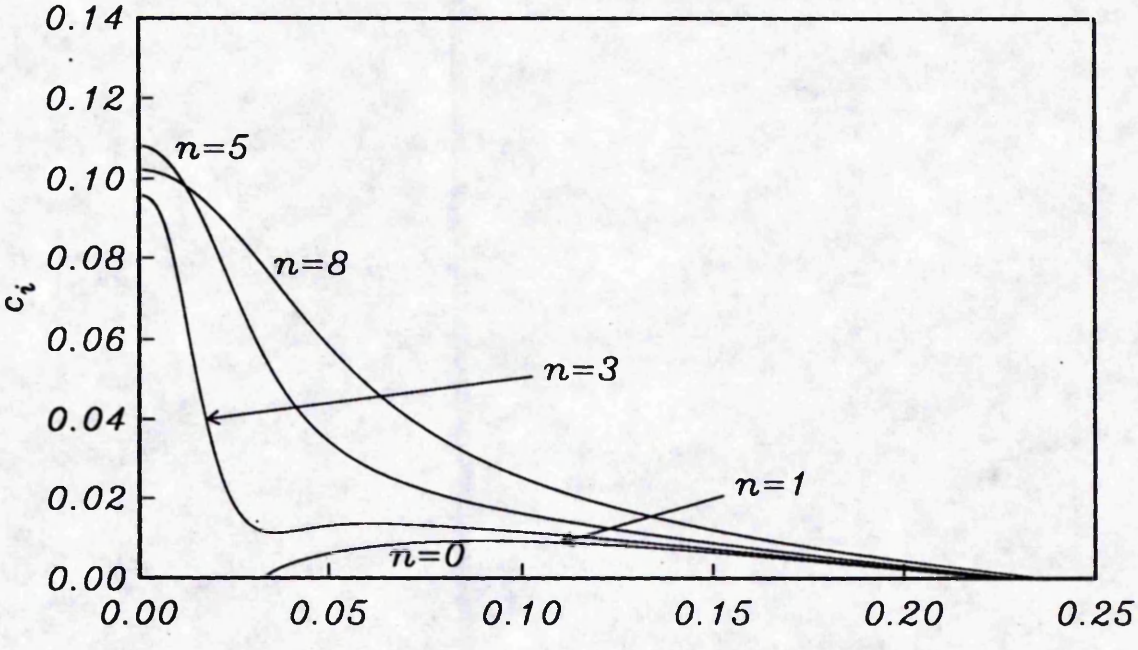


Figure 3.66: Variation of c_i with α for adiabatic cone, $M_\infty = 3.8$, $\zeta = 75.0$, Mode I.

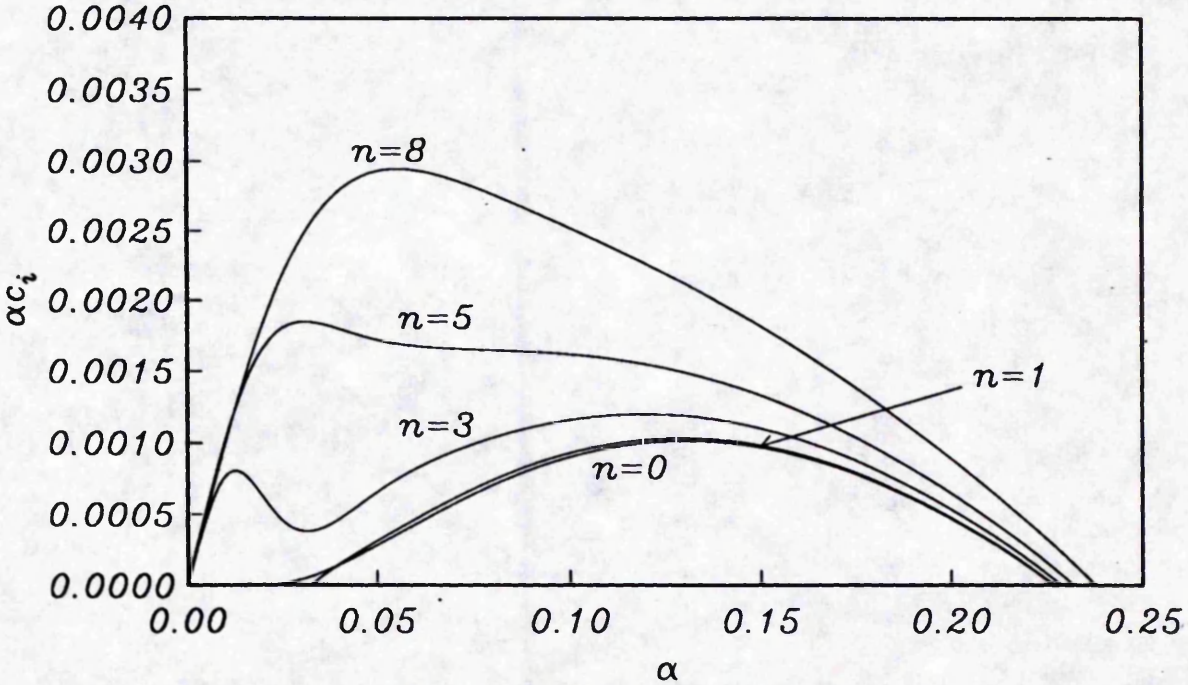


Figure 3.67: Variation of αc_i with α for adiabatic cone, $M_\infty = 3.8$, $\zeta = 75.0$, Mode I.

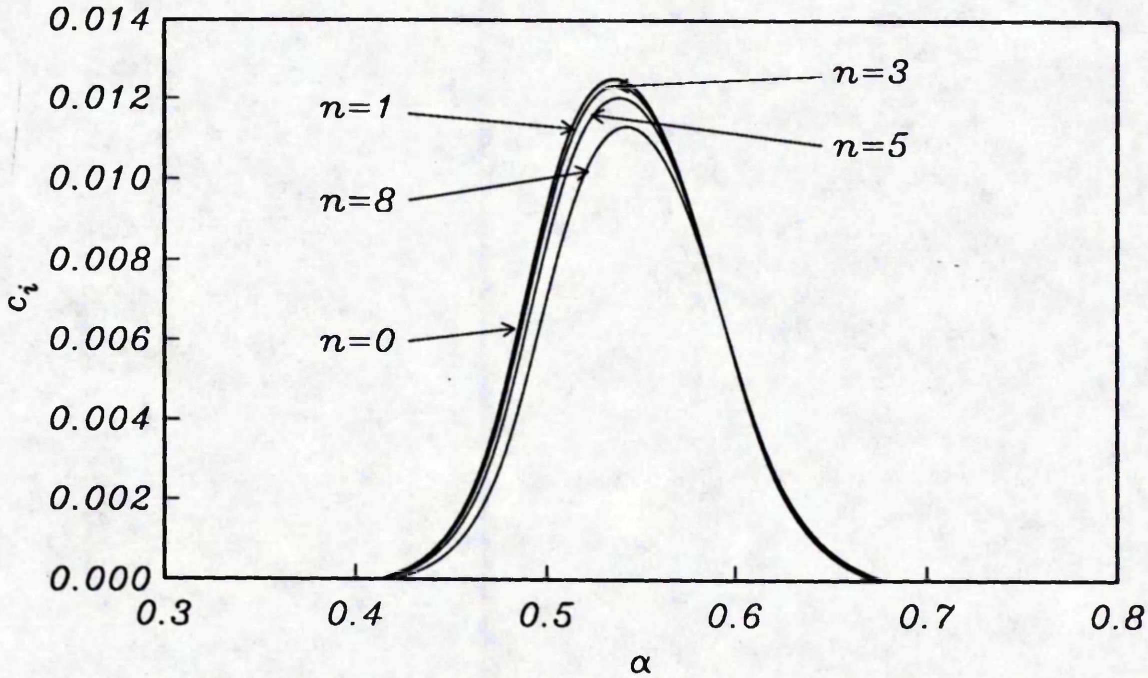


Figure 3.68: Variation of c_i with α for adiabatic cone, $M_\infty = 3.8$, $\zeta = 75.0$, Mode II.

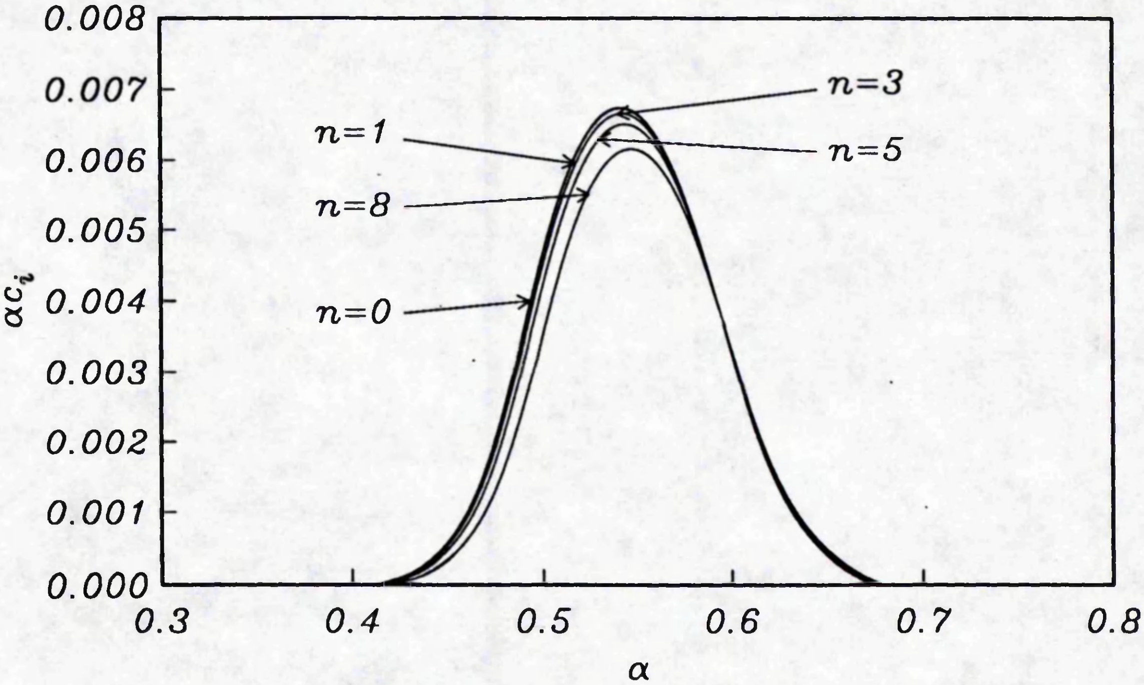


Figure 3.69: Variation of αc_i with α for adiabatic cone, $M_\infty = 3.8$, $\zeta = 75.0$, Mode II.

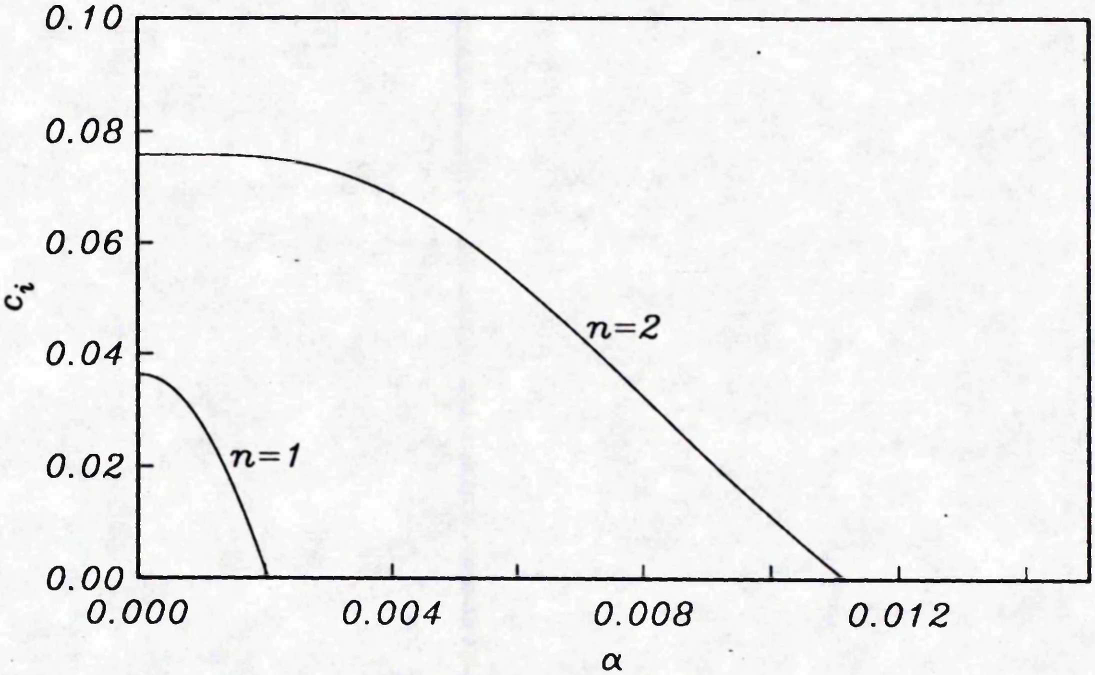


Figure 3.70: Variation of c_i with α for adiabatic cone, $M_\infty = 3.8$, $\zeta = 75.0$, Mode $\mathrm{I_A}$.

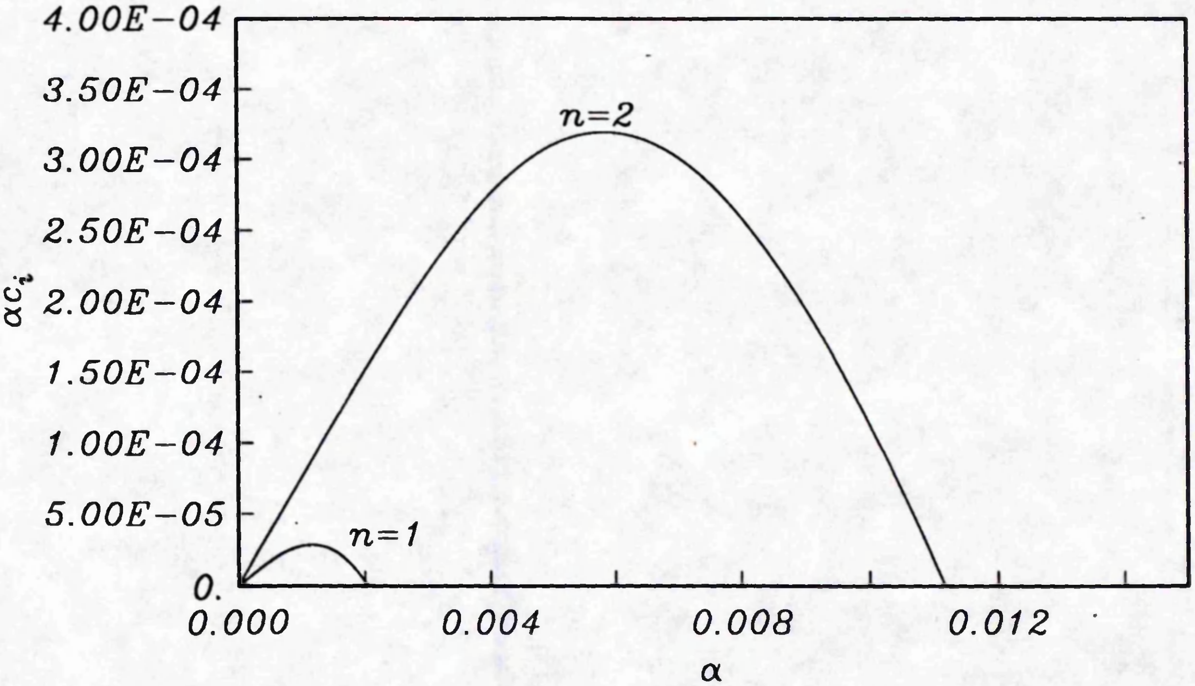


Figure 3.71: Variation of αc_i with α for adiabatic cone, $M_\infty = 3.8$, $\zeta = 75.0$, Mode $\mathrm{I_A}$.

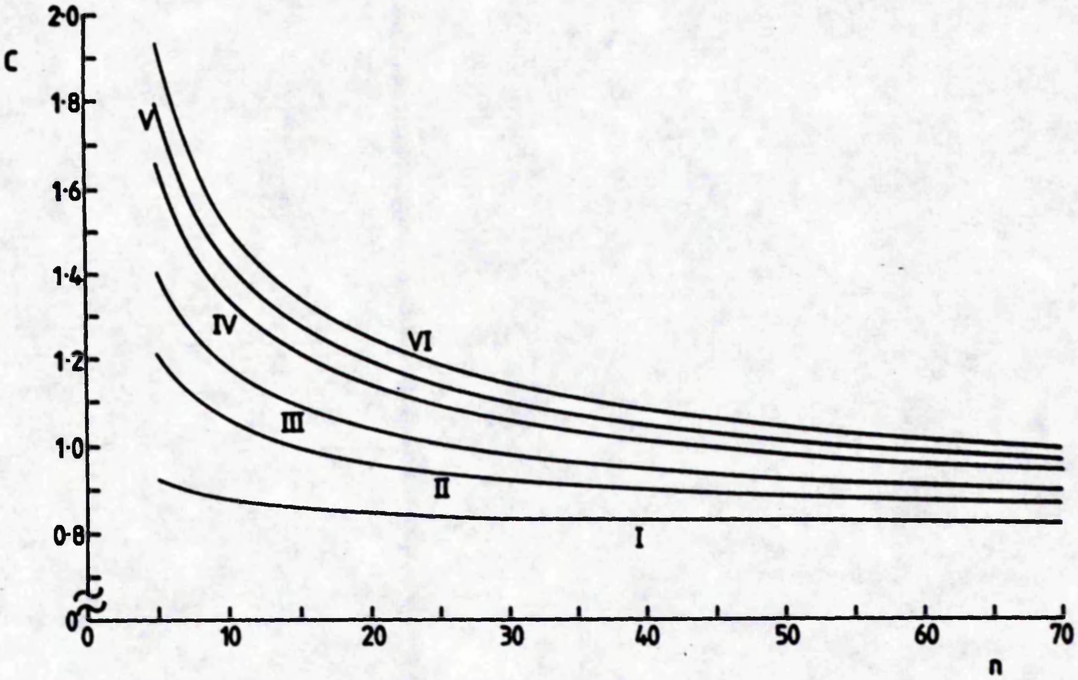


Figure 3.72: Variation of asymptotically determined value of c with n.

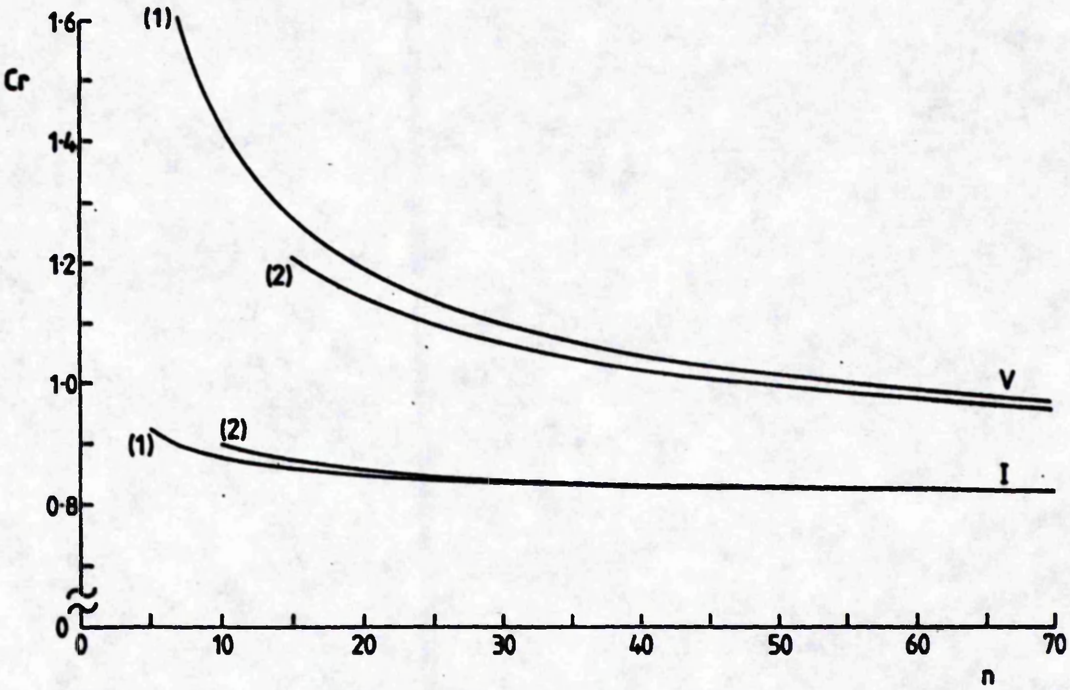


Figure 3.73: Comparison of computed c_r with asymptotic form.

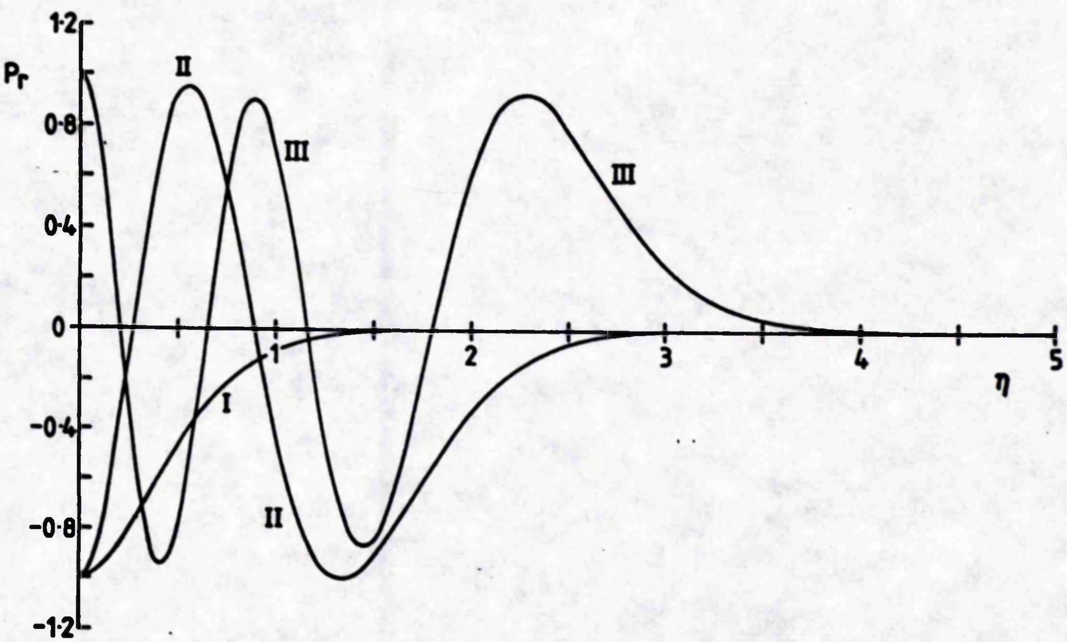


Figure 3.74: Variation of p_r = Real$\{p\}$ with η.

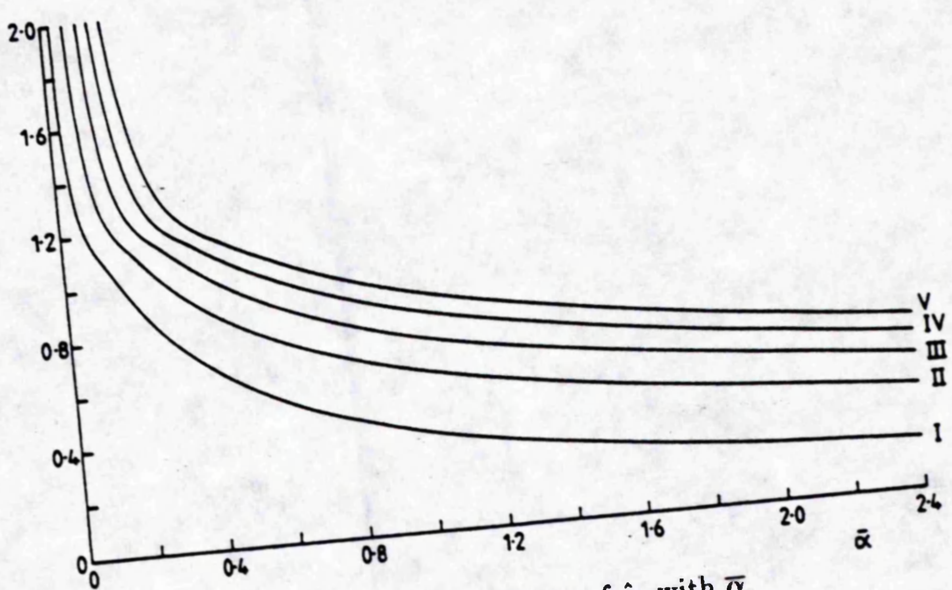


Figure 3.75: Variation of $\hat{c}_1$ with $\overline{\alpha}$.

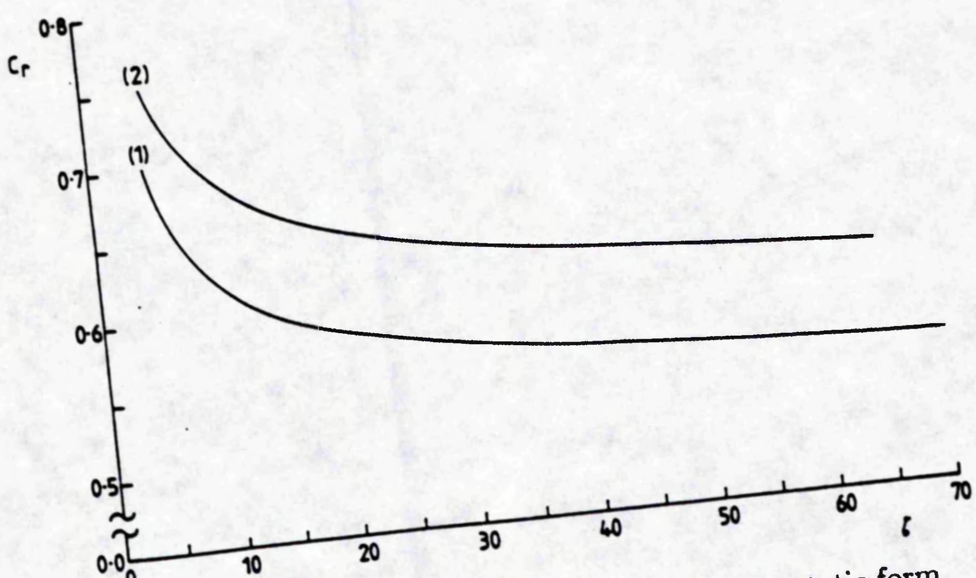


Figure 3.76: Variation of computed c_r with asymptotic form.

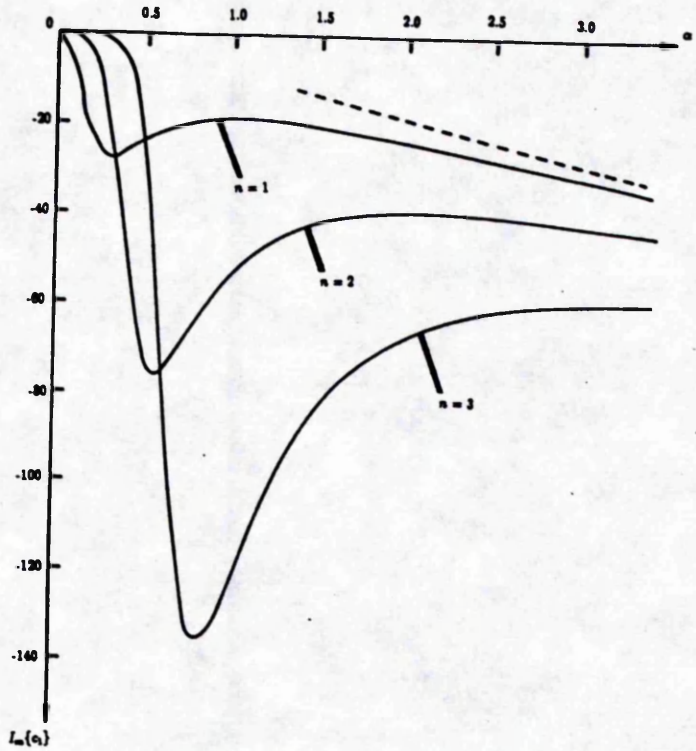


Figure 3.77: Variation of Im{c} with α.

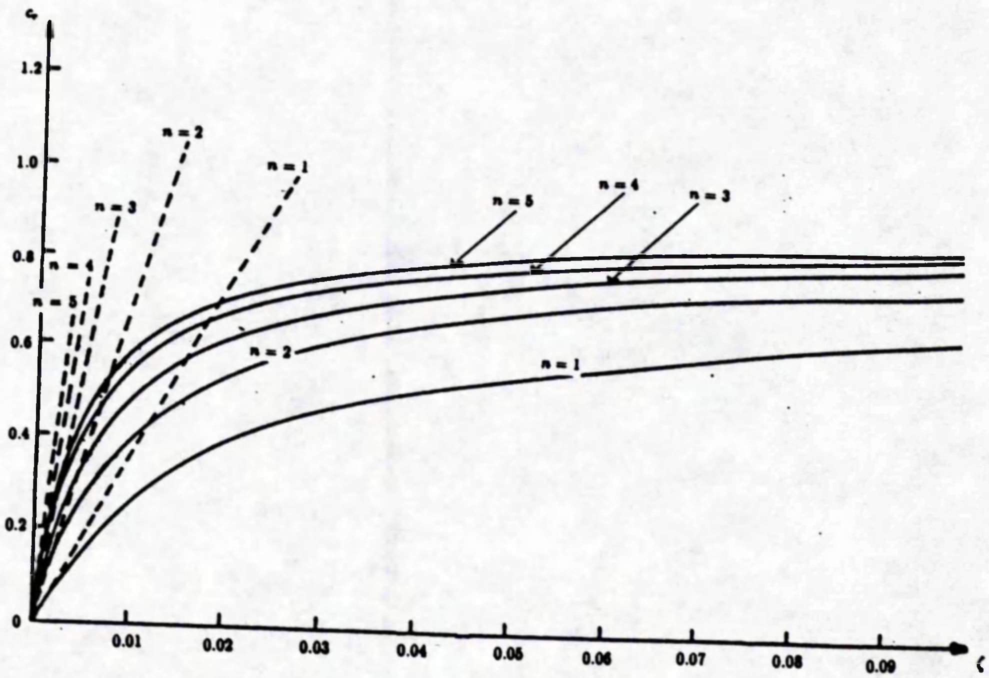


Figure 3.78: Comparison of computed $c_r(\alpha = 0)$ with asymptotic form for adiabatic axisymmetric body.

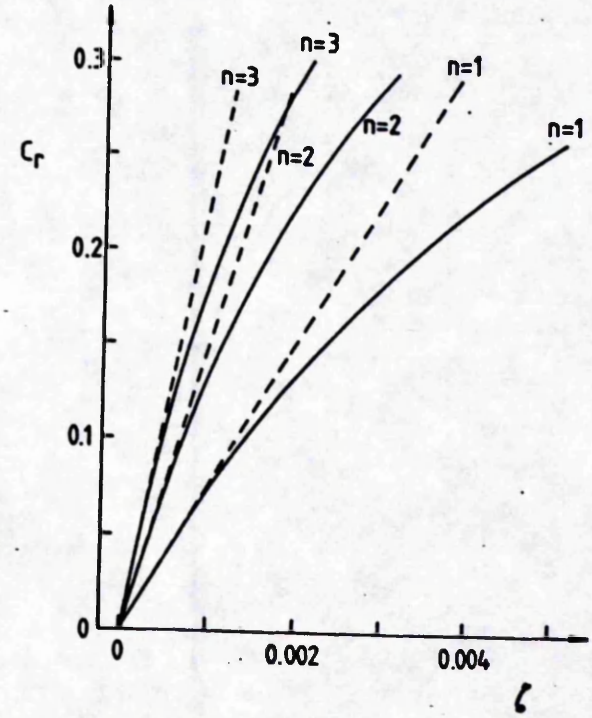


Figure 3.79: Comparison of computed $c_r(\alpha = 0)$ with asymptotic form, $M_\infty = 3.8$, $T_w = 5.0$.

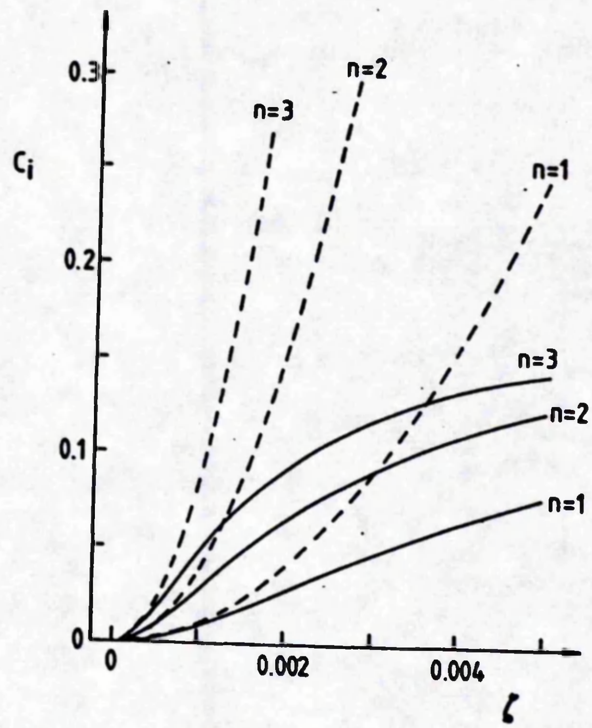


Figure 3.80: Comparison of computed $c_i(\alpha = 0)$ with asymptotic form, $M_\infty = 3.8$, $T_w = 5.0$.

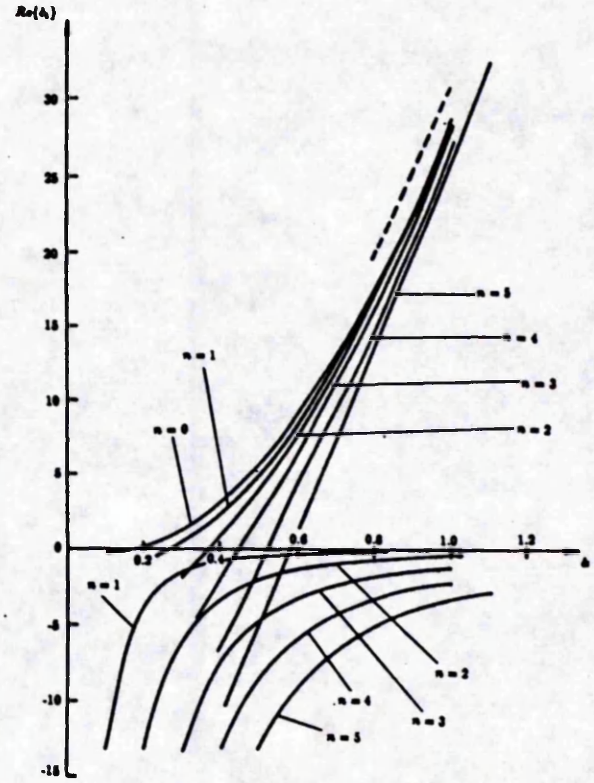


Figure 3.81: Variation of Re$\{\hat{c_1}\}$ with $\hat{\alpha}$.

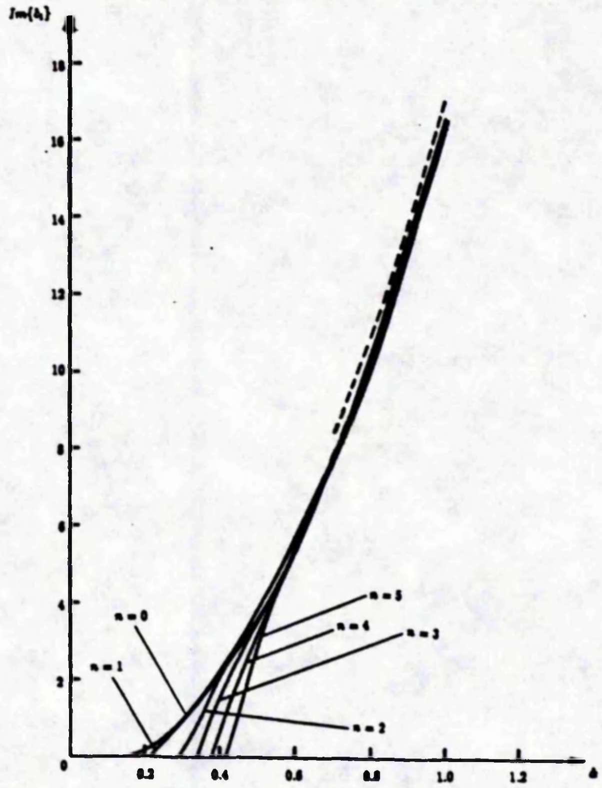


Figure 3.82: Variation of Im$\{\hat{c}_1\}$ with $\hat{\alpha}$ $M_\infty = 3.8$,$T_w = 5.0$.

Chapter 4

Critical Layer Theory

In this chapter we consider the effects of the critical layer on the temporal evolution of subsonic instability modes in the compressible boundary layer formed on a cylinder. Only two-dimensional disturbances are considered, i.e. axisymmetric disturbances only. As remarked upon in Chapter 1, two theories have been developed to determine inviscid, neutral stability characteristics of quasi-parallel flows. Both resolve difficulties which arise when the linearized inviscid problem is considered. The first is due essentially to Heisenberg (1924), Tollmien (1929) and Lin (1944, 1945) and is termed linear viscous theory. Lin (1955) summarizies the overall theory, the essence of which is the retention of viscous terms in the neighbourhood of the critical point. In the second method, the effects of finite amplitude are assumed to dominate over viscous terms: this theory is known as nonlinear theory. Benney and Bergeron (1969) and Davis (1969) independently suggested that the inviscid singularity could be resolved by including nonlinear terms in the critical layer.

We begin this chapter by applying the linear viscous theory to the stability problem we have developed, to determine whether curvature terms have any significant influence on the theory. It should be noted that the analysis developed in the following section is based on the work of Heisenberg (1924), Tollmien (1929) and Stuart (in Rosenhead's book (1963)).

4.1 Linear Critical Layer Equation Derivation

In Chapter 3 we have shown that for supersonic flow past axisymmetric bodies, the disturbance equations reduce to a compressible Rayleigh type equation which possesses a singularity at the critical point, i.e. a singularity exists at the point r_i, where $w_0(r_i) = c$. Since the disturbance equations are of the Rayleigh form, then in the neighbourhood of the critical point, applying the method of Frobenius, solutions of the Tollmien form can be obtained. In Appendix C the pressure disturbance term is found to have a solution of the form

$$\begin{aligned}\tilde{p}_A &= (r-r_i)^3 + a_1(r-r_i)^4 + \cdots,\\ \tilde{p}_B &= 1 + b_2(r-r_i)^2 + b_4(r-r_i)^4 + \cdots + \frac{\alpha^2}{3}\beta\tilde{p}_A ln(r-r_i),\end{aligned} \tag{4.1}$$

where

$$\beta = -\frac{w_{0rr}(r_i)}{w_{0r}(r_i)} + \frac{T_{0r}(r_i)}{T_0(r_i)} + \frac{1}{r_i}, \tag{4.2}$$

and the coefficients a_j and b_j are defined in Appendix C.

Clearly the first solution is regular, whilst the second is generally not, due to the presence of the logarithmic term. If an axisymmetric generalized inflexional mode exists somewhere in the boundary layer, i.e. if condition (3.37) holds (with $\lambda = n = 0$), then the term $ln(r-r_i)$ will be absent and the second solution becomes regular. However, generally the generalized inflexion condition will not be satisfied, giving rise to a a dilema - which branch of the logarithm should be taken on either side of $r = r_i$ if the eigenvalue problem associated with the compressible axisymmetric Rayleigh-type equation is to be solved. To allow for all possibilities, we now write

$$ln(r-r_i) = ln|r-r_i| + i\theta \qquad r < r_i, \tag{4.3}$$

where θ is to be determined.

A second consequence of the logarithmic term is that it gives rise to strong gradients in the perturbation pressure (and consequently the other disturbance terms) in the neighbourhood of the critical point.

Before proceeding to determine the form of the solution in the critical layer, we shall make use of (4.1) to determine the form of the normal velocity and temperature disturbance terms outside the critical layer. To achieve this we consider the behaviour of the disturbance equations, as derived in Chapter 3, as the critical layer is approached.

From equation (3.12) $\tilde{p}$ and $\tilde{v}_1$ are related by

$$\tilde{v}_1 = \frac{iT_0}{\gamma M_\infty^2 \alpha^2 (w_0 - c)} \tilde{p}_r, \tag{4.4}$$

where α refers to the unscaled form of the spatial wavenumber ($\alpha \equiv \overline{\alpha}$ of Chapter 3).

Therefore in the limit $r \to r_i$, $\tilde{v}_1$ will have the form

$$\begin{aligned} \tilde{v}_{1A} &= d_1(r - r_i) + d_2(r - r_i)^2 + d_3(r - r_i)^3 + \cdots, \\ \tilde{v}_{1B} &= e_0 + e_1(r - r_i) + e_2(r - r_i)^2 + \cdots + \frac{\alpha^2}{3}\beta \tilde{v}_{1A} ln(r - r_i), \end{aligned} \tag{4.5}$$

where

$$d_1 = \frac{3iT_0(r_i)}{\gamma M_\infty^2 \alpha^2 w_{0r}(r_i)}, \tag{4.6}$$

$$d_2 = \frac{i}{\alpha^2 \gamma M_\infty^2} \Big[\frac{4a_1 T_0(r_i)}{w_{0r}(r_i)} + 3\Big(\frac{T_{0r}(r_i)}{w_{0r}(r_i)} - \frac{T_0(r_i) w_{0rr}(r_i)}{2w_{0r}^2(r_i)}\Big)\Big], \tag{4.7}$$

$$d_3 = \frac{i}{\alpha^2\gamma M_\infty^2}\Big[\frac{5a_2T_0(r_i)}{w_{0r}(r_i)} + 4a_1\Big(\frac{T_{0r}(r_i)}{w_{0r}(r_i)} - \frac{T_0(r_i)w_{0rr}(r_i)}{2w_{0r}^2(r_i)}\Big) + 3\Big(\frac{w_{0rr}^2(r_i)T_0(r_i)}{4w_{0r}^3(r_i)}$$

$$-\frac{w_{0rrr}(r_i)T_0(r_i)}{6w_{0r}^2(r_i)} - \frac{T_{0r}(r_i)w_{0rr}(r_i)}{2w_{0r}^2(r_i)} + \frac{T_{0rr}(r_i)}{2w_{0r}(r_i)}\Big)\Big], \quad (4.8)$$

and

$$e_0 = \frac{2b_2iT_0(r_i)}{\gamma M_\infty^2\alpha^2 w_{0r}(r_i)}, \quad (4.9)$$

$$e_1 = \frac{i}{\alpha^2\gamma M_\infty^2}\Big[\frac{\alpha^2\beta T_0(r_i)}{3w_{0r}(r_i)} + 2b_2\Big(\frac{T_{0r}(r_i)}{w_{0r}(r_i)} - \frac{T_0(r_i)w_{0rr}(r_i)}{2w_{0r}^2(r_i)}\Big)\Big], \quad (4.10)$$

$$e_2 = \frac{i}{\alpha^2\gamma M_\infty^2}\Big[\frac{T_0(r_i)}{w_{0r}(r_i)}\Big(4b_4 + \frac{\alpha^2\beta}{3}a_1\Big) + \frac{\alpha^2\beta}{3}\Big(\frac{T_{0r}(r_i)}{w_{0r}(r_i)} - \frac{T_0(r_i)w_{0rr}(r_i)}{2w_{0r}^2(r_i)}\Big)$$

$$+2b_2\Big(\frac{w_{0rr}^2(r_i)T_0(r_i)}{4w_{0r}^3(r_i)} - \frac{w_{0rrr}(r_i)T_0(r_i)}{6w_{0r}^2(r_i)} - \frac{T_{0r}(r_i)w_{0rr}(r_i)}{2w_{0r}^2(r_i)} + \frac{T_{0rr}(r_i)}{2w_{0r}(r_i)}\Big)\Big]. \quad (4.11)$$

Considering the energy equation (3.8) in the limit $r \to r_i$, yields an outer solution for $\tilde{T}$ of the form

$$\tilde{T}_A = f_0 + f_1(r - r_i) + f_2(r - r_i)^2 + \cdots,$$

$$\tilde{T}_B = \frac{g_0}{r - r_i} + g_1 + g_2(r - r_i) + \cdots + \frac{\alpha^2}{3}\beta\tilde{T}_A ln(r - r_i), \quad (4.12)$$

where

$$f_0 = -\frac{3T_{0r}(r_i)T_0(r_i)}{\alpha^2\gamma M_\infty^2 w_{0r}^2(r_i)}, \quad (4.13)$$

$$f_1 = -\frac{1}{\alpha^2\gamma M_\infty^2}\Big[\frac{4a_1T_{0r}(r_i)T_0(r_i)}{w_{0r}^2(r_i)} + 3\Big(\frac{T_{0r}^2(r_i) + T_0(r_i)T_{0rr}(r_i)}{w_{0r}^2(r_i)}$$

$$-\frac{T_{0r}(r_i)T_0(r_i)w_{0rr}(r_i)}{w_{0r}^3(r_i)}\Big)\Big], \quad (4.14)$$

$$f_2 = -\frac{1}{\alpha^2\gamma M_\infty^2}\Big[\frac{5a_2T_{0r}(r_i)T_0(r_i)}{w_{0r}^2(r_i)} + 4a_1\Big(\frac{T_{0r}^2(r_i)+T_0(r_i)T_{0rr}(r_i)}{w_{0r}^2(r_i)}$$

$$-\frac{T_{0r}(r_i)T_0(r_i)w_{0rr}(r_i)}{w_{0r}^3(r_i)}\Big) + 3\Big(\frac{3T_{0r}(r_i)T_0(r_i)w_{0rr}^2(r_i)}{4w_{0r}^4(r_i)} - \frac{T_{0r}(r_i)T_0(r_i)w_{0rrr}(r_i)}{3w_{0r}^3(r_i)}$$

$$-\frac{w_{0rr}(r_i)(T_{0r}^2(r_i)+T_0(r_i)T_{0rr}(r_i))}{w_{0r}^3(r_i)} + \frac{3T_{0r}(r_i)T_{0rr}(r_i)+T_{0rrr}(r_i)T_0(r_i)}{2w_{0r}^2(r_i)}\Big)\Big], \tag{4.15}$$

and

$$g_0 = -\frac{2b_2T_{0r}(r_i)T_0(r_i)}{\alpha^2\gamma M_\infty^2 w_{0r}^2(r_i)}, \tag{4.16}$$

$$g_1 = -\frac{1}{\alpha^2\gamma M_\infty^2}\Big[\frac{\alpha^2\beta T_{0r}(r_i)T_0(r_i)}{3w_{0r}^2(r_i)} + 2b_2\Big(\frac{T_{0r}^2(r_i)+T_0(r_i)T_{0rr}(r_i)}{w_{0r}^2(r_i)}$$

$$-\frac{T_{0r}(r_i)T_0(r_i)w_{0rr}(r_i)}{w_{0r}^3(r_i)}\Big)\Big] + \frac{\gamma-1}{\gamma}T_0(r_i), \tag{4.17}$$

$$g_2 = -\frac{1}{\alpha^2\gamma M_\infty^2}\Big[\frac{T_{0r}(r_i)T_0(r_i)}{w_{0r}^2(r_i)}\Big(4b_4 + \frac{\alpha^2\beta}{3}a_1\Big) + \frac{\alpha^2\beta}{3}\Big(\frac{T_{0r}^2(r_i)+T_0(r_i)T_{0rr}(r_i)}{w_{0r}^2(r_i)}$$

$$-\frac{T_{0r}(r_i)T_0(r_i)w_{0rr}(r_i)}{w_{0r}^3(r_i)}\Big) + 2b_2\Big(\frac{3T_{0r}(r_i)T_0(r_i)w_{0rr}^2(r_i)}{4w_{0r}^4(r_i)} - \frac{T_{0r}(r_i)T_0(r_i)w_{0rrr}(r_i)}{3w_{0r}^3(r_i)}$$

$$-\frac{w_{0rr}(r_i)(T_{0r}^2(r_i)+T_0(r_i)T_{0rr}(r_i))}{w_{0r}^3(r_i)} + \frac{3T_{0r}(r_i)T_{0rr}(r_i)+T_{0rrr}(r_i)T_0(r_i)}{2w_{0r}^2(r_i)}\Big)\Big]$$

$$+\frac{\gamma-1}{\gamma}T_{0r}(r_i). \tag{4.18}$$

Note that the leading order term for $\tilde{T}_B$ possesses an algebraic singularity at its critical point.

Since viscous effects cannot be ignored in regions of rapid change as, for example, near the singularities of the inviscid equation, we wish to retain the leading order viscous terms in the neighbourhood of the critical point, to determine whether viscous effects will remove the unwanted singularities. For this purpose we now return to the full system of equations of continuity, momentum, energy and state, as defined by equations (1.1) - (1.5) and (1.20). Substituting disturbance terms (3.1) into this system, and collecting $O(\delta)$ terms, gives the following system

$$i(w_0 - c)\tilde{\rho} + \frac{\tilde{v}_{1r}}{T_0} - \frac{\tilde{v}_1 T_{0r}}{T_0^2} + i\frac{\tilde{v}_3}{T_0} + \frac{\tilde{v}_1}{rT_0} = 0, \tag{4.19}$$

$$\begin{aligned}\frac{i\alpha^2(w_0 - c)}{T_0}\tilde{v}_1 = &-\frac{1}{\gamma M_\infty^2}\tilde{p}_r + \frac{2\alpha}{Re}\Big\{\overline{\mu}_{0r}\tilde{v}_{1r} + \overline{\mu}_0\tilde{v}_{1rr} + \frac{\overline{\mu}_{0r}}{3}\Big[\tilde{v}_{1r} + \frac{\tilde{v}_1}{r} + i\tilde{v}_3\Big]\\
&+\frac{\overline{\mu}}{3}\Big[\tilde{v}_{1rr} + \frac{v_{1r}}{r} - \frac{\tilde{v}_1}{r^2} + i\tilde{v}_{3r}\Big]\Big\} + \frac{\alpha}{Re_2}\Big\{\overline{\mu}_{02r}\Big[\tilde{v}_{1r} + \frac{\tilde{v}_1}{r} + i\tilde{v}_3\Big] + \overline{\mu}_{02}\Big[\tilde{v}_{1rr} + \frac{\tilde{v}_{1r}}{r}\\
&-\frac{\tilde{v}_1}{r^2} + i\tilde{v}_{3r}\Big]\Big\} + \frac{i\alpha}{Re}\{w_{0r}\tilde{\mu} + \overline{\mu}_0(\tilde{v}_{3r} + i\alpha^2\tilde{v}_1)\} + \frac{2\alpha}{rRe}\Big\{\overline{\mu}_0\tilde{v}_{1r} + \frac{\overline{\mu}_0}{3}\Big[\tilde{v}_{1r}\\
&+\frac{\tilde{v}_1}{r} + i\tilde{v}_3\Big]\Big\} + \frac{\alpha}{rRe_2}\Big\{\overline{\mu}_{02}\Big[\tilde{v}_{1r} + \frac{v_1}{r} + i\tilde{v}_3\Big]\Big\},\end{aligned} \tag{4.20}$$

$$\begin{aligned}\frac{\alpha}{T_0}[i(w_0 - c)\tilde{v}_3 + w_{0r}\tilde{v}_1] = &-\frac{i\alpha}{\gamma M_\infty^2}\tilde{p} + \frac{1}{Re}\Big\{\overline{\mu}_{0r}[i\alpha^2\tilde{v}_1 + \tilde{v}_{3r}] + \overline{\mu}_0[i\alpha^2\tilde{v}_{1r} + \tilde{v}_{3rr}]\\
&+w_{0rr}\tilde{\mu} + w_{0r}\tilde{\mu}_r + 2i\alpha^2\Big(i\overline{\mu}_0\tilde{v}_3 + \frac{\overline{\mu}_0}{3}\Big[\tilde{v}_{1r} + \frac{\tilde{v}_1}{r} + i\tilde{v}_3\Big]\Big) + \frac{1}{r}(w_{0r}\tilde{\mu}\\
&+\overline{\mu}_0[\tilde{v}_{3r} + i\alpha^2\tilde{v}_1])\Big\} + \frac{i\alpha^2}{Re_2}\{\overline{\mu}_{02}[\tilde{v}_{1r} + \frac{v_1}{r} + i\tilde{v}_3]\},\end{aligned} \tag{4.21}$$

$$\begin{aligned}\frac{\alpha}{T_0}[i(w_0 - c)\tilde{T} + \tilde{v}_1 T_{0r}] - \alpha\Big(\frac{\gamma - 1}{\gamma}\Big)i(w_0 - c)\tilde{p} = &\frac{2(\gamma - 1)M_\infty^2}{Re}\{\overline{\mu}_0 w_{0r}(\tilde{v}_{3r}\\
+i\alpha^2\tilde{v}_1) + \frac{1}{2}w_{0r}^2\tilde{\mu}\} + \frac{1}{\sigma Re}\Big\{\frac{1}{r}(T_{0r}\tilde{\mu} + \overline{\mu}_0\tilde{T}_r) + T_{0rr}\tilde{\mu} &+ T_{0r}\tilde{\mu}_r + \overline{\mu}_{0r}\tilde{T}_r + \overline{\mu}_0\tilde{T}_{rr}\\
&-\alpha^2\overline{\mu}_0\tilde{T}\Big\},\end{aligned} \tag{4.22}$$

$$\tilde{p} = T_0\tilde{\rho} + \frac{\tilde{T}}{T_0}, \tag{4.23}$$

where it is assumed

$$\mu^* = \mu^*_\infty[\overline{\mu}_0 + \delta\tilde{\mu}(r)\exp(i\alpha(z-ct))] + O(\delta^2),$$
$$\mu^*_2 = \mu^*_{2\infty}[\overline{\mu}_{02} + \delta\tilde{\mu}_2(r)\exp(i\alpha(z-ct))] + O(\delta^2), \tag{4.24}$$

and

$$Re_2 = \frac{U^*_\infty a^* \rho^*_\infty}{\mu^*_{2\infty}}. \tag{4.25}$$

Near the critical layer we wish to retain viscous terms at leading order, therefore we now have to determine the necessary scales to ensure this. Consider the z-momentum equation (4.21). The viscous term which will be expected to undergo the most rapid change across the critical layer is

$$\frac{\overline{\mu}_0}{Re}\tilde{v}_{3rr}. \tag{4.26}$$

Balancing the term $\frac{\alpha}{T_0}[i(w_0 - c)\tilde{v}_3]$ with term (4.26), i.e. requiring them to be the same order, leads to the relation

$$(r - r_i) = O(Re^{-1/3}). \tag{4.27}$$

Therefore defining

$$(r - r_i) = \epsilon Y, \qquad \text{where} \qquad \epsilon = Re^{-1/3} \quad \text{and} \quad Y = O(1), \tag{4.28}$$

viscous terms are expected to be dominant in a critical layer of thickness ϵ .

Making use of the scaling (4.28) the z-momentum equation is expected to have the following leading order form

$$\frac{\alpha}{T_0(r_i)}[iY\tilde{v}_3 w_{0r}(r_i) + w_{0r}(r_i)\tilde{v}_1] = -\frac{i\alpha\tilde{p}}{\gamma M_\infty^2} + \overline{\mu}(r_i)\tilde{v}_{3YY}, \tag{4.29}$$

where $\tilde{v}_1$ and $\tilde{p}$ are assumed to be $O(\epsilon)$, $\tilde{v}_3 = O(1)$ and $\tilde{\mu} = O(1)$. However, taking the inner limit of the outer expansion for $\tilde{v}_1$, we determine $\tilde{v}_1 = O(1)$. Therefore to achieve a sensible balancing, $\tilde{p} = O(1)$ and $\tilde{v}_3 = O(\epsilon^{-1})$ (and $\tilde{\mu}$ will most likely be $O(\epsilon^{-1})$).

Turning our attention to the energy equation (4.22), we determine that for a sensible balancing to be achieved, it is required that $\tilde{T} = O(\epsilon^{-1})$, $\tilde{p} = O(\epsilon^{-1})$ and $\tilde{\mu} = O(\epsilon^{-1})$. However, taking the inner limit of the outer solution for $\tilde{p}$, it is determined that $\tilde{p} = O(1)$ and therefore it can be deduced that at leading order the pressure makes no contribution to the critical layer energy equation.

Examining the r-momentum equation (4.20), it is found that since the left-hand-side is $O(\epsilon)$, then $\tilde{p}_r = O(\epsilon)$, which in turn implies $\tilde{p}(Y) = O(\epsilon^2)$, i.e. pressure terms less than $O(\epsilon^2)$ will be independent of Y.

Guided by the form of the outer solution and the above work, we propose expansions for the inner (critical) layer of the form

$$\begin{aligned}
\tilde{v}_1 &= \tilde{U}_1(Y) + \epsilon ln\epsilon \tilde{U}_2(Y) + \epsilon \tilde{U}_3(Y) + \cdots, \\
\tilde{v}_3 &= \epsilon^{-1}\tilde{W}_0(Y) + ln\epsilon \tilde{W}_1(Y) + \tilde{W}_2(Y) + \cdots, \\
\tilde{p} &= \tilde{p}_1 + \epsilon\tilde{p}_2 + \epsilon^2\tilde{p}_3(Y) + \epsilon^3 ln\epsilon \tilde{p}_4(Y) + \cdots, \\
\tilde{T} &= \epsilon^{-1}\tilde{T}_0(Y) + ln\epsilon \tilde{T}_1(Y) + \tilde{T}_2(Y) + \cdots, \\
\tilde{\mu} &= \epsilon^{-1}\tilde{\mu}_0(Y) + ln\epsilon\tilde{\mu}_1(Y) + \tilde{\mu}_2 + \cdots,
\end{aligned} \tag{4.30}$$

where Y and ϵ are defined by (4.28) and it is noted that the pressure logarithmic term does not entry until $O(\epsilon^3 ln\epsilon)$, because of the form of the outer solution.

Substituting the expansions (4.30) into the equation of continuity (4.19) and

Taylor expanding mean flow terms about the critical point, to leading order gives

$$\tilde{U}_{1Y} + i\tilde{W}_0 = 0. \tag{4.31}$$

The next order equation ($O(ln\epsilon)$) has the form

$$\tilde{U}_{2Y} + i\tilde{W}_1 = 0. \tag{4.32}$$

The $O(1)$ equation is

$$-\frac{w_{0r}(r_i)}{T_0(r_i)} iY\tilde{T}_0 + \tilde{U}_{3Y} - \frac{T_{0r}(r_i)}{T_0(r_i)}\tilde{U}_1 + i\tilde{W}_2 + \frac{\tilde{U}_1}{r_i} = 0. \tag{4.33}$$

Substituting expansions (4.30) into the z-momentum equation and Taylor expanding mean flow terms yields the leading order equation

$$\frac{\alpha w_{0r}(r_i)}{T_0(r_i)}[iY\tilde{W}_0 + \tilde{U}_1] = -\frac{i\alpha}{\gamma M_\infty^2}\tilde{p}_1 + \overline{\mu}_0(r_i)\tilde{W}_{0YY}. \tag{4.34}$$

Using result (4.31), equation (4.34) can be re-written

$$\frac{\alpha w_{0r}(r_i)}{T_0(r_i)}[-Y\tilde{U}_{1Y} + \tilde{U}_1] = -\frac{i\alpha}{\gamma M_\infty^2}\tilde{p}_1 + i\tilde{U}_{1YYY}\overline{\mu}_0(r_i). \tag{4.35}$$

Differentiation with respect to Y yields

$$\tilde{U}_{1YYYY} - i\overline{\lambda}Y\tilde{U}_{1YY} = 0, \tag{4.36}$$

where

$$\overline{\lambda} = \frac{\alpha w_{0r}(r_i)}{T_0(r_i)\overline{\mu}_0(r_i)}. \tag{4.37}$$

Equation (4.36) is of course Airy's equation in the variable $\tilde{U}_{1YY}$, therefore

$$\tilde{U}_{1YY} = aAi[(i\overline{\lambda})^{1/3}Y] + bBi[(i\overline{\lambda})^{1/3}Y], \tag{4.38}$$

where a and b are constants. Since $\tilde{U}_1$ is required to be bounded in the far-field ($Y \to \infty$), then we must have $b = 0$. Therefore the leading order term for the radial velocity, in the critical layer, has the form

$$\tilde{U}_1 = \int_0^Y dY \int_0^Y aAi[(i\overline{\lambda})^{1/3}Y]dY + dY + e, \tag{4.39}$$

where a, c and d are constants which are determined by matching with the outer layer. In the limit $Y \to 0$ equation (4.5) can be written in the form

$$\begin{aligned} \tilde{v}_{1A} &= d_1\epsilon Y + d_2\epsilon^2 Y^2 + d_3\epsilon^3 Y^3 + \cdots, \\ \tilde{v}_{1B} &= e_0 + \frac{\alpha^2}{3}\beta d_1 Y\epsilon ln\epsilon + \epsilon\Big[e_1 Y + \frac{\alpha^2}{3}\beta d_1 Y lnY\Big] + \cdots. \end{aligned} \tag{4.40}$$

Making use of the asymptotic result

$$\int_0^Y \int_0^Y Ai[(i\overline{\lambda})^{1/3}Y]dYdY \sim \frac{Y}{3} \qquad \text{as} \qquad Y \to \infty, \tag{4.41}$$

and matching $O(1)$ terms of the inner and outer solutions yields

$$a = d = 0, \qquad e = e_0. \tag{4.42}$$

Therefore we have

$$\tilde{U}_1(Y) = e_0 \qquad \Rightarrow \qquad \tilde{W}_0(Y) = 0. \tag{4.43}$$

Substitution of expansions (4.30) into the energy disturbance equation yields the leading order equation

$$\tilde{T}_{0YY} - i\Omega w_{0r}(r_i) Y\tilde{T}_0 = \Omega e_0 T_{0r}(r_i), \tag{4.44}$$

where

$$\Omega = \frac{\alpha\sigma}{T_0(r_i)\overline{\mu}_0(r_i)}. \tag{4.45}$$

Equation (4.44) has a solution of the form

$$\tilde{T}_0(Y) = -\pi e^{-i\pi/3} e_0 T_{0r}(r_i)\left[\frac{\Omega}{w_{0r}^2(r_i)}\right]^{1/3} Gi[(i\Omega w_{0r}(r_i))^{1/3}Y], \tag{4.46}$$

where the function Gi is defined by Abramowitz and Stegun (1965, p.448).

Since

$$Gi(x) \sim \pi^{-1}x^{-1} \qquad \text{as} \qquad x \to \infty, \tag{4.47}$$

then in the limit $Y \to \infty$, equation (4.46) becomes

$$\tilde{T}_0(Y) \sim \frac{ie_0 T_{0r}(r_i)}{w_{0r}(r_i)Y} = -\frac{2b_2 T_{0r}(r_i)T_0(r_i)}{\alpha^2\gamma M_\infty^2 w_{0r}^2(r_i)Y}. \tag{4.48}$$

Comparing (4.48) with the outer solution taken in the limit $Y \to 0$, to leading order we have perfect agreement. Consequently, it is noted that the critical layer solution confirms that there exists an algebraic discontinuity in the temperature as the critical layer is crossed.

Returning to the $\tilde{v}_1$ inner expansion we now determine the next two terms in the series. The $O(\epsilon ln\epsilon)$ z-momentum equation has the form

$$\frac{\alpha w_{0r}(r_i)}{T_0(r_i)}[\tilde{U}_2 + iY\tilde{W}_1] = \overline{\mu}_0(r_i)\tilde{W}_{1YY}. \tag{4.49}$$

Making use of result (4.32) and differentiating with respect to Y yields

$$\tilde{U}_{2YYYY} - i\overline{\lambda}Y\tilde{U}_{2YY} = 0, \tag{4.50}$$

where $\overline{\lambda}$ is defined by (4.37). A solution to (4.50) has the form

$$\tilde{U}_2 = \int_0^Y dY \int_0^Y f Ai[(i\overline{\lambda})^{1/3} Y] dY + gY + h, \tag{4.51}$$

where again boundness in the far-field has been assumed and the constants f, g and h are determined by matching with the outer solution. Matching the inner limit of the outer solution with the critical layer solution as $Y \to \infty$, at order $O(\epsilon ln \epsilon)$ gives

$$\frac{\alpha^2}{3}\beta d_1 Y = (\frac{f}{3} + g)Y + h. \tag{4.52}$$

Therefore

$$h = 0 \qquad \text{and} \qquad \frac{f}{3} + g = \frac{\alpha^2}{3}\beta d_1, \tag{4.53}$$

which in turn implies

$$\tilde{U}_2 = \int_0^Y dY \int_0^Y f Ai[(i\overline{\lambda})^{1/3} Y] dY + \Big(\frac{\alpha^2}{3}\beta d_1 - \frac{f}{3}\Big)Y. \tag{4.54}$$

At this point it should be noted that in the inviscid region, well away from the critical layer, the velocity perturbation term has a solution of the form

$$\tilde{v}_1(r) = \tilde{v}_{10}(r) + \frac{1}{Re}\tilde{v}_{1Re}(r) + \cdots. \tag{4.55}$$

Therefore it can be clearly seen that in this region there will be no terms of order ϵ, $\epsilon ln \epsilon$, etc. As the critical point is approached it is the leading order term, namely $\tilde{v}_{10}(r)$, that is produced by the method of Frobenius, as this term satisfies the Rayleigh-type equation. The double integral of the Airy function, in the far-field, will overlap into the inviscid region where it will be required to be zero, since no $O(\epsilon ln \epsilon)$ terms exist in this region. This can only be achieved if $f = 0$, thus simplifing equation (4.54) to

$$\tilde{U}_2 = \frac{\alpha^2}{3}\beta d_1 Y. \tag{4.56}$$

We now consider the next order term in the $\tilde{v}_1$ expansion, namely $\tilde{U}_3(Y)$. The $O(\epsilon)$ z-momentum equation has the form

$$\frac{w_{0r}(r_i)\alpha}{T_0(r_i)}\Big[iY\tilde{W}_2+\tilde{U}_3+Y\Big(\frac{w_{0rr}(r_i)}{w_{0r}(r_i)}-\frac{T_{0r}(r_i)}{T_0(r_i)}\Big)e_0\Big]$$
$$=-\frac{i\alpha}{\gamma M_\infty^2}\tilde{p}_2+\overline{\mu}_0(r_i)\tilde{W}_{2YY}+w_{0r}(r_i)\tilde{\mu}_{0Y}, \tag{4.57}$$

where results (4.43) have been used to simplify matters.

In the limit $Y\to\infty$ using results (4.43) and (4.48), the O(1) continuity equation (4.33) has the form

$$i\tilde{W}_2=-\frac{e_0}{r_i}-\tilde{U}_{3Y}. \tag{4.58}$$

Therefore, in the limit $Y\to\infty$ (4.57) has the form

$$\frac{w_{0r}(r_i)\alpha}{T_0(r_i)}\Big\{-Y\tilde{U}_{3Y}+\tilde{U}_3+Y\Big[\frac{w_{0rr}(r_i)}{w_{0r}(r_i)}-\frac{T_{0r}(r_i)}{T_0(r_i)}-\frac{1}{r_i}\Big]e_0\Big\}$$
$$=-\frac{i\alpha}{\gamma M_\infty^2}\tilde{p}_2+i\overline{\mu}_0(r_i)\tilde{U}_{3YYY}+w_{0r}(r_i)\tilde{\mu}_{0Y}. \tag{4.59}$$

Differentiation of (4.59) with respect to Y yields

$$\frac{w_{0r}(r_i)\alpha}{T_0(r_i)}[-Y\tilde{U}_{3YY}-\beta e_0]=i\overline{\mu}_0(r_i)\tilde{U}_{3YYYY}+w_{0r}(r_i)\tilde{\mu}_{0YY}, \tag{4.60}$$

where β is defined by (4.2)

It is now necessary to determine the form of $\tilde{\mu}_{0YY}$ in the limit $Y\to\infty$. Write

$$\mu(T)=\overline{\mu}_0(T_0+\tilde{T}(r))=\overline{\mu}_0(T_0)+\tilde{T}\frac{\partial\overline{\mu}_0}{\partial T_0}+\cdots. \tag{4.61}$$

Comparing (4.61) with the non-dimensional $\mu(=\mu^*/\mu_\infty^*)$ equation in (4.24) yields

$$\tilde{\mu} = \tilde{T}\frac{\partial \overline{\mu}_0}{\partial T_0}. \tag{4.62}$$

Differentiating twice with respect to r gives

$$\tilde{\mu}_{rr} = \tilde{T}_{rr}\frac{\partial \overline{\mu}_0}{\partial T_0} + 2\tilde{T}_r\frac{\partial^2 \overline{\mu}_0}{\partial T_0^2}\frac{\partial T_0}{\partial r} + \tilde{T}\frac{\partial^3 \overline{\mu}_0}{\partial T_0^3}(\frac{\partial T_0}{\partial r})^2 + \tilde{T}\frac{\partial^2 \overline{\mu}_0}{\partial T_0^2}\frac{\partial^2 T_0}{\partial r^2}. \tag{4.63}$$

Substituting expansions (4.30) into (4.63) and transforming to the Y variable gives the leading order

$$\tilde{\mu}_{0YY} = \overline{\mu}_{0T_0}(r_i)\tilde{T}_{0YY}. \tag{4.64}$$

In the limit $Y \to \infty$, $Gi'' = O(\frac{1}{Y^3})$ (where Gi is defined by equation (4.47)), resulting in $\tilde{\mu}_{0YY} \to 0$ much faster than the other terms . Therefore equation (4.60) can be re-written in the simplified form

$$\tilde{U}_{3YYYY} - i\overline{\lambda}Y\tilde{U}_{3YY} = i\beta e_0\overline{\lambda}, \tag{4.65}$$

(where $\overline{\lambda}$ is defined by equation (4.37)) and has the solution (in the limit $Y \to \infty$)

$$\tilde{U}_{3YY} = -\pi e^{i\pi/6}\overline{\lambda}^{1/3}\beta e_0 Gi[(i\overline{\lambda})^{1/3}Y]. \tag{4.66}$$

Consequently, making use of the asymptotic form of Gi, i.e. relation (4.47), in the same limit, we can write

$$\tilde{U}_{3YY} \sim -\frac{\beta e_0}{Y}, \tag{4.67}$$

which in turn implies

$$\tilde{U}_{3Y} \sim -\beta e_0 lnY$$

$$\text{and} \qquad \tilde{U}_3 \sim -\beta e_0 Y lnY + Ys_1, \tag{4.68}$$

where s_1 is a constant.

Matching $O(\epsilon)$ terms of (4.40) with the $\tilde{U}_3$ equation in (4.68) reveals that we have perfect matching between the inner and outer $YlnY$ terms, since $-\beta e_0 = \frac{\alpha^2}{3}\beta d_1$ (see equations (4.6) and (4.9)). The constant s_1 will be equivalent to some linear combination of the ϵY terms occuring in the Frobenius solutions $\tilde{v}_{1A}$ and $\tilde{v}_{1B}$.

Therefore we conclude that when curvature terms are important in the linear stability problem, the retention of viscous terms in the neighbourhood of the critical point, is still an adequate method by which the singularity in the Rayleigh-type equation can be smoothed out. Indeed the determined results are found to closely resemble the compressible work of Lees and Lin (1946) and Lees and Reshotko (1962), although in the former case solutions are determined in terms of Hankel functions. In our work the effects of curvature on the results is found to be restricted to the constant β, which if the critical point coincided with an axisymmetric generalized inflexional mode is zero, anyway.

All that is left is to determine the form of $YlnY$ across the critical layer. For $Y > 0$ the solution is valid, we now seek the form of the solution that is valid for large negative Y. In the region of the singularity the term $YlnY$ is re-expressed as $N(Y)$. Following Lees and Lin (1946), any contour of integration present in the solutions, must be indented below the critical point since the solutions are only valid in certain regions of the complex plane. Consequently $N(Y)$ behaves like $YlnY$ for large positive Y and like $YlnY - \pi i$ for large negative Y, provide $\beta \neq 0$, i.e., provided the critical point is not a axisymmetric generalized inflexional point. Therefore it follows that $ln(r - r_i)$ tends to $ln|r - r_i| - \pi i$ for $r < r_i$, i.e., $\theta = -\pi$ in equation (4.3). For the eigenvalue problem it is sufficient to use the asymptotic forms of $ln(z - z_c)$ and $ln|z - z_c| - \pi i$, since on the critical scale, Y will be very large at the boundaries, and the asymptotic form of $N(Y)$ is valid. For $r < r_c$, the pressure perturbation term $\tilde{p}_B$ has the form

$$\tilde{p}_B = 1 + b_2(r-r_i)^2 + b_4(r-r_i)^4 + \cdots + \frac{\alpha^2}{3}\beta\tilde{p}_A[ln|r-r_i| - \pi i]. \qquad (4.69)$$

The other irregular disturbance perturbation terms occuring in (4.5) and (4.12) can be expressed in a similar manner.

We now consider the case where nonlinear effects are used to smooth out the singularity in our axisymmetric compressible inviscid equations.

4.2 Nonlinear Critical Layers

In the previous section we considered removing the singularity which occurs in the axisymmetric, compressible Rayleigh equation by restoring viscosity in the neighbourhood of the critical point. In this section, we consider instead, retaining nonlinear terms in the critical layer and what role, if any, curvature plays in the nonlinear problem. The method used is based on the method developed by Goldstein and his many co-authors (Goldstein *et al* (1987), Goldstein and Hultgren (1988), Goldstein and Leib (1988, 1989), Goldstein and Choi (1989), Goldstein and Wundrow (1990) and Leib (1991)), although all this work is based on Hickernell's (1984) work.

In this problem the temporal evolution of a growing, small amplitude instability wave (which is harmonic in space) is treated. To ease the analysis that will be carried out inside the critical layer, it is found convenient to work in terms of the streamwise coordinate $\overline{\xi}$, where

$$\overline{\xi} = z - c_0 t, \qquad (4.70)$$

represents a coordinate in the z direction moving downstream with the neutral phase velocity c_0. The streamwise velocity component, $\overline{v}_3$, as measured relative to this new coordinate, is related to the streamwise velocity as measured in the stationary frame of reference, by

$$\overline{v}_3 = v_3 - c_0. \tag{4.71}$$

Consequently, even though in the stationary frame of reference at a fixed point mean flow terms do not vary with time, in the moving frame, since we are moving downstream with respect to the stationary frame, at a fixed point mean flow terms appear to vary temporally, if sufficient time has evolved. It should be stressed that this variance of the mean flow terms actually occurs over long viscous lengthscales, but since the measurement is made with respect to the moving frame of reference is has the appearance of time variance. Therefore, in the following analysis when we talk of the boundary layer varying over given timescales, it should always be remembered that the changes are occuring within a moving frame of reference, and in reality the variance is occuring over lengthscales, i.e., as time evolves we move over these lengthscales.

For earlier times the wave amplitude will be small and is well described by the linear, inviscid temporal theory developed in Chapter 3. However, as time increases and the instability amplitude contains to grow, this will no longer be the case. The boundary layer is assumed to thicken over the long (when compared with the timescales over which the instability wave varys) viscous timescale and this mean flow spreading will act to reduce the local growth rate while the instability wave amplitude continues to grow temporally. After a long enough time interval, the amplitude is found to be sufficiently large and the growth rate sufficiently small, that nonlinear effects are of the same order as the instability growth rates. When this occurs the growth rate of the instability wave, which is otherwise governed by linear dynamics, is determined by the nonlinear effects developed within the critical layer.

We shall begin by determining the form of the solution outside the critical layer for the particular time when nonlinear critical layer effects are important.

4.2.1 The Outer Layer

Since the viscous timescale is assumed long (with respect to local timescales) then locally the mean flow will be nearly parallel and we can assume the inviscid limit. The full system of equations of continuity, momentum, energy and state in the axisymmetric cylindrical polar system, in non-dimensional form and in the inviscid limit are

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial r}[\rho v_1] + \frac{\partial}{\partial \overline{\xi}}[\rho \overline{v}_3] + \frac{\rho v_1}{r} = 0, \tag{4.72}$$

$$\rho \frac{D v_1}{Dt} = -\frac{1}{\gamma M_\infty^2} \frac{\partial P}{\partial r}, \tag{4.73}$$

$$\rho \frac{D v_3}{Dt} = -\frac{1}{\gamma M_\infty^2} \frac{\partial P}{\partial \overline{\xi}}, \tag{4.74}$$

$$\rho \frac{DT}{Dt} = \frac{\gamma - 1}{\gamma} \frac{DP}{Dt}, \tag{4.75}$$

$$P = \rho T, \tag{4.76}$$

where we are in the moving frame of reference as defined by (4.70), and in this frame the Eulerian operator has the form

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + v_1 \frac{\partial}{\partial r} + \overline{v}_3 \frac{\partial}{\partial \overline{\xi}}. \tag{4.77}$$

Since temporally growing waves are being considered and for the particular time being treated the growth rates will be assumed small, then nonlinear critical layer effects will be expected to cause changes to the flow over the slow timescale

$$t_1 = \delta^\mu t \qquad \Rightarrow \qquad \frac{d}{dt} = \delta^\mu \frac{d}{dt_1}, \tag{4.78}$$

where δ represents the disturbance amplitude and it is assumed that $\delta \ll 1$. μ is a number to be determined.

For early times before nonlinear terms have had any significant effect, the gradual thickening of the boundary layer due to the action of viscosity causes the temporal growth rate of the linear instability wave to gradually decrease, finally approaching its neutral stability condition (i.e. point of zero growth). We denote the linear, neutral spatial and temporal wavenumbers by α_0 and c_0, respectively. Nonlinear effects will first become important in the critical layer at the time when the local spatial wavenumber, α (where as in the previous section, $\alpha \equiv \bar{\tilde{\alpha}}$ of Chapter 3), differs from its neutral value by an amount of order δ^μ, so that

$$\alpha = \alpha_0 + \delta^\mu \alpha_1, \tag{4.79}$$

where $\alpha_1 = O(1)$.

Outside the critical layer the instability wave is assumed to continue behaving linearly (to leading order). Consequently the flow parameters are expected to expand in the following manner

$$\begin{aligned} v_1 &= \delta\alpha Rl[A^\dagger(t_1)\tilde{v}_1(r)e^{i\alpha_0\zeta}] + \cdots, \\ \overline{v}_3 &= W_0(r) + \delta Rl[A^\dagger(t_1)\tilde{v}_3(r)e^{i\alpha_0\zeta}] + \cdots, \\ T &= T_0(r) + \delta Rl[A^\dagger(t_1)\tilde{T}(r)e^{i\alpha_0\zeta}] + \cdots, \\ \rho &= 1/T_0 + \delta Rl[A^\dagger(t_1)\tilde{\rho}(r)e^{i\alpha_0\zeta}] + \cdots, \\ P &= 1 + \delta\gamma M_\infty^2 Rl[A^\dagger(t_1)\tilde{P}(r)e^{i\alpha_0\zeta}] + \cdots, \end{aligned} \tag{4.80}$$

where the mean flow velocity, as measured in the stationary frame. is given by $W_0(r) + c_0$ $(= w_0(r)$ of Chapter 3),

$$\zeta = \left(1 + \frac{\alpha_1}{\alpha_0}\delta^{\mu}\right)\overline{\xi}, \tag{4.81}$$

and the slowly varying amplitude function, $A^{\dagger}(t_1)$, is to be determined by matching with the nonlinear flow in the critical layer. This amplitude function, which is governed by the nonlinear dynamics of the critical layer is also required to match onto the initial linear solution. This matching process will be carried out once the evolution equation governing $A^{\dagger}(t_1)$ has been determined. Note, that the temporal growth rates are contained solely within the amplitude function, $A^{\dagger}(t_1)$, the phase velocity contribution to the exponential terms, as measured in the stationary frame of reference is the neutral value c_0 only. The pressure disturbance term has been normalized by the factor γM_{∞}^2, to ease the analysis and also aid the determination of numerical results when we come to compute the amplitude evolution equation.

Substituting expansions (4.79) and (4.80) into the system of equations (4.72) - (4.76) and making use of result (4.78) to the required level of approximation we obtain the following system of equations

$$i(W_0 - c)\tilde{\rho} + \frac{\tilde{v}_{1r}}{T_0} - \frac{T_{0r}}{T_0^2}\tilde{v}_1 + \frac{i\tilde{v}_3}{T_0} + \frac{\tilde{v}_1}{rT_0} = 0, \tag{4.82}$$

$$\frac{i\alpha^2}{T_0}(W_0 - c)\tilde{v}_1 = -\tilde{P}_r, \tag{4.83}$$

$$\frac{1}{T_0}[i(W_0 - c)\tilde{v}_3 + W_{0r}\tilde{v}_1] = -i\tilde{P}, \tag{4.84}$$

$$\frac{1}{T_0}[i(W_0 - c)\tilde{T} + T_{0r}\tilde{v}_1] = i(\gamma - 1)M_{\infty}^2(W_0 - c)\tilde{P}, \tag{4.85}$$

$$\gamma M_\infty^2 \tilde{P} = T_0 \tilde{\rho} + \frac{\tilde{T}}{T_0}, \tag{4.86}$$

where

$$c = -\frac{\delta^\mu}{i\alpha A^\dagger}\frac{dA^\dagger}{dt_1}. \tag{4.87}$$

Examining (4.87), it is noted that c represents the deviation of the phase speed from its neutral value (remembering that we are in a frame of reference moving with the neutral phase velocity), therefore the disturbance terms must have expansions of the form

$$\begin{aligned}
\tilde{v}_1 &= \tilde{v}_{10} + \delta^\mu \tilde{v}_{11} + \cdots, \\
\tilde{v}_3 &= \tilde{v}_{30} + \delta^\mu \tilde{v}_{31} + \cdots, \\
\tilde{T} &= \tilde{T}_0 + \delta^\mu \tilde{T}_1 + \cdots, \\
\tilde{P} &= \tilde{p}_0 + \delta^\mu \tilde{p}_1 + \cdots, \\
\tilde{\rho} &= \tilde{\rho}_0 + \delta^\mu \tilde{\rho}_1 + \cdots.
\end{aligned} \tag{4.88}$$

Substituting these expansions into system (4.82) - (4.86) and making use of (4.79) yields the leading order system

$$iW_0\tilde{\rho}_0 + \frac{\tilde{v}_{10r}}{T_0} - \frac{T_{0r}}{T_0^2}\tilde{v}_{10} + \frac{i\tilde{v}_{30}}{T_0} + \frac{\tilde{v}_{10}}{rT_0} = 0, \tag{4.89}$$

$$\frac{i\alpha_0^2}{T_0}W_0\tilde{v}_{10} = -\tilde{p}_{0r}, \tag{4.90}$$

$$\frac{1}{T_0}[iW_0\tilde{v}_{30} + W_{0r}\tilde{v}_{10}] = -i\tilde{p}_0, \tag{4.91}$$

$$\frac{1}{T_0}[iW_0\tilde{T}_0 + T_{0r}\tilde{v}_{10}] = i(\gamma-1)M_\infty^2 W_0\tilde{p}_0, \qquad (4.92)$$

$$\gamma M_\infty^2 \tilde{p}_0 = T_0\tilde{\rho}_0 + \frac{\tilde{T}_0}{T_0}. \qquad (4.93)$$

The system (4.89) - (4.93), as one would expect, corresponds to the linear axisymmetric disturbance equations as determined by Duck (1990) and can be reduced to the form

$$\tilde{v}_{10r} + \frac{\tilde{v}_{10}}{r} - \frac{W_{0r}}{W_0}\tilde{v}_{10} = \frac{i\tilde{p}_0}{W_0}[T_0 - M_\infty^2 W_0^2], \qquad (4.94)$$

$$\frac{i\alpha_0^2}{T_0}W_0\tilde{v}_{10} = -\tilde{p}_{0r}. \qquad (4.95)$$

Combining equations (4.94) and (4.95) to eliminate velocity terms, yields

$$W_0\frac{d}{dr}\Big[\frac{T_0\tilde{p}_{0r}}{W_0}\Big] - \Big(W_{0r} - \frac{W_0}{r}\Big)\frac{T_0\tilde{p}_{0r}}{W_0} - \alpha_0^2\Phi\tilde{p}_0 = 0, \qquad (4.96)$$

where

$$\Phi = T_0 - M_\infty^2 W_0^2. \qquad (4.97)$$

Defining the compressible Rayleigh-type operator for axisymmetric flows, '$\mathcal{L}^R$', to have the form

$$\mathcal{L}^R \equiv W_0\frac{d}{dr}\Big[\frac{T_0}{W_0}\frac{d}{dr}\Big] - \Big[W_{0r} - \frac{W_0}{r}\Big]\frac{T_0}{W_0}\frac{d}{dr} - \alpha_0^2\Phi, \qquad (4.98)$$

equation (4.96) reduces to

$$\mathcal{L}^R\tilde{p}_0 = 0. \qquad (4.99)$$

In the notation of Goldstein *et al.*, equation (4.96) can be re-written

$$\frac{1}{T_0}\frac{d}{dr}(T_0\tilde{p}_{0r}) - \Big[2W_{0r} - \frac{W_0}{r}\Big]\frac{\tilde{p}_{0r}}{W_0} - \alpha_0^2\Big[1 - M_\infty^2\frac{W_0^2}{T_0}\Big]\tilde{p}_0 = 0, \tag{4.100}$$

resulting in the Goldstein-Rayleigh type operator '$\mathcal{L}^G$' having the form

$$\mathcal{L}^G \equiv \frac{1}{T_0}\frac{d}{dr}\Big[T_0\frac{d}{dr}\Big] - \Big[2W_{0r} - \frac{W_0}{r}\Big]\frac{1}{W_0}\frac{d}{dr} - \alpha_0^2\Big[1 - M_\infty^2\frac{W_0^2}{T_0}\Big], \tag{4.101}$$

and equation (4.96) now becomes

$$\mathcal{L}^G\tilde{p}_0 = 0. \tag{4.102}$$

Making use of equations (4.90) and (4.91) the streamwise velocity perturbation term $\tilde{v}_{30}$ has the form

$$\alpha_0^2\tilde{v}_{30} = -\frac{T_0}{W_0}\Big[\frac{W_{0r}}{W_0}\tilde{p}_{0r} + \alpha_0^2\tilde{p}_0\Big]. \tag{4.103}$$

The next order system of equations obtained from substituting expansions (4.88) into equations (4.82) - (4.86) (this corresponds to the $O(\delta^{\mu+1})$ system for the outer solution) has the form

$$\tilde{\Omega}\tilde{\rho}_0 + i\tilde{\rho}_1 W_0 + \frac{\tilde{v}_{11r}}{T_0} - \frac{T_{0r}}{T_0^2}\tilde{v}_{11} + \frac{i\tilde{v}_{31}}{T_0} + \frac{\tilde{v}_{11}}{rT_0} = 0, \tag{4.104}$$

$$\frac{1}{T_0}[\alpha_0^2\tilde{\Omega}\tilde{v}_{10} + 2i\alpha_0\alpha_1 W_0\tilde{v}_{10} + i\alpha_0^2 W_0\tilde{v}_{11}] = -\tilde{p}_{1r}, \tag{4.105}$$

$$\frac{1}{T_0}[\tilde{\Omega}\tilde{v}_{30} + iW_0\tilde{v}_{31} + W_{0r}\tilde{v}_{11}] = -i\tilde{p}_1, \tag{4.106}$$

$$\frac{1}{T_0}[\tilde{\Omega}\tilde{T}_0 + iW_0\tilde{T}_1 + T_{0r}\tilde{v}_{11}] = (\gamma-1)M_\infty^2[\tilde{\Omega}\tilde{p}_0 + iW_0\tilde{p}_1], \tag{4.107}$$

$$\gamma M_\infty^2 \tilde{p}_1 = T_0 \tilde{\rho}_1 + \frac{\tilde{T}_1}{T_0}, \tag{4.108}$$

where

$$\tilde{\Omega} = \frac{1}{\alpha_0 A^\dagger} \frac{dA^\dagger}{dt_1}. \tag{4.109}$$

After a little algebra the system (4.104) - (4.108) can be reduced to

$$\tilde{v}_{11r} + \Big[-\frac{W_{0r}}{W_0} + \frac{1}{r}\Big]\tilde{v}_{11} - \frac{i\tilde{\Omega}W_{0r}}{W_0^2}\tilde{v}_{10} = \frac{i}{W_0}\big[T_0 - M_\infty^2 W_0^2\big]\tilde{p}_1 - \frac{\tilde{\Omega}}{W_0^2}\big[T_0 + M_\infty^2 W_0^2\big]\tilde{p}_0, \tag{4.110}$$

and

$$\frac{1}{T_0}[\alpha_0^2 \tilde{\Omega}\tilde{v}_{10} + 2i\alpha_0\alpha_1 W_0 \tilde{v}_{10} + i\tilde{v}_{11}\alpha_0^2 W_0] = -\tilde{p}_{1r}. \tag{4.111}$$

Equations (4.90), (4.110) and (4.111) can be combined to eliminate velocity terms, yielding an equation in terms of pressure disturbances terms only

$$\begin{aligned} i\alpha_0 W_0 \mathcal{L}^G \tilde{p}_1 = \alpha_0 \tilde{\Omega}\Big\{\frac{1}{T_0}\frac{d}{dr}(T_0\tilde{p}_{0r}) + \Big[-4W_{0r} + \frac{W_0}{r}\Big]\frac{\tilde{p}_{0r}}{W_0} - \alpha_0^2\Big[1 + \frac{M_\infty^2 W_0^2}{T_0}\Big]\tilde{p}_0\Big\} \\ +2i\alpha_1 W_0\Big\{\frac{1}{T_0}\frac{d}{dr}(T_0\tilde{p}_{0r}) + \Big(-2W_{0r} + \frac{W_0}{r}\Big)\frac{\tilde{p}_{0r}}{W_0}\Big\}. \end{aligned} \tag{4.112}$$

Making use of equations (4.91), (4.105) and (4.106) the streamwise velocity $\tilde{v}_{31}$ perturbation equation is found to have the form

$$\alpha_0^2 \tilde{v}_{31} = -\frac{T_0}{W_0}\Big\{\Big[\frac{W_{0r}}{W_0}\tilde{p}_{1r} + \alpha_0^2 \tilde{p}_1\Big] + 2\alpha_1\alpha_0\Big[\tilde{p}_0 + \frac{W_0}{T_0}\tilde{v}_{30}\Big] - i\tilde{\Omega}\alpha_0^2\Big[\frac{2\tilde{v}_{30}}{T_0} + \frac{\tilde{p}_0}{W_0}\Big]\Big\}. \tag{4.113}$$

To determine the form of the solution in the boundary layer, away from the critical point, both the linear and nonlinear systems of equations represented by

(4.89) -(4.93) and (4.104) - (4.108) respectively, must be solved numerically. In this study, fortunately, we only require to determine the behaviour of the solutions in the neighbourhood of the critical layer. Another assumption that shall be made is that at the critical point the axisymmetric generalized inflexion condition holds. Duck (1990) has shown that from linear axisymmetric compressible stability theory, if the critical point is inflexional in nature, then the following condition is assumed to hold for subsonic neutral modes

$$\frac{d}{dr}\Big[\frac{W_{0r}}{rT_0}\Big]\Big|_{r=r_i} = 0 \quad \Leftrightarrow \quad \frac{w_c''}{w_c'} - \frac{T_c'}{T_c} - \frac{1}{r_i} = 0, \tag{4.114}$$

which corresponds to setting $\lambda = n = 0$ in equation (3.37). The subscript 'c' implies evaluation at the critical level. It should be noted that the non-dimensional mean flow velocity $W_0(r)$ is monotonic, thus insuring that there only exists just one subsonic generalized inflexional point coinciding with a critical point in the boundary layer.

The form of the solution to the homogeneous equation (4.102) in the asymptotic limit $r \rightarrow r_i$ has already been considered in the previous section for the more complicated case of non-inflexional profiles, although this was carried out in the stationary frame of reference. This equation will have the same solution in a frame of reference moving with the neutral phase velocity, however, since $W_0 = w_0 - c_0$, yielding the Taylor expansion

$$W_0(y) = w_c' y + \frac{1}{2} w_c'' y^2 + \cdots. \tag{4.115}$$

From Appendix C, it can clearly be seen that setting $\beta = 0$ (which corresponds to the axisymmetric generalized inflexional condition being satisfied) logarithmic contributions are removed from the leading order pressure disturbance term solution, yielding the two completely non-singular linearly independent homogeneous solutions

$$\tilde{p}_0^{(1)} = 1 - \frac{\alpha_0^2}{2} y^2 + a_4 y^4 + \cdots, \tag{4.116}$$

$$\tilde{p}_0^{(2)} = y^3 + \cdots, \tag{4.117}$$

as $y \to 0$ where

$$y = (r - r_i), \tag{4.118}$$

and

$$a_4 = \frac{\alpha_0^2}{4}\left[\frac{T_c''}{T_c} - \frac{2w_c'''}{3w_c'} - \frac{1}{2}\left(\frac{w_c''}{w_c'} - \frac{2}{r_i}\right)^2 - \frac{\alpha_0^2}{2} - M_\infty^2 \frac{(w_c')^2}{T_c}\right]. \tag{4.119}$$

We shall assume that the pressure solution $\tilde{p}_0$ will be a linear combination of (4.116) and (4.117), i.e.

$$\tilde{p}_0 = \tilde{p}_0^{(1)} + b_1 \tilde{p}_0^{(2)}. \tag{4.120}$$

The constant b_1 must be generally determined numerically. Since both $\tilde{p}_0^{(1)}$ and $\tilde{p}_0^{(2)}$ will be unbounded as $y \to \pm\infty$, then b_1 is chosen to ensure that the linear combination $\tilde{p}_0$ satisifies the far-field conditions, namely, that for subsonic disturbances $\tilde{p}_0$ is bounded. In the case of generalized inflexion points, when the critical layer solution is examined, b_1 is found to take the same value above and below the critical layer. For non-inflexional profiles this is not true, causing b_1 to undergo a finite jump passing from one side of the critical layer to the other.

Making use of result (4.120) and considering equation (4.95) in the limit $y \to 0$ the radial velocity $\tilde{v}_{10}$ is determined to have the form

$$\tilde{v}_{10} = -\frac{iT_c}{w_c'}\Big\{1 - y\Big[\frac{3b_1}{\alpha_0^2} - \Big(\frac{T_c'}{T_c} - \frac{w_c''}{2w_c'}\Big)\Big] - \frac{y^2}{\alpha_0^2}\Big[4a_4 + 3b_1\Big(\frac{T_c'}{T_c} - \frac{w_c''}{2w_c'}\Big) - \alpha_0^2\chi\frac{w_c'}{T_c}\Big]\Big\} + O(y^3), \tag{4.121}$$

where

$$\chi = \frac{T_c'''}{2w_c'} - \frac{w_c''T_c'}{2(w_c')^2} - \frac{T_c w_c'''}{6(w_c')^2} + \frac{T_c(w_c'')^2}{4(w_c')^3}. \tag{4.122}$$

To simplify equation (4.121) we re-define the pressure solution equation (4.120) to have the form

$$\begin{aligned}\tilde{p}_0 &= \frac{w_c'}{T_c}\Big[\tilde{p}_0^{(1)} + \frac{\alpha_0^2}{3}\Big(b_1 + \frac{T_c'}{T_c} - \frac{w_c''}{2w_c'}\Big)\tilde{p}_0^{(2)}\Big] \\ &= \frac{w_c'}{T_c}\Big[\tilde{p}_0^{(1)} + \frac{\alpha_0^2}{3}\Big(b_1 + \frac{w_c''}{2w_c'} - \frac{1}{r_i}\Big)\tilde{p}_0^{(2)}\Big],\end{aligned} \tag{4.123}$$

where the generalized inflexion condition (4.114) has been applied. Equation (4.121) can now be re-expressed in the simplified form

$$\begin{aligned}\tilde{v}_{10} &= -i\Big\{1 - b_1 y - y^2\Big(\frac{4a_4}{\alpha_0^2} + \Big(\frac{T_c'}{T_c} - \frac{w_c''}{2w_c'}\Big)\Big[b_1 + \Big(\frac{T_c'}{T_c} - \frac{w_c''}{2w_c'}\Big)\Big] - \chi\frac{w_c'}{T_c}\Big)\Big\} \\ &\qquad + O(y^3) \\ &= -i\Big\{1 - b_1 y - y^2\Big(\frac{4a_4}{\alpha_0^2} + \Big(\frac{w_c''}{2w_c'} - \frac{1}{r_i}\Big)\Big[b_1 + \Big(\frac{w_c''}{2w_c'} - \frac{1}{r_i}\Big)\Big] - \chi\frac{w_c'}{T_c}\Big)\Big\} \\ &\qquad + O(y^3).\end{aligned} \tag{4.124}$$

Substituting equations (4.123) and (4.124) into equation (4.91) and taking the limit $y \to 0$ yields the following result for the streamwise velocity $\tilde{v}_{30}$

$$\begin{aligned}\tilde{v}_{30} = \Big(-b_1 + \frac{1}{r_i}\Big) + y\Big[&-\frac{T_c''}{T_c} - \frac{1}{2}\Big(\frac{w_c''}{w_c'}\Big)^2 + \frac{1}{2}\Big(\frac{w_c''}{w_c'} - \frac{2}{r_i}\Big)^2 + \alpha_0^2 \\ &+ M_\infty^2\frac{(w_c')^2}{T_c} - \frac{T_c'}{T_c}b_1 + \frac{T_c'}{T_c r_i} + \frac{w_c'''}{w_c'}\Big] + O(y^2),\end{aligned} \tag{4.125}$$

where again we have made use of the generalized inflexional condition (4.114). Substituting equation (4.123) and (4.124) into equation (4.92) taken in the limit $y \rightarrow 0$, the temperature disturbance term $\tilde{T}_0$, in the neighbourhood of the critical layer, is found to have the form

$$\tilde{T}_0 = \frac{T_c'}{y w_c'} + \Big(\frac{T_c'}{w_c'}\Big[\frac{T_c''}{T_c'} - \frac{w_c''}{2w_c'} - b_1\Big] + M_\infty^2(\gamma-1)w_c'\Big) + O(y)$$

$$= \frac{T_c'}{y w_c'} + \Big(\frac{T_c'}{w_c'}\Big[\frac{w_c''}{2w_c'} - \frac{1}{r_i} - b_1\Big] + M_\infty^2(\gamma-1)w_c'\Big) + O(y). \qquad (4.126)$$

As noted in the previous section, as the critical layer is approached the leading order temperature disturbance term develops an algebraic singularity (as noted also by Reshotko (1960, 1962), Goldstein and Leib (1989)). Consequently in the critical layer the energy equation and in particular temperature terms are expected to be important in the determination of the revelant scaling in this problem, since these terms will be large relative to the velocity components.

Before considering the form of the solution in the critical layer, it is found necessary to determine the next order term in the pressure expansion, namely $\tilde{p}_1$, as defined by (4.112). Making use of the homogeneous equation (4.102), the inhomogeneous equation (4.112) can be re-written as

$$\mathcal{L}^G\tilde{p}_1 = 2i\tilde{\Omega}\Big[\frac{W_{0y}}{W_0^2}\tilde{p}_{0y} + \alpha_0^2\frac{M_\infty^2 W_0}{T_0}\tilde{p}_0\Big] + \frac{2\alpha_1}{\alpha_0}\Big[\alpha_0^2\Big(1 - \frac{M_\infty^2(w_0-c_0)^2}{T_0}\Big)\tilde{p}_0\Big]. \qquad (4.127)$$

Defining

$$\tilde{p}_1 = \frac{2i\tilde{\Omega}w_c'}{T_c}\tilde{p}_{1A} + \frac{2\alpha_1 w_c'}{\alpha_0 T_c}\tilde{p}_{1B}, \qquad (4.128)$$

(4.127) simplifies to

$$\mathcal{L}^G\tilde{p}_1 = 2i\tilde{\Omega}\mathcal{L}^G\tilde{p}_{1A} + \frac{2\alpha_1}{\alpha_0}\mathcal{L}^G\tilde{p}_{1B}, \tag{4.129}$$

where

$$\mathcal{L}^G\tilde{p}_{1A} = \frac{T_c}{w'_c}\Big[\frac{W_{0y}}{W_0^2}\tilde{p}_{0y} + \alpha_0^2\frac{M_\infty^2 W_0}{T_0}\tilde{p}_0\Big], \tag{4.130}$$

and

$$\mathcal{L}^G\tilde{p}_{1B} = \frac{T_c}{w'_c}\alpha_0^2\Big(1 - \frac{M_\infty^2 W_0^2}{T_0}\Big)\tilde{p}_0. \tag{4.131}$$

The general solutions to (4.130) and (4.131) have the form

$$\tilde{p}_{1A} = \tilde{p}_{1,1} + C_{2,1}^{\pm}\tilde{p}_0^{(1)} + \frac{\alpha_0^2}{3}\Big(b_{2,1}^{\pm} + \frac{w''_c}{2w'_c} - \frac{1}{r_i}\Big)\tilde{p}_0^{(2)}, \tag{4.132}$$

and

$$\tilde{p}_{1B} = \tilde{p}_{1,2} + C_{2,2}^{\pm}\tilde{p}_0^{(1)} + \frac{\alpha_0^2}{3}\Big(b_{2,2}^{\pm} + \frac{w''_c}{2w'_c} - \frac{1}{r_i}\Big)\tilde{p}_0^{(2)}, \tag{4.133}$$

where $b_{2,n}^{\pm}$ and $C_{2,n}^{\pm}$ are constants, and $\tilde{p}_{1,1}$ and $\tilde{p}_{1,2}$ are particular solutions of (4.130) and (4.131), respectively, whose form can be determined by the method of variation of parameters.

Therefore the solution satisfying (4.127) has the form

$$\begin{aligned}\tilde{p}_1 = &\frac{2i\tilde{\Omega}w'_c}{T_c}\Big[\tilde{p}_{1,1} + C_{2,1}^{\pm}\tilde{p}_0^{(1)} + \frac{\alpha_0^2}{3}\Big(b_{2,1}^{\pm} + \frac{w''_c}{2w'_c} - \frac{1}{r_i}\Big)\tilde{p}_0^{(2)}\Big]\\ &+\frac{2\alpha_1 w'_c}{\alpha_0 T_c}\Big[\tilde{p}_{1,2} + C_{2,2}^{\pm}\tilde{p}_0^{(1)} + \frac{\alpha_0^2}{3}\Big(b_{2,2}^{\pm} + \frac{w''_c}{2w'_c} - \frac{1}{r_i}\Big)\tilde{p}_0^{(2)}\Big].\end{aligned} \tag{4.134}$$

The particular solutions $\tilde{p}_{1,1}$ and $\tilde{p}_{1,2}$ are assumed to be continuous, but are unbounded generally as $y \to \pm\infty$. As $y \to 0$, $\tilde{p}_{1,1}$ is expected to behave as

$$\frac{d^2\tilde{p}_{1,1}}{dy^2} - \frac{2}{y}\frac{d\tilde{p}_{1,1}}{dy} = \frac{c_1}{y} + c_2 + c_3 y + c_4 y^2 + \cdots, \tag{4.135}$$

giving a solution of the form

$$\tilde{p}_{1,1} = \tilde{e}_1 + \tilde{e}_2 y + \tilde{e}_3 y^2 + \tilde{e}_4 y^3 lny + \tilde{e}_5 y^3 + \cdots, \tag{4.136}$$

while $\tilde{p}_{1,2}$ must be regular as $y \to 0$.

The constants $b^{\pm}_{2,n}$ and $C^{\pm}_{2,n}$ arising from the complementary solutions may take different values either side of the critical point and this is indicated by the '+' and '−' superscripts. From the form of the inner solution it is found that $C^{+}_{2,n} \equiv C^{-}_{2,n}$ because the critical layer cannot support an $O(\delta^{1+\mu})$ pressure discontinuity (see the streamwise momentum critical layer equation in the subsection concerning the form of the solution the critical layer). Therefore we define

$$C_{2,n} = C^{+}_{2,n} = C^{-}_{2,n}, \qquad n = 1 \text{ or } 2. \tag{4.137}$$

As with the two functions that add to give the homogeneous solution, the particular solution and the two complementary functions that make up the inhomogeneous solutions (4.132) and (4.133), will generally be unbounded as $y \to \pm\infty$. Consequently, the constants $b^{\pm}_{2,n}$ and $C_{2,n}$ are determined to ensure that the far field conditions are satisfied (i.e. the modes decay exponentially since they are subsonic). This involves a complete numerical solution of the $O(\delta^{1+\mu})$ problem for the pressure. An alternative method, known as the modified solvability condition, exists to determine the constants in (4.132) and (4.133) which requires only the determination of the local behaviour of the pressure term $\tilde{p}_1$, near the critical point, and has been used by Benney and Maslowe (1975), Redekopp (1977), Heurre (1980), Hickernell (1984), Churilov and Shukhman (1987), and Leib (1991), among others. In this problem we make use of the modified solivability condition to determine the jump constants.

If the axisymmetric compressible Goldstein-Rayleigh type operator, $\mathcal{L}^G$, were non-singular at the critical point then the necessary and sufficient condition for the solvability of the non-homogeneous equations (4.130) and (4.131), would be the standard orthogonality condition

$$\int_{-\infty}^{\infty} \text{RHS}(z)\tilde{p}_0^{(1)}(z)dz = 0, \tag{4.138}$$

where 'RHS' refers to the respective right-hand-sides of (4.130) and (4.131). However, since $\mathcal{L}^G$ is singular at the critical point, this integral diverges, and a modified solvability condition must be determined. This is achieved by employing the far-field conditions. As $y \to \pm\infty$, pressure disturbance terms are required to be bounded, since only subsonic modes are being considered. Therefore, the $O(\delta^{1+\mu})$ pressure terms must satisfy the condition

$$\tilde{p}_1 \to 0 \qquad \text{as} \qquad y \to \pm\infty. \tag{4.139}$$

Since $\tilde{p}_1$ is a linear combination of $\tilde{p}_{1A}$ and $\tilde{p}_{1B}$ with respect to y and the factors multiplying these terms (which are y independent) cannot be specified by employing suitable boundary conditions on y, then we also have

$$\tilde{p}_{1A}, \tilde{p}_{1B} \to 0 \qquad \text{as} \qquad y \to \pm\infty. \tag{4.140}$$

By the method of variation of parameters, equation (4.132) can be re-written in the form

$$\tilde{p}_{1A} = \int^{y} dz\Big\{[\tilde{p}_0^{(1)}(y)\tilde{p}_0^{(2)}(z) - \tilde{p}_0^{(1)}(z)\tilde{p}_0^{(2)}(y)]\frac{T_c}{w_c'}\Big[\frac{W_{0z}}{W_0^2}\tilde{p}_{0z} + \frac{\alpha_0^2 M_\infty^2 W_0}{T_0}\tilde{p}_0\Big]\Big\}$$

$$+C_{2,1}\tilde{p}_0^{(1)} + \frac{\alpha_0^2}{3}\Big(b_{2,1}^{\pm} + \frac{w_c''}{2w_c'} - \frac{1}{r_i}\Big)\tilde{p}_0^{(2)}. \quad (4.141)$$

Applying the limit $y \to \pm\infty$, yields

$$\lim_{y\to\pm\infty}\int^{y}\frac{T_c}{w_c'}\tilde{p}_0^{(2)}(z)\Big[\frac{W_{0z}}{W_0^2}\tilde{p}_{0z} + \frac{\alpha_0^2 M_\infty^2 W_0}{T_0}\tilde{p}_0\Big]dz \to -C_{2,1}, \quad (4.142)$$

and

$$\lim_{y\to\pm\infty}\int^{y}\frac{T_c}{w_c'}\tilde{p}_0^{(1)}(z)\Big[\frac{W_{0z}}{W_0^2}\tilde{p}_{0z} + \frac{\alpha_0^2 M_\infty^2 W_0}{T_0}\tilde{p}_0\Big]dz \to \frac{\alpha_0^2}{3}\Big(b_{2,1}^{\pm} + \frac{w_c''}{2w_c'} - \frac{1}{r_i}\Big). \quad (4.143)$$

In the limit $y \to +\infty$ we choose the lower limit of integration to be just above the critical point, i.e. the point $y_c + \epsilon$ where $\epsilon << 1$, and for $y \to -\infty$ choose a point just below the critical point, i.e. $y_c - \epsilon$. Subtracting the $y \to +\infty$ equation from the $y \to -\infty$ equation in (4.142) gives

$$\lim_{y\to\infty}⨍_{-y}^{+y}\frac{T_c}{w_c'}\tilde{p}_0^{(2)}(z)\Big[\frac{W_{0z}}{W_0^2}\tilde{p}_{0z} + \frac{\alpha_0^2 M_\infty^2 W_0}{T_0}\tilde{p}_0\Big]dz = 0, \quad (4.144)$$

while in (4.143) yields

$$b_{2,1}^{+} - b_{2,1}^{-} = -\frac{3}{\alpha_0^2}\lim_{y\to\infty}⨍_{-y}^{+y}\tilde{p}_0^{(1)}(z)\frac{T_c}{w_c'}\Big[\frac{W_{0z}}{W_0^2}\tilde{p}_{0z} + \frac{\alpha_0^2 M_\infty^2 W_0}{T_0}\tilde{p}_0\Big]dz, \quad (4.145)$$

where $⨍$ denotes the Cauchy principle value integral.

Substituting (4.123) into (4.145) and making use of result (4.144) gives that the jump in the constant $b_{2,1}$ across the critical layer in terms of the leading order pressure disturbance has the form

$$b_{2,1}^{+} - b_{2,1}^{-} = \frac{3}{\alpha_0^2} \lim_{y\to\infty} \int_{-y}^{+y} \left(\frac{T_c}{w_c'}\right)^2 \tilde{P}_0(z)\left[\frac{W_{0z}}{W_0^2}\tilde{P}_{0z} + \frac{\alpha_0^2 M_\infty^2 W_0}{T_0}\tilde{p}_0\right] dz. \qquad (4.146)$$

Applying a similar agrument to the $\tilde{p}_{1B}$ terms, i.e. equation (4.133) yields the jump condition

$$b_{2,2}^{+} - b_{2,2}^{-} = 3 \lim_{y\to\infty} \int_{-y}^{+y} \left(\frac{T_c}{w_c'}\right)^2 \left[1 - \frac{M_\infty^2 W_0^2}{T_0}\right] \tilde{p}_0^2 dz. \qquad (4.147)$$

We now wish to determine the form of the streamwise velocity term $\tilde{v}_{31}$ which is defined by equation (4.113), as the critical layer is approached, by making use of results (4.123), (4.125) and (4.134). Firstly it is found necessary to determine the coefficients of the solution for $\tilde{p}_{1,1}$ in equation (4.136). Since $\tilde{p}_{1,1}$ is a particular solution of (4.130), then substituting expansion (4.136) into (4.130) taken in the limit $y \to 0$ and equating powers of y gives

$$\begin{aligned}
O(\tfrac{1}{y}): &\quad \tilde{e}_2 = \tfrac{\alpha_0^2}{2w_c'}, \\
O(1): &\quad 2\tilde{e}_3 + \alpha_0^2\tilde{e}_1 = -\tfrac{\alpha_0^2}{w_c'}\left[b_1 + \tfrac{w_c''}{2w_c'} - \tfrac{1}{r_i}\right], \\
O(y): &\quad 3\tilde{e}_4 + (\kappa - \alpha_0^2)\tilde{e}_2 = \tfrac{4a_4}{w_c'} - \alpha_0^2\left[\tfrac{w_c'''}{6(w_c')^2} - \tfrac{(w_c'')^2}{4(w_c')^3}\right] + \alpha_0^2 M_\infty^2 \tfrac{w_c'}{T_c}.
\end{aligned} \qquad (4.148)$$

where κ is defined by equation (C.5) in Appendix C.

It is also found necessary to obtain the first few terms in the $\tilde{p}_{1,2}$ solution, which by the form of equation (4.131) is assumed to be regular in the limit $y \to 0$ and of the general form

$$\tilde{p}_{1,2} = \tilde{f}_1 + \tilde{f}_2 y + \tilde{f}_3 y^2 + \tilde{f}_4 y^3 + \cdots. \qquad (4.149)$$

Again, substituting the particular expansion (4.149) into equation (4.131) in the limit $y \to 0$ and equating corresponding powers in y yields

$$O(\frac{1}{y}): \qquad \tilde{f}_2 = 0,$$

$$O(1): \qquad 2\tilde{f}_3 + \alpha_0^2\tilde{f}_1 = -\alpha_0^2,$$

$$O(y): \qquad \tilde{f}_2 = 0,$$

$$O(y^2): \; 4\tilde{f}_5 + (2\kappa - \alpha_0^2)\tilde{f}_3 + \tfrac{\alpha_0^2 M_\infty^2 (w_c')^2}{T_c}\tilde{f}_1 = -\alpha_0^2\Big[\tfrac{\alpha_0^2}{2} + \tfrac{M_\infty^2 (w_c')^2}{T_c}\Big]. \qquad (4.150)$$

Making use of results (4.116), (4.117), (4.136), (4.148) - (4.150) and solutions (4.123), (4.124), (4.134), in the limit $y \to 0$, the streamwise velocity is determined to have the form

$$\tilde{v}_{31} = -\Big[e_1 + e_2 ln|y| + 2i\tilde{\Omega} b_{2,1}^{\pm} + \frac{2\alpha_1}{\alpha_0} b_{2,2}^{\pm}\Big] + O(y). \qquad (4.151)$$

which, by the form of equation (4.109), can be re-expressed as

$$A^\dagger \tilde{v}_{31} = -\Big[\bar{e}_1(t_1) + \bar{e}_2(t_1) ln|y| + \frac{2i}{\alpha_0}\frac{dA^\dagger}{dt_1} b_{2,1}^{\pm} + \frac{2\alpha_1}{\alpha_0} A^\dagger b_{2,2}^{\pm}\Big] + O(y). \qquad (4.152)$$

It should be noted that the coefficients of the $O(\frac{1}{y^2})$ and $O(\frac{1}{y})$ terms are both zero, resulting in the leading term being $O(lny)$. It should also be noted that the undetermined function $\bar{e}_1(t_1)$ is the same order as the $2i\tilde{\Omega} b_{2,1}^{\pm}$ and $\frac{2\alpha_1}{\alpha_0} b_{2,2}^{\pm}$ terms.

The temperature perturbation term at this order, $\tilde{T}_1$, has the form

$$\tilde{T}_1 = \frac{1}{y^2}\Big[\frac{i\tilde{\Omega} T_c'}{(w_c')^2}\Big] + \frac{1}{y}\Big[\frac{i\tilde{\Omega} T_c'}{(w_c')^2}\Big(\frac{T_c''}{T_c'} - \frac{w_c''}{w_c'} - \frac{1}{r_i}\Big) + \frac{2i\tilde{\Omega} T_c'}{w_c'}\big(\tilde{e}_1 + C_{2,1}^{\pm}\big)$$

$$+ \frac{2\alpha_1 T_c'}{\alpha_0 w_c'}\big(\tilde{f}_1 + C_{2,2}^{\pm}\big)\Big] + O(y). \qquad (4.153)$$

We note that the leading order term is $O(\frac{1}{y^2})$. Since the $\tilde{T}_0$ temperature term was $O(\frac{1}{y})$, we see no reason to discount that the n^{th} perturbation term, $\tilde{T}_n$, would have the leading order term $O(\frac{1}{y^n})$. This presents no problem to the solution as we move away from the critical layer, since as $y \to \pm\infty$ it can be clearly seen that successively higher terms will tend to zero faster. However, as the critical layer is approached this means that the higher terms will approach the singular condition faster.

We now turn our attention to the form of the solution in the critical layer.

4.2.2 The Critical Layer

The form of the temperature expansion clearly shows that the complete outer expansion becomes singular at the critical layer. This has also been determined to be true by Reshotko (1960, 1962) for compressible boundary layers and Goldstein and Leib (1989) for compressible shear layers. Consequently, in the neighbourhood of the critical point the equations will have to be re-scaled to obtain the so-called critical-layer solution. Since the thickness of the linear, small-growth rate critical layer is of the same order as the growth rate of the disturbance wave, i.e. $O(\delta^\mu)$ in the present case, for the reasons given at the beginning of this section, then the appropriate scaled radial coordinate in this region is

$$Y = y/\delta^\mu = (r - r_i)/\delta^\mu, \qquad \text{where} \qquad y << 1. \tag{4.154}$$

To obtain the form of the critical layer expansions, the inner limit of the outer solution, as determined in the previous subsection, is re-expanded in terms of the critical layer radial coordinate Y, to give

$$v_1 = -\delta\alpha_0 Rl[iA^\dagger e^{i\alpha_0\zeta}] + \delta^{3\mu}(\text{bounded Y-independent terms}) + \cdots, \tag{4.155}$$

$$\overline{v}_3 = Yw_c'\delta^\mu + \frac{1}{2}Y^2 w_c''\delta^{2\mu} + \big(-b_1 + \frac{1}{r_i}\big)\delta Rl[A^\dagger e^{i\alpha_0\zeta}]$$

$$+\delta^{3\mu}\Big(\frac{Y^3}{6}w_c''' + \text{bounded Y-independent terms}\Big) + \delta^{1+\mu}ln\mu\Big(-Rl[\overline{e}_2 e^{i\alpha_0\zeta}]\Big)$$

$$+\delta^{1+\mu}Rl\Big\{\Big[YA^\dagger\Big(-\frac{T_c''}{T_c} - \frac{1}{2}\Big(\frac{w_c''}{w_c'}\Big)^2 + \frac{1}{2}\Big(\frac{w_c''}{w_c'} - \frac{2}{r_i}\Big)^2 + \alpha_0^2 + M_\infty^2\frac{(w_c')^2}{T_c} - \frac{T_c'}{T_c}b_1$$

$$+\frac{T_c'}{T_c}\Big(\frac{1}{r_i}\Big) + \frac{w_c'''}{w_c'}\Big) - \Big(\overline{e}_1(t_1) + \overline{e}_2(t_1)ln|Y| + \frac{2i}{\alpha_0}\frac{dA^\dagger}{dt_1}b_{2,1}^{\pm} + \frac{2\alpha_1}{\alpha_0}A^\dagger b_{2,2}^{\pm}\Big)\Big]e^{i\alpha_0\zeta}\Big\}$$

$$+O(\delta^{4\mu}), \quad (4.156)$$

$$P = 1 + \delta\gamma M_\infty^2\frac{w_c'}{T_c}Rl[A^\dagger e^{i\alpha_0\zeta}] + \delta^{3\mu}(\text{Y-independent terms}) + \cdots, \quad (4.157)$$

$$T = T_c + T_c'\delta^\mu Y + \delta^{1-\mu}Rl\Big\{\frac{T_c'}{w_c'}\Big[\frac{1}{Y} + \frac{i\tilde{\Omega}}{Y^2 w_c'} + \cdots\Big]A^\dagger e^{i\alpha_0\zeta}\Big\} + \frac{1}{2}T_c''\delta^{2\mu}Y^2 + \cdots, \quad (4.158)$$

where it is known from the previous section that

$$\zeta = \lambda(z - c_0 t), \quad (4.159)$$

$$t_1 = \delta^\mu t, \quad (4.160)$$

and

$$\lambda = 1 + \delta^\mu\frac{\alpha_1}{\alpha_0}. \quad (4.161)$$

Examining (4.158) it is noted that there exists the possibility of an infinite arithmetic series in the parameter Y^{-n}, at $O(\delta^{1-\mu})$, where n is a positive integer. Therefore as the critical layer is approached this term will tend to the singular condition very rapidly, as noted at the end of the previous subsection.

From the form of expansions (4.155) - (4.158) the critical layer solution is expected to expand in the following manner

$$v_1 = -\delta\alpha_0 Rl[iA^{\dagger}e^{i\alpha_0\zeta}] + \delta^{3\mu}\tilde{\psi}_1(\zeta, t_1) + \delta^{1+\mu}\tilde{\psi}_2 + \cdots, \tag{4.162}$$

$$\overline{v}_3 = Yw_c'\delta^{\mu} + \frac{Y^2}{2}w_c''\delta^{2\mu} + \delta\tilde{\phi}_1 + \delta^{3\mu}\tilde{\phi}_2 + \delta^{1+\mu}\tilde{\phi}_3 + \cdots, \tag{4.163}$$

$$P = 1 + \delta\tilde{p}_{11}(\zeta, t_1) + \delta^{3\mu}\tilde{p}_{12}(\zeta, t_1) + \delta^{1+\mu}\tilde{p}_{13} + \cdots, \tag{4.164}$$

$$T = T_c + T_c'\delta^{\mu}Y + \delta^{1-\mu}\tilde{T}_{11} + \delta^{2\mu}\tilde{T}_{12} + \delta\tilde{T}_{13} + \cdots, \tag{4.165}$$

where $\tilde{\psi}_n$, $\tilde{\phi}_n$, etc., are functions of ζ, t_1, and Y at most, and it is assumed that $\delta^{1-\mu} > \delta^{2\mu}$. The $ln\mu$ term occuring in (4.156) has been incorporated in the term $\tilde{\phi}_3$.

Each term in the expansions (4.162) -(4.165) is determined by solving the inviscid equations of momentum, energy, and continuity, where we begin by absorbing the equation of state into the other four equations. Since the disturbance terms are now dependent on ζ, t_1 and Y, it is found necessary to transform the Eulerian operator, '$\frac{D}{Dt}$', by applying the chain rule, i.e. since

$$P = P(\zeta, t_1, Y) \quad \Rightarrow \quad \frac{\partial P}{\partial x} = \frac{\partial P}{\partial \zeta}\frac{\partial \zeta}{\partial x} + \frac{\partial P}{\partial t_1}\frac{\partial t_1}{\partial x} + \frac{\partial P}{\partial Y}\frac{\partial Y}{\partial x}, \tag{4.166}$$

and so on, then the Eulerian operator in terms of ζ, t_1 and Y, where it should be remembered that we are in a frame of reference moving with velocity c_0, has the form

$$\overline{D} \equiv \overline{v}_3\Big(1 + \delta^{\mu}\frac{\alpha_1}{\alpha_0}\Big)\frac{\partial}{\partial \zeta} + \delta^{\mu}\frac{\partial}{\partial t_1} + v_1\delta^{-\mu}\frac{\partial}{\partial Y}. \tag{4.167}$$

Combining the energy equation (4.75) with the equation of continuity (4.72), both re-expressed in terms of ζ, t_1, and Y, yields

$$\frac{1}{\gamma P}\overline{D}P = \frac{1}{(\gamma-1)T}\overline{D}T = -\Big[\delta^{-\mu}\frac{\partial v_1}{\partial Y} + \lambda\frac{\partial \overline{v}_3}{\partial \zeta} + \frac{v_1}{r}\Big]. \tag{4.168}$$

The momenta equations (4.73), (4.74) in terms of ζ, t_1 and Y have the form

$$\frac{1}{T}\overline{D}v_1 = -\frac{1}{\gamma M_\infty^2}\frac{1}{\delta^\mu P}\frac{\partial P}{\partial Y}, \tag{4.169}$$

and

$$\frac{1}{T}\overline{D}\overline{v}_3 = -\frac{1}{\gamma M_\infty^2}\Big(\frac{1+\delta^\mu\frac{\alpha_1}{\alpha_0}}{P}\Big)\frac{\partial P}{\partial \zeta}. \tag{4.170}$$

It is found easier in this problem to work in terms of the vorticity vector which has the form

$$\text{curl}\ \underline{v} = \Omega\hat{\underline{\theta}}, \tag{4.171}$$

where

$$\Omega = \lambda\frac{\partial v_1}{\partial \zeta} - \delta^{-\mu}\frac{\partial \overline{v}_3}{\partial Y}, \tag{4.172}$$

and λ is defined by equation (4.161), $\hat{\underline{\theta}}$ represents the azimuthal coordinate. Therefore taking '$\lambda\frac{\partial}{\partial\zeta}$' of equation (4.169) and '$-\delta^{-\mu}\frac{\partial}{\partial Y}$' of equation (4.170) and adding yields

$$\overline{D}\Omega + \Omega\Big[\lambda\frac{\partial \overline{v}_3}{\partial \zeta} + \delta^{-\mu}\frac{\partial v_1}{\partial Y}\Big] = \frac{1}{\gamma M_\infty^2}\frac{\lambda}{P\delta^\mu}\Big[\frac{\partial T}{\partial Y}\frac{\partial P}{\partial \zeta} - \frac{\partial T}{\partial \zeta}\frac{\partial P}{\partial Y}\Big]. \tag{4.173}$$

Making use of (4.168) equation (4.173) can be simplified to

$$\overline{D}\Omega - \Omega\Big[\frac{1}{\gamma P}\overline{D}P + \frac{v_1}{r}\Big] = \frac{1}{\gamma M_\infty^2}\frac{\lambda}{P\delta^\mu}\Big[\frac{\partial T}{\partial Y}\frac{\partial P}{\partial \zeta} - \frac{\partial T}{\partial \zeta}\frac{\partial P}{\partial Y}\Big]. \tag{4.174}$$

Equation (4.174) can be regarded as the critical layer vorticity equation.

Substituting expansions (4.162) - (4.165) into (4.168) and equating equal powers yields (from the comparison between the pressure equation and right-hand-side)

$$O(\delta): \qquad \tilde{\psi}_{2Y} + \tilde{\phi}_{1\zeta} + \frac{\tilde{\psi}_0}{r_i} = 0, \tag{4.175}$$

$$O(\delta^{3\mu}): \qquad \tilde{\psi}_{3Y} + \tilde{\phi}_{2\zeta} + \frac{\tilde{\psi}_1}{r_i} = 0, \tag{4.176}$$

$$O(\delta^{1+\mu}): \qquad \tilde{\psi}_{4Y} + \tilde{\phi}_{3\zeta} + \frac{\alpha_1}{\alpha_0}\tilde{\phi}_{1\zeta} + \frac{1}{r_i}\big[\tilde{\psi}_2 - \frac{Y}{r_i}\tilde{\psi}_0\big] = -\frac{1}{\gamma}\tilde{\mathcal{L}}\tilde{p}_{11}, \tag{4.177}$$

where the operator $\tilde{\mathcal{L}}$ is defined to have the form

$$\tilde{\mathcal{L}} = Yw_c'\frac{\partial}{\partial\zeta} + \frac{\partial}{\partial t_1}. \tag{4.178}$$

For the temperature terms on the left-hand-side of equation (4.168) the leading order equation (i.e. $O(\delta)$) has the form

$$\frac{1}{(\gamma-1)T_c}[\tilde{\mathcal{L}}\tilde{T}_{11} + T_c'\tilde{\psi}_0] = -\big[\tilde{\psi}_{2Y} + \tilde{\phi}_{1\zeta} + \frac{\tilde{\psi}_0}{r_i}\big]. \tag{4.179}$$

However, by the form of (4.175) equation (4.179) simplifies to

$$\tilde{\mathcal{L}}\tilde{T}_{11} = T_c'\alpha_0 Rl[iA^\dagger e^{i\alpha_0\zeta}], \tag{4.180}$$

where we have defined

$$\tilde{\psi}_0 = -\alpha_0 Rl[iA^\dagger e^{i\alpha_0\zeta}]. \tag{4.181}$$

The next order temperature/continuity equation has the form

$$\frac{1}{(\gamma-1)T_c}\{\delta^{3\mu}[\tilde{\mathcal{L}}\tilde{T}_{12}+T_c'\tilde{\psi}_1]\}+\delta^{2-2\mu}\tilde{\psi}_0\tilde{T}_{11Y}=-\Big[\tilde{\psi}_{3Y}+\tilde{\phi}_{2\zeta}+\frac{\tilde{\psi}_1}{r_i}\Big]\delta^{3\mu}. \qquad (4.182)$$

Again the right-hand-side is just zero by equation (4.176).

In this problem it is required that nonlinear effects are present at the lowest possible level. This is possible only if we choose $\delta^{3\mu}=O(\delta^{2-2\mu})$, i.e. nonlinear effects will then be of the same order as the other terms in the equation. This implies that μ must take the value

$$\mu=\frac{2}{5}, \qquad (4.183)$$

which is the same as the scaling determined by Hickernell (1984) for time-dependent critical layers in shear flows on the beta-plane, by Goldstein and Leib (1989) and Leib(1991) in their work on compressible shear layers and recently by Shukhman (1991) in his study of spiral density waves generated by the instability of the shear layer in a rotating compressible fluid.

Applying the scale (4.183) simplifies (4.182) to

$$\tilde{\mathcal{L}}\tilde{T}_{12}=-T_c'\tilde{\psi}_1-\tilde{\psi}_0\tilde{T}_{11Y}. \qquad (4.184)$$

Substituting expansions (4.162) - (4.165) into (4.170) yields the following useful results

$$O(\delta): \qquad \frac{w_c'}{T_c}\tilde{\psi}_0=-\frac{1}{\gamma M_\infty^2}\tilde{p}_{11\zeta}, \qquad (4.185)$$

$$O(\delta^{6/5}): \qquad \frac{w_c'}{T_c}\tilde{\psi}_1=-\frac{1}{\gamma M_\infty^2}\tilde{p}_{12\zeta}, \qquad (4.186)$$

$$O(\delta^{7/5}): \ \frac{1}{T_c}\Big[\tilde{\mathcal{L}}\tilde{\phi}_1 + Yw_c''\tilde{\psi}_0 + w_c'\tilde{\psi}_2 - \frac{T_c'}{T_c}Yw_c'\tilde{\psi}_0\Big] = -\frac{1}{\gamma M_\infty^2}\Big[\tilde{p}_{13\zeta} + \frac{\alpha_1}{\alpha_0}\tilde{p}_{11\zeta}\Big]. \quad (4.187)$$

Substituting expansions (4.162) - (4.165) into (4.174) yields the leading order vorticity equation (i.e., $O(\delta)$)

$$-\tilde{\mathcal{L}}\tilde{\phi}_{1Y} - w_c''\tilde{\psi}_0 + \frac{w_c'\tilde{\psi}_0}{r_i} = \frac{T_c'\tilde{p}_{11\zeta}}{\gamma M_\infty^2}. \quad (4.188)$$

Making use of (4.185) and the doubly generalized inflexion condition obtained by Duck (1990) (i.e. (4.114)), (4.188) reduces to the form

$$\tilde{\mathcal{L}}\tilde{Q}_1 = 0, \quad (4.189)$$

where

$$\tilde{Q}_1 = \tilde{\phi}_{1Y}. \quad (4.190)$$

At the next order ($O(\delta^{6/5})$) the vorticity equation has the form

$$-\tilde{\mathcal{L}}[\tilde{\phi}_{2Y}] + \tilde{\psi}_1\Big[\frac{w_c'}{r_i} - w_c''\Big] - \tilde{\psi}_0\frac{\partial}{\partial Y}[\tilde{\phi}_{1Y}] = \frac{1}{\gamma M_\infty^2}[T_c'\tilde{p}_{12\zeta} + \tilde{T}_{11Y}\tilde{p}_{11\zeta}]. \quad (4.191)$$

Applying the results (4.185), (4.186), (4.190) and the doubly generalized inflexion condition, (4.191) simplifies to

$$\tilde{\mathcal{L}}[\tilde{\phi}_{2Y}] = \tilde{\psi}_0\Big[\frac{w_c'}{T_c}\tilde{T}_{11Y} - \tilde{Q}_{1Y}\Big]. \quad (4.192)$$

Making use of (4.184) equation (4.192) can be re-expressed as

$$\tilde{\mathcal{L}}\Big[\tilde{\phi}_{2Y} - \frac{w_c'}{T_c}\tilde{T}_{12} + \tau_1\Big] = \alpha_0 Rl(iA^{\dagger}e^{i\alpha_0\zeta})\Big[\tilde{Q}_{1Y} - 2\frac{w_c'}{T_c}\tilde{T}_{11Y}\Big] + \frac{w_c'}{T_c}\tilde{\psi}_1 T_c', \quad (4.193)$$

where equation (4.181) has been applied and term τ_1, which is assumed to be a function of Y only and therefore is independent of the operator, will be determined by the boundary conditions. Defining

$$\tilde{Q}_2 = \tilde{\phi}_{2Y} - \frac{w_c'}{T_c}\tilde{T}_{12} + \tau_1, \tag{4.194}$$

then (4.193) simplifies to

$$\tilde{\mathcal{L}}\tilde{Q}_2 = \alpha_0 Rl(iA^{\dagger}e^{i\alpha_0\zeta})\big[\tilde{Q}_{1Y} - 2\frac{w_c'}{T_c}\tilde{T}_{11Y}\big] + \frac{w_c'}{T_c}\tilde{\psi}_1 T_c'. \tag{4.195}$$

At the next order ($O(\delta^{7/5})$) we obtain the vorticity equation

$$\tilde{\mathcal{L}}\big[\tilde{\psi}_{0\zeta} - \tilde{\phi}_{3Y} + \frac{w_c'}{\gamma}\tilde{p}_{11}\big] - \big[\frac{Y^2}{2}w_c'' + \frac{\alpha_1}{\alpha_0}Yw_c'\big]\frac{\partial}{\partial\zeta}(\tilde{\phi}_{1Y}) - \tilde{\psi}_0\frac{\partial}{\partial Y}(\tilde{\phi}_{2Y})$$
$$-\tilde{\psi}_1\frac{\partial}{\partial Y}(\tilde{\phi}_{1Y}) - w_c''\tilde{\psi}_2 + \frac{1}{r_i}\big[w_c'\tilde{\psi}_2 + Yw_c''\tilde{\psi}_0 - w_c'\tilde{\psi}_0\frac{Y}{r_i}\big]$$
$$= \frac{1}{\gamma M_\infty^2}\big[T_c'\tilde{p}_{13\zeta} + \tilde{T}_{11Y}\tilde{p}_{12\zeta} + \tilde{T}_{12Y}\tilde{p}_{11\zeta} + T_c'\tilde{p}_{11\zeta}\frac{\alpha_1}{\alpha_0}\big]. \tag{4.196}$$

Making use of equations (4.185), (4.186) and (4.187) and the doubly generalized inflexion condition equation, (4.196) simplifies to

$$\tilde{\mathcal{L}}\tilde{Q}_3 = \alpha_0 Rl[iA^{\dagger}e^{i\alpha_0\zeta}]\tilde{Q}_{2Y} - \big[\frac{Y^2}{2}w_c'' + \frac{\alpha_1}{\alpha_0}Yw_c'\big]\tilde{Q}_{1\zeta}$$
$$+\tilde{\psi}_0\tau_{1Y} + \tilde{\mathcal{L}}\tau_2 - \tilde{\psi}_1\tilde{Q}_{1Y} + \frac{w_c'}{T_c}\big[\frac{2T_c'}{r_i}Y\tilde{\psi}_0 + \tilde{T}_{11Y}\tilde{\psi}_1\big], \tag{4.197}$$

where

$$\tilde{Q}_3 = \tilde{\phi}_{3Y} - \tilde{\psi}_{0\zeta} - \frac{w_c'}{\gamma}\tilde{p}_{11} - \frac{T_c'}{T_c}\tilde{\phi}_1 + \tau_2. \tag{4.198}$$

$\tilde{Q}_2$ is given by equation (4.194) and τ_2 which will be determined by applying the boundary and matching conditions, is generally a function of all three variables.

It is now found necessary to match the above vorticity results with the inner limit of the outer solution in order to determine τ_1 and τ_2. Clearly $\tilde{T}_{11} = O(\frac{1}{Y})$ as Y becomes large, therefore

$$\tilde{T}_{11} \rightarrow 0 \qquad \text{as} \qquad Y \rightarrow \pm\infty. \tag{4.199}$$

Since the leading order disturbance term in the $\overline{v}_3$ outer expansion is independent of Y then

$$\tilde{Q}_1 = \tilde{\phi}_{1Y} \rightarrow 0 \qquad \text{as} \qquad Y \rightarrow \pm\infty. \tag{4.200}$$

At the next order matching yields

$$\tilde{Q}_2 \rightarrow \frac{Y^2}{2} w_c' \left(\frac{w_c'''}{w_c'} - \frac{T_c''}{T_c}\right) + \tau_1 \quad \text{as} \quad Y \rightarrow \pm\infty, \tag{4.201}$$

and since it is required that $\tilde{Q}_2 \rightarrow 0$ as $Y \rightarrow \pm\infty$, then

$$\tau_1 = -\frac{Y^2}{2} w_c' \left(\frac{w_c'''}{w_c'} - \frac{T_c''}{T_c}\right). \tag{4.202}$$

At the $O(\delta^{7/5})$ order matching gives

$$\tilde{Q}_3 \rightarrow \left[-\frac{T_c''}{T_c} - \frac{1}{2}\left(\frac{w_c''}{w_c'}\right)^2 + \frac{1}{2}\left(\frac{w_c''}{w_c'} - \frac{2}{r_i}\right)^2 + \frac{w_c'''}{w_c'}\right] Rl(A^{\dagger} e^{i\alpha_0\zeta}) + \tau_2 \qquad \text{as } Y \rightarrow \pm\infty. \tag{4.203}$$

Again, requiring the boundary condition $\tilde{Q}_3 \rightarrow 0$, as $Y \rightarrow \pm\infty$, to be satisfied then

$$\tau_2 = -\Upsilon Rl[A^{\dagger} e^{i\alpha_0\zeta}], \tag{4.204}$$

where

$$\Upsilon = -\frac{T_c''}{T_c} - \frac{1}{2}\Big(\frac{w_c''}{w_c'}\Big)^2 + \frac{1}{2}\Big(\frac{w_c''}{w_c'} - \frac{2}{r_i}\Big)^2 + \frac{w_c'''}{w_c'}. \tag{4.205}$$

Making use of results (4.202) and (4.204) $\tilde{Q}_2$ and $\tilde{Q}_3$ are re-defined to have the form

$$\tilde{Q}_2 = \tilde{\phi}_{2Y} - \frac{w_c'}{T_c}\tilde{T}_{12} - \frac{Y^2}{2}w_c'\Big(\frac{w_c'''}{w_c'} - \frac{T_c''}{T_c}\Big), \tag{4.206}$$

and

$$\tilde{Q}_3 = \tilde{\phi}_{3Y} - \tilde{\psi}_{0\zeta} - \frac{w_c'}{\gamma}\tilde{p}_{11} - \frac{T_c'}{T_c}\tilde{\phi}_1 - \Upsilon Rl[A^\dagger e^{i\alpha_0\zeta}], \tag{4.207}$$

and equation (4.197) can be simplified to

$$\begin{aligned}\tilde{\mathcal{L}}\tilde{Q}_3 = \alpha_0 Rl[iA^\dagger e^{i\alpha_0\zeta}]\tilde{Q}_{2Y} - [\frac{Y^2}{2}w_c'' + \frac{\alpha_1}{\alpha_0}Yw_c']\tilde{Q}_{1\zeta} \\ -\Upsilon Rl[A_{t_1}^\dagger e^{i\alpha_0\zeta}] + \tilde{\psi}_1[\tilde{T}_{11Y}\frac{w_c'}{T_c} - \tilde{Q}_{1Y}].\end{aligned} \tag{4.208}$$

As was stated in the previous subsection, the slowly varying amplitude function is determined by matching the entire critical solution to the inner limit of the outer solution (4.155) - (4.158) (to the order of approximation of the analysis). Inspecting the outer solution, it is observed that the $O(\delta^{7/5})$ streamwise velocity undergoes a finite jump crossing the critical layer, so this term is expected to be important in the matching process. Inspecting, now, the inner solution determined above, it is noted that the $O(\delta^{7/5})$ streamwise velocity term is contained within the component $\tilde{Q}_3$. Since this term is required to match the outer solution, it is found necessary to consider $\tilde{Q}_3$, in terms of the inner expansions of the outer solution, i.e. the term

$$Rl\Big[\frac{\partial}{\partial Y}\Big\{YA^\dagger\Big(-\frac{T_c''}{T_c}-\frac{1}{2}\Big(\frac{w_c''}{w_c'}\Big)^2+\frac{1}{2}\Big(\frac{w_c''}{w_c'}-\frac{2}{r_i}\Big)^2+\alpha_0^2+M_\infty^2\frac{(w_c')^2}{T_c}-\frac{T_c'}{T_c}b_1+$$

$$\frac{T_c'}{T_c}\Big(\frac{1}{r_i}\Big)+\frac{w_c'''}{w_c'}\Big)-\Big(\overline{e}_1(t_1)+\overline{e}_2(t_2)ln|Y|+\frac{2i}{\alpha_0}\frac{dA^\dagger}{dt_1}b_{2,1}^{\pm}$$

$$+\frac{2\alpha_1}{\alpha_0}A^\dagger b_{2,2}^{\pm}\Big)\Big\}e^{i\alpha_0\zeta}\Big]-\alpha_0^2Rl[A^\dagger e^{i\alpha_0\zeta}]-\frac{(w_c')^2M_\infty^2}{T_c}Rl[A^\dagger e^{i\alpha_0\zeta}]$$

$$-\frac{T_c'}{T_c}\Big(-b_1+\frac{1}{r_i}\Big)Rl[A^\dagger e^{i\alpha_0\zeta}]-\Upsilon Rl[A^\dagger e^{i\alpha_0\zeta}],\quad (4.209)$$

which on applying result (4.205) simplifies to

$$Rl\Big[-\frac{\partial}{\partial Y}\Big(\overline{e}_1(t_1)+\overline{e}_2(t_1)ln|Y|+\frac{2i}{\alpha_0}\frac{dA^\dagger}{dt_1}b_{2,1}^{\pm}+\frac{2\alpha_1}{\alpha_0}A^\dagger b_{2,2}^{\pm}\Big)e^{i\alpha_0\zeta}\Big].\qquad (4.210)$$

Integrating (4.210) with respect to Y over the boundary layer and matching with the critical layer solution, it is found that since the $b_{2,1}^{\pm}$ and $b_{2,2}^{\pm}$ terms will be discontinuous across the critical layer (the other terms in (4.210) being continuous), then we have the matching condition

$$Rl\Big[-\Big\{\frac{2i}{\alpha_0}\frac{dA^\dagger}{dt_1}[b_{2,1}^{+}-b_{2,1}^{-}]+\frac{2\alpha_1}{\alpha_0}A^\dagger[b_{2,2}^{+}-b_{2,2}^{-}]\Big\}e^{i\alpha_0\zeta}\Big]=\int_{-\infty}^{\infty}\tilde{Q}_3dY.\qquad (4.211)$$

It is now found necessary to determine $\tilde{Q}_3$ (as defined in the critical layer) by solving equations (4.180), (4.184), (4.189) (4.195) and (4.208), which can be done seriatim, since $\tilde{\mathcal{L}}$ is a simple linear operator (Stewartson (1978,1981)).

The revelant solution to (4.189) is the trival solution

$$\tilde{Q}_1\equiv 0.\qquad (4.212)$$

It is found convenient to work in terms of the following normalized variables

$$\bar{t} = -\frac{1}{2}\alpha_1 w_c' t_1 - t_0, \tag{4.213}$$

$$X = \alpha_o \zeta - X_0, \tag{4.214}$$

$$\eta = -\frac{2\alpha_0}{\alpha_1} Y. \tag{4.215}$$

The coordinate shifts t_0 and X_0 are introduced to ensure that the slow varying amplitude function matches onto the initial linear wave. Their actual values will be calculated once the equation governing the amplitude has been determined.

From the form of equation (4.180) it is assumed that $\tilde{T}_{11}$ has the form

$$\begin{aligned}\tilde{T}_{11} &= Rl[\tilde{T}_{11}^{(0)}(\eta, \bar{t}) e^{iX}] \\ &= \frac{1}{2}\{\tilde{T}_{11}^{(0)}(\eta, \bar{t}) e^{iX} + [\tilde{T}_{11}^{(0)}(\eta, \bar{t})]^* e^{-iX}\}.\end{aligned} \tag{4.216}$$

Substituting (4.216) into (4.180) and collecting $O(e^{iX})$ terms yields

$$\tilde{T}_{11}^{(0)} = e^{-i\eta\bar{t}} \int_{-\infty}^{\bar{t}} -\frac{2i\alpha_0 T_c'}{\alpha_1 w_c'} A^\dagger(s_1) e^{iX_0} e^{i\eta s_1} ds_1. \tag{4.217}$$

It is found convenient to introduce the normalized variable

$$A = \frac{4\alpha_0^2 \mathcal{D}}{\alpha_1^2 w_c w_c'} A^\dagger e^{iX_0}. \tag{4.218}$$

The form of this normalized variable is based on the normalization carried out by Goldstein and Leib (1989) on their corresponding amplitude function (which in their case is a function of a slow spatial scale), although an arbitrary constant $\mathcal{D}$ has been introduced here, to allow for any differences which may occur. The actual value of $\mathcal{D}$ will be determined by numerical computations when the amplitude evolution equation is solved. This will be discussed in more detail once the evolution equation has been determined.

Making use of (4.218), equation (4.217) can be re-written in the form

$$\tilde{T}_{11}^{(0)} = -\frac{iT_c'w_c\alpha_1}{2\alpha_0\mathcal{D}}\int_{-\infty}^{\bar{t}} A(s_1)e^{-i\eta(\bar{t}-s_1)}ds_1. \tag{4.219}$$

From the form of equation (4.195) it is assumed that $\tilde{Q}_2$ has the form

$$\begin{aligned}\tilde{Q}_2 &= Rl[\tilde{Q}_2^{(0)}(\eta,\bar{t}) + \tilde{Q}_2^{(2)}(\eta,\bar{t})e^{2iX}] \\ &= Rl[\tilde{Q}_2^{(0)}(\eta,\bar{t})] + \frac{1}{2}\{\tilde{Q}_2^{(2)}(\eta,\bar{t})e^{2iX} + [\tilde{Q}_2^{(2)}(\eta,\bar{t})]^*e^{-2iX}\}.\end{aligned} \tag{4.220}$$

It is also assumed that $\tilde{\psi}_1$ has the form

$$\begin{aligned}\tilde{\psi}_1 &= Rl[\tilde{\psi}_1^{(0)}(\eta,\bar{t}) + \tilde{\psi}_1^{(2)}(\eta,\bar{t})e^{2iX}] \\ &= Rl[\tilde{\psi}_1^{(0)}(\eta,\bar{t})] + \frac{1}{2}\{\tilde{\psi}_1^{(2)}(\eta,\bar{t})e^{2iX} + [\tilde{\psi}_1^{(2)}(\eta,\bar{t})]^*e^{-2iX}\}.\end{aligned} \tag{4.221}$$

Substituting equations (4.220) and (4.221) into equation (4.195) and collecting O(1) terms gives

$$\begin{aligned}Rl\big[\tilde{Q}_2^{(0)}\big] = Rl\Big[\int_{-\infty}^{\bar{t}} &-\frac{2i\alpha_0T_c'w_c}{T_c\alpha_1\mathcal{D}}(A^\dagger)^*e^{-iX_0}\int_{-\infty}^{s_1} A(s_2)(s_1-s_2)e^{-i\eta(s_1-s_2)}ds_2ds_1 \\ &-\frac{2T_c'}{\alpha_1T_c}\int_{-\infty}^{\bar{t}}\tilde{\psi}_1^{(0)}(s_1)ds_1\Big].\end{aligned} \tag{4.222}$$

Making use of (4.218) equation (4.222) can be re-written in the form

$$\begin{aligned}Rl[\tilde{Q}_2^{(0)}] = Rl\Big[\int_{-\infty}^{\bar{t}} &-\frac{iT_c'w_c^2\alpha_1w_c'}{2\alpha_0T_c\mathcal{D}^2}A^*(s_1)\int_{-\infty}^{s_1} A(s_2)(s_1-s_2)e^{-i\eta(s_1-s_2)}ds_2ds_1 \\ &-\frac{2T_c'}{\alpha_1T_c}\int_{-\infty}^{\bar{t}}\tilde{\psi}_1^{(0)}(s_1)ds_1\Big].\end{aligned} \tag{4.223}$$

Introducing the normalizing variable,

$$Q^{(n)} = \frac{\alpha_0^2 \tilde{Q}_n}{(\alpha_1 w_c)^2 w_c'}, \tag{4.224}$$

equation (4.223) simplifies to

$$Rl[Q_0^{(2)}] = Rl\Big[-\frac{i\alpha_0 T_c'}{2\alpha_1 T_c \mathcal{D}^2} \int_{-\infty}^{\bar{t}} A^*(s_1) \int_{-\infty}^{s_1} A(s_2)(s_1 - s_2) e^{-i\eta(s_1 - s_2)} ds_2 ds_1$$
$$- \frac{2\alpha_0^2 T_c'}{\alpha_1^3 w_c^2 w_c' T_c} \int_{-\infty}^{\bar{t}} \tilde{\psi}_1^{(0)}(s_1) ds\Big]. \tag{4.225}$$

Turning our attention to the $O(e^{2iX})$ terms we obtain the equation

$$\tilde{Q}_2^{(2)} = e^{-2i\eta\bar{t}} \int_{-\infty}^{\bar{t}} e^{2i\eta s_1} \frac{2i\alpha_0 T_c' w_c}{T_c \alpha_1 \mathcal{D}} A^{\dagger}(s_1) e^{iX_0} \int_{-\infty}^{s_1} A(s_2)(s_1 - s_2) e^{-i\eta(s_1 - s_2)} ds_2 ds_1$$
$$- \frac{2T_c'}{\alpha_1 T_c} \int_{-\infty}^{\bar{t}} \tilde{\psi}_2^{(2)} e^{-2i\eta(\bar{t} - s_1)} ds_1. \tag{4.226}$$

Making use of equations (4.218) and (4.224) equation (4.226) can be re-written

$$Q_2^{(2)} = \frac{i\alpha_0 T_c'}{2T_c \alpha_1 \mathcal{D}^2} e^{-2i\eta\bar{t}} \int_{-\infty}^{\bar{t}} A(s_1) e^{i\eta s_1} \int_{-\infty}^{s_1} A(s_2)(s_1 - s_2) e^{i\eta s_2} ds_2 ds_1$$
$$- \frac{2\alpha_0^2 T_c'}{\alpha_1^3 w_c^2 w_c' T_c} \int_{-\infty}^{\bar{t}} \tilde{\psi}_1^{(2)} e^{-2i\eta(\bar{t} - s_1)} ds_1. \tag{4.227}$$

From the form of (4.208) $\tilde{Q}_3$ is assumed to have the form

$$\begin{aligned} \tilde{Q}_3 &= Rl[\tilde{Q}_3^{(1)} e^{iX} + \tilde{Q}_3^{(3)} e^{3iX}] \\ &= \frac{1}{2}\{\tilde{Q}_3^{(1)} e^{iX} + [\tilde{Q}_3^{(1)}]^* e^{-iX}\} + \frac{1}{2}\{\tilde{Q}_3^{(3)} e^{3iX} + [\tilde{Q}_3^{(3)}]^* e^{-3iX}\}. \end{aligned} \tag{4.228}$$

Substitution of (4.228) into (4.208) and collecting $O(e^{iX})$ terms yields

$$Q_1^{(3)} = -\frac{\Upsilon}{4w_c\mathcal{D}}\int_{-\infty}^{\bar{t}} e^{-i\eta(\bar{t}-s_1)}A_{s_1}ds_1 - \frac{i\alpha_0 T_c' w_c}{4\alpha_1 T_c \mathcal{D}^3}\times$$

$$\int_{-\infty}^{\bar{t}} e^{-i\eta(\bar{t}-s_1)}\Big\{2A(s_1)Rl\Big[\int_{-\infty}^{s_1} A^*(s_2)\int_{-\infty}^{s_2} A(s_3)(s_2-s_3)^2 e^{-i\eta(s_2-s_3)}ds_3ds_2ds_1\Big]$$
$$+A^*(s_1)\int_{-\infty}^{s_1} A(s_2)\int_{-\infty}^{s_2} A(s_3)(s_2-s_3)(2s_1-s_2-s_3)e^{-i\eta(2s_1-s_2-s_3)}ds_3ds_2ds_1\Big\}$$

$$-\frac{iT_c'\alpha_0^2}{\alpha_1^3 w_c w_c' T_c\mathcal{D}}\Big\{2\int_{-\infty}^{\bar{t}} e^{-i\eta(\bar{t}-s_1)}A(s_1)Rl\Big[\int_{-\infty}^{s_1}\tilde{\psi}_{1\eta}^{(0)}ds_2ds_1\Big]$$
$$-\int_{-\infty}^{\bar{t}} e^{-i\eta(\bar{t}-s_1)}A^*(s_1)\int_{-\infty}^{s_1}\frac{\partial}{\partial\eta}[\tilde{\psi}_1^{(2)}e^{-2i\eta(s_1-s_2)}]ds_2ds_1\Big\}$$
$$+\frac{4\alpha_0^3}{\alpha_1^4 w_c^2 w_c' T_c}\int_{-\infty}^{\bar{t}} e^{-i\eta(\bar{t}-s_1)}(Rl[\tilde{\psi}_1^{(0)}]\tilde{T}_{11\eta}^{(0)} + \frac{1}{2}\tilde{\psi}_1^{(2)}[\tilde{T}_{11\eta}^{(0)}]^*)ds_1, \quad (4.229)$$

where again equations (4.218) and (4.224) have been used to simplify matters. Following the approach of Goldstein and Leib (1989), it is now found necessary to evaluate $\int_{-\infty}^{\infty} Q_1^{(3)}d\eta$, in order to match with the outer solution as defined by (4.211) (which is, of course, in terms of the un-normalized variables). Since equation (4.229) is rather cumbersome this shall be carried out in stages. Firstly consider

$$\int_{-\infty}^{\infty} -\frac{\Upsilon}{4w_c\mathcal{D}}\int_{-\infty}^{\bar{t}} e^{-i\eta(\bar{t}-s_1)}A_{s_1}ds_1d\eta. \quad (4.230)$$

This equation can be re-written in the form

$$-\frac{\Upsilon}{4w_c\mathcal{D}}\int_{-\infty}^{\infty}\int_{-\infty}^{\bar{t}} A_{s_1}\{\cos\eta(\bar{t}-s_1) - i\sin\eta(\bar{t}-s_1)\}ds_1d\eta. \quad (4.231)$$

Remembering that $\sin\beta\eta$ is odd in η and employing integration by parts with respect to s_1, (4.231) becomes

$$-\frac{\Upsilon}{4w_c\mathcal{D}}\int_{-\infty}^{\infty}\Big\{\Big[A_{s_1}\Big(-\frac{\sin\eta(\bar{t}-s_1)}{\eta}\Big)\Big]_{-\infty}^{\bar{t}} + \frac{1}{\eta}\int_{-\infty}^{\bar{t}} A_{s_1s_1}\sin\eta(\bar{t}-s_1)ds_1\Big\}d\eta. \quad (4.232)$$

Assuming that $A(s_1 \to -\infty) \to A_{s_1}(s_1 \to -\infty) \to 0$ and changing the order of integration yields

$$-\frac{\Upsilon}{4w_c\mathcal{D}}\int_{-\infty}^{\bar{t}}\frac{\partial^2 A}{\partial s_1^2}\int_{-\infty}^{\infty}\frac{\sin\eta(\bar{t}-s_1)}{\eta}d\eta ds_1, \tag{4.233}$$

which in turn gives the result

$$-\frac{\pi\Upsilon}{4w_c\mathcal{D}}A_{\bar{t}}, \tag{4.234}$$

where it has been assumed that $sgn(\bar{t}-s_1) > 0$, $\forall s_1 \in (-\infty, \bar{t}\,]$.

Equation (4.230) can be evaluated by another method. Firstly re-write the equation in the form

$$-\frac{\Upsilon}{4w_c\mathcal{D}}\int_{-\infty}^{\bar{t}}A_{s_1}\int_{-\infty}^{\infty}e^{-i\eta(\bar{t}-s_1)}d\eta ds_1. \tag{4.235}$$

Making use of the delta function, (4.235) becomes

$$-\frac{\Upsilon}{4w_c\mathcal{D}}\int_{-\infty}^{\bar{t}}A_{s_1}(s_1)2\pi\delta(\bar{t}-s_1)ds_1, \tag{4.236}$$

which when evaluated has the form

$$-\frac{\pi\Upsilon}{4w_c\mathcal{D}}A_{\bar{t}}, \tag{4.237}$$

where we have made use of the result

$$\int_{-\infty}^{0}f(x)\delta(x)dx = \frac{1}{2}f(0). \tag{4.238}$$

Clearly there is agreement between the two different methods.

The next integral considered is

$$\int_{-\infty}^{\infty}v_1\int_{-\infty}^{\bar{t}}e^{-i\eta(\bar{t}-s_1)}A^*(s_1)\int_{-\infty}^{s_1}A(s_2)\int_{-\infty}^{s_2}f(s_3)e^{-i\eta(2s_1-s_2-s_3)}ds_3ds_2ds_1d\eta, \tag{4.239}$$

where

$$f(s_3) = A(s_3)(s_2 - s_3)(2s_1 - s_2 - s_3), \tag{4.240}$$

and

$$v_1 = -\frac{i\alpha_0 T_c' w_c}{4\alpha_1 T_c \mathcal{D}^3}, \tag{4.241}$$

which can be re-written in the form

$$v_1 \int_{-\infty}^{\bar{t}} A^*(s_1) \int_{-\infty}^{s_1} A(s_2) \int_{-\infty}^{\infty} \int_{-\infty}^{s_2} f(s_3) e^{-i\eta(\bar{t}+s_1-s_2-s_3)} ds_3 d\eta ds_2 ds_1. \tag{4.242}$$

Considering the two inner integrations of (4.242), applying integration by parts yields

$$\int_{-\infty}^{\infty} \Big\{ \Big[f(s_3)\Big(-\frac{\sin\eta(\bar{t}+s_1-s_2-s_3)}{\eta}\Big)\Big]_{-\infty}^{s_2}$$
$$+ \int_{-\infty}^{s_2} \Big(\frac{\sin\eta(\bar{t}+s_1-s_2-s_3)}{\eta}\Big)\frac{\partial f}{\partial s_3} ds_3 \Big\} d\eta. \tag{4.243}$$

Applying the boundary conditions and changing the order of integration yields

$$\int_{-\infty}^{s_2} \frac{\partial f}{\partial s_3} \int_{-\infty}^{\infty} \frac{\sin\eta(\bar{t}+s_1-s_2-s_3)}{\eta} d\eta ds_3, \tag{4.244}$$

which is zero.

Alternatively, (4.239) could be re-written in the form

$$v_1 \int_{-\infty}^{\bar{t}} A^*(s_1) \int_{-\infty}^{s_1} A(s_2) \int_{-\infty}^{s_2} f(s_3) \int_{-\infty}^{\infty} e^{-i\eta(\bar{t}+s_1-s_2-s_3)} d\eta ds_3 ds_2 ds_1. \tag{4.245}$$

Again, using the properties of the delta-function (4.245) becomes

$$v_1 \int_{-\infty}^{\bar{t}} A^*(s_1) \int_{-\infty}^{s_1} A(s_2) \int_{-\infty}^{s_2} f(s_3) 2\pi\delta(\bar{t} + s_1 - (s_2 + s_3)) ds_3 ds_2 ds_1. \tag{4.246}$$

In the range of integration, $\bar{t}+s_1 \geq s_2+s_3$, but since at the point $\bar{t} = s_1 = s_2 = s_3$ the integrand is zero, then clearly equation (4.246) is just zero , as before.

We now consider the integration

$$\int_{-\infty}^{\infty} v_1 \int_{-\infty}^{\bar{t}} e^{-i\eta(\bar{t}-s_1)} 2A(s_1) Rl\Big\{ \int_{-\infty}^{s_1} A^*(s_2) \int_{-\infty}^{s_2} g(s_3) e^{-i\eta(s_2-s_3)} ds_3 ds_2 ds_1 d\eta \Big\}, \tag{4.247}$$

where

$$g(s_3) = A(s_3)(s_2 - s_3)^2, \tag{4.248}$$

which can be expressed in the form

$$v_1 \int_{-\infty}^{\bar{t}} A(s_1) \int_{-\infty}^{s_1} A^*(s_2) \int_{-\infty}^{s_2} g(s_3) \int_{-\infty}^{\infty} e^{-i\eta(\bar{t}-s_1+s_2-s_3)} d\eta ds_3 ds_2 ds_1 +$$
$$v_1 \int_{-\infty}^{\bar{t}} A(s_1) \int_{-\infty}^{s_1} A(s_2) \int_{-\infty}^{s_2} g^*(s_3) \int_{-\infty}^{\infty} e^{-i\eta(\bar{t}-s_1-(s_2-s_3))} d\eta ds_3 ds_2 ds_1. \tag{4.249}$$

Consider the integral

$$\int_{-\infty}^{s_2} A(s_3)(s_2 - s_3)^2 \int_{-\infty}^{\infty} e^{-i\eta(\bar{t}-s_1+s_2-s_3)} d\eta ds_3. \tag{4.250}$$

Integrating once gives

$$\int_{-\infty}^{s_2} A(s_3)(s_2 - s_3)^2 2\pi\delta(\bar{t} - s_1 + s_2 - s_3) ds_3. \tag{4.251}$$

Since $\bar{t} - s_1 + s_2 - s_3 > 0$ except at the point $\bar{t} = s_1$, $s_2 = s_3$, where the integrand is zero anyway, then the first integration term in (4.249) is zero.

The integral

$$\int_{-\infty}^{s_2} A^*(s_3)(s_2 - s_3)^2 \int_{-\infty}^{\infty} e^{-i\eta(\bar{t}-s_1-(s_2-s_3))} d\eta ds_3, \tag{4.252}$$

can be evaluated to be

$$2\pi A^*(s_2 + s_1 - \bar{t})(\bar{t} - s_1)^2, \tag{4.253}$$

and therefore (4.231) can be simplified to

$$2\pi v_1 \int_{-\infty}^{\bar{t}} A(s_1) \int_{-\infty}^{s_1} A(s_2) A^*(s_1 - s_2 - \bar{t})(\bar{t} - s_1)^2 ds_2 ds_1. \tag{4.254}$$

Looking at the form of the inner limit of the outer solution corresponding to the critical layer term, $\tilde{Q}_3$, it is observed that there appears to be no outer terms for which the critical layer term, $\hat{\psi}_1$ (or any of its derivatives), can match onto, i.e. there are no $\delta^{6/5}$, radial velocity terms in the inner limit of the outer solution. Indeed, it is found that outer radial velocity terms at this order are purely viscous in nature and since we are investigating inviscid nonlinear terms in this section (implying that over a local timescale/lengthscale viscous effects are negligible to this order), then outside the critical layer there will be no contributions at this ordering. Consequently, there is nothing for the other integration terms in (4.229), namely the $\hat{\psi}_1$ integrations, to match onto. Therefore, it can be concluded that these terms will make no contribution to the $\int \tilde{Q}_3 d\eta$ integration. In fact the only way that matching can be achieved is if $\tilde{\psi}_1$ is zero. Therefore, we can now write

$$\frac{i}{\pi} \int_{-\infty}^{\infty} Q_1^{(3)} d\eta = -\frac{i\Upsilon}{4w_c\mathcal{D}} A_{\bar{t}} +$$

$$\frac{\alpha_0 T_c' w_c}{2\alpha_1 T_c \mathcal{D}^3} \int_{-\infty}^{\bar{t}} A(s_1) \int_{\infty}^{s_1} A(s_2) A^*(s_1 + s_2 - \bar{t})(\bar{t} - s_1)^2 ds_2 ds_1. \tag{4.255}$$

Note that the curvature terms are contained within the constant Υ only, which is defined by (4.205).

Substituting (4.255) into equation (4.241) yields the amplitude evolution equation

$$A_{\bar{t}} = -\kappa A - \frac{1}{\Gamma}\int_{-\infty}^{\bar{t}} A(s_1)\int_{-\infty}^{s_1} A(s_2)A^*(s_1+s_2-\bar{t})(\bar{t}-s_1)^2 ds_2 ds_1, \qquad (4.256)$$

where

$$\Gamma = \frac{i\alpha_1 T_c \mathcal{D}^2}{2\alpha_0 T_c' w_c^2}\Big[-\Upsilon + \frac{2iw_c'}{\pi}(b_{2,1}^+ - b_{2,1}^-)\Big], \qquad (4.257)$$

and

$$\kappa = -\frac{2i\alpha_1 T_c \mathcal{D}^2}{\pi\alpha_0 T_c' w_c^2 \Gamma}(b_{2,2}^+ - b_{2,2}^-). \qquad (4.258)$$

Equation (4.256) is required to match onto the initial linear solution. Assuming for small times nonlinear terms are insignificant in the evolution equation, then we have

$$A(\bar{t}) \rightarrow e^{-\kappa\bar{t}} \qquad \text{as} \qquad \bar{t} \rightarrow -\infty. \qquad (4.259)$$

Therefore, in terms of the unshifted time t_1, the un-normalized amplitude $A^\dagger$ is expected to behave like

$$A^\dagger \rightarrow \frac{\alpha_1^2 w_c w_c'}{4\alpha_0^2\mathcal{D}} e^{-iX_0} e^{-\kappa(-\frac{1}{2}\alpha_1 w_c' t_1 - t_0)} \qquad \text{as} \qquad t_1 \rightarrow -\infty. \qquad (4.260)$$

Inspecting (4.260) the growth rate of the linear wave is expected to be

$$\frac{1}{2}\kappa_r \alpha_1 w_c', \qquad (4.261)$$

where $\kappa_r = Rl\{\kappa\}$. Therefore the coordinate shifts X_0 and t_0 are chosen to ensure the following condition is achieved, namely

$$A^\dagger \rightarrow a^\dagger e^{\frac{1}{2}\kappa\alpha_1 w_c' t_1}, \tag{4.262}$$

where $a^\dagger$ represents some initial amplitude of the initial linear growth wave, which will be complex. Therefore, substituting (4.262) into (4.260) yields

$$a^\dagger e^{\frac{1}{2}\kappa\alpha_1 w_c' t_1} = \frac{\alpha_1^2 w_c w_c'}{4\alpha_0^2 \mathcal{D}} e^{-iX_0} e^{-\kappa(-\frac{1}{2}\alpha_1 w_c' t_1 - t_0)}, \tag{4.263}$$

which on rewritting $a^\dagger$ in terms of its complex modulus/argument representation, i.e.

$$a^\dagger = |a^\dagger| e^{i\theta}, \qquad \text{where} \qquad \theta = \arg a^\dagger, \tag{4.264}$$

gives the conditions

$$t_0 = \frac{1}{\kappa_r} ln \Big| \frac{4\alpha_0^2 |a^\dagger| \mathcal{D}}{\alpha_1^2 w_c w_c'} \Big|, \tag{4.265}$$

and

$$X_0 = \kappa_i t_0 - \theta. \tag{4.266}$$

κ_i is of course the imaginary part of κ.

Equation (4.256) is the main result of the nonlinear theory developed in this Chapter and represents the governing equation for the growth of the instability wave. It is found to be of the Hickernell type, in that the nonlinearity occurs through a type of integral convolution and can be regarded as a cubic nonlinearity in the slow varying amplitude function. On comparing (4.256) with the evolution equation obtained by Goldstein and Leib (1989) the nonlinear term is found to have exactly the same form - although Goldstein and Leib's equation is in terms of the slow streamwise distance, $\bar{x}$, by virtue of their spatial approach, as opposed to the slow time, $\bar{t}$, in our temporal approach - the only difference being in the coefficient terms κ and Γ. In his study of the nonlinear evolution of a mixing layer in a rotating compressible fluid, Shukhman's (1991) critical layer is fundamently different from that presented here (and also that treated by Goldstein and Leib) in that the nonlinearity is developed from the irregular pressure disturbance logarithmic contribution resulting from the critical point not coinciding with a generalized inflexional point. However, the nonlinear term in his evolution equation is found to be exactly the same form as that determined here and by Goldstein and Leib. Shukhman noted that the only difference occurs in the coefficient terms which are determined by neutral mode analysis peculiar to the fluid dynamic problem being considered, but these turn out only to be constants, thus resulting in an even greater universality between Hickernell's (1984), Goldstein and Leib's (1989) and Shukhman's (1991) results. The results obtained in this section add even more weight to the validity of Shukhman's statement.

It is now required to solve (2.255). This is achieved by using a straightforward Crank-Nicolson scheme to advance the solution in time starting from the initial linear solution (4.259). The double integrals are solved using the trapezoidal rule with early time 'tails' evaluated analytically from the initial linear solution. To determine the values of κ and Γ, suitable subsonic, axisymmetric, generalized inflexional neutral modes from Chapter 3 are chosen and the constants appearing in equations (4.257) and (4.258) are determined for each of these modes. The jump constants, $(b_{2,1}^{+} - b_{2,1}^{-})$ and $(b_{2,2}^{+} - b_{2,2}^{-})$, appearing in (4.257) and (4.258) are determined by solving (4.146) and (4.147) respectively, for each of the chosen neutral modes. It is found that by adjusting the term $\mathcal{D}$, the nonlinear constant $\frac{1}{\Gamma}$ can be varied relative to the linear constant κ - which remains fixed - thus allowing us to control when nonlinearity becomes important, i.e. we are varying the level of competitive nonlinearity. The axisymmetric generalized inflexional neutral mode occuring for the axisymmetric mode I instability, on an adiabatic cylinder at the location $\zeta = 0.05$ (see Figure 3.13) was chosen as our standard. By numerical experimentation it is found that arbitrarily setting $\mathcal{D} \equiv w_c'$, causes $\frac{1}{\Gamma}$ to be large enough relative κ to allow for nonlinearity to produce a significant effect after a reasonable enough timescale and thus allow us to study its effects. It should be emphasised that the choice for $\mathcal{D}$ is completely arbitrary - any value may be taken.

Figures 4.1, 4.2 and 4.3 display variations of $Rl\{A\}$, $\mathrm{Im}\{A\}$ and $|A|$ with scaled time, $\bar{t}$, respectively. For all curves presented the constants κ and Γ are determined with respect to axisymmetric subsonic neutral modes only, where (1) and (4) correspond to adiabatic neutral modes at $\zeta = 0.01$ and $\zeta = 0.05$ respectively, and (2) and (3) correspond to neutral modes at $\zeta = 0.01$ for wall temperatures of $T_w = 4.0$ and $T_w = 4.5$ respectively. It is found that in all the cases presented the amplitude growth rates terminate explosively in a singularity after a finite time evolution, as expected.

The explosive growth of the slow varying amplitude terms, as noted above, is attributrible to nonlinear effects which are in characterized by critical layer effects only, and are independent of the constant coefficient terms. Since the aim of this study was to see what difference including curvature terms would have on Goldstein and Leib's (1989) results, it is concluded that since curvature term contributions are confined to the constant coefficient terms only, then there is no direct effect on the nonlinearity within the problem, i.e. nonlinearity effects here have exactly the same form as the Goldstein and Leib (1989) results. However, when the numerical study was conducted it was found the the level of competitive nonlinearity, i.e. the ratio of κ to $1/\Gamma$, is important. It is found that $1/\Gamma$ must be large enough relative to κ in order for nonlinear effects to be observed over the timescales being considered. For some of the neutral modes considered (but not presented here) $\kappa\Gamma >> 1$, resulting in the linear terms completely dominating nonlinear contributions. However, even in these cases it is felt that if sufficient time were allowed to evolve - and provided the numerical scheme remains accurate - cumulative history effects of the nonlinear terms should eventually cause explosive growth. Care must be taken with regard to this statement, however, since the timescales involved may be of the same ordering as the viscous timescale, and if this where so then the effects of viscosity could no longer be ignored. It appears that in such a case the explosive growth conditions may never be achieved since viscous effects may supress any rapid growth, but this can only be answered by conducting a full viscous, nonlinear critical layer analysis.

In the problem treated here we have introduced a parameter $\mathcal{D}$ to help control the product $\kappa\Gamma$. For a given neutral mode values of $\mathcal{D}$ can be varied until $\kappa\Gamma = O(1)$, but once $\mathcal{D}$ is fixed, then there will still exist neutral modes where $\kappa\Gamma >> 1$.

In light of the numerical results and what we have said above it can be clearly seen that even though the explosive growth of the amplitude terms is independent of the particular fluid dynamical problem being treated (as stated by Shukhman (1991)), it is found that since the product $\kappa\Gamma$ controls when it occurs, if at all, then indirectly the constant coefficient terms are still important when considering the amplitude evolution. Therefore, curvature will be important in the problem through its effects on the constant coefficient terms and the resultant effects on the values of competitive nonlinearity.

Since the nonlinear term calculated here has the same form as that determined by Goldstein and Leib (1989) then it is expected that if explosive growth occurs, in the neighbourhood of the singularity the amplitude terms will have the same asymptotic form as that determined by Goldstein and Leib, i.e. in the limit $\bar{t} \to \bar{t}_s$ where $\bar{t}_s$ represents the time when the singularity will occur,

$$A = \frac{b}{(\bar{t}_s - \bar{t})^{(\frac{5}{2}+i\sigma)}}, \tag{4.267}$$

where σ is a real constant and b is a complex constant. As yet values for σ and b for our particular problem have yet to be determined, but this will be conducted sometime in the near future.

In the next chapter of this thesis we consider the problem of a viscous nonlinear critical layer to see whether or not the observed growth rate blow ups can be eliminated.

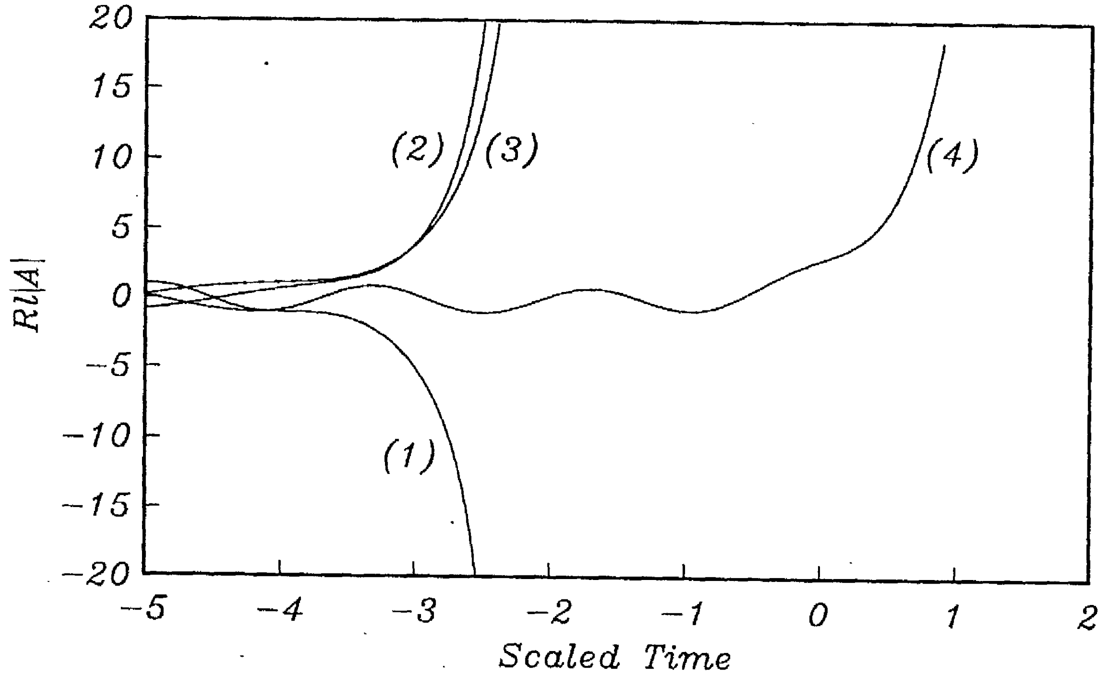


Figure 4.1: Variation of $Rl\{A\}$ with scaled time $\bar{t}$.

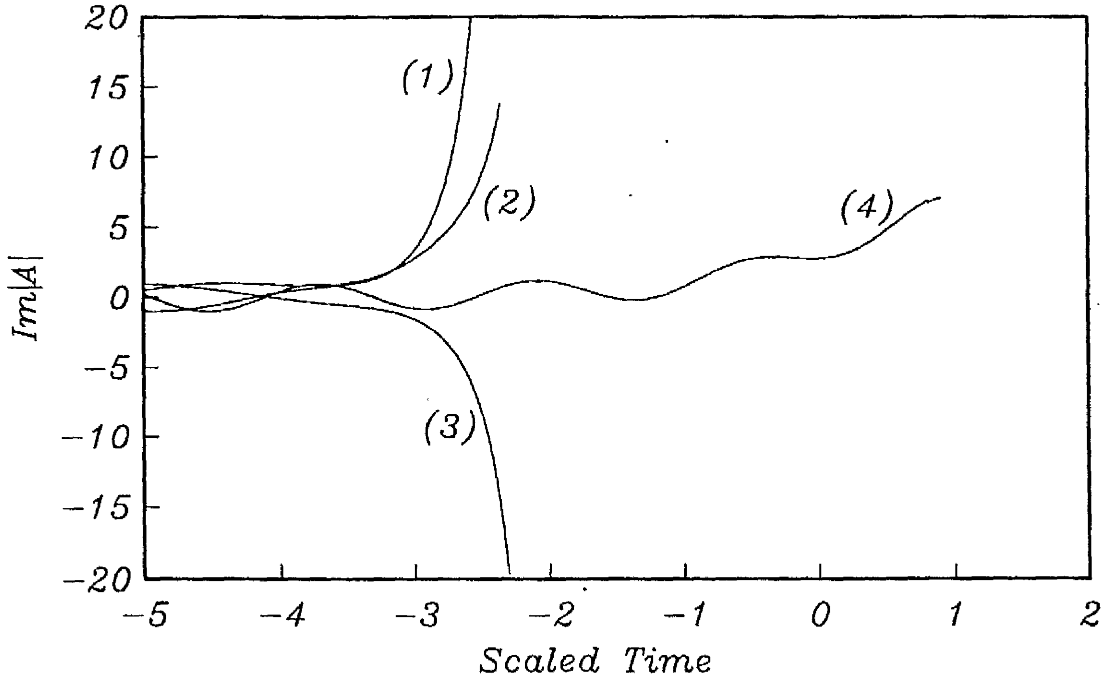


Figure 4.2: Variation of $\mathrm{Im}\{A\}$ with scaled time $\bar{t}$.

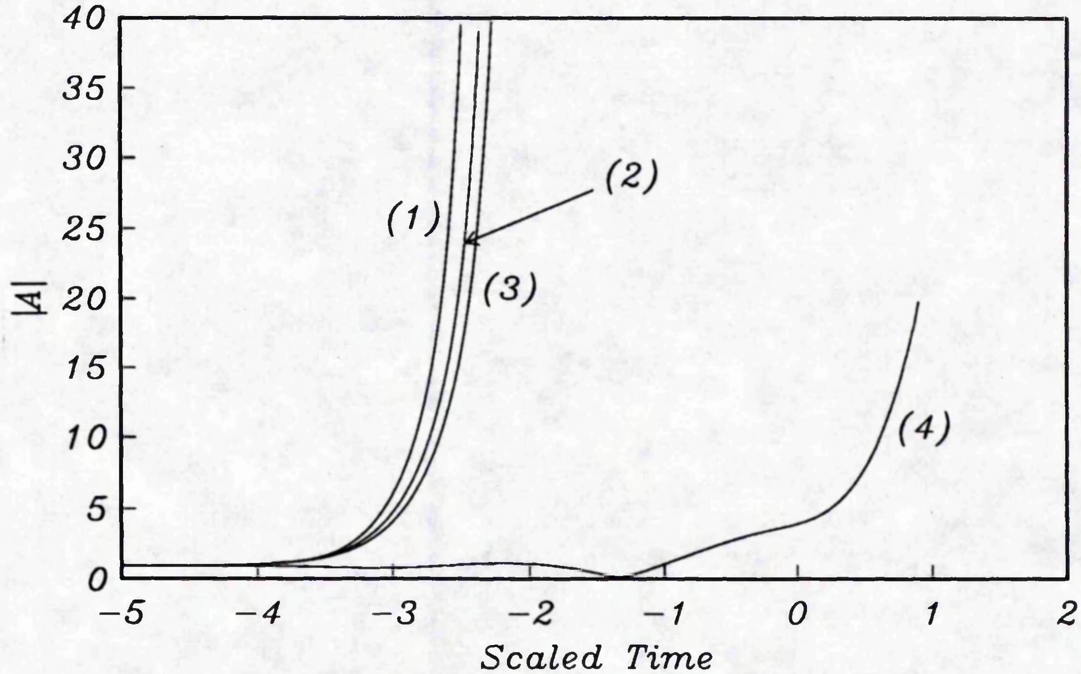


Figure 4.3: Variation of $|A|$ with scaled time $\bar{t}$.

Chapter 5

The Viscous Nonlinear Critical Layer Solution

In the previous chapter it was found that when nonlinear terms were retained in favour of viscous terms, for the particular unsteady critical layer problem being treated in this thesis, after sufficient time had evolved the growth rate of the instability wave was found to terminate in a singularity. The reason for this is that the nonlinear term is a form of convolution integral in which past histories are important and the cumulative effect of this integral results in explosive growth after a finite time evolution. This occured because in the formulation of the nonlinear critical layer problem important physics has been disregarded, or more precisely, in the neighbourhood of the critical point viscous effects have been totally ignored. In this chapter we consider the case where viscous effects are of the same order as the nonlinearity developed within the critical layer. Examining the critical layer solution of Chapter 4, it is noted that nonlinearity first becomes important in the temperature equation at $O(\delta^{6/5})$, therefore viscous effects are now required to be important at the same ordering, implying that the Reynolds number, Re, now has the form

$$Re \sim \delta^{-6/5}. \tag{5.1}$$

If Re is chosen smaller than $\delta^{-6/5}$ viscous effects will dominate the nonlinear critical layer effects, while if Re is larger then viscosity will not significantly effect the solution.

We introduce the scaled viscous parameter

$$\hat{\lambda} = \frac{1}{Re\delta^{6/5}}, \tag{5.2}$$

where $\hat{\lambda} = O(1)$ and the inviscid solution obtained in the previous chapter can be retrieved by setting $\hat{\lambda} = 0$.

Again, it is found convenient to work in terms of a frame of reference moving with streamwise velocity c_0, namely (4.70). Therefore, we again stress that even though the boundary layer varies over long viscous lengthscales in the stationary frame of reference, since fixed points in the moving frame translate downstream as time evolves with respect to the stationary frame, then mean flow terms can be regarded as functions of time in the moving frame. Since viscous effects are no longer ignored, then mean flow terms are expected to vary as the boundary layer spreads over the long viscous timescale (as measured in the moving frame of reference)

$$t_2 = \frac{t}{Re} = \hat{\lambda} t_1 \delta^{4/5}, \tag{5.3}$$

which is of course longer than the nonlinear timescale, t_1, defined in the previous chapter.

Locally, i.e. for short time intervals, the mean flow terms can be Taylor expanded about $t_2 = 0$, to obtain

$$u_0(r, t_2) = u_0(r) + \frac{\partial u_0}{\partial t_2}(r, 0)\hat{\lambda} t_1 \delta^{4/5} + \cdots, \tag{5.4}$$

$$w_0(r, t_2) = w_0(r) + \frac{\partial w_0}{\partial t_2}(r, 0)\hat{\lambda} t_1 \delta^{4/5} + \cdots, \tag{5.5}$$

$$T_0(r,t_2) = T_0(r) + \frac{\partial T_0}{\partial t_2}(r,0)\hat{\lambda}t_1\delta^{4/5} + \cdots, \tag{5.6}$$

$$\rho_0(r,t_2) = \frac{1}{T_0(r)} - \frac{1}{T_0^2(r)}\frac{\partial T_0}{\partial t_2}(r,0)\hat{\lambda}t_1\delta^{4/5} + \cdots, \tag{5.7}$$

where $u_0(r,t_2)$ represents mean flow velocity terms in the radial direction and the $O(\delta^{4/5})$ terms represent the effects on the basic flow due to viscous spreading.

Consequently the solutions outside the critical layer now expand in the form

$$v_1 = \delta\alpha Rl[A^\dagger(t_1)\tilde{v}_1(r)e^{i\alpha_0\zeta}] + \delta^{6/5}v_{1m} + \cdots, \tag{5.8}$$

$$\overline{v}_3 = W_0(r) + a(r)\hat{\lambda}t_1\delta^{4/5} + \delta Rl[A^\dagger(t_1)\tilde{v}_3(r)e^{i\alpha_0\zeta}] + \delta^{6/5}\overline{v}_{3m} + \cdots, \tag{5.9}$$

$$T = T(r) + d(r)\hat{\lambda}t_1\delta^{4/5} + \delta Rl[A^\dagger(t_1)\tilde{T}(r)e^{i\alpha_0\zeta}] + \delta^{6/5}T_m + \cdots, \tag{5.10}$$

$$\rho = \frac{1}{T_0(r)} + e(r)\hat{\lambda}t_1\delta^{4/5} + \delta Rl[A^\dagger(t_1)\tilde{\rho}(r)e^{i\alpha_0\zeta}] + \delta^{6/5}\rho_m + \cdots, \tag{5.11}$$

$$P = 1 + \delta\gamma M_\infty^2 Rl[A^\dagger(t_1)\tilde{p}(r)e^{i\alpha_0\zeta}] + \delta^{6/5}P_m + \cdots, \tag{5.12}$$

where $a(r)$, $d(r)$ and $e(r)$ can be determined from the basic flow, $W_0(r)+c_0 = w_0(r)$ since we are in a frame of reference moving with streamwise velocity c_0, and

$$\zeta = (1 + \delta^{2/5}\frac{\alpha_1}{\alpha_0})\overline{\xi} = (1 + \delta^{2/5}\frac{\alpha_1}{\alpha_0})(z - c_0 t). \tag{5.13}$$

The $O(\delta^{6/5})$ terms have been introduced in expansions (5.8) - (5.12) to match with the solution within the critical layer - which we know exists at this ordering by the form of the inviscid solution - and to account for the slowly varying basic flow terms.

Fortunately, outside the critical layer the additional viscous terms do not effect the solutions already determined in the inviscid case, i.e. to the required order, outside the critical layer viscous effects are not important.

We are not so fortunate with the solutions inside the critical layer. Re-expanding the inner limit of the outer solution in terms of the critical layer radial coordinate Y, namely (4.154), and including viscous terms, suggests that in the case of a viscous nonlinear critical layer the flow parameters possess the following expansions

$$v_1 = -\delta\alpha_0 Rl[iA^{\dagger}e^{i\alpha_0\zeta}] + \delta^{6/5}\hat{\lambda}u_c + \delta^{7/5}\hat{\psi}_2 + \cdots, \qquad (5.14)$$

$$\overline{v}_3 = Yw_c'\delta^{2/5} + \delta^{4/5}\Big(\frac{Y^2}{2}w_c'' + a_c\hat{\lambda}t_1\Big) + \delta\hat{\phi}_1 + \delta^{6/5}\hat{\phi}_2 + \delta^{7/5}\hat{\phi}_3 + \cdots, \qquad (5.15)$$

$$P = 1 + \delta\hat{p}_{11}(\zeta, t_1) + \delta^{6/5}\hat{p}_{12}(\zeta, t_1) + \delta^{7/5}\hat{p}_{13} + \cdots, \qquad (5.16)$$

$$T = T_c + T_c'\delta^{2/5}Y + \delta^{3/5}\hat{T}_{11} + \delta^{4/5}\hat{T}_{12} + \delta\hat{T}_{13} + \cdots. \qquad (5.17)$$

Examining the radial velocity expansion (5.14), it is noted that the $O(\delta^{6/5})$ term has been set to $u_c\hat{\lambda}$. This needs some explanation. Since by the locally parallel flow assumption it is assumed that there is no basic flow contribution in the radial direction, this implies that the first order correction term to this assumption will have the form $u_0(r)/Re$. Clearly, a Taylor expansion about the critical point, will yield a correction term of the form $u_c\hat{\lambda}\delta^{6/5}$.

It is now found necessary to consider the full viscous equations of momenta and energy (as presented in Chapter 2), which in non-dimensional form are

$$\rho\frac{Dv_1}{Dt} = -\frac{1}{\gamma M_\infty^2}\frac{\partial P}{\partial r} + \frac{\partial}{\partial r}\Big[\frac{2\mu}{Re}\frac{\partial v_1}{\partial r} + \Big(\frac{\mu_2}{Re_2} + \frac{2}{3}\frac{\mu}{Re}\Big)\underline{\nabla}\cdot\underline{\mathbf{v}}\Big] +$$

$$\frac{1}{Re}\frac{\partial}{\partial z}\Big[\mu\Big(\frac{\partial \overline{v}_3}{\partial r} + \frac{\partial v_1}{\partial z}\Big)\Big] + \frac{1}{r}\Big[\frac{2\mu}{Re}\frac{\partial v_1}{\partial r} + \Big(\frac{\mu_2}{Re_2} + \frac{2}{3}\frac{\mu}{Re}\Big)\underline{\nabla}\cdot\underline{\mathbf{v}}\Big], \tag{5.18}$$

$$\rho\frac{D\overline{v}_3}{Dt} = -\frac{1}{\gamma M_\infty^2}\frac{\partial P}{\partial z} + \frac{1}{Re}\frac{\partial}{\partial r}\Big[\mu\Big(\frac{\partial \overline{v}_3}{\partial r} + \frac{\partial v_1}{\partial z}\Big)\Big] + \frac{\partial}{\partial z}\Big[\frac{2\mu}{Re}\frac{\partial \overline{v}_3}{\partial z}$$

$$+\Big(\frac{\mu_2}{Re_2} + \frac{2}{3}\frac{\mu}{Re}\Big)\underline{\nabla}\cdot\underline{\mathbf{v}}\Big] + \frac{1}{Re}\frac{\mu}{r}\Big(\frac{\partial \overline{v}_3}{\partial r} + \frac{\partial v_1}{\partial z}\Big), \tag{5.19}$$

$$\rho\frac{DT}{Dt} - \frac{\gamma-1}{\gamma}\frac{DP}{Dt} = (\gamma-1)M_\infty^2\Big\{\frac{2\mu}{Re}\Big[\Big(\frac{\partial v_1}{\partial r}\Big)^2 + \Big(\frac{\partial \overline{v}_3}{\partial z}\Big)^2 + \frac{1}{2}\Big(\frac{\partial v_1}{\partial z} +$$

$$\frac{\partial \overline{v}_3}{\partial r}\Big)^2\Big] + \frac{\mu_2}{Re_2}(\underline{\nabla}\cdot\underline{\mathbf{v}})^2\Big\} + \frac{1}{\sigma Re}\frac{1}{r}\frac{\partial}{\partial r}\Big(\mu r\frac{\partial T}{\partial r}\Big) + \frac{1}{\sigma Re}\frac{\partial}{\partial z}\Big(\mu\frac{\partial T}{\partial z}\Big), \tag{5.20}$$

where

$$Re_2 = \frac{\mu_{2\infty}^*}{\rho_\infty^* U_\infty^* a^*}. \tag{5.21}$$

From the inviscid nonlinear critical layer analysis developed in the previous chapter, we note that we are only interested in terms up to $O(\delta^{7/5})$, therefore to the required order of approximation we can write (in terms of the transform variables ζ, Y and t_1, which are defined by (4.159), (4.154) and (4.160), respectively)

$$\overline{D}v_1 = -\frac{1}{\gamma M_\infty^2}\delta^{-2/5}\frac{T}{P}\frac{\partial P}{\partial Y} + O(\delta^{9/5}), \tag{5.22}$$

$$\overline{D}\overline{v}_3 = -\frac{\lambda}{\gamma M_\infty^2}\frac{T}{P}\frac{\partial P}{\partial \zeta} + \frac{\hat{\lambda}\delta^{2/5}CT}{P}\Big[\frac{\partial T}{\partial Y}\frac{\partial \overline{v}_3}{\partial Y} + T\frac{\partial^2 \overline{v}_3}{\partial Y^2}\Big] + \frac{\hat{\lambda}\delta^{4/5}CT^2}{rP}\frac{\partial \overline{v}_3}{\partial Y} + O(\delta^{11/5}), \tag{5.23}$$

$$\overline{D}T - \frac{(\gamma-1)T}{\gamma P}\overline{D}P = M_\infty^2(\gamma-1)\frac{CT^2\hat{\lambda}}{P}\delta^{2/5}\Big(\frac{\partial \overline{v}_3}{\partial Y}\Big)^2$$

$$+\frac{\hat{\lambda}C\delta^{2/5}T}{\sigma P}\Big[T_Y^2 + \frac{\delta^{2/5}TT_Y}{r} + TT_{YY}\Big] + O(\delta^{9/5}), \tag{5.24}$$

where the non-dimensional linear Chapman viscosity law has been assumed, i.e.

$$\mu = CT, \tag{5.25}$$

C being assumed constant and λ is defined by equation (4.161). The '$\overline{D}$' operator is the same as defined in the inviscid case (equation (4.167)).

Substituting the equation of continuity (4.72) into equation (5.24), gives

$$\frac{\overline{D}T}{\gamma T} + \frac{\gamma-1}{\gamma}\Big(\delta^{-2/5}\frac{\partial v_1}{\partial Y} + \lambda\frac{\partial \overline{v}_3}{\partial \zeta} + \frac{v_1}{r}\Big) = \frac{\overline{D}P}{\gamma P} + \Big(\delta^{-2/5}\frac{\partial v_1}{\partial Y} + \lambda\frac{\partial \overline{v}_3}{\partial \zeta} + \frac{v_1}{r}\Big)$$

$$= M_\infty^2(\gamma-1)\frac{CT\hat{\lambda}}{P}\delta^{2/5}\Big(\frac{\partial \overline{v}_3}{\partial Y}\Big)^2 + \frac{\hat{\lambda}C\delta^{2/5}}{\sigma P}\Big(T_Y^2 + \delta^{2/5}\frac{TT_Y}{r} + TT_{YY}\Big). \tag{5.26}$$

Following in the manner of the theory developed for the inviscid case, it is again found convenient to use the azimuthal vorticity component,

$$\Omega = \lambda\frac{\partial v_1}{\partial \zeta} - \delta^{-2/5}\frac{\partial \overline{v}_3}{\partial Y}, \tag{5.27}$$

as a dependent variable. The critical-layer vorticity equation now has the form

$$\overline{D}\Omega - \frac{\Omega}{\gamma P}\overline{D}P = -\frac{\lambda\delta^{-2/5}}{\gamma M_\infty^2}\frac{1}{P}\Big[\frac{\partial P}{\partial Y}\frac{\partial T}{\partial \zeta} - \frac{\partial P}{\partial \zeta}\frac{\partial T}{\partial Y}\Big] - \frac{\Omega M_\infty^2(\gamma-1)CT\hat{\lambda}}{P}\delta^{2/5}\Big(\frac{\partial \overline{v}_3}{\partial Y}\Big)^2$$

$$-\Omega\frac{\hat{\lambda}C\delta^{2/5}}{\sigma P}\Big[T_Y^2 + \delta^{2/5}\frac{TT_Y}{r} + TT_{YY}\Big] - \hat{\lambda}C\frac{\partial}{\partial Y}\Big\{\frac{T}{P}\frac{\partial}{\partial Y}[T\overline{v}_{3Y}]\Big\} + \frac{v_1\Omega}{r}$$

$$-\hat{\lambda}C\delta^{2/5}\frac{\partial}{\partial Y}\Big[\frac{T^2}{rP}\frac{\partial \overline{v}_3}{\partial Y}\Big]. \tag{5.28}$$

(5.26)

Substituting expansions (5.14) - (5.17) into equation (5.26) and equating powers in δ gives a sequence of partial differential equations for the temperature terms of the form

$$\hat{\mathcal{L}}_1 \overline{T}_{11} = -\hat{\psi}_0 T_c', \tag{5.29}$$

$$\hat{\mathcal{L}}_1 \overline{T}_{12} = -\hat{\psi}_0 \overline{T}_{11Y}, \tag{5.30}$$

$$\begin{aligned}\hat{\mathcal{L}}_1 \overline{T}_{13} = {} & \frac{T_c'}{w_c'}\Big(\frac{2}{r_i} - \frac{w_c''}{2w_c'}\Big) Rl\{e^{i\alpha_0\zeta}[i\alpha_0 Y w_c' A^\dagger + A_{t_1}^\dagger]\} + \overline{\mathcal{L}}_s \overline{T}_{11} \\ & -\tilde{\psi}_0 \overline{T}_{12Y} + T_c' Y\Big(\frac{w_c''}{w_c'} - \frac{T_c''}{T_c'}\Big)\hat{\psi}_0 + \frac{T_c' T_c}{w_c' \gamma M_\infty^2}\Big[\hat{p}_{13\zeta} + \frac{\alpha_1}{\alpha_0}\hat{p}_{11\zeta}\Big] \\ & + \frac{T_c' \hat{\lambda} C T_c^2}{w_c' \sigma}\hat{Q}_{1Y} - \frac{\hat{\lambda} C T_c' T_c}{w_c'}[T_c \hat{Q}_{1Y} + w_c' \overline{T}_{11Y}] + \frac{\hat{\lambda} C T_c}{\sigma}\Big[2T_c' \overline{T}_{11Y} \\ & + \frac{T_c \overline{T}_{11Y}}{r_i} + T_c' Y \overline{T}_{11YY}\Big],\end{aligned} \tag{5.31}$$

where

$$\overline{T}_{11} = \hat{T}_{11}, \tag{5.32}$$

$$\begin{aligned}\overline{T}_{12} = \hat{T}_{12} - \frac{1}{2}T_c'' Y^2 - \hat{\lambda} C t_1\Big[& -\frac{u_c T_c'}{C} + \frac{T_c^2 T_c''}{\sigma} + M_\infty^2(\gamma - 1)T_c^2 (w_c')^2 \\ & + \frac{T_c^2}{\sigma}\Big(\frac{T_c'}{r_i} + \frac{(T_c')^2}{T_c}\Big)\Big],\end{aligned} \tag{5.33}$$

$$\overline{T}_{13} = \hat{T}_{13} - \frac{\gamma - 1}{\gamma} T_c \hat{p}_{11} - \frac{T_c'}{w_c'}\hat{\phi}_1 + \frac{T_c'}{w_c'}\Big[\frac{2}{r_i} - \frac{w_c''}{2w_c'}\Big] Rl[A^\dagger e^{i\alpha_0\zeta}], \tag{5.34}$$

and we have defined the operators

$$\hat{\mathcal{L}}_1 = \tilde{\mathcal{L}} - \frac{\hat{\lambda} C T_c^2}{\sigma} \frac{\partial^2}{\partial Y^2}, \tag{5.35}$$

$$\overline{\mathcal{L}}_s = \Big[-\frac{Y^2}{2} w_c'' - a_c \hat{\lambda} t_1 + Y w_c' \Big(\frac{T_c'}{T_c} Y - \frac{\alpha_1}{\alpha_0} \Big) \Big] \frac{\partial}{\partial \zeta} + \frac{T_c'}{T_c} Y \frac{\partial}{\partial t_1} - \hat{\lambda} u_c T_c' \frac{\partial}{\partial Y}, \tag{5.36}$$

the operator $\tilde{\mathcal{L}}$ having been defined in Chapter 4, namely (4.178).

Substituting expansions (5.14) - (5.17) into the critical-layer vorticity equation (5.28) yields the following sequence of partial differential equations

$$\hat{\mathcal{L}}_2 \hat{Q}_1 = \hat{\lambda} C T_c \Big(\frac{\sigma - 1}{\sigma} \Big) w_c' \overline{T}_{11YY}, \tag{5.37}$$

$$\hat{\mathcal{L}}_2 \hat{Q}_2 = -\hat{\psi}_0 \hat{Q}_{1Y} + \frac{2 w_c'}{T_c} \overline{T}_{11Y} \hat{\psi}_0 + 2 T_c w_c' \hat{\lambda} C \Big(\frac{\sigma - 1}{\sigma} \Big) \overline{T}_{12YY}, \tag{5.38}$$

$$\begin{aligned} \hat{\mathcal{L}}_2 \hat{Q}_3 = {} & -\hat{Q}_{1\zeta} \Big[\frac{Y^2}{2} w_c'' + a_c \hat{\lambda} t_1 + \frac{\alpha_1}{\alpha_0} Y w_c' \Big] - \hat{\psi}_0 \hat{Q}_{2Y} \\ & + \frac{a_c \hat{\lambda}}{T_c} \overline{T}_{11Y} - \Upsilon Rl\{ A_{t_1}^{\dagger} e^{i\alpha_0 \zeta} \} + \hat{\lambda} C B_1 - \frac{\hat{\lambda} C}{\sigma} B_2, \end{aligned} \tag{5.39}$$

where

$$\hat{Q}_1 = \hat{\phi}_{1Y}, \tag{5.40}$$

$$\begin{aligned} \hat{Q}_2 = {} & \hat{\phi}_{2Y} - \frac{w_c'}{T_c} \hat{T}_{12} + \hat{\lambda} C t_1 \Big[-\frac{w_c' u_c T_c'}{C T_c} + 2 M_\infty^2 (\gamma - 1)(w_c')^3 T_c - T_c w_c' T_c'' \Big(\frac{\sigma - 2}{\sigma} \Big) \\ & - T_c^2 w_c''' - 2 T_c T_c' w_c' \Big(\frac{\sigma - 1}{\sigma} \frac{w_c''}{w_c'} - \frac{1}{r_i} \Big) \Big] - \frac{T_c'}{T_c} a_c \hat{\lambda} t_1 + \frac{Y^2}{2} \Big(\frac{w_c' T_c''}{T_c} - w_c''' \Big), \end{aligned} \tag{5.41}$$

$$\hat{Q}_3 = \hat{\phi}_{3Y} - \hat{\psi}_{0\zeta} - \frac{w_c'}{\gamma} \hat{p}_{11} - \frac{T_c'}{T_c} \hat{\phi}_1 - \Upsilon Rl[A^{\dagger} e^{i\alpha_0 \zeta}], \tag{5.42}$$

Υ is defined, as before, by

$$\Upsilon = -\frac{T_c''}{T_c} - \frac{1}{2}\Big(\frac{w_c''}{w_c'}\Big)^2 + \frac{1}{2}\Big(\frac{w_c''}{w_c'} - \frac{2}{r_i}\Big)^2 + \frac{w_c'''}{w_c'}, \tag{5.43}$$

and

$$B_1 = T_c^2\Big(\frac{3T_c'}{T_c} + \frac{1}{r_i}\Big)\hat{Q}_{1Y} + YT_c w_c'\Big(\frac{w_c''}{w_c'} + \frac{T_c'}{T_c}\Big)\overline{T}_{11YY} + T_c w_c' \hat{T}_{13YY}$$

$$+2T_c'YT_c\hat{Q}_{1YY} + T_c w_c'\Big[\frac{2w_c''}{w_c'} + \frac{1}{r_i}\Big]\overline{T}_{11Y} + \frac{u_c}{C}\Big[\frac{w_c'}{T_c}\overline{T}_{11Y} - \hat{Q}_{1Y}\Big], \tag{5.44}$$

$$B_2 = T_c w_c'\Big[\frac{2T_c'}{T_c} + \frac{1}{r_i}\Big]\overline{T}_{11Y} + w_c'T_c\hat{T}_{13YY} + Yw_c'T_c\Big[\frac{T_c'}{T_c} + \frac{w_c''}{w_c'}\Big]\overline{T}_{11YY}, \tag{5.45}$$

$$B_2 = T_c w_c'\Big[\frac{2T_c'}{T_c} + \frac{1}{r_i}\Big]\overline{T}_{11Y} + w_c'T_c\hat{T}_{13YY} + Yw_c'T_c\Big[\frac{T_c'}{T_c} + \frac{w_c''}{w_c'}\Big]\overline{T}_{11YY}, \qquad (5.45)$$

whilst we have defined the operator

$$\hat{\mathcal{L}}_2 = \tilde{\mathcal{L}} - \hat{\lambda} C T_c^2 \frac{\partial^2}{\partial Y^2}. \tag{5.46}$$

Inspecting (5.32) - (5.34) and (5.40) - (5.42) it is noted that these terms have been matched with the appropriate inner expansions of the outer solution and far-field conditions. As before, $\overline{T}_{11}$, $\overline{T}_{13}$, $\hat{Q}_1$ and $\hat{Q}_3$ have been matched in such a way as to ensure all four terms tend to zero as $Y \to \pm\infty$. In the case of the $\overline{T}_{12}$ and $\hat{Q}_2$ terms, because of the additional viscous contributions, we have to be more careful in our matching procedure. Matching is used to remove Y-dependent terms, ensuring that both terms tend to a constant as $Y \to \pm\infty$, i.e. are bounded. It is found, however, that there is no matching procedure by which these respective constants can be removed, as this results in the solution for these respective terms acquiring unwanted singularities. It is found necessary, therefore, to carry these constants through our solution procedure.

Following in the manner of the inviscid theory the normalized variables $\bar{t}$, X and η, as defined by (4.213) - (4.215) are introduced to aid the analysis, where the coordinate shifts X_0 and t_0 again ensure that the instability wave matches correctly with the initial linear upstream disturbance term.

From inviscid theory $\overline{T}_{11}$ is assumed to have a solution of the form

$$\overline{T}_{11} = Rl[\overline{T}_{11}^{(1)}(\eta, \bar{t})e^{iX}]. \tag{5.47}$$

Substitution of equation (5.47) into equation (5.29) and equating $O(e^{iX})$ terms yields

$$\overline{T}_{11\bar{t}}^{(1)} + i\eta\overline{T}_{11}^{(1)} - \frac{\hat{\Omega}}{\sigma}\overline{T}_{11\eta\eta}^{(1)} = -\frac{iT_c'\alpha_1 w_c}{2\alpha_0 w_c'}A(\bar{t}), \tag{5.48}$$

where

$$\hat{\Omega} = -\frac{8\alpha_0^2\hat{\lambda}CT_c^2}{w_c'\alpha_1^3}, \tag{5.49}$$

and $A(\bar{t})$ is defined by (4.218), and we have set $\mathcal{D} \equiv w_c'$.

To solve the above equation the Fourier transform method of Hickernell (1984) must be employed. Defining

$$z_1(K, \bar{t}) = \int_{-\infty}^{\infty} \overline{T}_{11}^{(1)}(\eta, \bar{t})e^{-iK\eta}d\eta, \tag{5.50}$$

equation (5.48) can be re-expressed in the form

$$\int_{-\infty}^{\infty}\left\{\overline{T}_{11\bar{t}}^{(1)} + i\eta\overline{T}_{11}^{(1)} - \frac{\hat{\Omega}}{\sigma}\overline{T}_{11\eta\eta}^{(1)}\right\}e^{-iK\eta}d\eta = \int_{-\infty}^{\infty}\left\{-\frac{iT_c'\alpha_1 w_c}{2\alpha_0 w_c'}A(\bar{t})\right\}e^{-iK\eta}d\eta. \tag{5.51}$$

Integrating term by term we have

$$\int_{-\infty}^{\infty} \overline{T}_{11\bar{t}}^{(1)}e^{-iK\eta}d\eta = \frac{\partial}{\partial\bar{t}}\left[\int_{-\infty}^{\infty} \overline{T}_{11}^{(1)}e^{-iK\eta}d\eta\right] = z_{1\bar{t}}, \tag{5.52}$$

and

$$\int_{-\infty}^{\infty} i\eta \overline{T}_{11}^{(1)} e^{-iK\eta} d\eta = -\frac{\partial}{\partial K}\Big[\int_{-\infty}^{\infty} \overline{T}_{11}^{(1)} e^{-iK\eta} d\eta\Big] = -z_{1K}. \tag{5.53}$$

Integrating by parts twice and making use of the result

$$\overline{T}_{11\eta}^{(1)},\ \overline{T}_{11}^{(1)} \to 0, \qquad \text{as} \qquad \eta \to \pm\infty, \tag{5.54}$$

gives

$$\int_{-\infty}^{\infty} \overline{T}_{11\eta\eta}^{(1)} e^{-iK\eta} d\eta = (iK)^2\Big[\int_{-\infty}^{\infty} \overline{T}_{11}^{(1)} e^{-iK\eta} d\eta\Big] = -K^2 z_1. \tag{5.55}$$

Thus equation (5.51) can be re-written as

$$z_{1\bar{t}} - z_{1K} + \frac{\hat{\Omega}K^2}{\sigma} z_1 = -\frac{iT_c'\alpha_1 w_c}{2\alpha_0 w_c'} A(\bar{t})\delta(K)2\pi. \tag{5.56}$$

Switching to the variables K and $\tilde{t} = \bar{t} + K$, reduces equation (5.56) to the first-order ordinary differential equation

$$z_{1K} - \frac{\hat{\Omega}K^2}{\sigma} z_1 = \frac{i\pi T_c'\alpha_1 w_c}{\alpha_0 w_c'} A(\tilde{t} - K)\delta(K), \tag{5.57}$$

where we have made use of the result

$$\frac{\partial z_1}{\partial K} = \frac{\partial z_1}{\partial \tilde{t}}\frac{\partial \tilde{t}}{\partial K} + \frac{\partial z_1}{\partial K}\frac{\partial K}{\partial K} = \frac{\partial z_1}{\partial \tilde{t}} + \frac{\partial z_1}{\partial K}. \tag{5.58}$$

Solving equation (5.57) gives

$$\begin{aligned} z_1 &= \hat{B}_1 e^{\hat{\Omega}K^3/3\sigma} && \text{if} \quad K < 0, \\ z_1 &= \Big\{\frac{i\pi T_c'\alpha_1 w_c}{\alpha_0 w_c'} A(\tilde{t}) + \hat{B}_1\Big\} e^{\hat{\Omega}K^3/3\sigma} && \text{if} \quad K > 0, \end{aligned} \tag{5.59}$$

where $\hat{B}_1$ is a constant to be determined. Since $\alpha_1 < 0$, then by (5.49) $\hat{\Omega} > 0$, implying that z_1 will grow exponentially as $K \to \infty$. However, in the far field z_1 is required to be bounded and this can only be achieved if

$$\hat{B}_1 = -\frac{i\pi T_c' \alpha_1 w_c}{\alpha_0 w_c'} A(\hat{t}). \tag{5.60}$$

Therefore equation (5.57) has a solution of the form

$$z_1(K, \bar{t}) = -\frac{i\pi T_c' \alpha_1 w_c}{\alpha_0 w_c'} e^{\hat{\Omega} K^3/3\sigma} A(\bar{t} + K) H(-K), \tag{5.61}$$

where $H(K)$ represents the Heaviside function.

Defining the inverse Fourier transform of $z_1(K, \bar{t})$ to have the form

$$\overline{T}_{11}^{(1)}(\eta, \bar{t}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} z_1(K, \bar{t}) e^{iK\eta} dK, \tag{5.62}$$

then by the form of (5.61) we have

$$\overline{T}_{11}^{(1)}(\eta, \bar{t}) = -\frac{iT_c' \alpha_1 w_c}{2\alpha_0 w_c'} \int_{-\infty}^{0} A(\bar{t} + K) e^{\hat{\Omega} K^3/3\sigma} e^{iK\eta} dK. \tag{5.63}$$

Introducing the transform variable $s_1 = \bar{t} + K$, equation (5.63) can be re-written

$$\overline{T}_{11}^{(1)}(\eta, \bar{t}) = -\frac{iT_c' \alpha_1 w_c}{2\alpha_0 w_c'} \int_{-\infty}^{\bar{t}} e^{-\hat{\Omega}(\bar{t} - s_1)^3/3\sigma} A(s_1) e^{-i(\bar{t} - s_1)} ds_1. \tag{5.64}$$

Comparing (5.64) with the corresponding inviscid result, namely (4.219), it is noted that the only difference is the term $\exp(-\hat{\Omega}(\bar{t} - s_1)^3/3\sigma)$.

Guided by the inviscid work of Chapter 4, it is assumed that $\hat{Q}_1$ can be expressed in the form

$$\hat{Q}_1 = Rl[\hat{Q}_1^{(1)}(\eta, \bar{t}) e^{iX}]. \tag{5.65}$$

Substituting equation (5.65) into equation (5.37), collecting $O(e^{iX})$ terms, applying Fourier transforms and switching to the variables K and $\tilde{t}$, yields

$$Y_{1K} - \hat{\Omega}K^2 Y_1 = \frac{\alpha_0^2\hat{\Omega}}{w_c^2\alpha_1^2 T_c}\Big(\frac{\sigma-1}{\sigma}\Big)K^2 z_1, \tag{5.66}$$

where

$$Y_1(K,\bar{t}) = \int_{-\infty}^{\infty} Q_1^{(1)}(\eta,\bar{t})e^{-iK\eta}d\eta. \tag{5.67}$$

Note that $\hat{Q}_1^{(1)}$ has been normalized by employing relation (4.224), where 'tilde' terms have been replaced by 'hatted' terms here.

Integrating equation (5.66) gives

$$Y_1 = \Big\{ -\frac{i\pi T_c'\alpha_0\hat{\Omega}}{w_c\alpha_1 T_c w_c'}\Big(\frac{\sigma-1}{\sigma}\Big)A(\tilde{t})\int_{-\infty}^{K} l_1^2 H(-l_1)\exp\Big[-\frac{\hat{\Omega}l_1^3}{3}\Big(\frac{\sigma-1}{\sigma}\Big)\Big]dl_1 + \hat{B}_2\Big\}e^{\hat{\Omega}K^3/3}, \tag{5.68}$$

where $\hat{B}_2$ is a constant. Integrating (5.68) by parts yields

$$\begin{aligned} Y_1 &= \Big\{\frac{i\pi T_c'\alpha_0}{w_c\alpha_1 T_c w_c'}A(\tilde{t})\exp\Big[-\frac{\hat{\Omega}l_1^3}{3}\Big(\frac{\sigma-1}{\sigma}\Big)\Big]\Big|_{-\infty}^{K} + \hat{B}_2\Big\}e^{\hat{\Omega}K^3/3} \quad \text{if} \quad K<0, \\ Y_1 &= \Big\{\frac{i\pi T_c'\alpha_0}{w_c\alpha_1 T_c w_c'}A(\tilde{t})\exp\Big[-\frac{\hat{\Omega}l_1^3}{3}\Big(\frac{\sigma-1}{\sigma}\Big)\Big]\Big|_{-\infty}^{0} + \hat{B}_2\Big\}e^{\hat{\Omega}K^3/3} \quad \text{if} \quad K>0. \end{aligned} \tag{5.69}$$

Since Y_1 is required to be bounded as $K \to \infty$, then

$$\hat{B}_2 = -\frac{i\pi T_c'\alpha_0}{\alpha_1 w_c T_c w_c'}A(\tilde{t}), \tag{5.70}$$

where $\sigma \leq 1$ is a necessary condition if Y_1 is to be zero for $K \to -\infty$. Note, if $\sigma = 1$ then the first exponential term in (5.69) disappears. Therefore equation (5.66) has a solution of the form

$$Y_1(K,\bar{t}) = -\frac{i\pi T_c'\alpha_0}{\alpha_1 w_c T_c w_c'} e^{\hat{\Omega}K^3/3} A(\bar{t}+K)H(-K)\Big\{1 - \exp\Big[-\frac{\hat{\Omega}K^3}{3}\Big(\frac{\sigma-1}{\sigma}\Big)\Big]\Big\}. \quad (5.71)$$

By the form of equation (5.30) $\overline{T}_{12}$ is assumed to have the form

$$\overline{T}_{12} = Rl[\overline{T}_{12}^{(0)}(\eta,\bar{t}) + \overline{T}_{12}^{(2)}(\eta,\bar{t})e^{2iX}]. \quad (5.72)$$

Substituting equation (5.72) into equation (5.30), and collecting $O(1)$ terms yields

$$Rl[\overline{T}_{12\bar{t}}^{(0)} - \frac{\hat{\Omega}}{\sigma}\overline{T}_{12\eta\eta}^{(0)}] = -\frac{w_c}{2w_c'} Rl[iA^*(\bar{t})\overline{T}_{11\eta}^{(1)}]. \quad (5.73)$$

Because the real parts of the respective sides of (5.73) are equated, great care must be taken when applying Fourier transforms. Defining

$$z_2(K,\bar{t}) = \int_{-\infty}^{\infty} \overline{T}_{12}^{(0)}(\eta,\bar{t})e^{-iK\eta}d\eta, \quad (5.74)$$

then clearly

$$z_2^*(-K,\bar{t}) = \int_{-\infty}^{\infty} \left[\overline{T}_{12}^{(0)}(\eta,\bar{t})\right]^* e^{-iK\eta}d\eta. \quad (5.75)$$

Therefore the Fourier transform of (5.73) is

$$\begin{aligned} z_{2\bar{t}}(K,\bar{t}) + \frac{\hat{\Omega}}{\sigma}K^2 z_2(K,\bar{t}) + z_{2\bar{t}}^*(-K,\bar{t}) + \frac{\hat{\Omega}}{\sigma}K^2 z_2^*(-K,\bar{t}) = \\ \frac{w_c}{2w_c'}[KA^*(\bar{t})z_1(K,\bar{t})] - \frac{w_c}{2w_c'}[KA(\bar{t})z_1^*(-K,\bar{t})], \end{aligned} \quad (5.76)$$

where it has been assumed that $\overline{T}_{12}^{(0)}$ and $\overline{T}_{12\eta}^{(0)}$ (and their respective complex conjugates) are bounded as $\eta \to \pm\infty$, respectively tending towards the same constant in both limits, giving zero resultant.

Discarding complex conjugate terms in z_i (where $i = 1, 2$) occuring in (5.76) and integrating with respect to K yields

$$z_2(K, \bar{t}) = -\frac{\pi w_c^2 T_c' \alpha_1}{2\alpha_0 (w_c')^2} \Big\{ \int_{-\infty}^{\bar{t}} H(-K) F(K, \bar{t}) dr_1 \Big\}, \tag{5.77}$$

where

$$F(K, \bar{t}) = iK \exp\Big[-\frac{\hat{\Omega} K^2}{3\sigma} (3(\bar{t} - r_1) - K) \Big] A^*(r_1) A(r_1 + K). \tag{5.78}$$

Note, if the complex conjugate terms in (5.76) are integrated instead, the resultant equation for $z_2^*(-K, \bar{t})$ confirms the identity (5.75).

The Fourier transformed $O(e^{2iX})$ equation has the form

$$z_{3K} - \frac{\hat{\Omega} K^2}{2\sigma} z_3 = \frac{w_c}{4w_c'} A(\hat{t} - \frac{K}{2}) K z_1, \tag{5.79}$$

where

$$z_3(K, \bar{t}) = \int_{-\infty}^{\infty} \overline{T}_{12}^{(2)}(\eta, \bar{t}) e^{-iK\eta} d\eta, \tag{5.80}$$

and we have introduced the transform variable

$$\hat{t} = \bar{t} + \frac{K}{2}. \tag{5.81}$$

Integrating equation (5.79) gives

$$z_3 = \frac{i\pi T_c' w_c^2 \alpha_1}{4\alpha_0 (w_c')^2} H(-K) \int_K^0 l_1 \exp\Big[\frac{\hat{\Omega}}{6\sigma} (K^3 + l_1^3) \Big] A(\hat{t} - \frac{l_1}{2}) A(\hat{t} + \frac{l_1}{2}) dl_1, \tag{5.82}$$

where $z_3 = z_3(K, \hat{t} - \frac{K}{2})$.

By the form of (5.38), we assume $\hat{Q}_2$ can be expressed in the form

$$\hat{Q}_2 = Rl[\hat{Q}_2^{(0)}(\eta, \bar{t}) + \hat{Q}_2^{(2)}(\eta, \bar{t}) e^{2iX}]. \tag{5.83}$$

Substituting equation (5.83) into equation (5.38), collecting $O(1)$ terms, and Fourier transforming yields

$$Y_{2\bar{t}}(K,\bar{t}) + \hat{\Omega}K^2 Y_2(K,\bar{t}) = \frac{w_c}{2w_c'} K A^*(\bar{t}) Y_1(K,\bar{t})$$

$$-\frac{\alpha_0^2}{w_c T_c \alpha_1^2 w_c'} K A^*(\bar{t}) z_1(K,\bar{t}) - \frac{2\alpha_0^2 \hat{\Omega}}{\alpha_1^2 w_c^2 T_c}\Big(\frac{\sigma-1}{\sigma}\Big) K^2 z_2(K,\bar{t}), \tag{5.84}$$

while the complex conjugate equation has the form

$$Y_{2\bar{t}}^*(-K,\bar{t}) + \hat{\Omega}K^2 Y_2^*(-K,\bar{t}) = -\frac{w_c}{2w_c'} K A(\bar{t}) Y_1^*(-K,\bar{t})$$

$$+\frac{\alpha_0^2}{w_c T_c \alpha_1^2 w_c'} K A(\bar{t}) z_1^*(-K,\bar{t}) - \frac{2\alpha_0^2 \hat{\Omega}}{\alpha_1^2 w_c^2 T_c}\Big(\frac{\sigma-1}{\sigma}\Big) K^2 z_2^*(-K,\bar{t}), \tag{5.85}$$

where

$$Y_2(K,\bar{t}) = \int_{-\infty}^{\infty} Q_0^{(2)}(\eta,\bar{t}) e^{-iK\eta} d\eta. \tag{5.86}$$

Integrating equation (5.84) with respect to $\bar{t}$ gives

$$Y_2(K,\bar{t}) = \frac{i\pi\alpha_0 T_c'}{2\alpha_1 T_c (w_c')^2} \int_{-\infty}^{\bar{t}} e^{-\hat{\Omega}K^2(\bar{t}-r_1)} \Big[2KA^*(r_1)A(r_1+K)H(-K)\exp\Big\{\frac{\hat{\Omega}K^3}{3\sigma}\Big\}$$

$$-KA^*(r_1)\exp\Big(\frac{\hat{\Omega}K^3}{3}\Big)A(r_1+K)H(-K)\Big\{1-\exp\Big[-\frac{\hat{\Omega}K^3}{3}\Big(\frac{\sigma-1}{\sigma}\Big)\Big]\Big\}\Big]dr_1$$

$$+\frac{\pi T_c'\alpha_0\hat{\Omega}}{T_c\alpha_1(w_c')^2}\Big(\frac{\sigma-1}{\sigma}\Big)\int_{-\infty}^{\bar{t}} K^3 e^{-\hat{\Omega}K^2(\bar{t}-r_1)} H(-K)\Big\{\int_{-\infty}^{r_1} \hat{v}_1 dr_2\Big\} dr_1, \tag{5.87}$$

where

$$\hat{v}_1 = iA^*(r_2)A(r_2+K)\exp\Big(-\frac{\hat{\Omega}K^2}{3\sigma}[3(r_1-r_2)-K]\Big). \tag{5.88}$$

Before proceeding it is found necessary to simplify the third contribution to (5.87), i.e. the term involving the double integral. Changing the order of integration this term can be expressed in the form

$$\frac{\pi T_c' \alpha_0 \hat{\Omega}}{T_c \alpha_1 (w_c')^2} \Big(\frac{\sigma-1}{\sigma}\Big) \int_{-\infty}^{\bar{t}} \hat{v}_2 \int_{r_2}^{\bar{t}} \exp\Big[\hat{\Omega} K^2 \Big(\frac{\sigma-1}{\sigma}\Big) r_1\Big] dr_1 dr_2, \tag{5.89}$$

where

$$\hat{v}_2 = iH(-K)A^*(r_2)A(r_2+K)K^3 e^{-\hat{\Omega}K^2\bar{t}} \exp\Big[\frac{\hat{\Omega}K^2}{3\sigma}(3r_2+K)\Big]. \tag{5.90}$$

On carrying out the inner integration, (5.89) simplifies to

$$\frac{\pi T_c' \alpha_0}{T_c \alpha_1 (w_c')^2} \int_{-\infty}^{\bar{t}} iH(-K)A^*(r_1)A(r_1+K)K\Big\{\exp\Big[-\frac{\hat{\Omega}K^2}{3\sigma}(3(\bar{t}-r_1)-K)\Big] - \exp\Big[\hat{\Omega}K^2(\bar{t}-r_1)\Big]\exp\Big[\frac{\hat{\Omega}K^3}{3\sigma}\Big]\Big\} dr_1, \tag{5.91}$$

where we have set $r_2 \equiv r_1$.

Equation (5.87) can now be re-expressed in the simplified form

$$Y_2(K,\bar{t}) = \frac{\pi T_c' \alpha_0}{2T_c \alpha_1 (w_c')^2} \int_{-\infty}^{\bar{t}} iKH(-K)A^*(r_1)A(r_1+K)\Big\{2\exp\Big[-\frac{\hat{\Omega}K^2}{3\sigma}(3(\bar{t}-r_1)-K)\Big] - \Big\{1-\exp\Big[-\frac{\hat{\Omega}K^3}{3}\Big(\frac{\sigma-1}{\sigma}\Big)\Big]\Big\}\exp\Big[-\frac{\hat{\Omega}K^2}{3}(3(\bar{t}-r_1)-K)\Big]\Big\} dr_1. \tag{5.92}$$

The Fourier transformed $O(e^{2iX})$ equation in terms of the variables K and $\hat{t}$, has the form

$$Y_{3K} - \frac{\hat{\Omega}K^2}{2}Y_3 = -\frac{\alpha_0^2}{2w_c T_c \alpha_1^2 w_c'}A(\hat{t} - \frac{K}{2})Kz_1 + \frac{w_c}{4w_c'}A(\hat{t} - \frac{K}{2})KY_1$$

$$+\frac{\alpha_0^2\hat{\Omega}}{w_c^2\alpha_1^2 T_c}\Big(\frac{\sigma - 1}{\sigma}\Big)K^2 z_3, \qquad (5.93)$$

where

$$Y_3(K,\bar{t}) = \int_{-\infty}^{\infty} Q_2^{(2)}(\eta,\bar{t})e^{-iK\eta}d\eta. \qquad (5.94)$$

Integrating equation (5.93) yields

$$Y_3 = \frac{i\pi T_c'\alpha_0}{4\alpha_1 T_c(w_c')^2}H(-K)\int_K^0 \Big\{ -2e^{\frac{\hat{\Omega}}{6}(K^3 - l_1^3)}A(\hat{t} - \frac{l_1}{2})l_1 A(\hat{t} + \frac{l_1}{2})\exp\Big(\frac{\hat{\Omega}l_1^2}{3\sigma}\Big)$$

$$+\Big\{1 - \exp\Big[-\frac{\hat{\Omega}l_1^3}{3}\Big(\frac{\sigma - 1}{\sigma}\Big)\Big]\Big\}A(\hat{t} - \frac{l_1}{2})l_1 A(\hat{t} + \frac{l_1}{2})e^{\frac{\hat{\Omega}}{6}(K^3 + l_1^3)}\Big\}dl_1$$

$$-\frac{i\pi\alpha_0 T_c'\hat{\Omega}}{4T_c\alpha_1(w_c')^2}\Big(\frac{\sigma - 1}{\sigma}\Big)H(-K)\int_K^0 l_1^2\exp\Big[\frac{\hat{\Omega}}{6}(K^3 - l_1^3)\Big]\int_{l_1}^0 \hat{v}_3 ds_2 dl_1, \quad (5.95)$$

where

$$\hat{v}_3 = s_2\exp\Big[\frac{\hat{\Omega}}{6\sigma}(l_1^3 + s_2^3)\Big]A(\hat{t} - \frac{s_2}{2})A(\hat{t} + \frac{s_2}{2}), \qquad (5.96)$$

and the boundary condition $Y_3 \to 0$ as $K \to \infty$ has been applied.

Again, changing the order of integration in the third contribution to the right-hand-side of (5.95), yields the simplified equation

$$Y_3 = \frac{i\pi T_c'\alpha_0}{4\alpha_1 T_c(w_c')^2}H(-K)\int_K^0 l_1 A(\hat{t} - \frac{l_1}{2})A(\hat{t} + \frac{l_1}{2})\Big\{ -2\exp\Big[\frac{\hat{\Omega}}{6\sigma}(l_1^3 + K^3)\Big]$$

$$+\Big\{1 - \exp\Big[-\frac{\hat{\Omega}l_1^3}{3}\Big(\frac{\sigma - 1}{\sigma}\Big)\Big]\Big\}\exp\Big[\frac{\hat{\Omega}}{6}(K^3 + l_1^3)\Big]\Big\}dl_1, \quad (5.97)$$

where $Y_3 = Y_3(K, \hat{t} - \frac{K}{2})$.

Following in the manner of the above work, it is assumed that $\overline{T}_{13}$ can be expressed in the form

$$\overline{T}_{13} = Rl[\overline{T}_{13}^{(1)}(\eta,\bar{t})e^{iX} + \overline{T}_{13}^{(3)}(\eta,\bar{t})e^{3iX}]. \tag{5.98}$$

Substituting equation (5.98) into (5.31), collecting the $O(e^{iX})$ terms, and Fourier transforming yields the following system in terms of K and $\tilde{t}$

$$\begin{aligned}
z_{4K} - \frac{\hat{\Omega}K^2}{\sigma}z_4 = {} & \frac{T_c'\pi\alpha_1^2 w_c}{2\alpha_0^2 w_c'}\Big(\frac{w_c''}{2w_c'} - \frac{2}{r_i}\Big)[\delta(K)A_{\tilde{t}} - A(\tilde{t}-K)\delta_K] + \frac{4\alpha_0^2 T_c'\hat{\lambda}u_c}{w_c'\alpha_1^3}iKz_1 \\
& + \frac{iT_c'\alpha_1}{2\alpha_0 T_c}(z_{1\tilde{t}K} + z_{1\tilde{t}\tilde{t}}) + \frac{2\alpha_0}{\alpha_1 w_c'}\Big\{\frac{2a_c\hat{\lambda}}{\alpha_1 w_c'}(\tilde{t} - K + t_0)iz_1 - \frac{\alpha_1^2 w_c'}{2\alpha_0^2}(z_{1K} + z_{1\tilde{t}}) \\
& - i\Big(\frac{\alpha_1^2 T_c' w_c'}{4\alpha_0^2 T_c} - \frac{w_c''\alpha_1^2}{8\alpha_0^2}\Big)(z_{1KK} + 2z_{1K\tilde{t}} + z_{1\tilde{t}\tilde{t}})\Big\} \\
& - \frac{\alpha_1^2 w_c T_c' A(\tilde{t}-K)\pi}{2\alpha_0^2 w_c'}\Big(\frac{w_c''}{w_c'} - \frac{T_c''}{T_c'}\Big)\Big[\delta_K + \frac{2i\alpha_0}{w_c'}\delta(K)\Big] \\
& - \frac{iw_c}{w_c'}\Big[A(\tilde{t}-K)\frac{1}{2}iK\tilde{z}_2 - \frac{A^*(\tilde{t}-K)}{2}iKz_3\Big] - \frac{T_c' w_c^2\alpha_1^3}{2\alpha_0^3}iKY_1\hat{\Omega}\Big(\frac{\sigma-1}{\sigma}\Big) \\
& + \frac{\alpha_1\hat{\Omega}}{2\alpha_0 T_c\sigma}\Big[\frac{T_c}{r_i}iKz_1 + \frac{2\alpha_0 T_c'}{w_c'}K^2 z_1 - iT_c'K^2(z_{1K} + z_{1\tilde{t}})\Big] - \frac{\alpha_1 T_c'\hat{\Omega}}{2\alpha_0 T_c}iKz_1 \\
& + \frac{2iT_c'T_c\alpha_0}{\alpha_1(w_c')^2\gamma M_\infty^2}\Big(\hat{P}_{13}^{(1)}(\tilde{t}-K) + \frac{\alpha_1}{\alpha_0}\hat{P}_{11}^{(1)}(\tilde{t}-K)\Big)2\pi\delta(K),
\end{aligned} \tag{5.99}$$

where

$$z_4(K,\bar{t}) = \int_{-\infty}^{\infty} \overline{T}_{13}^{(1)}(\eta,\bar{t})e^{-iK\eta}d\eta, \tag{5.100}$$

$$\tilde{z}_2(K,\tilde{t}-K) = z_2(K,\tilde{t}-K) + z_2^*(-K,\tilde{t}-K), \tag{5.101}$$

and it has been assumed that

$$\hat{p}_{11} = Rl[\hat{P}_{11}^{(1)}(\bar{t})e^{iX}], \tag{5.102}$$

and

$$\hat{p}_{13} = Rl[\hat{P}_{13}^{(1)}(\bar{t})e^{iX} + \hat{P}_{13}^{(3)}(\bar{t})e^{3iX}]. \tag{5.103}$$

It should be noted that $\hat{p}_{13}$ is determined to independent of Y, from equation (5.22).

Equation (5.99) is now integrated with respect to K and the far-field condition $z_4 \to 0$ as $K \to \infty$ is applied. It is found that great care must be taken when applying this condition. Defining

$$\tilde{z}_4(K, \tilde{t} - K) = H(-K)\hat{z}_4(K, \tilde{t} - K), \tag{5.104}$$

where

$$\hat{z}_4(K, \tilde{t} - K) = \Big\{\frac{4\alpha_0^2 T_c' \hat{\lambda} u_c}{w_c' \alpha_1^3} iK\hat{z}_1 + \frac{iT_c'\alpha_1}{2\alpha_0 T_c}(\hat{z}_{1\tilde{t}K} + \hat{z}_{1\tilde{t}\tilde{t}}) + \frac{2\alpha_0}{\alpha_1 w_c'}\Big\{\frac{2a_c\hat{\lambda}}{\alpha_1 w_c'}(\tilde{t} - K$$
$$+t_0)i\hat{z}_1 - \frac{\alpha_1^2 w_c'}{2\alpha_0^2}(\hat{z}_{1K} + \hat{z}_{1\tilde{t}}) - i\Big(\frac{\alpha_1^2 T_c' w_c'}{4\alpha_0^2 T_c} - \frac{w_c''\alpha_1^2}{8\alpha_0^2}\Big)(\hat{z}_{1KK} + 2\hat{z}_{1K\tilde{t}} + \hat{z}_{1\tilde{t}\tilde{t}})\Big]\Big\}$$
$$-\frac{iw_c}{w_c'}\Big[\frac{1}{2}A(\tilde{t} - K)iK\hat{z}_2 - \frac{A^*(\tilde{t} - K)}{2}iK\hat{z}_3\Big] - \frac{T_c' w_c^2 \alpha_1^3}{2\alpha_0^3} iK\hat{Y}_1\hat{\Omega}\Big(\frac{\sigma - 1}{\sigma}\Big)$$
$$+\frac{\alpha_1\hat{\Omega}}{2\alpha_0 T_c \sigma}\Big[\frac{T_c}{r_i}iK\hat{z}_1 + \frac{2\alpha_0 T_c'}{w_c'}K^2\hat{z}_1 - iT_c'K^2(\hat{z}_{1K} + \hat{z}_{1\tilde{t}})\Big] - \frac{\alpha_1 T_c'\hat{\Omega}}{2\alpha_0 T_c}iK\hat{z}_1\Big\}, \tag{5.105}$$

and

$$\hat{z}_1(K, \bar{t}) = -\frac{i\pi T_c'\alpha_1 w_c}{\alpha_0 w_c'} e^{\hat{\Omega}K^3/3\sigma} A(\bar{t} + K), \tag{5.106}$$

$$\hat{z}_2(K, \bar{t}) = -\frac{\pi w_c^2 T_c' \alpha_1}{2\alpha_0 (w_c')^2}\int_{-\infty}^{\bar{t}} F(K, \bar{t})dr_1, \tag{5.107}$$

$$\hat{z}_3(K,\hat{t}-\frac{K}{2})=\frac{i\pi T_c' w_c^2\alpha_1}{4\alpha_0(w_c')^2}\int_K^0 l_1\exp\Big[\frac{\hat{\Omega}}{6\sigma}(K^3+l_1^3)\Big]A(\hat{t}-\frac{l_1}{2})A(\hat{t}+\frac{l_1}{2})dl_1, \quad (5.108)$$

$$\hat{Y}_1(K,\bar{t})=-\frac{i\pi T_c'\alpha_0}{\alpha_1 w_c T_c w_c'}e^{\hat{\Omega}K^3/3}A(\bar{t}+K)\Big\{1-\exp\Big[-\frac{\hat{\Omega}K^3}{3}\Big(\frac{\sigma-1}{\sigma}\Big)\Big]\Big\}, \quad (5.109)$$

the integrand $F(K,\bar{t})$ being defined by (5.78). For $K<0$, equation (5.99) has a solution of the form

$$z_4=\exp\Big(\frac{\hat{\Omega}K^3}{3\sigma}\Big)\Big\{\int_{-\infty}^K \hat{z}_4(l_1,\tilde{t}-l_1)\exp\Big(-\frac{\hat{\Omega}l_1^3}{3\sigma}\Big)dl_1+\hat{B}_4\Big\}, \quad (5.110)$$

while for $K>0$ the solution is given by

$$z_4=\exp\Big(\frac{\hat{\Omega}K^3}{3\sigma}\Big)\Big\{\Lambda(K,\tilde{t})+\int_{-\infty}^0 \hat{z}_4(l_1,\tilde{t}-l_1)\exp\Big(-\frac{\hat{\Omega}l_1^3}{3\sigma}\Big)dl_1$$
$$+\int_0^K \frac{w_c}{2w_c'}A(\tilde{t}-l_1)l_1\hat{z}_2^*(-l,\tilde{t}-l_1)\exp\Big(-\frac{\hat{\Omega}l_1^3}{3\sigma}\Big)dl_1+\hat{B}_4\Big\}, \quad (5.111)$$

where

$$\Lambda(K,\tilde{t})=\frac{4i\pi T_c' T_c\alpha_0}{\alpha_1(w_c')^2\gamma M_\infty^2}\Big(\hat{P}_{13}^{(1)}(\tilde{t})+\frac{\alpha_1}{\alpha_0}\hat{P}_{11}^{(1)}(\tilde{t})\Big),$$
$$+\frac{T_c'\pi\alpha_1^2 w_c}{2\alpha_0^2 w_c'}\Big(\frac{w_c''}{2w_c'}-\frac{2}{r_i}\Big)\Big\{\frac{\partial}{\partial\tilde{t}}[A(\tilde{t}-l_1)]\Big|_{l_1=0}-A(\tilde{t}-K)\exp\Big(-\frac{\hat{\Omega}K^3}{3\sigma}\Big)\delta(K)$$
$$+\frac{\partial}{\partial l_1}\Big[A(\tilde{t}-l_1)\exp\Big(-\frac{\hat{\Omega}l_1^3}{3\sigma}\Big)\Big]\Big|_{l_1=0}\Big\}-\frac{\alpha_1^2 w_c T_c'\pi}{2\alpha_0^2 w_c'}\Big(\frac{w_c''}{w_c'}-\frac{T_c''}{T_c'}\Big)\Big\{\frac{2i\alpha_0}{w_c'}A(\tilde{t})$$
$$+A(\tilde{t}-K)\exp\Big(-\frac{\hat{\Omega}K^3}{3\sigma}\Big)\delta(K)-\frac{\partial}{\partial l_1}\Big[A(\tilde{t}-l_1)\exp\Big(-\frac{\hat{\Omega}l_1^3}{3\sigma}\Big)\Big]\Big|_{l_1=0}\Big\}, \quad (5.112)$$

and $\hat{B}_4$ is a constant of integration.

Applying the far-field condition yields

$$\hat{B}_4 = -\exp\Big(\frac{\hat{\Omega}K^3}{3\sigma}\Big)\Big\{\Lambda(K,\tilde{t}) + \int_{-\infty}^{0} \hat{z}_4(l_1,\tilde{t}-l_1)\exp\Big(-\frac{\hat{\Omega}l_1^3}{3\sigma}\Big)dl_1$$

$$+\int_0^\infty \frac{w_c}{2w_c'}A(\tilde{t}-l_1)l_1\hat{z}_2^*(-l,\tilde{t}-l_1)\exp\Big(-\frac{\hat{\Omega}l_1^3}{3\sigma}\Big)dl_1\Big\}. \qquad (5.113)$$

Therefore equation (5.99) has a solution of the form

$$z_4 = -H(K)\int_K^\infty \frac{w_c}{2w_c'}l_1A(\tilde{t}-l_1)\hat{z}_2^*(-l_1,\tilde{t}-l_1)\exp\Big[-\frac{\hat{\Omega}}{3\sigma}(l_1^3-K^3)\Big]dl_1$$

$$-H(-K)\exp\Big(\frac{\hat{\Omega}K^3}{3\sigma}\Big)\Big\{\int_K^0 \hat{z}_4(l_1,\tilde{t}-l_1)\exp\Big(-\frac{\hat{\Omega}l_1^3}{3\sigma}\Big)dl_1 + \Lambda(K,\tilde{t})\Big\}. \quad (5.114)$$

It is now found necessary to determine the $O(\delta^{7/5})$ vorticity component. We define $\hat{Q}_3$ to have the form

$$\hat{Q}_3 = Rl[\hat{Q}_3^{(1)}(\eta,\bar{t})e^{iX} + \hat{Q}_3^{(3)}(\eta,\bar{t})e^{3iX}]. \qquad (5.115)$$

Substituting equation (5.115) into equation (5.39), collecting $O(e^{iX})$ terms and Fourier transforming gives in terms of K and $\tilde{t}$

$$
\begin{aligned}
Y_{4K} - \hat{\Omega}K^2Y_4 = {} & \frac{\Upsilon\pi}{2w_cw_c'}\frac{\partial}{\partial\tilde{t}}\big[A(\tilde{t}-K)\big]\delta(K) + \frac{2i\alpha_0}{\alpha_1 w_c'}\Big[\frac{2a_c\hat{\lambda}}{\alpha_1 w_c'}(\tilde{t}-K+t_0)Y_1 \\
& + \frac{i\alpha_1^2 w_c'}{2\alpha_0^2}\big(Y_{1\tilde{t}} + Y_{1K}\big) + \frac{w_c''\alpha_1^2}{8\alpha_0^2}(Y_{1\tilde{t}\tilde{t}} + 2Y_{1\tilde{t}K} + Y_{1KK})\Big] \\
& -\frac{iw_c}{w_c'}\Big\{A(\tilde{t}-K)\frac{1}{2}iK\tilde{Y}_2 - \frac{i}{2}A^*(\tilde{t}-K)KY_3\Big\} + \frac{\alpha_0 a_c\hat{\Omega}}{2\alpha_1 w_c' w_c^2 T_c^3 C}iKz_1, \\
& + \frac{\alpha_0^2\hat{\Omega}}{T_c\alpha_1^2 w_c^2}\Big(\frac{\sigma-1}{\sigma}\Big)K^2 z_4 - \frac{\alpha_0\hat{\Omega}}{2\alpha_1 T_c w_c^2}\Big[\Big(\frac{\sigma-1}{\sigma}\Big)\frac{2w_c''}{w_c'} - \Big(\frac{3\sigma-1}{\sigma}\Big)\frac{1}{r_i}\Big]iKz_1 \\
& + \frac{\alpha_0^2\hat{\Omega}}{\alpha_1 w_c' w_c^2 T_c}\Big(\frac{2w_c''}{w_c'} - \frac{1}{r_i}\Big)\Big(\frac{\sigma-1}{\sigma}\Big)\Big[K^2 z_1 - \frac{iw_c'}{2\alpha_0}K^2(z_{1K} + z_{1\tilde{t}})\Big] \\
& \frac{\alpha_1\hat{\Omega}}{2\alpha_0}\Big[\frac{w_c''}{w_c'} - \frac{T_c'}{T_c}\Big(\frac{\sigma+1}{\sigma}\Big)\Big]iKY_1 - \frac{2T_c'c_1\hat{\Omega}}{w_c'T_c}\Big(-K^2Y_1 + \frac{iw_c'}{2\alpha_0}K^2(Y_{1K} + Y_{1\tilde{t}})\Big) \\
& - \frac{\alpha_1\hat{\Omega}u_c}{\alpha_0 C T_c^2}\Big[\frac{\alpha_0^2}{T_c\alpha_1^2 w_c^2}iKz_1 - iKY_1\Big], \qquad (5.116)
\end{aligned}
$$

where

$$
Y_4(K,\bar{t}) = \int_{-\infty}^{\infty} Q_1^{(3)}(\eta,\bar{t})e^{-iK\eta}d\eta, \qquad (5.117)
$$

and

$$
\tilde{Y}_2(K,\tilde{t}-K) = Y_2(K,\tilde{t}-K) + Y_2^*(-K,\tilde{t}-K). \qquad (5.118)
$$

Integrating (5.116) and requiring that Y_4 is bounded as $K \to \infty$ yields

$$Y_4 = -\exp\Big(\frac{\hat{\Omega}K^3}{3}\Big)\Big\{H(K)\int_K^\infty \frac{w_c}{2w_c'} l_1 A(\tilde{t}-l_1)\hat{Y}_2^*(-l_1,\tilde{t}-l_1)\exp\Big(-\frac{\hat{\Omega}l_1^3}{3}\Big)dl_1$$

$$-\frac{\alpha_0^2\hat{\Omega}}{T_c\alpha_1^2 w_c^2}\Big(\frac{\sigma-1}{\sigma}\Big)H(K)\int_K^\infty l_1^2\hat{v}_4(s_1,\tilde{t})ds_1 + H(-K)\frac{\Upsilon\pi}{2w_c w_c'}\frac{\partial}{\partial\tilde{t}}\big[A(\tilde{t}-l_1)\big]\Big|_{l_1=0}$$

$$+H(-K)\Big[\frac{\alpha_0^2}{T_c\alpha_1^2 w_c^2}B(\tilde{t})\Big\{1-\exp\Big[-\frac{\hat{\Omega}K^3}{3}\Big(\frac{\sigma-1}{\sigma}\Big)\Big]\Big\}\Big]$$

$$+H(-K)\int_K^0\Big\{\frac{2i\alpha_0}{\alpha_1 w_c'}\Big[\frac{2a_c\hat{\lambda}}{\alpha_1 w_c'}(\tilde{t}-l_1+t_0)\hat{Y}_1 + \frac{i\alpha_1^2 w_c'}{2\alpha_0^2}(\hat{Y}_{1\tilde{t}}+\hat{Y}_{1l_1})$$

$$+\frac{w_c''\alpha_1^2}{8\alpha_0^2}(\hat{Y}_{1\tilde{t}\tilde{t}}+2\hat{Y}_{1\tilde{t}l_1}+\hat{Y}_{1l_1l_1})\Big] + \frac{\alpha_0 a_c\hat{\Omega}}{2\alpha_1 w_c' w_c^2 T_c^3 C} il_1\hat{z}_1$$

$$-\frac{\alpha_0^2\hat{\Omega}}{T_c\alpha_1^2 w_c^2}\Big(\frac{\sigma-1}{\sigma}\Big)l_1^2\exp\Big(\frac{\hat{\Omega}l_1^3}{3\sigma}\Big)\int_{l_1}^0 \hat{z}_4(l_2,\tilde{t}-l_2)\exp\Big(-\frac{\hat{\Omega}l_2^3}{3\sigma}\Big)dl_2$$

$$-\frac{\alpha_0\hat{\Omega}}{2\alpha_1 T_c w_c^2}\Big[\Big(\frac{\sigma-1}{\sigma}\Big)\frac{2w_c''}{w_c'} - \Big(\frac{3\sigma-1}{\sigma}\Big)\frac{1}{r_i}\Big]il_1\hat{z}_1 - \frac{\alpha_1\hat{\Omega}u_c}{\alpha_0 C T_c^2}\Big[\frac{\alpha_0^2}{T_c\alpha_1^2 w_c^2}iK\hat{z}_1 - iK\hat{Y}_1\Big]$$

$$+\frac{\alpha_0^2\hat{\Omega}}{\alpha_1 w_c' w_c^2 T_c}\Big(\frac{2w_c''}{w_c'}-\frac{1}{r_i}\Big)\Big(\frac{\sigma-1}{\sigma}\Big)\Big[l_1^2\hat{z}_1 - \frac{iw_c'}{2\alpha_0}l_1^2(\hat{z}_{1l_1}+\hat{z}_{1\tilde{t}})\Big]$$

$$\frac{\alpha_1\hat{\Omega}}{2\alpha_0}\Big[\frac{w_c''}{w_c'} - \frac{T_c'}{T_c}\Big(\frac{\sigma+1}{\sigma}\Big)\Big]il_1\hat{Y}_1 - \frac{2T_c'c_1\hat{\Omega}}{w_c'T_c}\Big(-l_1^2\hat{Y}_1 + \frac{iw_c'}{2\alpha_0}l_1^2(\hat{Y}_{1l_1}+\hat{Y}_{1\tilde{t}})\Big)$$

$$-\frac{iw_c}{w_c'}\Big\{A(\tilde{t}-l_1)\frac{1}{2}il_1\hat{Y}_2] - \frac{i}{2}A^*(\tilde{t}-l_1)l_1\hat{Y}_3\Big\}\Big\}\exp\Big(-\frac{\hat{\Omega}l_1^3}{3}\Big)dl_1\Big\}, \quad (5.119)$$

where $\hat{z}_1$, $\hat{z}_4$, and $\hat{Y}_1$ are defined by (5.106), (5.105) and (5.109), respectively, and

$$\hat{Y}_2(K,\bar{t}) = \frac{\pi T_c'\alpha_0}{2T_c\alpha_1(w_c')^2}\int_{-\infty}^{\bar{t}} iKA^*(r_1)A(r_1+K)\Big\{2\exp\Big[-\frac{\hat{\Omega}K^2}{3\sigma}(3(\bar{t}-r_1)-K)\Big]$$

$$-\Big\{1-\exp\Big[-\frac{\hat{\Omega}K^3}{3}\Big(\frac{\sigma-1}{\sigma}\Big)\Big]\Big\}\exp\Big[-\frac{\hat{\Omega}K^2}{3}(3(\bar{t}-r_1)-K)\Big]\Big\}dr_1, \quad (5.120)$$

$$\hat{Y}_3(K,\hat{t}-\frac{K}{2})=\frac{i\pi T_c'\alpha_0}{4\alpha_1 T_c(w_c')^2}\int_K^0 l_1 A(\hat{t}-\frac{l_1}{2})A(\hat{t}+\frac{l_1}{2})\Big\{-2\exp\Big[\frac{\hat{\Omega}}{6\sigma}(l_1^3+K^3)\Big]$$

$$+\Big\{1-\exp\Big[-\frac{\hat{\Omega}l_1^3}{3}\Big(\frac{\sigma-1}{\sigma}\Big)\Big]\Big\}\exp\Big[\frac{\hat{\Omega}}{6}(K^3+l_1^3)\Big]\Big\}dl_1, \tag{5.121}$$

$$\hat{v}_4(K,\tilde{t})=\exp\Big(-\frac{\hat{\Omega}K^3}{3}\Big)\int_K^\infty \frac{w_c}{2w_c'}l_1 A(\tilde{t}-l_1)\hat{z}_2^*(-l_1,\tilde{t}-l_1)\exp\Big[-\frac{\hat{\Omega}}{3\sigma}(l_1^3-K^3)\Big]dl_1, \tag{5.122}$$

$$B(\tilde{t})=\frac{4i\pi T_c''T_c\alpha_0}{\alpha_1(w_c')^2\gamma M_\infty^2}\Big(\hat{P}_{13}^{(1)}(\tilde{t})+\frac{\alpha_1}{\alpha_0}\hat{P}_{11}^{(1)}(\tilde{t})\Big)$$

$$+\frac{T_c'\pi\alpha_1^2 w_c}{2\alpha_0^2 w_c'}\Big(\frac{w_c''}{2w_c'}-\frac{2}{r_i}\Big)\Big\{\frac{\partial}{\partial\tilde{t}}[A(\tilde{t}-l_1)]\Big|_{l_1=0}+\frac{\partial}{\partial l_1}\Big[A(\tilde{t}-l_1)\exp\Big(-\frac{\hat{\Omega}l_1^3}{3\sigma}\Big)\Big]\Big|_{l_1=0}\Big\}$$

$$-\frac{\alpha_1^2 w_c T_c'\pi}{2\alpha_0^2 w_c'}\Big(\frac{w_c''}{w_c'}-\frac{T_c''}{T_c'}\Big)\Big\{\frac{2i\alpha_0}{w_c'}A(\tilde{t})-\frac{\partial}{\partial l_1}\Big[A(\tilde{t}-l_1)\exp\Big(-\frac{\hat{\Omega}l_1^3}{3\sigma}\Big)\Big]\Big|_{l_1=0}\Big\}. \tag{5.123}$$

Now that the term Y_4 has been determined, it is found necessary to determine the corresponding value of $Q_1^{(3)}(\eta,\bar{t})$. We define the inverse Fourier transform

$$Q_1^{(3)}(\eta,\bar{t})=\frac{1}{2\pi}\int_{-\infty}^{\infty}Y_4(K,\bar{t})e^{iK\eta}dK, \tag{5.124}$$

where it is noted from (5.119) that Y_4 is a function of K and $\tilde{t}-K$, therefore care must be taken when applying the inverse Fourier transform of the right-hand-side. Secondly, (5.119) as it stands is a fairly lengthy expression and carrying out the inverse Fourier transform of every term on the right-hand-side can be a very tedious and time consuming process. However, since it is only the term $\int_{-\infty}^{\infty}Q_1^{(3)}(\eta,\bar{t})d\eta$ which we are ultimately interested in, as it is this term that is matched with the outer solution (see the inviscid theory in Chapter 4), it is found more convenient, and indeed the analysis is greatly simplified, to consider the double integration,

$$\int_{-\infty}^{\infty} Q_1^{(3)}(\eta, \bar{t})d\eta = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Y_4(\eta, \bar{t})e^{iK\eta}dK d\eta, \tag{5.125}$$

in one step.

On carrying out this process all terms in (5.119) contained within the l_1 integration are found to be zero; for example, consider the term

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{ik\eta}H(-K)\int_K^0 \frac{w_c}{2w_c'}A^*(\bar{t}+K-l_1)l_1\hat{Y}_3(l_1, \bar{t}+K-l_1)e^{\frac{\hat{\Omega}}{3}(K^3-l_1^3)}dl_1 dK d\eta, \tag{5.126}$$

where it is noted that in the inner integration the variables have been converted back to $\bar{t}$, K (although in terms of the inner integration this coordinate is now l_1) space to facilitate the application of the inverse Fourier transform.

Substituting in the value for $\hat{Y}_3$ as defined by (5.121) (where again care must be applied since $\hat{Y}_3$ is a function of K and $\hat{t}-\frac{K}{2}$ in (5.121)), and changing the order of integration yields

$$\frac{iw_c T_c'\alpha_0}{16\alpha_1 T_c(w_c')^3}\int_{-\infty}^{0}\int_{l_1}^{0}\int_{-\infty}^{l_1} l_1 l_2 \hat{v}_5(\bar{t}, K, l_1, l_2)\int_{-\infty}^{\infty} e^{iK\eta}d\eta dK dl_2 dl_1, \tag{5.127}$$

where

$$\hat{v}_5(\bar{t}, K, l_1, l_2) = A(\bar{t}+K-\frac{l_1}{2}-\frac{l_2}{2})A(\bar{t}+K-\frac{l_1}{2}+\frac{l_2}{2})\Big\{-2\exp\Big[\frac{\hat{\Omega}}{6\sigma}(l_2^3+l_1^3)\Big]$$
$$+\Big\{1-\exp\Big[-\frac{\hat{\Omega}l_2^3}{3}\Big(\frac{\sigma-1}{\sigma}\Big)\Big]\Big\}\exp\Big[\frac{\hat{\Omega}}{6}(l_1^3+l_2^3)\Big]\Big\}A^*(\bar{t}+K-l_1)\exp\Big[\frac{\hat{\Omega}}{3}(K^3-l_1^3)\Big]. \tag{5.128}$$

On carrying out the inner-most integration we have

$$\int_{-\infty}^{l_1} l_1 l_2 \hat{v}_5(\bar{t}, K, l_1, l_2) 2\pi\delta(-K) dK. \tag{5.129}$$

Since $l_1 \leq 0$, the delta function is only switched on when the upper limit of integration equals zero, but clearly the integrand will equal zero at this point. Moreover, the l_2 integration in (5.127) disappears when this occurs since the upper and lower limits of integration will now be equal. Therefore (5.126) is zero for all values, i.e. this term makes no contribution to the jump term on the left-hand-side of (5.125).

Similar arguments applied to each term within the l_1 integration appearing in (5.119) yields that each one is zero.

Consider now the term

$$-\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{iK\eta}\exp\left(\frac{\hat{\Omega}K^3}{3}\right)H(-K)\frac{\Upsilon\pi}{2w_c w_c'}\frac{\partial}{\partial\bar{t}}[A(\bar{t}+K-l_1)]\Big|_{l_1=0} dK d\eta. \tag{5.130}$$

Changing the order of integration yields

$$-\frac{\Upsilon}{4w_c w_c'}\int_{-\infty}^{0}\exp\left(\frac{\hat{\Omega}K^3}{3}\right)\frac{\partial}{\partial\bar{t}}[A(\bar{t}+K-l_1)]\Big|_{l_1=0}\int_{-\infty}^{\infty} e^{iK\eta} d\eta dK. \tag{5.131}$$

Carrying out both integrations gives

$$-\frac{\Upsilon\pi}{4w_c w_c'}\frac{\partial}{\partial\bar{t}}[A(\bar{t}+K-l_1)]\Big|_{K,l_1=0}. \tag{5.132}$$

Consequently this gives

$$-\frac{\Upsilon\pi}{4w_c w_c'}A_{\bar{t}}, \tag{5.133}$$

where $A = A(\bar{t})$.

Comparing this result with the inviscid theory developed in the previous chapter it is found that this term corresponds to the linear inviscid term.

This next term which is considered has the form

$$-\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{iK\eta}\exp\Big(\frac{\hat{\Omega}K^3}{3}\Big)H(-K)\frac{\alpha_0^2}{T_c\alpha_1^2 w_c^2}B(\bar{t}+K)\Big\{1-$$

$$\exp\Big[-\frac{\hat{\Omega}K^3}{3}\Big(\frac{\sigma-1}{\sigma}\Big)\Big]\Big\}dK\,d\eta. \qquad (5.134)$$

Again, changing the order of integration and carrying out the inner integration yields

$$-\frac{\alpha_0^2}{T_c\alpha_1^2 w_c^2}\int_{-\infty}^{0} B(\bar{t}+K)\exp\Big(\frac{\hat{\Omega}K^3}{3}\Big)\Big\{1-\exp\Big[-\frac{\hat{\Omega}K^3}{3}\Big(\frac{\sigma-1}{\sigma}\Big)\Big]\Big\}\delta(-K)dK. \quad (5.135)$$

Clearly on conducting the K-integration the term inside the curly-brackets will equal zero, resulting in (5.135) making no contribution to the jump across the critical layer.

The next term examined has the form

$$-\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{iK\eta}H(K)\int_{K}^{\infty}\frac{w_c l_1}{2w_c'}A(\bar{t}+K-l_1)\hat{Y}_2^*(-l_1,\bar{t}+K-l_1)e^{\frac{\hat{\Omega}}{3}(K^3-l_1^3)}dl_1\,dK\,d\eta. \qquad (5.136)$$

Substituting for $\hat{Y}_2^*$ where $\hat{Y}_2(K,\bar{t})$ is defined by (5.120) yields

$$-\frac{iw_cT_c'\alpha_0}{8\alpha_1T_c(w_c')^3}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{K}^{\infty}\int_{-\infty}^{\bar{t}+K-l_1} l_1^2\hat{v}_6(\bar{t},K,l_1,r_1)e^{ik\eta}dr_1\,dl_1\,dK\,d\eta, \qquad (5.137)$$

where

$$\hat{v}_6(\bar{t},K,l_1,r_1) = A(\bar{t}+K-l_1)A^*(r_1-l_1)\Big\{2\exp\Big[-\frac{\hat{\Omega}l_1^3}{3\sigma}(3(\bar{t}+K-r_1)-2l_1)\Big]$$

$$-\Big\{1-\exp\Big[\frac{\hat{\Omega}l_1^3}{3}\Big(\frac{\sigma-1}{\sigma}\Big)\Big]\Big\}\exp\Big[-\frac{\hat{\Omega}l_1^2}{3}(3(\bar{t}+K-r_1)-2l_1)\Big]\Big\}A(r_1)e^{-\frac{\hat{\Omega}}{3}(l_1^3-K^3)}. \qquad (5.138)$$

Changing the order of integration gives

$$-\frac{iw_cT_c'\alpha_0}{8\alpha_1T_c(w_c')^3}\Big\{\int_0^\infty\int_{-\infty}^{\bar{t}-l_1}\int_0^{l_1}l_1^2\hat{v}_6\int_{-\infty}^\infty e^{iK\eta}d\eta dKdr_1dl_1$$
$$+\int_0^\infty\int_{\bar{t}-l_1}^{l_1}\int_{r_1+l_1-\bar{t}}^{l_1}l_1^2\hat{v}_6\int_{-\infty}^\infty e^{iK\eta}d\eta dKdr_1dl_1\Big\}. \quad (5.139)$$

Considering the second quadruple integration first, conducting the inner-most integration yields

$$-\frac{iw_cT_c'\alpha_0}{8\alpha_1T_c(w_c')^3}\int_0^\infty\int_{\bar{t}-l_1}^{l_1}\int_{r_1+l_1-\bar{t}}^{l_1}l_1^2\hat{v}_6 2\pi\delta(-K)dKdr_1dl_1. \quad (5.140)$$

The delta function will be switched on if $K = 0$ at some point within the K-integration range. If $l_1 = 0$, the integrand will be zero, therefore $l_1 > 0$, and it is required that $r_1 + l_1 - \bar{t} \leq 0$. Examining the r_1-integration it is noted that the lower limit of integration is $\bar{t} - l_1$. Therefore, the smallest value that the lower limit of the K-integration can take is zero. But, as already stated, this corresponds to the lower limit of the r_1 integration and since there is no delta function multiplying this integrand, then this integration makes no contribution to (5.140). Consequently the second of the quadruple integrations in (5.139) makes no contribution to the jump across the critical layer.

Consider now the first quadruple integration in (5.139), which on carrying out the inner integration yields

$$-\frac{iw_cT_c'\alpha_0}{8\alpha_1T_c(w_c')^3}\int_0^\infty\int_{-\infty}^{\bar{t}-l_1}\int_0^{l_1}l_2^2\hat{v}_6 2\pi\delta(-K)dKdr_1dl_1. \quad (5.141)$$

Since the Heaviside function in (5.136) is switched on at $K = 0$, then the full contribution of the delta function will be felt at the lower limit of integration in (5.141). This statement was checked by Fourier transforming the second integration term occuring in equation (4.249), where the η integration is omitted here. On carrying out this transform, the inner integrand was found be multiplied by the delta function, $\delta(K + \bar{t} - s_1 - (s_2 - s_3))$. Over the integration range for the inner s_3-integration, the delta funtion will only be switched on if $K \geq 0$. This corresponds to introducing the Heaviside function $H(K)$, with the understanding that $K = 0$ is included. Employing a variable change it is easy to show that when the inverse Fourier transform of the Fourier transformed inviscid equation is applied and the resultant is integrated over all η, then this equation will have exactly the same form as (5.136) if $\hat{\Omega} = 0$ (corresponding to the inviscid case for (5.136)). Clearly, therefore, applying the inverse Fourier transform in (5.136) and integrating over all η, the delta function contribution will be exactly the same as in the inviscid case.

Carrying out the inner integration in (5.141) yields

$$-\frac{iw_c T_c' \alpha_0 \pi}{4\alpha_1 T_c (w_c')^3} \int_0^\infty \int_{-\infty}^{\bar{t}-l_1} l_1^2 \hat{v}_6(\bar{t}, 0, l_1, r_1) dr_1 dl_1. \tag{5.142}$$

Introducing the transform variables s_1 and s_2 where

$$\begin{aligned} s_1 &= \bar{t} - l_1, \\ s_2 &= r_1, \end{aligned} \tag{5.143}$$

equation (5.142) can be written in the form

$$\begin{aligned} -\frac{iw_c T_c' \alpha_0 \pi}{4\alpha_1 T_c (w_c')^3} \int_{-\infty}^{\bar{t}} \int_{-\infty}^{s_1} (\bar{t} - s_1)^2 A(s_1) A(s_2) A^*(s_2 - \bar{t} + s_1) \exp\Big[-\frac{\hat{\Omega}}{3}(\bar{t} - s_1)^3\Big] \\ \times\Big\{2\exp\Big[-\frac{\hat{\Omega}(\bar{t} - s_1)^3}{3\sigma}(3(\bar{t} - s_2) - 2(\bar{t} - s_1))\Big]\Big\{1 - \\ \exp\Big[\frac{\hat{\Omega}(\bar{t} - s_1)^3}{3}\Big(\frac{\sigma - 1}{\sigma}\Big)\Big] \exp\Big[-\frac{\hat{\Omega}(\bar{t} - s_1)^2}{3}(3(\bar{t} - s_2) - 2(\bar{t} - s_1)\Big]\Big\} ds_2 ds_1. \end{aligned} \tag{5.144}$$

The last term considered has the form

$$\frac{\alpha_0^2\hat{\Omega}}{T_c\alpha_1^2 w_c^2}\Big(\frac{\sigma-1}{\sigma}\Big)\exp\Big(\frac{\hat{\Omega}K^3}{3}\Big)H(K)\int_K^\infty l_1^2\exp\Big(-\frac{\hat{\Omega}l_1^3}{3}\Big)\int_{l_1}^\infty \frac{w_c}{2w_c'}$$

$$\times l_2 A(\bar{t}+K-l_2)\hat{z}_2^*(-l_2,\bar{t}+K-l_2)\exp\Big[-\frac{\hat{\Omega}}{3\sigma}(l_2^3-l_1^3)\Big]dl_2dl_1. \qquad (5.145)$$

Substituting for $\hat{z}_2^*$, where $\hat{z}_2(K,\bar{t})$ is defined by (5.107) yields

$$\beta e^{\frac{\hat{\Omega}K^3}{3}}H(K)\int_K^\infty\int_{l_1}^\infty\int_{-\infty}^{\bar{t}+K-l_2} l_1^2\hat{v}_7(\bar{t},K,l_2,r_1)\exp\Big[-\frac{\hat{\Omega}l_1^3}{3}\Big(\frac{\sigma-1}{\sigma}\Big)\Big]dr_1dl_2dl_1, \quad (5.146)$$

where

$$\beta=-\frac{i\pi\alpha_0 w_c T_c'\hat{\Omega}}{4T_c\alpha_1(w_c')^3}\Big(\frac{\sigma-1}{\sigma}\Big) \qquad (5.147)$$

and

$$\hat{v}_7(\bar{t},K,l_2,r_1)=l_2^2A(\bar{t}+K-l_2)A(r_1)A^*(r_2-l_2)\exp\Big[-\frac{\hat{\Omega}l_2^2}{3\sigma}(3(\bar{t}+K-r_1)-2l_2)\Big]e^{-\frac{\hat{\Omega}l_2^3}{3\sigma}}. \qquad (5.148)$$

Changing the order of integration in (5.146) gives

$$\beta e^{\frac{\hat{\Omega}K^3}{3}}H(K)\int_K^\infty\int_{-\infty}^{\bar{t}+K-l_2}\hat{v}_7(\bar{t},K,l_2,r_1)\int_K^{l_2} l_1^2\exp\Big[-\frac{\hat{\Omega}l_1^3}{3}\Big(\frac{\sigma-1}{\sigma}\Big)\Big]dr_1dl_2dl_1. \quad (5.149)$$

Carrying the inner integration, and then setting $l_2\equiv l_1$ yields

$$\frac{i\pi\alpha_0T_c'w_c}{4T_c\alpha_1(w_c')^3}\exp\Big(\frac{\hat{\Omega}K^3}{3}\Big)H(K)\int_K^\infty\int_{-\infty}^{\bar{t}+K-l_1} l_1^2A(r_1)A^*(r_1-l_1)A(\bar{t}+K-l_1)$$

$$\times\exp\Big[-\frac{\hat{\Omega}l_1^2}{3\sigma}(3(\bar{t}+K-r_1)-2l_1)\Big]\exp\Big(-\frac{\hat{\Omega}l_1^3}{3\sigma}\Big)$$

$$\times\Big\{\exp\Big(-\frac{\hat{\Omega}l_1^3}{3}\Big(\frac{\sigma-1}{\sigma}\Big)\Big]-\exp\Big[-\frac{\hat{\Omega}K^3}{3}\Big(\frac{\sigma-1}{\sigma}\Big)\Big]\Big\}dr_1dl_1. \quad (5.150)$$

Inverse Fourier transforming (5.150) and then integrating over all η, following arguments similar those developed in (5.136) - (5.144) yields the result

$$-\frac{i\pi\alpha_0 T_c' w_c}{4T_c\alpha_1(w_c')^3}\int_{-\infty}^{\bar{t}}\int_{-\infty}^{s_1}(\bar{t}-s_1)^2 A(s_1)A(s_2)A^*(s_2-\bar{t}+s_1)\Big\{1$$
$$-\exp\Big[-\frac{\hat{\Omega}(\bar{t}-s_1)^3}{3}\Big(\frac{\sigma-1}{\sigma}\Big)\Big]\Big\}\exp\Big[-\frac{\hat{\Omega}(\bar{t}-s_1)^2}{3\sigma}(3(\bar{t}-r_1)-(\bar{t}-s_1))\Big]ds_2ds_1, \tag{5.151}$$

where s_1 and s_2 are defined by (5.143).

Substituting the relevant results obtained between equations (5.126) and (5.151) into (5.125) yields that the jump across the critical layer is given by

$$\int_{-\infty}^{\infty}Q_1^{(3)}(\eta,\bar{t})d\eta = -\frac{\Upsilon\pi}{4w_c w_c'}A_{\bar{t}} - \frac{i\pi\alpha_0 T_c' w_c}{4T_c\alpha_1(w_c')^3}\int_{-\infty}^{\bar{t}}\int_{-\infty}^{s_1}(\bar{t}-s_1)^2 A(s_1)A(s_2)$$
$$\times A^*(s_2-\bar{t}+s_1)\mathcal{A}(\bar{t},s_1,s_2)\exp\Big[-\frac{\hat{\Omega}(\bar{t}-s_1)^2}{3}(3(\bar{t}-s_2)-(\bar{t}-s_1))\Big]ds_2ds_1, \tag{5.152}$$

where

$$\mathcal{A}(\bar{t},s_1,s_2) = 2\exp\Big[-\frac{\hat{\Omega}(\bar{t}-s_1)^2}{3}(3(\bar{t}-s_2)-2(\bar{t}-s_1))\Big(\frac{1-\sigma}{\sigma}\Big)\Big]$$
$$-1+\exp\Big[\frac{\hat{\Omega}(\bar{t}-s_1)^3}{3}\Big(\frac{\sigma-1}{\sigma}\Big)\Big]+\Big\{1-\exp\Big[\frac{\hat{\Omega}(\bar{t}-s_1)^3}{3}\Big(\frac{\sigma-1}{\sigma}\Big)\Big]\Big\}$$
$$\times\exp\Big[-\frac{\hat{\Omega}(\bar{t}-s_1)^2}{3}(3(\bar{t}-s_2)-(\bar{t}-s_1))\Big(\frac{\sigma-1}{\sigma}\Big)\Big]. \tag{5.153}$$

Matching (5.152) with the outer solution, namely (4.211) (where it is remembered that this equation is in terms of the non-normalized variables), gives that the amplitude evolution equation for the viscous nonlinear critical layer, as considered in this problem, has the form

$$A_{\bar{t}} = -\kappa A - \frac{1}{2\Gamma}\int_{-\infty}^{\bar{t}} A(s_1)\int_{-\infty}^{s_1} A(s_2)A^*(s_1+s_2-\bar{t})(\bar{t}-s_1)^2\mathcal{A}(\bar{t},s_1,s_2)$$
$$\times \exp\Big[-\frac{\hat{\Omega}(\bar{t}-s_1)^2}{3}(3(\bar{t}-s_2)-(\bar{t}-s_1))\Big]ds_2ds_1, \quad (5.154)$$

where κ and Γ are defined by (4.258) and (4.257) respectively.

Equation (5.154) represents the main result of this chapter. Setting $\sigma \equiv 1$ in (5.154) retrieves the viscous nonlinear amplitude evolution equation as determined by Goldstein and Leib (1989) for the compressible shear layer problem (although, as already noted at the end of Chapter 4, the constants κ and Γ are dependent on neutral mode characteristics and consequently will be different in both cases). As noted by Leib (1991), when $\sigma \neq 1$ then there is an extra contribution to the nonlinear term resulting from an interaction between the radial change in the $O(\delta^{4/5})$ temperature solution, i.e. the term $\tilde{T}_{12Y}$, and the leading order term in the radial velocity expansion, namely $\tilde{\psi}_0$. It is also noted, that there are additional exponential terms multiplying the nonlinear terms, due to the assumption $\sigma \neq 1$. Since Leib (1991) assumed that his normalized viscosity has a power-law dependence with temperature, then setting the constant n in his work to one (which corresponds to the linear Chapman law, as assumed throughout this thesis), retrieves the result (5.154) (although the constants κ and Γ are different, of course). It should be noted that although Leib mentions the $O(\delta)$ temperature contribution to the nonlinear term, it has actually been omitted from his equation (3.37).

As yet, a numerical study of (5.154) has not been conducted. However, owing to the similarity between our results and those obtained by Goldstein and Leib (1989) and Leib (1991) it is expected that for certain parameter ranges, if $\hat{\Omega}$ is large enough, then the singularity which occurs in the inviscid nonlinear theory can be eliminated and the solution will tend to a finite-amplitude equilibrium. In his work, Leib (1991) determined a necessary condition for the existence of an equilbrium solution which is equivalent to requiring $\hat{\Omega}$ to be large enough, resulting in the exponential term being small enough, to damp the history effects of the convolution integral, forcing the solution to become more local. In the case of subsonic disturbance terms, he determined that unless the temperature ratio of the low-to high-speed streams (for the compressible shear layer model he was treating) exceeds a critical value, no equilibrium solution is possible. Leib concluded that cooling the low-speed stream and decreasing the Mach number appears to destabilize the nonlinear evolution of subsonic disturbance terms, in that only explosive growth is observed, while increasing both the temperature ratio and Mach number appear to have the converse effect, with equilibrium solutions being achieved.

In the near future we shall determine what parameter range, if any, equilibrium solutions exist for our amplitude equation.

Chapter 6

Conclusions

In this thesis the supersonic boundary layer flow over axisymmetric bodies for the particular cases of a long thin, straight circular cylinder subject to heated, cooled or adiabatic wall conditions and a sharp cone subject to adiabatic wall conditions , has been investigated. The basic boundary layer flow has been obtained, and it is noted that in the case of the sharp cone that it is seen to evolve from one planar state to a second, as predicted by the Mangler (1946) transform.

A linear inviscid temporal non-axisymmetric stability analysis of this boundary layer flow is conducted and a 'triply generalized inflexional condition' is derived, this being the necessary condition for the existence of subsonic neutral modes, and is a (second) generalization of the well-known generalized inflexion condition, as obtained by Lees and Lin (1946). The importance of condition (3.37) is because it is possible to predict, *a priori*, whether subsonic neutral modes exist (and if so the corresponding wavespeeds). However, it is somewhat more difficult to use than the planar generalized inflexion condition of Lees and Lin (1946), since in our case the azimuthal and axial wavenumbers (which are of course unknown) are present in the condition, and so prediction of subsonic neutral modes must be made on a trial basis (i.e., to determine if any value of the ratio n/α satisfies (3.37)). However, if solutions do exist, the value of α (for a given n) must still be determined by a full numerical solution of the disturbance equations, as was carried out in obtaining the results presented in Tables 3.1 - 3.3.

In the case of cylindrical bodies, the effect of surface body curvature is seen to immediately (and significantly) reduce the importance of the 'first mode' of inviscid instability, which for axisymmetric disturbances is completely eradicated a relatively short distance down the axis of the cylinder. The maximum growth rate of the 'second mode' of invisid instability also suffers substantial reduction at locations increasingly further down the axis of the cylinder, although all the evidence suggests that it is not completely stabilized. These observations are very similar to those obtained by Duck (1990).

Non-axisymmetric 'first modes' of instability are found to be generally more important that the corresponding axisymmetric modes, and are found to persist well downstream of the location where the axisymmetric mode is completely stabilized. It is found that these non-axisymmetric modes, however, will be completely stabilized for sufficiently large azimuthal wavenumber, or far enough downstream location. In the case of the 'second mode' of instability, non-axisymmetric modes are generally less important than the corresponding axisymmetric modes, and it is found that increasing n results in further stabilization, although all the numerical evidence suggests that they are still present, with much diminished growth rates, for very large n.

The effect of cooled-wall conditions on cylindrical bodies is generally seen to reduce the importance of the 'first mode' of instability, while the amplification rates of the 'second mode' of instability are generally increased. Therefore we have agreement with the effect wall cooling has on planar boundary layers (Mack (1987), for example).

The converse effect is observed with wall heating. The amplification rates of the 'first mode' of instability increase, while wall-heating causes the 'second mode' of instability to be stabilized. Wall-heating may also cause the formation of a second subsonic generalized inflexional mode, which results in the appearance of an additional mode of instability, not found in adiabatic or cooled-wall studies.

In the case of the sharp cone, for locations not too far from the cone tip, since the body divergence will be small, the results closely resemble those obtained for the cylinder. It is found, however, that as a result of the Mangler transform, results far downstream from the cone tip mirror those in the neighbourhood of the tip except for a multiplicative factor of $\sqrt{3}$ in α. Varying the wall conditions and the introduction of non-axisymmetric terms is found to have the same effect on both modes of instability, as encountered in the cylindrical body case.

Significantly, the numerical results point to the occurrence of an extra mode of instability, not found in similar planar studies - this mode is found to be present for both cooled, heated and adiabatic wall conditions. An asymptotic study of this mode suggests this mode to be linked to a viscous mode found by Duck and Hall (1990), a study based on triple-deck theory.

The 'sonic' neutral mode (which is the genesis of the planar 'first mode' of instability) is found to be altered by curvature (and in fact becomes a supersonic neutral mode as revealed by the asymptotic analysis as $\zeta \to 0$ and $\zeta \to \infty$, as carried out in Chapter 3).

Significantly, our results show that the 'second mode' of instability is not always the most unstable, at least in the case of non-axisymmetric modes, and indeed our results indicate the new mode that occurs as $\alpha \to 0$, as found in this work, may possess the largest growth rates, and therefore is the most significant from a practical point of view.

Asymptotic studies for cylindrical bodies valid for large azimuthal wavenumbers reveal that the eigenvalue c is non-unique in this limit, suggesting that there exists an infinite number of discrete possible values for the real part of c, although the corresponding values of the imaginary part of c are exceedingly small. One question that still remains is the ultimate behaviour of c as ζ increases in the case of cylindrical bodies. Asymptotic analysis suggests that in this limit, $c \rightarrow T_w^{1/2} M_\infty$. However, this analysis failed to provide an estimate for the scale of c_i (except $c_i = o(1)$, although all the numerical evidence presented here, and also found in other computations performed, strongly pointed to $(\alpha c_i)_{\mathrm{max}} \rightarrow 0$) as $\zeta \rightarrow \infty$, i.e., a diminishingly small growth rate with increasing distance downstream.

On completing the linear stability analysis, a weakly nonlinear stability investigation of the compressible boundary formed on a long straight circular cylinder, is conducted. The method by which nonlinearity is introduced into this specific boundary-layer problem is through the interaction of the linear disturbance terms, with nonlinear effects developed within the critical layer. Considering initially a linear viscous critical layer, curvature is found only to effect the constant multiplying the logarithmic singularity in the neighbourhood of the critical layer. Since this constant generally corresponds to the generalized inflexion condition relevant to the flow being considered, then it is not unexpected that in our anaylsis we determine it is equilvalent to the axisymmetric generalized inflexion condition of Duck (1990).

Upon considering a nonlinear critical layer (in which viscous terms are assumed negligible), matching between the critical layer and the outer solution leads to an integro-differential equation governing the evolution of the slowly varying amplitude of the instability wave, which possesses a cubic nonlinearity term. The nonlinear term has the form of the convolution integral of the Hickernell (1984) type, and since past histories are important the growth rates are found to terminate explosively in a singularity after a finite time evolution. Comparing our results to those of Goldstein and Leib (1989) and Shukhman (1991), adds weight to Shukhman's observation, that the nonlinear term is independent of the particular fluid dynamical problem being considering. The only difference between our results and those of Goldstein and Leib (1989) and Shukhman (1991) are the values of the constant terms multiplying the linear and nonlinear contributions, which are dependent upon neutral-mode characteristics peculiar to the flow being treated. However, the numerical observations indicate that the constant coefficient terms are important in that they control when nonlinear effects, if at all, are first felt.

Lastly the case of a viscous nonlinear critical layer is treated and the corresponding amplitude evolution equation is obtained. As yet a full numerical study of this equation has not been carried out, but it is expected that the additional exponential term will behave in the same way as in the Goldstein and Leib case, causing the solution to become more local through its damping action, and for certain parameter ranges resulting in the equilbrium state being achieved.

Appendix A

Large n amplitude function

In this appendix the amplitude function $f(\eta)$ in the WKBJ solution (3.49) is determined. Substituting (3.49) into (3.47), gives to leading order

$$f'' + \Big[\frac{T_{0\eta}}{T_0} + \frac{\zeta}{1+\eta\zeta} - \frac{2w_{0\eta}}{w_0 - c_0}\Big] f' = 0. \tag{A.1}$$

Integrating once with respect to η yields

$$f' = A \exp\Big[-\int \Big(\frac{T_{0\eta}}{T_0} + \frac{1}{\eta + \frac{1}{\zeta}} - \frac{2w_{0\eta}}{w_0 - c_0}\Big) d\eta\Big], \tag{A.2}$$

where A is a constant. Making use of the result

$$\int \Big(\frac{T_{0\eta}}{T_0} + \frac{1}{\eta + \frac{1}{\zeta}} - \frac{2w_{0\eta}}{w_0 - c_0}\Big) d\eta = ln\Big|\frac{T_0(1+\zeta\eta)}{\zeta(w_0 - c_0)^2}\Big| + B. \tag{A.3}$$

A.2 simplifies to

$$f' = C\frac{\zeta(w_0 - c_0)^2}{T_0(1+\zeta\eta)} \tag{A.4}$$

where B and C are constants. A further integration yields

$$f(\eta) \sim \int^{\eta} \frac{\zeta(w_0 - c_0)^2}{T_0(1+\zeta\eta)} d\eta. \tag{A.5}$$

Appendix B

The Zero Wavenumber limit

In the case of axisymmetric modes (and indeed of planar modes, as considered by Lees and Lin (1946) and Mack (1965a,b, 1984, 1987a), for example), as $\alpha \to 0$, the wavespeed c approaches the sonic limit, i.e.,

$$c \to 1 \pm \frac{1}{M_\infty}. \tag{B.1}$$

In the case of nonasymmetric modes, however, this is no longer the case. As the numerical results clearly indicate, when $n \neq 0$, $c_i \not\to 0$ as $\alpha \to 0$. The explanation for this is as follows:

Considering the simple limit $\alpha \to 0$ (assuming $n \neq 0$), then (3.116) reduces to

$$\frac{d}{d\eta}\Big\{\frac{(w_o - c)}{T_0}[1 + \lambda\zeta^2 + \zeta\eta][(1 + \lambda\zeta^2 + \zeta\eta)\phi_\eta + \zeta\phi] - \frac{w_{0\eta}[1 + \lambda\zeta^2 + \zeta\eta]^2\phi}{T_0}\Big\}$$
$$= \frac{n^2\zeta^2\phi}{T_0}(w_0 - c). \tag{B.2}$$

As $\eta \to \infty$, this system has the form

$$\frac{d}{d\eta}[\phi_\eta\eta^2 + \eta\phi] = n^2\phi, \tag{B.3}$$

which clearly admits solutions of the form

$$\phi \sim \eta^{-n-1} \tag{B.4}$$

which can be shown to be completely compatible with the outer solution, where $\hat{\eta} = O(1)$ ($\hat{\eta}$ defined by (3.19)), namely (3.18), by considering the Bessel function ascending series expression (3.153). System (B.2) also satisfies the impermeability wall condition, i.e. $\phi = 0$ on $\eta = 0$.

Equation (B.2) then represents a reduced problem as $\alpha \to 0$, and in this limit the triply generalized inflexion condition also has a reduced form, namely

$$\frac{\partial}{\partial \eta}\left\{\frac{w_{0\eta}(1+\lambda\zeta^2+\zeta\eta)^2}{T_0}\right\}_{\eta=\eta_i} = 0. \tag{B.5}$$

The system (B.2) and (B.4) was solved in a number of cases (in an identical manner to the $\alpha = O(1)$ eigenvalue system) and its correctness was confirmed. Notice, however, since the actual temporal growth rate is αc_i, this still reduces to zero as $\alpha \to 0$.

Appendix C

Outer Frobenius solution

From (3.15) the linear pressure equation for axisymmetric disturbances in the compressible boundary layer formed on a cylinder can be written in the form

$$\frac{d^2\tilde{p}}{dr^2} - \Big[\frac{2w_{0r}}{w_0 - c} - \frac{T_{0r}}{T_0} - \frac{1}{r}\Big]\frac{d\tilde{p}}{dr} - \alpha^2\Big[1 - M_\infty^2\frac{(w_0 - c)^2}{T_0}\Big]\tilde{p} = 0. \qquad \text{(C.1)}$$

Clearly this equation possesses a singularity at the critical point, i.e. the point where the phase speed of the disturbance is equal to the mean flow velocity. In the limit $r \to r_i$ (where r_i represents the critical point) we expand all basic flow terms as Taylor series about the critical point resulting in the pressure disturbance equation having the form

$$\frac{d^2\tilde{p}_0}{dy^2}+(-\frac{2}{y}+\beta+y\kappa+\lambda y^2+\Gamma y^3+O(y^4))\frac{d\tilde{p}_0}{dy}-\alpha^2[1-(\Lambda y^2+\epsilon y^3+O(y^4))]\tilde{p}_0 = 0, \quad \text{(C.2)}$$

where

$$y = r - r_i = \Delta r, \qquad \text{(C.3)}$$

and

$$\beta = -\frac{w_c''}{w_c'} + \frac{T_c'}{T_c} + \frac{1}{r_i}, \qquad \text{(C.4)}$$

$$\kappa = -\Big(\frac{T'_c}{T_c}\Big)^2 + \frac{T''_c}{T_c} - \frac{2w'''_c}{3w'_c} + \frac{1}{2}\Big(\frac{w''_c}{w'_c}\Big)^2 - \frac{1}{r_i^2}, \tag{C.5}$$

$$\lambda = -\frac{3T'_cT''_c}{2T_c^2} + \Big(\frac{T'_c}{T_c}\Big)^3 + \frac{T'''_c}{2T_c} - \frac{w_c^{IV}}{4w'_c} + \frac{w'''_c w''_c}{2(w'_c)^2} - \frac{1}{4}\Big(\frac{w''_c}{w'_c}\Big)^3 + \frac{1}{r_i^3}, \tag{C.6}$$

$$\Gamma = -\Big(\frac{T'_c}{T_c}\Big)^4 - \frac{(T''_c)^2}{2T_c^2} + \frac{2(T'_c)^2T''_c}{T_c^3} - \frac{2T'''_cT'_c}{3T_c^2} + \frac{T_c^{IV}}{6T_C} - \frac{w_c^V}{15w'_c} + \frac{w_c^{IV}w''_c}{6(w'_c)^2}$$
$$+\frac{1}{9}\Big(\frac{w'''_c}{w'_c}\Big)^2 - \frac{(w''_c)^2w'''_c}{3(w'_c)^3} + \frac{1}{8}\Big(\frac{w''_c}{w'_c}\Big)^4 - \frac{1}{r_i^4}, \tag{C.7}$$

$$\Lambda = M_\infty^2\frac{(w'_c)^2}{T_c}, \tag{C.8}$$

$$\epsilon = M_\infty^2\Big[-\frac{(w'_c)^2T'_c}{T_c^2} + \frac{w'_cw''_c}{T_c}\Big], \tag{C.9}$$

where subscript c denotes evaluation at the critical layer.

The expansion for $\tilde{p}$ is obtained by applying the method of Frobenius, i.e. we substitute into (C.2) a solution of the form

$$\tilde{p} = \sum_{n=0}^{\infty} a_n y^{n+s}, \tag{C.10}$$

where the number s and the coefficients a_n are to be determined. For $n = 0$, we have, since $a_0 \neq 0$, an indicial equation of the form

$$s(s-3) = 0. \tag{C.11}$$

Considering the larger root of (C.11) first, i.e. $s = 3$, substituting (C.10) (with $s = 3$) into (C.2) and equating powers in y, yields

$$a_0 = 1, \qquad a_1 = -\tfrac{3}{4}a_0\beta, \qquad a_2 = \tfrac{1}{10}[3\beta^2 + \alpha^2 - 3\kappa]a_0,$$
$$a_3 = \frac{1}{72}[-6\beta^3 - 5\beta\alpha^2 + 18\beta\kappa - 12\lambda]a_0. \tag{C.12}$$

Thus, as $y \to 0$ $(r \to r_i)$ equation (C.1) has one solution of the form

$$\tilde{p}_1 = y^3 + \frac{3}{4}\Big(\frac{w_c''}{w_c'} - \frac{T_c'}{T_c} - \frac{1}{r_i}\Big)y^4 + a_2 y^5 + \cdots. \tag{C.13}$$

We now turn our attention to the second solution. Since the roots of (C.11) differ by an integer this implies the second solution will have the form

$$\tilde{p}_2 = B\ln y \sum_{n=0}^{\infty} a_n y^{n+3} + \sum_{n=0}^{\infty} b_n y^n, \tag{C.14}$$

where the coefficients a_n are defined by (C.12) and the constant B and coefficients b_n are to be determined. Substituting (C.14) into (C.2) and equating powers in y, from the $O(y)$ equation gives the relation

$$B = -\frac{2\beta}{3a_0}b_2. \tag{C.15}$$

Equating the other powers in y yields

$$b_0 = 1, \qquad b_1 = 0, \qquad b_2 = -\tfrac{\alpha^2}{2}b_0,$$

$$b_4 = \frac{\alpha^2}{4}\Big\{\Big[\frac{T_c''}{T_c} - \frac{2w_c'''}{3w_c'} - \frac{1}{2}\Big(\frac{w_c''}{w_c'}\Big)^2 - \Big(\frac{T_c'}{T_c}\Big)^2 - \frac{1}{r_i^2} - \frac{\alpha^2}{2} - M_\infty^2\frac{(w_c')^2}{T_c} + \frac{11}{12}\beta^2\Big]b_0 - \frac{12b_3\beta}{\alpha^2}\Big\},$$

$$b_5 = \frac{1}{10}\{-B[7a_2 + a_1\beta + \kappa a_0] - 4b_4\beta + (\alpha^2 - 3\kappa)b_3 + \alpha^2[\lambda - \epsilon]b_0\}. \tag{C.16}$$

Thus making use of (C.12) and (C.16), relation (C.15) can be rewritten as

$$B = \frac{\alpha^2\beta}{3}. \tag{C.17}$$

Inspection of (C.16) yields that there exists no means by which the coefficient b_3 can be determined. Consequently it is expected, as most, $b_3 = 1$. However, since the coefficient of the y^3 term in the regular solution is also unity, setting $b_3 = 1$ will only re-generate terms in the regular solution. Therefore we must have $b_3 = 0$.

Thus, in the limit $y \to 0$, $(r \to r_i)$ the second solution to (C.1) takes the form

$$\tilde{p}_2 = 1 - \frac{\alpha^2}{2}y^2 + \frac{\beta\alpha^2}{3}y^3 ln y - \frac{\beta^2\alpha^2}{4}y^4 ln y + b_4 y^4 + \cdots . \tag{C.18}$$

References

Abramowitz, M. and Stegun, I.A. (1965). Handbook of Mathematical functions. Dover Publications, Inc., New York.

Barry, M.D.J. and Ross, M.A.S. (1970). The flat plate boundary layer. Part 2. The effect of increasing thickness on stability, *J. Fluid Mech.* **43**, 813-818.

Bassom, A.P. and Gajjar, J.S.B. (1988). Non-stationary cross-flow vortices in three dimensional boundary layer flows, *Proc. R. Soc. Lond.* A **417**, 179-212.

Benney, D.J. and Bergeron, R.F. (1969). A new class of non-linear waves in parallel flows, *Stud. Appl. Maths* **48**, 181-204.

Benney, D.J. and Maslowe, S.A. (1975). The evolution in space and time of nonlinear waves in parallel shear flows, *Stud. Appl. Maths.* **54**, 181-205.

Bodonyi, R.J. and Smith, F.T. (1981). The upper branch stability of the Blasius boundary layer, including non-parallel flow effects, *Proc. R. Soc. Lond.* **A375**, 65-92.

Bodonyi, R.J., Smith, F.T. and Gajjar, J.S.B. (1983). Amplitude-dependent stability of boundary-layer flow with a strongly nonlinear critical layer, *I.M.A. J. Appl. Maths.* **30**, 1-19.

Bouthier, M. (1972). Stabilité linéaire des écoulements presque parallèles, *J. de Mécanique* **11**, 599-621.

Bouthier, M. (1973). Stabilité linéaire des écoulements presque parallèles, Partie II, *J. de Mécanique* **12**, 75-95.

Brown, S.N. and Stewartson, K. (1978a). The evolution of the critical layer of a

Rossby wave. Part II, *Geophys. Astrophys. Fluid Dyn.* **10**, 1-24.

Brown, S.N. and Stewartson, K. (1978b). The evolution of a small inviscid disturbance to a marginally unstable stratified shear flow: stage two, *Proc. R. Soc. Lond.* A **363**, 174-194.

Brown, S.N. and Stewartson, K. (1980). On the nonlinear reflection of a gravity wave at a critical level. Part 1, *J. Fluid Mech.* **100**, 577-595.

Brown, S.N. and Stewartson, K. (1982a). On the nonlinear reflection of a gravity wave at a critical level. Part 2, *J. Fluid Mech.* **115**, 217-230.

Brown, S.N. and Stewartson, K. (1982b). On the nonlinear reflection of a gravity wave at a critical level. Part 3, *J. Fluid Mech.* **115**, 231-250.

Brown, W.B. (1959). Numerical calculation of the stability of cross flow profiles in laminar boundary layers on a rotating disc and on a swept wing and an exact calculation of the stability of the Blasius velocity profile, Report No. NAI-59-5, Northrop Aircraft Inc., Hawthorne, L.A.

Brown, W.B. (1962). Exact numerical solutions of the complete linearized equations for the stability of compressible boundary layers, Norair Report No. NOR-62-15, Northrop Aircraft Inc., Hawthorne, L.A.

Bush, W.B. (1976). Axial incompressible viscous flow past a slender body of revolution, *Rocky Mountain J. Maths.* **6**, 527-550.

Carpenter, P.W. and Gajjar, J.S.B. (1990). A general theory for two and three dimensional wall-mode instabilities in boundary layers over isotropic and anisotropic compliant walls, *Theor. Comp. Fluid Dynamics* **1**, 349.

Chang, C.-L., Malik, M.R. and Hussaini, M.Y. (1990) AIAA Paper 90-1448.

Cheng, Sin-I (1953). On the stability of laminar boundary layer flow, *Quart. Appl. Math.* **11**, 346-350.

Churilov, S.M. and Shukhman, I.G. (1987). Nonlinear stability of a stratified shear flow: a viscous critical layer, *J. Fluid Mech.* **180**, 1-20.

Churilov, S.M. and Shukhman, I.G. (1988). Nonlinear stability of a stratified shear flow in a regime with an unsteady critical layer, *J. Fluid Mech.* **194**, 187-217.

Corner, D., Houston, D.J.R., and Ross, M.A.S. (1976). Higher eigenstates in boundary layer stability theory, *J. Fluid Mech.* **77**, 81-103.

Cowley, S. and Hall. P. (1990). On the instability of hypersonic flow past a wedge, *J. Fluid Mech.* **214**, 17-42.

Davis, R.E. (1969). On the high Reynolds number flow over a wavy boundary, *J. Fluid Mech.* **36**, 337-346.

De Villiers, J.M. (1975). Aymptotic solutions of the Orr-Sommerfeld equation, *Phil. Trans. Roy. Soc.* **A280**, 271-316.

Demetriades, A. (1958). An experimental investigation of the stability of the laminar hypersonic boundary layer, GALCIT Hypersonic Res. Project, Memo, 43.

Demetriades, A. (1960). An experiment on the stability of the laminar hypersonic boundary layer, *J. Fluid Mech.* **7**, 385-396.

Demetriades, A. (1977). Laminar boundary layer stability measurements at Mach 7 including wall temperature effects, AFOSR TR-77-1311.

Dougherty, N.S. and Fisher, D.F. (1980). Boundary layer transition on a 10^0 cone: wind tunnel/flight data correlations, AIAA Paper 80-0154 (also NASA TP 1971 1982).

Drazin, P.G. and Howard, L.N. (1966). Hydrodynamic stability of parallel flow of inviscid fluid, *Adv. Appl. Mech.* **9**, 1-89.

Drazin, P.G. and Reid, W.H. (1981). *Hydrodynamic Stability*, Cambridge Univ. Press, Cambridge.

Duck, P.W. (1990). The inviscid axisymmetric stability of the supersonic flow along a circular cylinder, *J. Fluid Mech.* **214**, 611-637 (also ICASE Report No. 89-19, 1989).

Duck, P.W. and Bodonyi, R.J. (1986). The wall jet on an axisymmetric body, *Quart.*

J. Mech. Appl. Maths. **39**, 467-483.

Duck, P.W. and Hall, P. (1989). On the interaction of Tollmien-Schlichting waves in axisymmetric supersonic flows, *Quart. J. Mech. Appl. Math.* **42**, 115-130 (also ICASE Report No. 88-10, 1988).

Duck, P.W. and Hall, P. (1990). Non-axisymmetric viscous lower branch modes in axisymmetric supersonic flows, *J. Fluid Mech.* **213** 191-201 (also ICASE Report No. 88-42, 1988).

Duck, P.W. and Shaw, S.J. (1990). The inviscid stability of supersonic flow past a shrap cone, *Theoret. Comput. Fluid Dynamics* **2**, 139-163.

Dunn, D.W. (1953). On the stability of the laminar boundary layer in a compressible fluid, Ph.D. Thesis, M.I.T.

Dunn, D.W. and Lin, C.C. (1955). On the stability of the laminar boundary layer in a compressible fluid, *J. Aero. Sci.* **22**, 455-477.

Eagles, P.M. (1969). Composite series in the Orr-Sommerfeld problem for symmetric channel flow, *Quart. J. Mech. Appl. Math.* **22**, 129-182.

Fjørtoft, R. (1950). Application of integral theorems in deriving criteria of stability for laminar flows and for the baroclinic circular vortex, *Geofys. Publ., Oslo* **17**, No. 6, 1-52.

Fraenkel, L.E. (1969). On the method of matched asymptotic expansions. Part I: A matching principle, *Proc. Camb. Phil. Soc.* **65**, 209-231.

Gajjar, J.S.B. (1991a). Nonlinear evolution of modes in compressible boundary layers, in *Proc. of the Royal Aero. Soc. Meeting on Boundary Layer Transition and Control*, Cambridge, 12.1.

Gajjar, J.S.B. (1991b). Nonlinear evolution of modes in the flow over compliant surfaces, *Recent Developments in Turbulence Management*, ed. K.-S. Choi, 223-239, Kluwer-Academic Publishers.

Gajjar, J.S.B. and Cole, J.W. (1989). The upper-branch stability of compressible boundary layer flows, *Theoret. Comput. Fluid Dynamics* **1**, 105-123.

Gajjar, J.S.B. and Smith, F.T. (1985). On the global instability of free disturbances with a time-dependent nonlinear viscous critical layer, *J. Fluid Mech.* **157**, 53-77.

Gaster, M. (1974). On the effects of boundary layer growth on flow stability, *J. Fluid Mech.* **66**, 465-480.

Gill, A.E. (1965). Instabilities of 'Top-hat' jets and wakes in compressible fluids, *Phys. Fluids* **8**, 1428-1430.

Glauert, M.B. and Lighthill, M.J. (1955). The axisymmetric boundary layer on a long thin cylinder, *Proc. R. Soc. Lond.* **A 230**, 188.

Goldstein, M.E. and Choi, S.-W. (1989). Nonlinear evolution of interacting oblique waves on two-dimensional shear layers, *J. Fluid Mech.* **207** 97-120.

Goldstein, M.E. and Durbin, P.A. (1986). Nonlinear critical layers eliminate the upper branch of spatially growing Tollmien-Schlichting waves, *Phys. Fluids* **29**, 2344-2345.

Goldstein, M.E., Durbin, P.A. and Leib, S.J. (1987). Roll-up of vorticity in adverse-pressure-gradient boundary layers, *J. Fluid Mech.* **183**, 325-342.

Goldstein, M.E. and Hultgren, L.S. (1988). Nonlinear spatial evolution of an externally excited instability wave in a free shear layer, *J. Fluid Mech.* **197**, 295-330.

Goldstein, M.E. and Leib, S.J. (1988). Nonlinear roll-up of externally excited free shear layers, *J. Fluid Mech.* **191**, 481-515.

Goldstein, M.E. and Leib, S.J. (1989). Nonlinear evolution of oblique waves on compressible shear layers, *J. Fluid Mech.* **207**, 73-96.

Goldstein, M.E. and Wundrow, D.W. (1990). Spatial evolution of nonlinear acoustic mode instabilities on hypersonic boundary layers, *J. Fluid Mech.* **219**, 585-607.

Goldstein, S. (1930). Concerning some equations of the boundary-layer equations in hydrodynamics, *Proc. Camb. Phil. Soc.* **26**, 1.

Graebel, W.P. (1966). On determination of the characterisitic equations for the stability of parallel flows, *J. Fluid Mech.* **24**, 497-508.

Gropengeisser, H. (1969). Study of the stability of boundary layers and compressible fluids, *Deutsche Luft- und Raunfahnt Rep.* DLR-FB-69-25, *NASA translations* TT-F-12, 786.

Grosch, C.E. and Salwen, H. (1978). The continuous spectrum of the Orr- Sommerfeld equation. Part I. The spectrum and the eigenfunctions, *J. Fluid Mech.* **87**, 33-54.

Haberman, R. (1972). Critical layers in parallel flows, *Stud. Appl. Maths.* **51**, 139-161.

Helmholtz, H. von (1868) Über discontinuirliche Flüssigkeitsbewegungen, *Monats. Königl. Preuss. Akad. Wiss. Berlin* **23**, 215-228. (also *Wissenschaftliche Abhandlungen* (1882) **I**, 146-157. Leipzig: J.A. Barth. Translated into English by F. Guthrie as 'On discontinuous movements of fluids', *Phil. Mag.* (4) **36**, 337-346 (1868)).

Heisenberg, W. (1924). Über Stabilität und Turbulenz von Flüssigkeitsströmen, *Ann. Phys. Lpz.(4)* **74**, 577-627.

Huerre, P. (1980). The nonlinear stability of a free shear layer in the viscous critical layer regime, *Phil. Trans. R. Soc. Lond.* A**293**, 643-675.

Heurre, P. and Scott, J.F. (1980). Effects of critical layer structure on the nonlinear evolution of waves in free shear layers, *Proc. R. Soc. London* **A 371**, 509-524.

Hickernell, F.J. (1984). Time-dependent critical layers in shear flows on the beta-plane, *J. Fluid Mech.* **142**, 431-449.

Jordinson, R. (1970). The flat plate boundary layer. Part 1. Numerical integration of the Orr-Sommerfeld equation, *J. Fluid Mech.* **43**, 801-811.

Jordinson, R. (1971). Spectrum of eigenvalues of the Orr-Sommerfeld equation for Blasius flow, *Phys. Fluids* **14**, 2535-2537.

Kaplan, R.E. (1964). The stability of laminar incompressible boundary layers in the presence of compliant boundaries, ASRL-TR-116-1, Cambridge Aeroelastic and

Structures Research Lab. Report M.I.T. (STAR N64-29052).

Kelvin, Lord. (1871). Hydrokinetic solutions and observations, *Phil. Mag.* (4) **42**, 362-377 (also *Mathematical and physical papers* (1910), vol. IV, 69-85, Cambridge University Press).

Kendall, J.M. (1967). Supersonic boundary layer stability experiments, *Proceedings of Boundary Layer Transition Study Group Metting* (W.D. McCauley, ed.), Aerospace Corp. Report No. BSD-TR-67-213, **II**.

Kendall, J.M. (1975). Wind tunnel experiments relating to supersonic and hypersonic boundary layer transition, AIAA **13** 290-299.

Kluwick, A., Gittler, P. and Bodonyi, R.J. (1984). Viscous-inviscid interactions on axisymmetric bodies of revolution in supersonic flow, *J. Fluid Mech.* **140**, 281-301.

Küchemann, D. (1938). Störungsbewegungen in einer Gasströmung mit Grenzschicht, ZAMM **18**, 207-222.

Kurtz, E.F., Jr. (1961). A study of the stability of laminar parallel flows, PhD. Thesis, M.I.T., Cambridge, M.A.

Kurtz, E.F., Jr. and Crandall, S.H. (1962). Computer-aided analysis of hydrodynamic stability, *J. Math. Phys.* **41**, 264-279.

Lakin, W.D., Ng, B.S. and Reid, W.H. (1978). Approximation to the eigenvalue relation for the Orr-Sommerfeld equation, *Phil. Trans. Roy. Soc.* **A289**, 347-371.

Lakin, W.D. and Reid, W.H. (1970). Stokes multipliers for the Orr-Sommerfeld equation, *Phil Trans. Roy. Soc.* **A268**, 325-349.

Langer, R.E. (1957). On the asymptotic solutions of a class of ordinary differential equations of the fourth order, with special reference to an equation of hydrodynamics, *Trans. Amer. Math. Soc.* **84**, 144-191.

Langer, R.E. (1959). Formal solutions and a related equation for a class of fourth order differential equations of a hydrodynamic type, *Trans. Amer. Math. Soc.* **92**, 371-410.

Laufer, J. and Vrebalovich, T. (1958). Stability of a supersonic laminar boundary layer of a flat plate, *Jet Prop. Lab. Cal. Inst. Tech. Rep.* No. 20-116.

Laufer, J. and Vrebalovich, T. (1960). Stability and transition of a supersonic laminar boundary layer of an insulated flat plate, *J. Fluid Mech.* **9**, 257-299.

Lees, L. (1947). The stability of the laminar boundary layer in a compressible fluid, NACA Tech. Report No. 876.

Lees, L. and Lin, C.C. (1946). Investigation of the stability of the laminar boundary layer in a compressible fluid, NACA Tech. Note No. 1115.

Lees, L. and Reshotko, E. (1962). Stability of the compressible laminar boundary layer, *J. Fluid Mech.* **12**, 555-590.

Leib, S.J. (1991). Nonlinear evolution of subsonic and supersonic disturbances on a compressible free shear layer, *J. Fluid Mech.* **224**, 551-578.

Lekoudis, S. (1980). Stability of the boundary layer on a swept wing with wall cooling, AIAA J. **18**, 1029-1035.

Lilly, D.K. (1966). On the instability of Ekman boundary layer flow, *J. Atmos. Sci.* **23**, 481-494.

Lin, C.C. (1945). On the stability of two-dimensional parallel flows, Parts I, II, III, *Quart. Appl. Math.* **3**, 117-142, 218-234, 277-301.

Lin, C.C. (1955). *The Theory of Hydrodynamic Stability*, Cambridge University Press, London and New York.

Lin, C.C. (1957a). On uniformly valid asymptotic solutions of the Orr-Sommerfeld equation, *Proceedings of 9th International Congress on Applied Mechanics* (Brussels), **I**, 136-48.

Lin, C.C. (1957b). On the stability of the laminar boundary layer, *Proceedings of the Symposium on Naval Hydrodynamics*, 353-71, (National Research Council publication 515), Washington DC.

Lin, C.C. (1958). On the stability of laminar flow and its transition to turbulence, *Proceedings of the Symposium on Boundary layer Research* (Freiburg), (H. Görtler, ed.), 144-60, Berlin: Springer-Verlag.

Lin, C.C. and Rabenstein, A.L. (1960). On the asymptotic solutions of a class of ordinary differential equations of the fourth order. I. Existence of regular formal solutions, *Trans. Amer. Math. Soc.* **94**, 24-57.

Lin, C.C. and Rabenstein, A.L. (1969). On the asymptotic theory of a class of ordinary differential equations of fourth order. II. Existence of solutions which are approximated by the formal solutions, *Studies in Appl. Math.* **48**, 311-40.

Mack, L.M. (1963). The inviscid stability of the compressible laminar boundary layer, Space Programs Summary, No. 37-23, 297, JPL, Pasadena, CA.

Mack, L.M. (1964). The inviscid stability of the compressible laminar boundary layer: Part II, Space Programs Summary, No. 37-26 **IV**, 165, JPL, Pasadena, CA.

Mack, L.M. (1965a). Computation of the stability of the laminar compressible boundary layer, in *Methods in Computational Physics* (B. Alder, S. Fernbach, and M. Rotenberg, eds.) **4**, 247-299, Academic Press, New York.

Mack, L.M. (1965b). Stability of the laminar boundary layer according to a direct numerical solution, AGARDograph **97**, Part I, 329-362.

Mack, L.M. (1969). Boundary layer stability theory, Document No. 900-277, Rev. A, JPL, Pasadena, CA.

Mack, L.M. (1976). A numerical study of the temporal eigenvalue spectrum of the Blasius boundary layer, *J. Fluid Mech.* **73**, 497-520.

Mack, L.M. (1977). Transition predication and linear stability theory, in AGARD Conference Proceedings No. 224, 1-13 to 1-22, NATO, Paris.

Mack, L.M. (1979a). On the stability of the boundary layer on a transonic swept wing, AIAA Paper No. 79-0264.

Mack, L.M. (1979b). Three-dimensional effects in boundary layer stability, in *Proceedings of Twelfth Symposium on Naval Hydrodynamics*, National Academy of Sciences, Washington, D.C., 63-76.

Mack, L.M. (1980). On the stabilization of laminar boundary layers by suction and cooling, *Proceedings of IUTAM Symposium on Laminar-Turbulent Transition*, Springer-Verlag, Berlin, 223-238.

Mack, L.M. (1982). Compressible boundary layer stability calculations for swept-back wings with suction, AIAA J. **20**, 363-369.

Mack, L.M. (1984). Boundary layer stability theory in 'Special Course on Stability and Transition of Laminar flow', AGARD Report No.709. 3-1.

Mack, L.M. (1987a). Review of linear compressible stability theory, in *Proc. ICASE Workshop on the Stability of Time Dependent and Spatially Varying Flows* (ed. D.L. Dwoyer and M.Y. Hussaini). Springer-Verlag. New York.

Mack L.M. (1987b). AIAA Paper 87-1413.

Mangler, W. (1946). Great Britain Ministry of Air Production, Volkernode Rept. and Trans., Vol. 55.

Maslowe, S.A. and Redekopp, L.G. (1979). Solitary waves in stratified shear flows, *Geophys. Astrophys. Fluid Dyn.* **13**, 185-196.

Maslowe, S.A. and Redekopp, L.G. (1980). Long nonlinear waves in stratified shear flows, *J. Fluid Mech.* **101**, 321-348.

Messiter, A.F. (1970). Boundary layer flow near the trailing edge of a flat plate, *SIAM J. Appl. Math.* **18**, 241.

Michalke, A. (1971). *Z. Flugwiss.* **19**, 319.

Moore, F.K. (1951). Unsteady laminar boundary layer flow. NACA. Tech. Note 2471.

Orr, W.McF. (1907). The stability or instability of the steady motions of a perfect liquid and of a viscous liquid. Part I: A perfect liquid, *Proc. R. Irish Acad.* **27 A**, 9-27, 69-138.

Osborne, M.R. (1967). Numerical methods for hydrodynamic stability problems, *SIAM J. Appl. Math.* **15**, 539-557.

Prandtl, L. (1921). Bemerkungen über die Entstehung der Turbulenz, *Z. angew. Math. Mech.* **1**, 431-6.

Pretsch, J. (1942). Die Anfachung instabler Störungen in einer laminaren Reibungsschicht, *Jb. deutsch. Luftfahrtf.*, 154-171.

Rayleigh, Lord (1880). On the stability, or instability, of certain fluid motions, in *Scientific Papers* **1**, 474-487, Cambridge Univ. Press, Cambridge.

Rayleigh, Lord (1887). On the stability, or instability, of certain fluid motions. II, in *Scientific Papers* **3**, 2-23, Cambridge Univ. Press, Cambridge.

Rayleigh, Lord (1892). On the question of the stability of the flow of fluids, in *Scientific Papers* **3**, 575-584, Cambridge Univ. Press, Cambridge.

Rayleigh, Lord (1895). On the stability or instability of certain fluid motions. III, in *Scientific Papers* **4**, 203-219, Cambridge Univ. Press, Cambridge.

Rayleigh, Lord (1913). On the stability of the laminar motion of an inviscid fluid, in *Scientific Papers* **6**, 197-204, Cambridge Univ. Press, Cambridge.

Rayleigh, Lord (1916). On the dynamics of revolving fluids, in *Scientific Papers* **6**, 447-53, Cambridge Univ. Press, Cambridge.

Reid, W.H. (1965). In *Basic developments in fluid dynamics* (ed. M. Holt) **1**, 249-307. New York: Academic Press.

Reid, W.H. (1972). Composite approximations to the solutions of the Orr-Sommerfeld equation, *Studies in Appl. Maths.* **51**, 341-368.

Redekopp, L.G. (1977). On the theory of solitary Rossby waves, *J. Fluid Mech.* **82**, 725-745.

Reshotko, E. (1960). Stability of the compressible laminar boundary layer, *Cal. Inst. Tech. Ph.D. Thesis.* (also *GALCIT Hypersonic Res. Project, Memo.* No. 52).

Reshotko, E. (1962). Stability of three dimensional compressible boundary layers, NASA Tech. Note No. D-1220

Reynolds, O. (1883). An experimental investigation of the circumstances which determine whether the motion of water shall be direct or sinuous, and of the law of resistance in parallel channels, *Phils. Trans. Roy. Soc.* **174**, 935-982.

Robinson, J.L. (1974). The inviscid nonlinear instability of parallel shear flows, *J. Fluid Mech.* **63**, 723-752.

Rosenhead, L. (1963). *Laminar Boundary Layers*, Oxford Univ. Press, Oxford.

Ross, J.A., Barnes, F.H., Burns, J.G. and Ross, M.A.S. (1970). The flat plate boundary layer. Part 3. Comparison of theory with experiment, *J. Fluid Mech.* **43**, 819-832.

Saric, W.S., and Nayfeh, A.H. (1975). Non-parallel stability of boundary layer flows, *Phys. Fluids* **18**, 945-950. 224, NATO, Paris.

Schade, H. (1964). Contribution to the nonlinear stability theory of inviscid shear layers, *Phys. Fluids* **7**, 623-628.

Schlichting, H. (1933a). Zur Entstehung der Turbulenz bei der Plattenstörmung, *Nachr. Ges. Wiss. Göttingen, Math.-Phys. Klasse*, 182-203.

Schlichting, H. (1933b). Berechnung der Anfachung kleiner Störungen bei der Plattenströmung, *Z. angew. Math. Mech.* **13**, 171-174.

Schlichting, H. (1935). Amplitudenverteilung und Energiebilanz der kleinen Störungen bei der Plattengrenzschicht, *Nachr. Ges. Wiss. Göttingen, Math.-Phys. Klasse*, 47-78.

Schlichting, H. (1940). Über die theoretische Berechnung der kritischen Reynoldschen Zahl einer Reibungsschicht in beschleunigter und verzögerter Strömung, *Jb. deutsch. Luftfahrtf.* **I**, 97-112.

Schubauer, G.B. and Skramstad, H.K. (1947). Laminar boundary layer oscillations and stability of laminar flow, *J. Aero. Sci.* **14**, 69-78.

Seban, R.A. and Bond, R. (1951). Skin friction and heat transfer characteristics of a laminar boundary layer on a cylinder in axial incompressible flow, *J. Aero. Sci.* **18**, 671-675.

Shaw, S.J. and Duck, P.W. (1992). The inviscid stability of supersonic flow past heated or cooled axisymmetric bodies, *Phys. Fluids*, to appear.

Shukham, I.G. (1989). Nonlinear stability of a weakly supercritical mixing layer in a rotating fluid, *J. Fluid Mech.* **200**, 425-450.

Shukham, I.G. (1991). Nonlinear evolution of spiral density waves generated by the instability of the shear layer in a rotating compressible fluid, *J. Fluid Mech.* **223**, 587-612.

Smith, F.T. (1979a). On the non-parallel flow stability of the Blasius boundary layer, *Proc. R. Soc. Lond.* **A366**, 91-109.

Smith, F.T. (1979b). Nonlinear stability of boundary layers for disturbances of various sizes, *Proc. R. Soc. Lond.* **A368**, 573-589.

Smith, F.T. and Burggraf, O.R. (1985). On the development of large-sized short-scaled disturbances in boundary layers, *Proc. R. Soc. Lond.* **A399**, 25-55.

Sommerfeld, A. (1908). Ein Beitrag zur hydrodynamischen Erklärung der turbulenten Flüssigkeitsbewegung, *Proc. 4th. Int. Congr. Math.*, Rome, 116-24.

Squire, H.B. (1933). On the stability of three dimensional disturbances of viscous flow between parallel walls, *Proc. Roy. Soc.* **A142**, 621-628.

Srokowski, A.J. and Orszag, S.A. (1977). Mass flow requirements for LFC wing design, AIAA Paper No. 77-1222.

Stetson. K.F., Thompson, E.R., Donaldson, J.C. and Siler, L.G. (1983). Laminar boundary layer stability experiments on a cone at Mach 8. Part I: Sharp cone, AIAA Paper No. 83-1761.

Stetson. K.F., Donaldson, J.C. and Thompson, E.R. (1984). Laminar boundary layer stability experiments on a cone at Mach 8. Part 2: Blunt cone, AIAA Paper

No. 84-0006.

Stewartson, K. (1951). On the impulsive motion of a flat plate in a viscous fluid, *Quart. J. Appl. Maths.* **4**, 182-198.

Stewartson, K. (1955). The asymptotic boundary layer on a circular cylinder in axial incompressible flow, *Quart. Appl. Math.* **13**, 113-122.

Stewartson, K. (1964). *The Theory of Laminar Boundary Layers in Compressible Fluids.* Oxford University Press, Oxford.

Stewartson, K. (1969). On the flow near the trailing edge of a flat plate - II, *Mathematika* **16**, 106.

Stewartson, K. (1978). The evolution of the critical layer of a Rossby wave, *Geophys. Astrophys. Fluid Dyn.* **9**, 185-200.

Stewartson, K. (1981). Marginally stable inviscid flows with critical layers, *I.M.A. J. Appl. Maths.* **27**, 133-175.

Stuart, J.T. (1960). On the nonlinear mechanics of wave disturbances in stable and unstable parallel flows. Part 1. The basic behaviour in plane Poiseuille flow, *J. Fluid Mech.* **9**, 353-370.

Taylor, G.I. (1915). Eddy motion in the atmosphere, *Phil. Trans. Roy. Soc* **A215**, 1-26.

Thomas, L.H. (1953). The stability of plane Poiseuille flow, *Phys. Rev.* **91**, 780-783.

Thompson, P.A. (1972). *Compressible Fluid Dynamics.* McGraw-Hill, New York.

Tietjens, O. (1925). Beiträge zur Entstehung der Turbulenz, *Z. angew Math. Mech.* **5**, 200-17.

Tollmien, W. (1929). Über die Entstehung der Turbulenz, *Nachr. Ges. Wiss. Göttingen. Math.-Phys. Klasse*, 21-44.

Tollmien, W. (1935). Ein allgemeines Kriterium der Instabiltät Laminarer Geschwdigkeitsverteilungen, *Nachr. Ges. Wiss. Göttingen, Math.-Phys. Klasse* **50**, 79-114.

Tollmien, W. (1947). Asymptoticshe Integration der Störungsdifferential-gleichung ebener laminarer Strömungen bei hohen Reynoldsschen Zahlen, *Z. angew. Math. Mech.*, 25/27, 33-50.

Troitskaya, Yu.I. (1991). The viscous-diffusion nonlinear critical layer in a stratified shear flow, *J. Fluid Mech.* **233**, 25-48.

Van Driest, E.R. (1952). Calculation of the stability of the laminar boundary layer in a compressible fluid on a flat plate with heat transfer, *J. Aero. Sci.* **19**, 801-12.

Van Dyke, M.D. (1964). *Perturbation methods in fluid dynamics.* New York: Academic Press.

Van Stijn, TH.L. and Van De Vooren, A.I. (1980). An accurate method for solving the Orr-Sommerfeld equation, *J. Eng. Math.* **14**, 17-26.

Ward, G.N. (1955). *Linearised Theory of High Speed Flow.* Cambridge University Press, Cambridge.

Warn, T. and Warn, H. (1978). The evolution of a nonlinear critical level, *Stud. Appl. Maths.* **59**, 37-71.

Wasow, W. (1948). The complex asymptotic theory of a fourth order differential equation of hydrodynamics, *Ann. Math.* (2) **49**, 852-871.

Wasow, W. (1953). Asymptotic solution of the differential equation of hydrodynamic stability in a domain containing a transition point, *Ann. Math.* **58**, 222-252.

Watson, J. (1960). On the non-linear mechanics of wave disturbances in stable and unstable parallel flows. Part2, *J. Fluid Mech.* **9** 371-389.

Wazzan, A.R., Okamura, T.T., amd Smith. A.M.D. (1968). Spatial and temporal stability charts for the Falkner-Skan boundary layer profiles, Report No. DAC-67086, McDonnel-Douglas Aircraft Co., Long Beach, C.A.

Zaat, J.A. (1958). Numerische Beiträge zur Stabilitätstheorie der Grenzschichten, *Proc. Symp. boundary layer Res.*, IUTAM, 127-138, Springer, Berlin.